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Adapted by

Amit Gupta

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CHAPTER 1

Units and Measurement

1.1 | WHAT IS PHYSICS?

Science and engineering are based on measurements and comparisons. Thus, we need rules about how things are measured and compared, and we need experiments to establish the units for those measurements and comparisons. One purpose of physics (and engineering) is to design and conduct those experiments.

For example, physicists strive to develop clocks of extreme accuracy so that any time or time interval can be precisely determined and compared. You may wonder whether such accuracy is actually needed or worth the effort. Here is one example of the worth: Without clocks of extreme accuracy, the Global Positioning System (GPS) that is now vital to worldwide navigation would be useless.

1.2 | MEASURING THINGS

Key Concept

- Physics is based on measurement of physical quantities. Certain physical quantities have been chosen as base quantities (such as length, time, and mass); each has been defined in terms of a standard and given a unit of measure (such as meter, second, and kilogram). Other physical quantities are defined in terms of the base quantities and their standards and units.

We discover physics by learning how to measure the quantities involved in physics. Among these quantities are length, time, mass, temperature, pressure, and electric current.

We measure each physical quantity in its own units, by comparison with a **standard**. The **unit** is a unique name we assign to measures of that quantity—for example, meter (m) for the quantity length. The standard corresponds to exactly 1.0 unit of the quantity. As you will see, the standard for length, which corresponds to exactly 1.0 m, is the distance traveled by light in a vacuum during a certain fraction of a second. We can define

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a unit and its standard in any way we care to. However, the important thing is to do so in such a way that scientists around the world will agree that our definitions are both sensible and practical.

Once we have set up a standard—say, for length—we must work out procedures by which any length whatsoever, be it the radius of a hydrogen atom, the wheelbase of a skateboard, or the distance to a star, can be expressed in terms of the standard. Rulers, which approximate our length standard, give us one such procedure for measuring length. However, many of our comparisons must be indirect. You cannot use a ruler, for example, to measure the radius of an atom or the distance to a star.

Base Quantities. There are so many physical quantities that it is a problem to organize them. Fortunately, they are not all independent; for example, speed is the ratio of a length to a time. Thus, what we do is pick out—by international agreement—a small number of physical quantities, such as length and time, and assign standards to them alone. We then define all other physical quantities in terms of these *base quantities* and their standards (called *base standards*). Speed, for example, is defined in terms of the base quantities length and time and their base standards.

The base physical quantities from which all other physical quantities can be derived are known as **fundamental quantities**. All physical quantities, which can be derived by suitable multiplication and division of fundamental quantities are called derived quantities.

Base standards must be both accessible and invariable. If we define the length standard as the distance between one's nose and the index finger on an outstretched arm, we certainly have an accessible standard—but it will, of course, vary from person to person. The demand for precision in science and engineering pushes us to aim first for invariability. We then exert great effort to make duplicates of the base standards that are accessible to those who need them.

1.3 | THE INTERNATIONAL SYSTEM OF UNITS

Key Concepts

- The unit system emphasized in this book is the International System of Units (SI). The three physical quantities displayed in Table 1-1 are used in the early chapters. Standards, which must be both accessible and invariable, have been established for these base quantities by international agreement. These standards are used in all physical measurement, for both the base quantities and the quantities derived from them. Scientific notation and the prefixes of Table 1-2 are used to simplify measurement notation.
- Conversion of units may be performed by using chain-link conversions in which the original data are multiplied successively by conversion factors written as unity and the units are manipulated like algebraic quantities until only the desired units remain.

In 1971, the 14th General Conference on Weights and Measures picked seven quantities as base quantities, there by forming the basis of the International System of Units, abbreviated SI from its French name and popularly known as the *metric system**. The units for the three base quantities—length, mass, and time—that we use in the early chapters of this book, were defined to be on a "human scale."

Any unit which cannot be expressed in terms of other units is called **fundamental unit** (e.g., unit of mass, length and time in mechanics). Other units which can be expressed in terms of fundamental units, are called **derived units**. Table 1-1 represents base units of seven fundamental quantities in SI units. There are also two supplementary quantities in addition to the seven fundamental quantities. They are:

1. **Plane angle:** It is equal to the ratio of length of arc to the radius and its unit is radian. Mathematically, plane angle is given by relation

$$\text{Plane angle, } d\theta = \frac{ds}{r}$$

where, ds is the length of the arc, r is the radius of the circle and $d\theta$ is the plane angle.

Table 1-1 Units for Three SI Base Quantities

Quantity	Unit Name	Unit Symbol
Length	meter	m
Time	second	s
Mass	kilogram	kg
Temperature	kelvin	K
Current	ampere	A
Luminous intensity	candela	Cd
Amount of substance	mole	mol

*In November 2018, the 26th General Conference on Weights and Measures has adopted a resolution to redefine four of the seven base units—the kilogram, kelvin, mole and ampere.

2. **Solid angle:** It is equal to the ratio of the intercepted area of the spherical surface described about the apex as the center, to the square of its radius r. The SI unit of solid angle is **steradian** and its symbol is **sr**. Mathematically, solid angle is given by the relation

$$d\Omega = \frac{dA}{r^2}$$

where dA is the intercepted area of the sphere, r is the radius of the sphere and $d\Omega$ is the solid angle. Both radian and steradian are dimensionless quantities.

Many SI *derived units* are defined in terms of these base units. For example, the SI unit for power, called the **watt** (W), is defined in terms of the base units for mass, length, and time. Thus, as you will see in Chapter 8,

$$1 \text{ watt} = 1 \text{ W} = 1 \text{ kg} \cdot \text{m}^2/\text{s}^3, \tag{1-1}$$

where the last collection of unit symbols is read as kilogram-meter squared per second cubed.
The other common systems of units are:

1. **CGS system:** The system is also called Gaussian system of units. In this system, length, mass, and time have been chosen as the fundamental quantities and their corresponding fundamental units are centimeter (cm), gram (g), and second (s), respectively.
2. **MKS system:** The system is also called Giorgi system. In this system also, length, mass, and time have been taken as fundamental quantities, and their corresponding fundamental units are meter, kilogram, and second.
3. **FPS system:** In this system, foot, pound, and second are used for measurements of length, mass, and time, respectively.

Scientific Notation

To express the very large and very small quantities we often run into in physics, we use **scientific notation**, which employs powers of 10. In this notation,

$$3\,560\,000\,000 \text{ m} = 3.56 \times 10^9 \text{ m} \tag{1-2}$$

and

$$0.000\,000\,492 \text{ s} = 4.92 \times 10^{-S} \text{ s}. \tag{1-3}$$

Scientific notation on computers sometimes takes on an even briefer look, as in 3.56 E9 and 4.92 E–7, where E stands for "exponent of ten." It is briefer still on some calculators, where E is replaced with an empty space.

As a further convenience when dealing with very large or very small measurements, we use the prefixes listed in Table 1-2. As you can see, each prefix represents a certain power of 10, to be used as a multiplication factor. Attaching a prefix to an SI unit has the effect of multiplying by the associated factor. Thus, we can express a particular electric power as

$$1.27 \times 10^9 \text{ watts} = 1.27 \text{ gigawatts} = 1.27 \text{ GW} \tag{1-4}$$

or a particular time interval as

$$2.35 \times 10^{-9} \text{ s} = 2.35 \text{ nanoseconds} = 2.35 \text{ ns}. \tag{1-5}$$

Some prefixes, as used in milliliter, centimeter, kilogram, and megabyte, are probably familiar to you.

Table 1-2 Prefixes for SI Units

Factor	Prefix[a]	Symbol
10^{24}	yotta-	Y
10^{21}	zetta-	Z
10^{18}	exa-	E
10^{15}	peta-	P
10^{12}	tera-	T
10^{9}	**giga-**	**G**
10^{6}	**mega-**	**M**
10^{3}	**kilo-**	**k**
10^{2}	hecto-	h
10^{1}	deka-	da
10^{-1}	deci-	d
10^{-2}	**centi-**	**c**
10^{-3}	**milli-**	**m**
10^{-6}	**micro-**	**μ**
10^{-9}	**nano-**	**n**
10^{-12}	**pico-**	**p**
10^{-15}	femto-	f
10^{-18}	atto-	a
10^{-21}	zepto-	z
10^{-24}	yocto-	y

[a]The most frequently used prefixes are shown in bold type.

Changing Units

We often need to change the units in which a physical quantity is expressed. We do so by a method called *chain-link conversion*. In this method, we

multiply the original measurement by a **conversion factor** (a ratio of units that is equal to unity). For example, because 1 min and 60 s are identical time intervals, we have

$$\frac{1\ \text{min}}{60\ \text{s}} = 1 \quad \text{and} \quad \frac{60\ \text{s}}{1\ \text{min}} = 1.$$

Thus, the ratios (1 min)/(60 s) and (60 s)/(1 min) can be used as conversion factors. This is *not* the same as writing $\frac{1}{60} = 1$ or $60 = 1$; each *number* and its *unit* must be treated together.

Because multiplying any quantity by unity leaves the quantity unchanged, we can introduce conversion factors wherever we find them useful. In chain-link conversion, we use the factors to cancel unwanted units. For example, to convert 2 min to seconds, we have

$$2\,\text{min} = (2\ \text{min})(1) = \left(2\ \cancel{\text{min}}\right)\left(\frac{60\ \text{s}}{1\ \cancel{\text{min}}}\right) = 120\ \text{s}. \tag{1-6}$$

If you introduce a conversion factor in such a way that unwanted units do *not* cancel, invert the factor and try again. In conversions, the units obey the same algebraic rules as variables and numbers.

Appendix D gives conversion factors between SI and other systems of units, including non-SI units still used in the United States. However, the conversion factors are written in the style of "1 min = 60 s" rather than as a ratio. So, you need to decide on the numerator and denominator in any needed ratio.

1.4 | FUNDAMENTAL SI QUANTITIES

Key Concepts

- The meter is defined as the distance traveled by light during a precisely specified time interval.
- The second is defined in terms of the oscillations of light emitted by an atomic (cesium-133) source. Accurate time signals are sent worldwide by radio signals keyed to atomic clocks in standardizing laboratories.
- The kilogram is defined in terms of a platinum–iridium standard mass kept near Paris. For measurements on an atomic scale, the atomic mass unit, defined in terms of the atom carbon-12, is usually used.
- The density ρ of a material is the mass per unit volume:

$$\rho = \frac{m}{V}.$$

Length

In 1792, the newborn Republic of France established a new system of weights and measures. Its cornerstone was the meter, defined to be one ten-millionth of the distance from the north pole to the equator. Later, for practical reasons, this Earth standard was abandoned and the meter came to be defined as the distance between two fine lines engraved near the ends of a platinum–iridium bar, the **standard meter bar**, which was kept at the International Bureau of Weights and Measures near Paris. Accurate copies of the bar were sent to standardizing laboratories throughout the world. These **secondary standards** were used to produce other, still more accessible standards, so that ultimately every measuring device derived its authority from the standard meter bar through a complicated chain of comparisons.

Eventually, a standard more precise than the distance between two fine scratches on a metal bar was required. In 1960, a new standard for the meter, based on the wavelength of light, was adopted. Specifically, the standard for the meter was redefined to be 1 650 763.73 wavelengths of a particular orange-red light emitted by atoms of krypton-86 (a particular isotope, or type, of krypton) in a gas discharge tube that can be set up anywhere in the world. This awkward number of wavelengths was chosen so that the new standard would be close to the old meter-bar standard.

By 1983, however, the demand for higher precision had reached such a point that even the krypton-86 standard could not meet it, and in that year a bold step was taken. The meter was redefined as the distance traveled by light in a specified time interval. In the words of the 17th General Conference on Weights and Measures:

The meter is the length of the path traveled by light in a vacuum during a time interval of 1/299 792 458 of a second.

This time interval was chosen so that the speed of light c is exactly

$$c = 299\,792\,458 \text{ m/s}.$$

Measurements of the speed of light had become extremely precise, so it made sense to adopt the speed of light as a defined quantity and to use it to redefine the meter.

Table 1-3 shows a wide range of lengths, from that of the universe (top line) to those of some very small objects.

Table 1-3 Some Approximate Lengths

Measurement	Length in Meters
Distance to the first galaxies formed	2×10^{26}
Distance to the Andromeda galaxy	2×10^{22}
Distance to the nearby star Proxima Centauri	4×10^{16}
Distance to Pluto	6×10^{12}
Radius of Earth	6×10^{6}
Height of Mt. Everest	9×10^{3}
Thickness of this page	1×10^{-4}
Length of a typical virus	1×10^{-8}
Radius of a hydrogen atom	5×10^{-11}
Radius of a proton	1×10^{-15}

However, the following practical units of length are also conveniently used and are expressed in terms of SI system of units.

1. **Micron** is a small unit for measurement of length. 1 micron = 1μm = 10^{-6} m
2. **Angstrom** is a unit of length in which the size of an atom is measured and is used in atomic physics. 1 Angstrom = 1Å = 10^{-10} m.
3. **Light year** is a unit of distance travelled by light in 1 year free space and is used in astrophysics. 1 Light year = 3×10^{8} m/s × 365 × 24 × 60 × 60 = 9.5×10^{15} m.
4. **Fermi** is a unit of distance which the size of a nucleus is measured. 1 Fermi = 10 m^{-15}

PROBLEM-SOLVING TACTICS

Tactic 1: Order of Magnitude The order of magnitude of a number is the power of 10 when the number is expressed in scientific notation. For example, if $A = 2.3 \times 10^{4}$ and $B = 7.8 \times 10^{4}$, then the orders of magnitude of both A and B are 4. Often the result of calculation is estimated to the *nearest* order of magnitude. For example, the nearest order of magnitude is 4 for A and 5 for B. Such estimation is common when detailed or precise data required in the calculation are not known or easily found. Worked Problem 1.1 is a perfect case.

Measurement of Length

Both direct and indirect methods can be used for measurement of length. For measurement of length, we use both direct and indirect methods. Direct methods involve use of measuring instruments such as a meter scale, Vernier caliper, a spherometer, and a screw gauge to measure different ranges of length. For example, a meter scale is used to measure lengths ranging between 0.001 m and 100 m; Vernier caliper to measure lengths up to 0.0001 m, and screw gauge and spherometer to measure lengths up to 0.00001 m. To measure length beyond these limits we use indirect methods that include the parallax method for measuring large astronomical distances and method for measurement of very small distances such as size of molecules.

Measurement of Large Distances (Parallax Method)

To understand parallax, let us perform an experiment: Take a pencil and hold it in front of our eyes. First, let us close our left eye and observe the position of the pencil carefully. Next, let us close our right eye and observe the position of the pencil. We can observe that there is a shift in the position of the pencil when we view it with a different eye. This shift in position of an object, say, a pencil, when viewed from two different eyes, keeping one eye closed, is known as **parallax**.

Let us use the parallax method to determine the distance between the Earth and the Moon. In Fig. 1-1, consider C as the center of the Earth and M as the center of the Moon. Then AB = b represents the diameter of Earth and MC = D represents the distance between Earth and Moon.

Let A and B be two diametrically opposite points on the surface of the Earth. From A and B, the parallaxes θ_1 and θ_2, respectively, of the center of the Moon M with respect to some distant star are determined with the help of an astronomical telescope.

Thus, the total parallax the Moon subtends on Earth is $\theta = \theta_1 + \theta_2$ (say). If θ is measured in radians, then

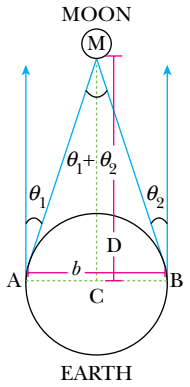


Figure 1-1 Schematic representation of measurement of the distance between the Moon and the Earth using the parallax method.

$$\theta = \frac{\text{AB}}{\text{AM}}.$$

Also, AM ≈ MC. Therefore,

$$\theta = \frac{\text{AB}}{\text{MC}} = \frac{b}{D}$$

or,

$$D = \frac{b}{\theta}$$

Knowing the values of b and θ, we can calculate the distance D of the Moon from the Earth.

The parallax method is used to determine distances of nearby stars. As the distance of the star from the Earth increases, the parallax angle decreases.

Estimation of Very Small Distances (Size of Molecules)

To measure a very small distance, we use optical microscopes in which the wave nature of light is exploited. Electron microscope is an example of such devices. In an electron microscope, a highly energized electron beam is used instead of visible light, and the beam is focused properly by electric and magnetic fields. The resolution of an electron microscope is of the order of 0.6 Å and it can be used to measure the size of atoms and molecules in a given material. More recently, tunneling microscopy has been developed in which the limit of resolution is even better and distances less than an angstrom (1 Å = 10^{-10} m) can be measured.

SAMPLE PROBLEM 1.01

Radius of the Earth by parallax method

Suppose that while lying on a beach watching the Sun set over a calm ocean, you start a stopwatch just as the top of the Sun disappears. You then stand, elevating your eyes by a height h = 1.70 m, and stop the watch when the top of the Sun again disappears. If the elapsed time on the watch is t = 11.1 s, what is the radius r of Earth?

KEY IDEA

Just as the Sun disappears, your line of sight to the top of the Sun is tangent to Earth's surface at point A while lying and point B while standing as shown in Fig. 1-2. Your eyes are located at point A while you are lying, and at height h above point A while you are standing.

Calculation: Let d represent the distance between point B and the location of your eyes when you are standing, and r the radius of Earth. From Pythagorean theorem, we have

$$d^2 + r^2 = (r+h)^2 = r^2 + 2rh + h^2$$
$$d^2 = 2rh + h^2. \tag{1-7}$$

Because the height h is so much smaller than Earth's radius r, the term h^2 is negligible compared to the term $2rh$, and we can rewrite Eq. 1-7 as

$$d^2 = 2rh. \tag{1-8}$$

In Fig. 1-2, the angle between the radii to the two-tangent points A and B is θ, which is also the angle through which the Sun moves about Earth during the measured time t = 11.1 s. During a full day, which is approximately 24 h, the Sun moves through an angle of 360° about Earth. Thus, we can write

$$\frac{\theta}{360^\circ} = \frac{t}{24\text{ h}}$$

t = 11.1 s gives us

$$\theta = \frac{(360^\circ)(11.1\text{ s})}{(24\text{ h})(60\text{ min/h})(60\text{ s/min})} = 0.04625^\circ.$$

In Fig. 1-2, we see that $d = r\tan\theta$. Substituting this for d in Eq. 1-8 gives us

$$r^2\tan^2\theta = 2rh.$$
$$r = \frac{2h}{\tan^2\theta}.$$

Substituting $\theta = 0.04625^\circ$ and $h = 1.70$ m, we find

$$r = \frac{(2)(1.70\text{ m})}{\tan^2 0.04625^\circ} = 5.22\times10^6\text{ m}. \qquad \text{(Answer)}$$

This radius is within 20% of the accepted value (6.37×10^6 m) for the (mean) radius of Earth.

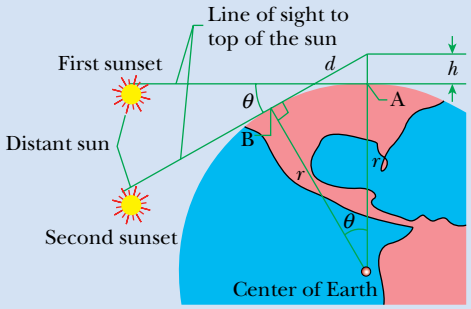


Figure 1-2 Your line of sight to the top of the setting Sun rotates through the angle θ when you stand up at point A and elevate your eyes by a distance h. (Angle θ and distance h are exaggerated here for clarity.)

SAMPLE PROBLEM 1.02

Estimating order of magnitude, ball of string

The world's largest ball of string is about 2 m in radius. To the nearest order of magnitude, what is the total length L of the string in the ball?

KEY IDEA

We could, of course, take the ball apart and measure the total length L, but that would take great effort and make the ball's builder most unhappy. Instead, because we want only the nearest order of magnitude, we can estimate any quantities required in the calculation.

Calculation: Let us assume the ball is spherical with radius R = 2 m. The string in the ball is not closely packed (there are uncountable gaps between adjacent sections of string). To allow for these gaps, let us somewhat overestimate the cross-sectional area of the string by assuming the cross section is square, with an edge length d = 4 mm. Then, with a cross-sectional area of d^2 and a length L, the string occupies a total volume of

$$V = (\text{cross-sectional area})(\text{length}) = d^2 L.$$

This is approximately equal to the volume of the ball, given by $\frac{4}{3}\pi R^3$, which is about $4R^3$ because π is about 3. Thus, we have the following:

$$d^2 L = 4R^3,$$

or
$$L = \frac{4R^3}{d^2} = \frac{4(2\text{ m})^3}{(4\times10^{-3}\text{ m})^2}$$
$$= 2\times10^6\text{ m} \approx 10^6\text{ m} = 10^3\text{ km}. \qquad \text{(Answer)}$$

To the nearest order of magnitude, the ball contains about 1000 km of string!

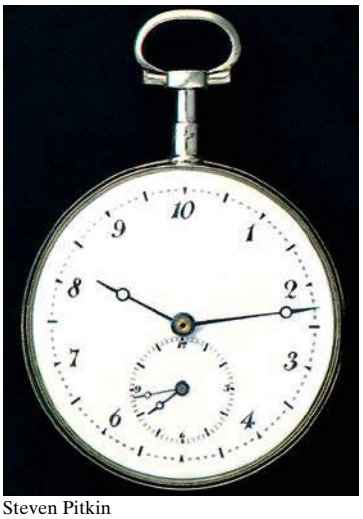
Time

Time has two aspects. For civil and some scientific purposes, we want to know the time of day so that we can order events in sequence. In much scientific work, we want to know how long an event lasts. Thus, any time standard must be able to answer two questions: "*When* did it happen?" and "What is its *duration*?" Table 1-4 shows some time intervals.

Table 1-4 Some Approximate Time Intervals

Measurement	Time Interval in Seconds	Measurement	Time Interval in Seconds
Lifetime of the proton (predicted)	3×10^{40}	Time between human heartbeats	8×10^{-1}
Age of the universe	5×10^{17}	Lifetime of the muon	2×10^{-6}
Age of the pyramid of Cheops	1×10^{11}	Shortest lab light pulse	1×10^{-16}
Human life expectancy	2×10^{9}	Lifetime of the most unstable particle	1×10^{-23}
Length of a day	9×10^{4}	The Planck time[a]	1×10^{-43}

[a]This is the earliest time after the big bang at which the laws of physics as we know them can be applied.



Steven Pitkin

Figure 1-3 When the metric system was proposed in 1792, the hour was redefined to provide a 10-hour day. The idea did not catch on. The maker of this 10-hour watch wisely provided a small dial that kept conventional 12-hour time. Do the two dials indicate the same time?

Any phenomenon that repeats itself is a possible time standard. Earth's rotation, which determines the length of the day, has been used in this way for centuries; Fig. 1-3 shows one novel example of a watch based on that rotation. A quartz clock, in which a quartz ring is made to vibrate continuously, can be calibrated against Earth's rotation via astronomical observations and used to measure time intervals in the laboratory. However, the calibration cannot be carried out with the accuracy called for by modern scientific and engineering technology.

To meet the need for a better time standard, atomic clocks have been developed. An atomic clock at the National Institute of Standards and Technology (NIST) in Boulder, Colorado, is the standard for Coordinated Universal Time (UTC) in the United States. Its time signals are available by shortwave radio (stations WWV and WWVH) and by telephone (303-499-7111). Time signals (and related information) are also available from the United States Naval Observatory at website http://tycho.usno.navy.mil/time.html. (To set a clock extremely accurately at your particular location, you would have to account for the travel time required for these signals to reach you.)

Figure 1-4 shows variations in the length of one day on Earth over a 4-year period, as determined by comparison with a cesium (atomic) clock. Because the variation displayed by Fig. 1-4 is seasonal and repetitious, we suspect the rotating Earth when there is a difference between Earth and atom as timekeepers. The variation is due to tidal effects caused by the Moon and to large-scale winds.

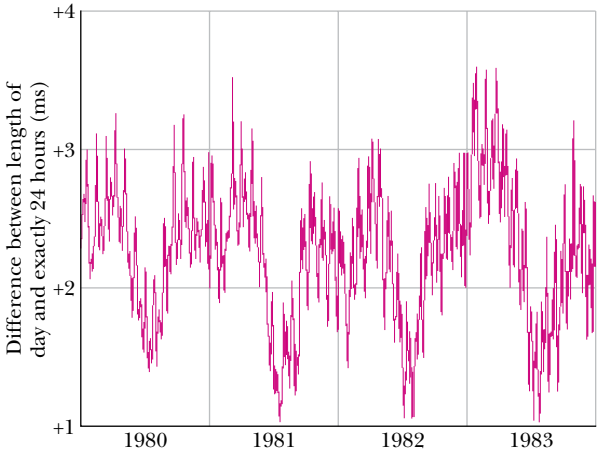


Figure 1-4 Variations in the length of the day over a 4-year period. Note that the entire vertical scale amounts to only 3 ms (= 0.003 s).

The 13th General Conference on Weights and Measures in 1967 adopted a standard second based on the cesium clock:

One second is the time taken by 9 192 631 770 oscillations of the light (of a specified wavelength) emitted by a cesium-133 atom.

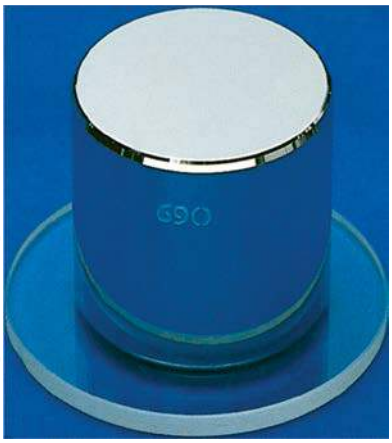
Atomic clocks are so consistent that, in principle, two cesium clocks would have to run for 6000 years before their readings would differ by more than 1 s. Even such accuracy pales in comparison with that of clocks currently being developed; their precision may be 1 part in 10^{18}—that is, 1 s in 1×10^{18} s (which is about 3×10^{10} y).

Mass

The Standard Kilogram

The SI standard of mass is a cylinder of platinum and iridium (Fig. 1-5) that is kept at the International Bureau of Weights and Measures near Paris and assigned, by international agreement, a mass of 1 kilogram. Accurate copies have been sent to standardizing laboratories in other countries, and the masses of other bodies can be determined by balancing them against a copy. Table 1-5 shows some masses expressed in kilograms, ranging over about 83 orders of magnitude.

The U.S. copy of the standard kilogram is housed in a vault at NIST. It is removed, no more than once a year, for the purpose of checking duplicate copies that are used elsewhere. Since 1889, it has been taken to France twice for recomparison with the primary standard.



Courtesy Bureau International des Poids et Mesures. Reproduced with permission of the BIPM.

Figure 1-5 The international 1 kg standard of mass, a platinum–iridium cylinder 3.9 cm in height and in diameter.

Table 1-5 Some Approximate Masses

Object	Mass in Kilograms
Known universe	1×10^{53}
Our galaxy	2×10^{41}
Sun	2×10^{30}
Moon	7×10^{22}
Asteroid Eros	5×10^{15}
Small mountain	1×10^{12}
Ocean liner	7×10^{7}
Elephant	5×10^{3}
Grape	3×10^{-3}
Speck of dust	7×10^{-10}
Penicillin molecule	5×10^{-17}
Uranium atom	4×10^{-25}
Proton	2×10^{-27}
Electron	9×10^{-31}

A Second Mass Standard

The masses of atoms can be compared with one another more precisely than they can be compared with the standard kilogram. For this reason, we have a second mass standard. It is the carbon-12 atom, which, by international agreement, has been assigned a mass of 12 **atomic mass units** (u). The relation between the two units is

$$1 \text{ u} = 1.660\,538\,86 \times 10^{-27} \text{ kg}, \qquad (1\text{-}9)$$

with an uncertainty of ±10 in the last two decimal places. Scientists can, with reasonable precision, experimentally determine the masses of other atoms relative to the mass of carbon-12. What we presently lack is a reliable means of extending that precision to more common units of mass, such as a kilogram.

Ampere

Consider two long straight wires with negligible cross-section placed parallel to each other in vacuum. Their separation is 1 m with electric currents established in the two in the same direction. It is found that they attract each other. The **ampere** is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 meter apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} newton per meter of length.

Kelvin

The fraction 1/273.16 of the thermodynamic temperature of triple point of water is called **1 Kelvin**.

Mole

The **mole** is the amount of substance in a system, which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon-12.

The number of atoms in 1 mole (i.e., the number of atoms in 0.012 kg of carbon-12) is called **Avogadro constant** and its best value available is 6.022045×10^{23} with an uncertainty of about 0.000031×10^{23}.

Candela

The **candela** is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} Hz and that has a radiant intensity in that direction of 1/683 W per steradian.

1.5 | SIGNIFICANT FIGURES AND DECIMAL PLACES

Key Concepts

- Significant figures set the number of digits that can be used in reporting the final answer.
- The results of calculations are rounded-off to match the least number of significant figures in the given data.

Suppose that you work out a problem in which each value consists of two digits. Those digits are called **significant figures** and they set the number of digits that you can use in reporting your final answer. With data given in two significant figures, your final answer should have only two significant figures. However, depending on the mode setting of your calculator, many more digits might be displayed. Those extra digits are meaningless.

In this book, final results of calculations are often rounded to match the least number of significant figures in the given data. (However, sometimes an extra significant figure is kept.) When the leftmost of the digits to be discarded is 5 or more, the last remaining digit is rounded up; otherwise it is retained as is. For example, 11.3516 is rounded to three significant figures as 11.4 and 11.3279 is rounded to three significant figures as 11.3. (The answers to sample problems in this book are usually presented with the symbol = instead of ≈ even if rounding is involved.)

When a number such as 3.15 or 3.15×10^3 is provided in a problem, the number of significant figures is apparent, but how about the number 3000? Is it known to only one significant figure (3×10^3)? Or is it known to as many as four significant figures (3.000×10^3)? In this book, we assume that all the zeros in such given numbers as 3000 are significant, but you had better not make that assumption elsewhere.

Don't confuse *significant figures* with *decimal places*. Consider the lengths 35.6 mm, 3.56 m, and 0.00356 m. They all have three significant figures but they have one, two, and five decimal places, respectively.

Larger the number of significant figures obtained in a measurement, greater is the accuracy of the measurement. The reverse is also true. The following rules are observed in counting the number of significant figures in a given measured quantity:

1. **All non-zero digits are significant.** For example, 42.3 has three significant figures; 243.4 has four significant figures; and 24.123 has five significant figures.
2. **A zero becomes significant figure if it appears between two non-zero digits.** For example, 5.03 has three significant figures; 5.604 has four significant figures; and 4.004 has four significant figures.
3. **Leading zeros or the zeros placed to the left of the number are never significant.** For example, 0.543 has three significant figures; 0.045 has two significant figures; and 0.006 has one significant figure.
4. **Trailing zeros or the zeros placed to the right of the number are significant.** For example, 4.330 has four significant figures; 433.00 has five significant figures; and 343.000 has six significant figures.
5. **In exponential notation, the numerical portion gives the number of significant figures.** For example, 1.32×10^{-2} has three significant figures and 1.32×10^4 has three significant figures.

Rounding Off

To reduce the number of significant digits in the measurement of a physical quantity, rounding off is used. After rounding off, generally, we express the measurement in one less number of significant digits as there are in original measurement before rounding off. While rounding off the measurement values, we use the following rules by convention:

1. **If the digit to be dropped is less than 5, then the preceding digit is left unchanged.** For example, $x = 7.82$ is rounded off to 7.8 and $x = 3.94$ is rounded off to 3.9.
2. **If the digit to be dropped is more than 5, then the preceding digit is raised by one.** For example, $x = 6.87$ is rounded off to 6.9 and $x = 12.78$ is rounded off to 12.8.
3. **If the digit to be dropped is 5 followed by digits other than zero, then the preceding digit is raised by one.** For example, $x = 16.351$ is rounded off to 16.4 and $x = 6.758$ is rounded off to 6.8.
4. **If the digit to be dropped is 5 or 5 followed by zeros, then the preceding digit, if it is even, is left unchanged.** For example, $x = 3.250$ becomes 3.2 on rounding off and $x = 12.650$ becomes 12.6 on rounding off.
5. **If the digit to be dropped is 5 or 5 followed by zeros, then the preceding digit, if it is odd, is raised by one.** For example, $x = 3.750$ is rounded off to 3.8, again $x = 16.150$ is rounded off to 16.2.

Significant Figures in Calculation

The following two rules should be followed to obtain the proper number of significant figures in any calculation.

Rule for Addition and Subtraction of Physical Quantities

The result of an addition or subtraction in the number having different precisions should be reported to the same number of decimal places as are present in the number having the least number of decimal places. For example:

```
  3.1421
  0.241
+ 0.09        ← (has two decimal places)
  3.4731      ← (answer should be reported to two decimal places after rounding off)
```

Answer: 3.47.

When numbers are added or subtracted, the last significant figure in the answer occurs in the last column (counting from left to right) containing a number that results from a combination of digits that are all significant.

Rule for Multiplication and Division of Physical Quantities

The answer to a multiplication or division is rounded off to the same number of significant figures as is possessed by the least precise term used in the calculation. For example:

```
  51.028
× 1.31        ← (three significant figures)
  66.84668    ← (the answer should have three significant figures after rounding off)
```

Answer: 66.8.

When numbers are multiplied or divided, the number of significant figures in the final answer equals the smallest number of significant figures in any of the original factors.

Rules for Determining the Uncertainty in the Result of Arithmetic Calculations

Rule 1: If a set of experimental data is specified to n significant figures, a result obtained by combining the data will also be valid to n significant figures.

Rule 2: The relative error of a physical quantity depends both on the number of significant digits in the physical quantity and the value of the physical quantity.

Rule 3: Intermediate results in multi-step computation should be calculated to one more significant figure in every.

SAMPLE PROBLEM 1.03

Significant figures in addition

The mass of a box is 2.3 kg. Two marble stones of masses 2.15 g and 12.39 g are added to it. Calculate the total mass of the box to the correct number of significant figures.

KEY IDEA

In the case of addition, we always express our result in the least precise measurement, which in our case is 2.3 kg and has two significant digits. Hence, the total mass should be expressed in two significant digits.

Calculation:

Mass of the box = 2.3 kg.

Mass of the first marble stone = 2.15 g = 0.00215 kg.

Mass of the second marble stone = 12.39 g = 0.01239 kg.

Total mass of the box is

$$\begin{array}{r} 2.3 \\ +\ 0.00215 \\ +\ 0.01239 \\ \hline 2.31454 \end{array}$$

Hence, the correct mass of the box in correct significant digits is 2.3 kg.

SAMPLE PROBLEM 1.04

Significant figures in multiplication

The length of a rectangular sheet is 1.5 cm and its breadth is 1.203 cm. Calculate the area of the face of the rectangular sheet to the correct number of significant figures.

KEY IDEA

As per the rule, in the case of multiplication of physical quantities, we should always express our final product in the same number of significant digits as are in the physical quantity having the least significant digit (in this case, it is length 1.5 cm with two significant digits).

Calculation:

Length of the rectangular sheet = 1.5 cm.

Breadth of the rectangular sheet = 1.203 cm.

Area of the rectangular sheet = $1.5 \times 1.203 = 1.8045\ \text{cm}^2$.

Hence, the area in the correct number of significant digits would be $1.8\ \text{cm}^2$

1.6 | ERROR ANALYSIS

Key Concepts

- The uncertainty in measurement of fundamental quantity using an instrument is known as error. Every calculated (derived) physical quantity which depends upon the measured fundamental quantity will also contain error.
- Accuracy of a measurement means how close the measured value is to the true value of a physical quantity whereas the limit or resolution to which a physical quantity is measured by a measuring instrument is known as its precision.
- Errors can be of three types—systemic, random and least count errors.
- Errors are measured as absolute, relative and percentage errors.
- Errors propagate in through mathematical operations.

Measurement is a useful tool for doing experiments in science. However, when we measure a physical quantity with the help of a measuring device, there is always some uncertainty in measurement. The uncertainty in measurement of an instrument is known as **error**. Therefore, every calculated physical quantity (derived physical

quantity), which depends upon the measured fundamental quantity will also contain error. Due to errors of measurement, the measured value of a physical quantity would show a deviation from its true value. For example, if we make an error when measuring the side of a cube with the help of meter scale, then in the volume of the cube calculated using this side length (equal to side^3), there would be uncertainty in the measurement in the form of an error.

Precision and Accuracy

While measuring physical quantities, we often use two terms: accuracy and precision. **Accuracy** of a measurement means how close the measured value is to the true value of a physical quantity. The limit or resolution to which a physical quantity is measured by a measuring instrument is known as its **precision**. The least count of a measuring instrument determines the precision. Smaller the least count, greater is the precision.

Suppose the true value of a physical quantity is 7.502 m. Two persons A and B perform two experiments to measure the value of this physical quantity using different instruments. The value measured by person A is 7.5 m with a resolution of 0.1 m and that by person B is 7.54 m with a resolution of 0.01 m. The value measured by person A is more accurate because it is closer to the true value, that is, 7.502 m. On the other hand, the value measured by person B is less accurate but more precise because its resolution is 0.01 m is higher than that for the value measured by person A.

Types of Errors

The errors that occur during the measurement of physical quantities are broadly divided into three categories:

1. **Systematic errors:** The measurement errors that are unidirectional in nature, that is, either positive or negative, are known as **systematic errors**. For a given setup, systematic errors may be estimated and necessary corrections can be done to the measured values. Some sources due to which systematic errors get introduced are as follows:
   - **(a)** *Personal errors*: These measurement errors occur due to the improper setting of the apparatus or an individual's carelessness in taking observations without following the required precautionary measures. These errors differ from person to person.
   - **(b)** *Instrumental errors*: These errors occur due to faulty or wrongly calibrated measuring instruments. For example, a meter scale worn off at one end would cause error in measurement. Similarly, if the zero of the Vernier scale and main scale do not coincide, it may cause error in measurement and such type of error is called **zero error**.
   - **(c)** *Imperfection in experimental technique or procedure*: This is an error occurring in the measurement of a physical quantity due to a wrong experimental technique or procedure. For example, to measure a person's body temperature, if the thermometer is placed in his/her armpit then the measured temperature will always be less than the person's actual body temperature.

   We can minimize the effect of systematic errors by using proper experimental techniques, selecting correct instruments and removing personal errors by following the required precautionary measures.
2. **Random errors:** These are errors occurring due to unpredictable variations in experimental conditions, such as fluctuations in temperature, voltage supply, and mechanical vibrations of experimental setup. These errors occur irregularly and vary in size and sign (positive or negative). For example, if a person measures a physical quantity twice using the same experimental setup and procedure then he may get different readings due to random errors.
3. **Least count error:** The least count error is the error associated with the resolution of the measuring instrument. Least count of the instrument is the smallest value that can be measured by a measuring instrument. For example, the least count of a Vernier caliper is 0.01 cm and that of a spherometer is 0.001 cm. These errors can be minimized by using high-precision instruments, improving experimental techniques and taking observations several times and then taking the arithmetic mean of all observations.

Smaller the least count of a measuring device, higher is the accuracy of measurement.

Absolute Error, Relative Error, and Percentage Error

Absolute Error

An **absolute error** in the measurement of a physical quantity is the magnitude of the difference between the true value and the measured value of the physical quantity. Let a physical quantity be measured n times. Let the measured values be $a_1, a_2, a_3, \ldots, a_n$. The arithmetic mean of these values is

$$a_m = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Usually, a_m is taken as the true value of the physical quantity if the same is unknown otherwise. By definition, the absolute errors in the measured values of the physical quantity are

$$\begin{aligned} \Delta a_1 &= a_m - a_1, \\ \Delta a_2 &= a_m - a_2, \\ &\vdots \\ \Delta a_n &= a_m - a_n. \end{aligned}$$

The absolute errors would be positive in certain cases and negative in other cases.

Hence the result of measurement after including the absolute error may be written as $a = a_m \pm \overline{\Delta a}$, where $\overline{\Delta a}$ is the mean absolute error. This implies that any measurement of the physical quantity is likely to lie between $(a_m + \overline{\Delta a})$ and $(a_m - \overline{\Delta a})$.

Mean absolute error is the arithmetic mean of the magnitudes of the absolute errors in all the measurements of a physical quantity. It is represented by $\overline{\Delta a}$ and is mathematically equal to

$$\overline{\Delta a} = \frac{|\Delta a_1| + |\Delta a_1| + \cdots + |\Delta a_n|}{n}.$$

Relative Error or Fractional Error

The **relative error** or **fractional error** of a measurement is defined as the ratio of the mean absolute error to the mean value of the physical quantity measured.

$$\text{Relative error or fractional error} = \frac{\text{Mean absolute error}}{\text{Mean value}} = \frac{\overline{\Delta a}}{a_m}.$$

Percentage Error

When the relative/fractional error is expressed in percentage, we call it percentage error. It is represented by the symbol δa. Thus, the percentage error is given as

$$\text{Percentage error}(\delta a) = \frac{\overline{\Delta a}}{a_m} \times 100\%.$$

Propagation of Errors

When we perform an experiment, we take several measurements and each measurement may have an error, which gets combined in various mathematical operations performed for measurement of derived quantities. For example, if values of mass and volume are determined for measuring, a certain error is used in each measurement which will propagate when density is measure as a ration of mass and volume.

Combination of Errors in the Sum of Quantities

Let us consider a physical quantity (x) which is equal to the sum of two physical quantities a and b such that $x = a + b$.

Let Δa = absolute error in the measurement of a,
Δb = absolute error in the measurement of b, and
Δx = absolute error in the calculation of x, that is, the sum of a and b.

The maximum absolute error in x is $\Delta x = \pm(\Delta a + \Delta b)$

Percentage error in the value of $x = \dfrac{(\Delta a + \Delta b)}{a+b} \times 100\%$.

The maximum absolute error in the sum of two quantities is equal to the sum of the absolute errors in the individual quantities.

Combination of Errors in the Difference of Quantities

Let us consider a physical quantity (x) which is equal to the difference of two physical quantities a and b such that $x = a - b$.

Let Δa = absolute error in the measurement of a,
Δb = absolute error in the measurement of b, and
Δx = absolute error in the calculation of x, that is, the difference of a and b.

The maximum absolute error in x is $\Delta x = \pm(\Delta a + \Delta b)$.

Percentage error in the value of $x = \dfrac{(\Delta a + \Delta b)}{a-b} \times 100\%$.

The maximum absolute error in the difference of two quantities is equal to the sum of the absolute errors in the individual quantities.

Combination of Errors in the Product of Quantities

Let us consider a physical quantity (x) which is equal to the product of two physical quantities a and b such that $x = a \times b$.

Let Δa = absolute error in the measurement of a,
Δb = absolute error in the measurement of b, and
Δx = absolute error in the calculation of x, that is, the product of a and b.

The maximum fractional error in x is $\dfrac{\Delta x}{x} = \pm\left(\dfrac{\Delta a}{a} + \dfrac{\Delta b}{b}\right)$.

Percentage error in the value of x = (Percentage error in value of a) + (Percentage error in value of b).

The maximum relative error or fractional error in the product of two quantities is equal to the sum of the relative error or fractional error in the individual quantities.

Combination of Errors in the Division of Quantities

Let us consider a physical quantity (x) which is equal to ratio of two physical quantities a and b such that $x = \dfrac{a}{b}$.

Let Δa = absolute error in the measurement of a,
Δb = absolute error in the measurement of b, and
Δx = absolute error in the calculation of x, that is, division of a and b.

The maximum fractional error in x is $\dfrac{\Delta x}{x} = \pm\left(\dfrac{\Delta a}{a} + \dfrac{\Delta b}{b}\right)$.

Percentage error in the value of x = (Percentage error in value of a) + (Percentage error in value of b).

The maximum relative error or fractional error in the division of two quantities is equal to the sum of the relative error or fractional error in the individual quantities.

Combination of Errors in Quantities Raised to Some Power

Let us consider a physical quantity (x) which is given as

$$x = \frac{a^n}{b^m}$$

Let Δa = absolute error in the measurement of a,
Δb = absolute error in the measurement of b, and
Δx = absolute error in the calculation of x.

The maximum relative error in x is

$$\frac{\Delta x}{x} = \pm\left(n \times \frac{\Delta a}{a} + m \times \frac{\Delta b}{b}\right).$$

Percentage error in the value of $x = n$ (Percentage error in value of a) + m (Percentage error in value of b).

Relative error or fractional error in a quantity raised to a power n is equal to n times the relative error or fractional error in the individual quantity.

SAMPLE PROBLEM 1.05

Combination of errors in sum of quantities

The lengths of two rods are recorded as $l_1 = 25.2 \pm 0.1$ cm and $l_2 = 16.8 \pm 0.1$ cm. Find the sum of the lengths of the two rods with the limits of error.

KEY IDEA

The absolute error in the measurement of total length will be equal to the sum of the absolute errors in the individual lengths

Calculation:

Length of the first rod, $l_1 = 25.2$ cm.
Absolute error in measurement of the length of the first rod, $\Delta l_1 = 0.1$ cm.
Length of the second rod, $l_2 = 16.8$ cm.
Absolute error in measurement of the length of the second rod, $\Delta l_2 = 0.1$ cm.

The sum of the lengths of the two rods will be

$$l = l_1 + l_2 = 25.2 + 16.8 = 42 \text{ cm}.$$

The absolute error is given by

$$\Delta l = \Delta l_1 + \Delta l_2 = 0.1 + 0.1 = 0.2 \text{ cm}$$

Hence, the sum of the lengths of the two rods will be (42 ± 0.2) cm.

SAMPLE PROBLEM 1.06

Combination of errors in multiplication of quantities

The length and the width of a rectangular plate are (16.30 ± 0.05) m and (13.80 ± 0.05) m, respectively. Calculate the area of the plate and also find the uncertainty in the area.

KEY IDEA

In the case of multiplication or division of physical quantities, the total error in the final result is equal to the sum of the relative errors of the individual quantities.

Calculations:

Length of the rectangular plate = (16.30 ± 0.05) cm.
Absolute error in the measurement of length, $\Delta l = 0.05$ cm.
Breadth of the rectangular plate = (13.80 ± 0.05) cm.
Absolute error in the measurement of breadth, $\Delta b = 0.05$ cm.

Relative error in the measurement of length is

$$\frac{\Delta l}{l} = \frac{0.05}{16.30}$$

Relative error in the measurement of breadth is

$$\frac{\Delta b}{b} = \frac{0.05}{13.80}$$

The relative error in area is

$$\frac{\Delta A}{A} = \frac{0.05}{16.30} + \frac{0.05}{13.80} = \frac{1.50}{224.94} = 0.0067.$$

Therefore, the relative percentage error is 0.67% and the area of the plate with uncertainty is (224.9 ± 1.5) cm^2.

1.7 | LENGTH MEASURING INSTRUMENTS

Key Concept

- For measuring length accurately to 0.1 to 0.01 mm, instruments commonly used are Vernier calipers and screw gauge.

The fundamental quantity length can be measured with the accuracy of 1mm using meter scale which has a least count of 1mm. To measure length accurately to one-tenth or one-hundredth (1/100) of millimeter, Vernier calipers, micrometer and screw gauge.

Vernier Calipers

As shown in Fig. 1-6, this instrument is made up of three parts: (i) main scale (M), graduated in mm and cm, carrying two jaws, A and C; (ii) Vernier scale (V) that slides on metallic strip M and can be fixed at any point with the help of screw S. It has 10 divisions over a length of 9 mm; and (iii) a thin metallic strip E attached to the back side of M and connected with Vernier scale. When jaws A and B touch each other, the edge of E touches the edge of M.

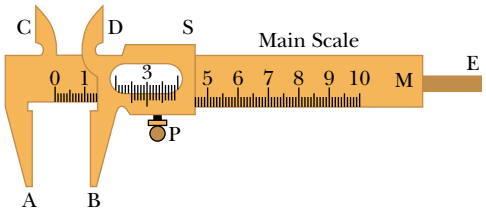


Figure 1-6 Vernier calipers.

Let n Vernier scale divisions (VSD) coincide with $(n-1)$ main scale divisions (MSD). Then,

$$n\text{VSD} = (n-1)\ \text{MSD};$$

$$1\ \text{VSD} = \left(\frac{n-1}{n}\right)\ \text{MSD};$$

$$1\ \text{MSD} - 1\ \text{VSD} = 1\ \text{MSD} - \left(\frac{n-1}{n}\right)\ \text{MSD} = \frac{1}{n}\text{MSD}$$

The difference between the values of one main scale division and one Vernier scale division is known as Vernier constant (VC) or the **least count** (LC). It is the smallest distance that can be accurately measured with the Vernier scale. Thus,

$$\text{VC} = \text{LC} = 1\ \text{MSD} - 1\ \text{VSD} = \left(\frac{1}{n}\right)\text{MSD} = \frac{\text{Smallest division on main scale}}{\text{Number of divisions on vernier scale}}$$

In a Vernier caliper, one main scale division is 1 mm and 10 Vernier scale divisions coincide with 9 main scale divisions, so

$$1\ \text{VSD} = \frac{9}{10}\text{MSD} = 0.9\ \text{mm};$$

$$\text{VC} = 1\ \text{MSD} - 1\ \text{VSD} = 1\ \text{mm} - 0.9\ \text{mm} = 0.1\ \text{mm} = 0.01\ \text{cm}$$

SAMPLE PROBLEM 1.07

Measuring volume of a cylinder

The length of a cylinder is measured with a meter rod having least count 0.1 cm. Its diameter is measured with Vernier calipers having least count 0.01 cm. Given that the length is 5.0 cm and the radius is 2.0 cm, calculate the percentage error in the calculated value of the volume.

KEY IDEA

The volume of cylinder is related to radius by the relation $V = \pi r^2 l$.

Calculation: The percentage error in volume is

$$\frac{\Delta V}{V}\times 100 = \frac{2\Delta r}{r}\times 100 + \frac{\Delta l}{l}\times 100$$

$$= \left(2\times\frac{0.01}{2.0}\times 100 + \frac{0.1}{5.0}\times 100\right) = (1+2)\% = 3\%.$$

Micrometer Screw

Vernier calipers generally have least count of 0.01 cm, so for measuring lengths with higher accuracy, say up to 0.001 cm, screw gauge and spherometer are used which are based on the principle of micrometer screw (Fig. 1-7). It is based on the principle that if an accurately cut single threaded screw is rotated in a closely fitted nut, then in addition to the circular motion of the screw there is also a linear motion of the screw head in the forward or backward direction, along the axis of the screw. The linear distance moved by the screw, when it is given one complete rotation, is called the pitch (p) of the screw. The pitch is basically equal to the distance between two consecutive threads as measured along the axis of the screw.

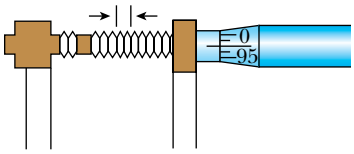


Figure 1-7 Micrometer screw gauge.

The screw moves forward or backward by 1/100 or 1/50 of the pitch, if the circular scale is rotated through one circular division, which is the minimum distance that can be accurately measured or the least count. Thus, the least count of the screw is given by

$$\text{Least count} = \frac{\text{Pitch}}{\text{Number of divisions on circular scale}}$$

If pitch is 1 mm and there are 100 divisions on the circular scale, then

$$\text{Least count} = \frac{\text{Pitch}}{\text{Number of divisions on circular scale}}$$

$$\text{L.C.} = \frac{1\text{ mm}}{100} = 0.01\text{ mm} = 0.001\text{ cm} = 10\ \mu\text{m}$$

The least count is of the order of 10 μm, hence the name of the meter.

Screw Gauge

Screw gauge (Fig. 1-8) consists of a U-shaped metal frame M to which is fixed stud (small metal piece) on one end and the other end N carries a cylindrical hum, H. It is graduated in millimeters or half millimeters based on the pitch of the screw. This scale is called a linear scale or a pitch scale. The surface E of the cap K is divided into 50 or 100 equal parts. It is called the circular scale or the head scale. In an accurately adjusted instrument when the

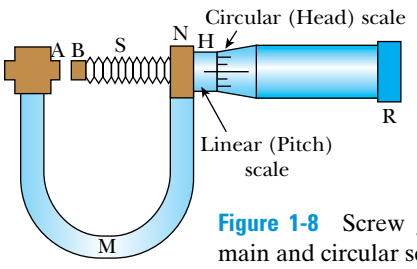


Figure 1-8 Screw gauge with main and circular scale.

faces A and B are just touching each other, then the zero of the circular scale should coincide with the zero of the linear scale.

To measure the diameter of the wire, it is placed between plane faces A and B. The edge of the cap lies ahead of N^{th} division of the linear scale, and nth division of circular scale lies over reference line then,

Total reading = $N + n \times \text{LC}$

SAMPLE PROBLEM 1.08

Measuring diameter of a metal rod

Using a screw gauge, the diameter of a metal rod was measured. The observations are given as follows: 0.39 mm, 0.38 mm, 0.37 mm, 0.41 mm, 0.38 mm, 0.37 mm, 0.40 mm, 0.39 mm. Calculate (a) the most accurate value of the diameter (b) the relative error, and (c) the percentage error in the measurement of the diameter.

Calculations:

Mean diameter $\overline{d}$

$$= \frac{d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 + d_8}{8}$$

$$= \frac{0.39 + 0.38 + 0.37 + 0.41 + 0.38 + 0.37 + 0.40 + 0.39}{8}$$

$$= 0.38875 \text{ mm} \approx 0.39 \text{ mm}.$$

Absolute error in the first reading

$$= \Delta d_1 = |0.39 - 0.39| = 0.0 \text{ mm}.$$

Absolute error in the second reading

$$= \Delta d_2 = |0.39 - 0.38| = 0.01 \text{ mm}.$$

Absolute error in the third reading

$$= \Delta d_3 = |0.39 - 0.37| = 0.02 \text{ mm}.$$

Absolute error in the fourth reading

$$= \Delta d_4 = |0.39 - 0.41| = 0.02 \text{ mm}.$$

Absolute error in the fifth reading

$$= \Delta d_5 = |0.39 - 0.38| = 0.01 \text{ mm}.$$

Absolute error in the sixth reading

$$= \Delta d_6 = |0.39 - 0.37| = 0.02 \text{ mm}.$$

Absolute error in the seventh reading

$$= \Delta d_7 = |0.38 - 0.40| = 0.01 \text{ mm}.$$

Absolute error in the eighth reading

$$= \Delta d_8 = |0.39 - 0.39| = 0.0 \text{ mm}.$$

(a) Mean absolute error is given by

$$\overline{\Delta d} = \frac{\Delta d_1 + \Delta d_2 + \Delta d_3 + \Delta d_4 + \Delta d_5 + \Delta d_6 + \Delta d_7 + \Delta d_8}{8}$$

$$= \frac{0.00 + 0.01 + 0.02 + 0.01 + 0.02 + 0.01 + 0.00}{8}$$

$$= 0.001125$$

$$= 0.01 \text{ mm}.$$

(b) Relative error $\dfrac{\overline{\Delta d}}{d} = \dfrac{0.01}{0.39} = 0.0256.$

(c) Percentage error $\delta d = \left(\dfrac{\overline{\Delta d}}{d} \times 100\right)\%$

$$= 0.0256 \times 100\%$$

$$= 2.56\%$$

$$= 2.6\%.$$

1.8 | DIMENSIONAL ANALYSIS

Key Concepts

- Dimensions of all the derived physical quantities can be expressed in terms of the seven fundamental dimensions of the physical world.
- Dimensional formula of a given physical quantity is the expression which shows how and which of the base quantities represent the dimensions of that physical quantity. An equation obtained by equating a physical quantity with its dimensional formula is called the dimensional equation of the physical quantity.
- Dimensional analysis can be used to derive physical relationship between different physical quantities and check the correctness of a given relationship between different physical quantities.

Dimension of physical quantity describes the nature of the physical quantity. All the derived physical quantities can be expressed in terms of some combination of seven fundamental or base quantities. The seven fundamental quantities are called as the seven dimensions of physical world and are denoted by square brackets [ ]. If a physical quantity is written in square brackets it means that we are dealing with the dimensions of the quantity. The seven dimensions according to the seven fundamental quantities are:

1. Dimension of length is [L].
2. Dimension of mass is [M].
3. Dimension of time is [T].
4. Dimension of electric current is [A].
5. Dimension of thermodynamic temperature is [K].
6. Dimension of luminous intensity is [cd].
7. Dimension of amount of substance is [mol].

Dimensions of all the derived physical quantities can be expressed in terms of the seven fundamental dimensions of the physical world. For example, area occupied by an object is equal to product of the length and breadth of the object. Therefore, the dimensions of area are $[L] \times [L] = [L^2]$. Since area is independent of mass and time it has zero dimension in mass, that is, $[M^0]$ and zero dimension in time, that is, $[T^0]$. Similarly, the dimensions in all the other base quantities are zero for area.

Density of an object can be expressed in terms of mass and volume as

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}}$$
$$= \frac{\text{Mass}}{\text{Length} \times \text{breadth} \times \text{height}}$$
$$= \frac{\text{Mass}}{\text{Length} \times \text{length} \times \text{length}} \quad \text{[since breadth and height are special types of length]}$$

Thus, the dimensions of density are

$$\frac{[M]}{[L] \times [L] \times [L]} = \frac{[M]}{[L^3]} = [ML^{-3}]$$

Thus, density has one dimension in mass, –3 dimensions in length and zero dimension in time. The dimensions in all other base quantities (time, electric current, luminous intensity, amount of substance, and thermodynamic temperature) are zero.

In case of dimensions, the quality of the physical quantity matters and not its magnitude. This means that change in velocity, average velocity, final velocity, and speed all have the same dimensions $[LT^{-1}]$ because all these quantities can be expressed as ratio of length and time.

The argument of a special function, such as the logarithmic, trigonometric and exponential functions, is a pure number which is a ratio of similar physical quantities and therefore must be dimensionless.

Dimensional Formulas and Dimensional Equations

The expression which shows how and which of the base quantities represent the dimensions of a physical quantity is called the **dimensional formula** of the given physical quantity. For example, the dimensional formula of area is $[M^0L^2T^0]$ and dimensional formula of density is $[ML^{-3}T^0]$. Similarly, the dimensional formula of acceleration is $[M^0LT^{-2}]$.

An equation obtained by equating a physical quantity with its dimensional formula is called the **dimensional equation** of the physical quantity. When a derived quantity is expressed in terms of fundamental quantities, it is

written as a product of different powers of the fundamental quantities. To understand this better, let us consider the physical quantity force. Force is expressed as

$$\text{Force} = \text{mass} \times \text{acceleration} = \frac{\text{mass} \times \text{length/time}}{\text{time}} = \text{mass} \times \text{length} \times (\text{time})^{-2} \tag{1-10}$$

Thus, the dimensions of force are 1 in mass, 1 in length, and –2 in time. Here the physical quantity that is expressed in terms of the base quantities is enclosed in square brackets to indicate that the equation is among the dimensions and not among the magnitudes. Thus Eq. (1-10) can be written as

$$[\text{Force}] = [\text{MLT}^{-2}].$$

Such an expression for a physical quantity in terms of the fundamental quantities is called the dimensional equation. If we consider only the RHS of the equation, the expression is termed as dimensional formula. Thus, dimensional formula for force is $[\text{MLT}^{-2}]$.

The equations which represent the dimensions of a physical quantity in terms of the base quantities are called the dimensional equations. For example the dimensional equations of area (A), density (ρ), force (F) and acceleration may be expressed as

$$[A] = [\text{M}^0\text{L}^2\text{T}^0]$$
$$[\rho] = [\text{M}^0\text{L}^2\text{T}^0]$$
$$[F] = [\text{MLT}^{-2}]$$
$$[a] = [\text{M}^0\text{LT}^{-2}]$$

Applications of Dimensional Analysis

Physical quantities represented by symbols on both sides of a mathematical equation must have the same dimensions. For example, force is represented by the relation $F = ma$. The dimension of force (F) should be similar to the product of the dimensions of mass (m) and acceleration (a). Dimensional analysis is used to check whether a given physical equation is correct, to derive relations among different physical quantities, and also to convert the units of physical quantities from one system of units to another. When the magnitudes of two or more physical quantities are multiplied, one can cancel identical dimensions in the numerator and denominator.

Conversion of Units

Dimensional analysis can be used to convert a physical quantity from one system of units to another. This application of dimensional analysis is based upon the fact that the magnitude of a physical quantity remains the same whatever be the system of its measurement. This can be expressed mathematically as

$$n_1 u_1 = n_2 u_2,$$

where u_1 and u_2 are two units of measurement of a physical quantity P and n_1 and n_2 are the numerical values of physical quantity in the two systems of units. The method of converting units of a physical quantity using dimensional analysis can be understood with help of the following example.

Young's modulus of steel is 19×10^{10} N/m^2. Let us see how we can express Young's modulus of steel in dyne/cm^2 using application of dimensional analysis. Here, dyne is the CGS unit of force.

The SI unit of Young's modulus is N/m^2.

This suggests that it has dimensions of force/(distance)^2.

Thus,
$$[Y] = \frac{[F]}{\text{L}^2} = \frac{\text{MLT}^{-2}}{\text{L}^2} = \text{ML}^{-1}\text{T}^{-2}.$$

Checking the Dimensional Consistency of Equations

Checking the correctness of an equation is based on the **principle of homogeneity** which states that the magnitudes of physical quantities may be added together or subtracted from one another only if they have the same dimensions. The statement implies that force cannot be added to mass and that electric current cannot be subtracted from temperature because all the four physical quantities have different dimensions.

The principle of homogeneity when applied to equations is useful in determining whether the given equation is correct or not. If the dimensions of LHS of an equation is not equal to the dimensions of RHS, the equation is wrong and if the dimensions of each term on both sides are same, the equation is dimensionally correct, otherwise not. A dimensionally correct equation may or may not be physically correct. To understand this, see the following two examples:

1. $F = mv^2/r^2$
   The dimensions of force are $[F] = [MLT^{-2}]$, dimensions of velocity (v) are $[v] = [LT^{-1}]$, dimension of mass (m) is $[m] = [M]$ and dimension of radius(r) is $[r] = [L]$. By substituting the dimensions of the physical quantities in the above relation, we get

   $$\text{Dimensions of LHS} = [MLT^{-2}].$$

   $$\text{Dimensions of RHS} = [M][LT^{-1}]^2/[L]^2 = [MT^{-2}].$$

   That is, $[MLT^{-2}] \neq [MT^{-2}]$.

   In the above equation the dimensions of both sides are not the same. Therefore, this formula is not correct dimensionally, and it can never be physically correct.
2. $s = ut - (1/2)at^2$
   The dimension of displacement (s) is $[s] = [L]$, dimensions of initial velocity (u) are $[v] = [LT^{-1}]$, dimension of time (t) is $[t] = [T]$ and dimensions of acceleration (a) are $[a] = [LT^{-2}]$.

   $$\text{Dimensions of LHS} = [L].$$

   $$\text{Dimensions of RHS} = [LT^{-1}][T] - [LT^{-2}][T^2] = [L] - [L] = [L].$$

   Thus dimensions of LHS are equal to dimensions of RHS. Therefore, this equation is dimensionally correct. However, from the equations of motion, we know that $s = ut + (1/2)at^2$. So the given equation is dimensionally correct but physically incorrect.

   A dimensionally correct equation need not be an exact (correct) equation physically. However, a dimensionally wrong (incorrect) or inconsistent equation must be wrong.

Deducing Relations among Physical Quantities

If one knows the dependency of a physical quantity on other quantities and if the dependency is of the product type, then by using the method of dimensional analysis, the relation between the quantities can be derived. To understand this, consider the following two examples.

1. **Time period of a simple pendulum:** The time period of a simple pendulum is a function of the mass of the bob (m), effective length (l), and acceleration due to gravity (g). Thus,

   $$T = km^x l^y g^z$$

   where k is the dimensionless constant. If this relation is dimensionally correct then by substituting the dimensions of the quantities, we get

Dimensions of LHS = Dimensions of RHS

$$[\mathrm{T}] = [\mathrm{M}]^x[\mathrm{L}]^y[\mathrm{LT}^{-2}]^z$$
$$[\mathrm{M}^0\mathrm{L}^0\mathrm{T}^1] = [\mathrm{M}^x\mathrm{L}^{y+z}\mathrm{T}^{-2z}].$$

On equating the dimensions on both the sides of this equation, we get

$$x = 0 \tag{1-11}$$
$$y + z = 0 \tag{1-12}$$
$$-2z = 1 \tag{1-13}$$

Solving Eqs. (1-11), (1-12), and (1-13), we get

$$x = 0,\ y = \frac{1}{2} \quad \text{and} \quad z = -\frac{1}{2}.$$

Substituting the values of x, y, and z in $T = km^x l^y g^z$ we get the required physical relation as

$$T = km^0 l^{(1/2)} g^{-(1/2)}$$
$$T = k\sqrt{\frac{l}{g}}.$$

The value of the dimensionless constant (2π) is found through experiments; hence

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

2. **Stokes' law:** When a small sphere moves at a low speed through a fluid, the viscous force F opposing the motion is found experimentally to depend on the radius r and the velocity v of the sphere, and the viscosity η of the fluid. So, $F = f(\eta, r, v)$. If the function is a product of power functions of η, r, and v, then,

$$F = k\eta^x r^y v^z,$$

where, k is the dimensionless constant.

If the above relation is dimensionally correct, then we have

Dimensions of LHS = Dimensions of RHS

$$[\mathrm{MLT}^{-2}] = [\mathrm{ML}^{-1}\mathrm{T}^{-1}]^x[\mathrm{L}]^y[\mathrm{LT}^{-1}]^z$$

or

$$[\mathrm{MLT}^{-2}] = [\mathrm{M}^x\mathrm{L}^{-x+y+z}\mathrm{T}^{-x-z}].$$

Equating the exponents of similar quantities, we have $x = 1$, $-x + y + z = 1$, and $-x - z = -2$. Solving these for x, y, and z, we get $x = y = z = 1$. Substituting the values of x, y and z in $F = k\eta^x r^y v^z$ we get

$$F = k(\eta)^1(r)^1(v)^1$$
$$F = k\eta rv.$$

On experimental grounds, $k = 6\pi$; hence

$$F = 6\pi\eta rv.$$

which is the Stokes' law.

SAMPLE PROBLEM 1.09

Checking dimensional consistency of equations

Using the dimensional analysis, check whether the following equation is correct or not: $T = 2\pi\sqrt{R^3/GM}$.

Calculation:

We have the following dimensions

$$[T] = [M^0L^0T^1]$$
$$[R] = [M^0L^1T^0]$$
$$[G] = [M^{-1}L^3T^{-2}]$$
$$[M] = [M^1L^0T^0]$$

Now the given relation is

$$T = 2\pi\sqrt{\frac{R^3}{GM}} \qquad (1\text{-}14)$$

Dimensions of LHS of Eq. (1-14) are $[M^0L^0T^1]$

Dimensions of RHS of Eq. (1-14) are

$$= [M^0L^3T^0]^{1/2}[M^{-1}L^3T^{-2}]^{-1/2}[M^1L^0T^0]^{-1/2} = [M^0L^0T^1]$$

Since LHD = RHD, thus Eq. (1-14) is dimensionally correct.

SAMPLE PROBLEM 1.10

Deducing relation between physical quantities

If the time period (T) of vibration of a liquid drop depends on surface tension (S) and radius (r) of the drop, and density (ρ) of the liquid, derive an expression for T using dimensional analysis.

Calculation:

Dimensions of time period $= [M^0L^0T^1]$

Dimensions of surface tension $= [M^1L^0T^{-2}]$

Dimensions of radius $= [M^0L^1T^0]$

Dimensions of density $= [M^1L^{-3}T^0]$

Now we have

$$T = kS^a r^b \rho^c$$
$$[M^0L^0T^1] = k[M^1L^0T^{-2}]^a[M^0L^1T^0]^b[M^1L^{-3}T^0]^c$$

On comparing the coefficients, we get

$$a = \frac{-1}{2},\ b = \frac{3}{2},\ c = \frac{1}{2}$$

$$T = k\sqrt{\frac{\rho r^3}{S}}$$

REVIEW AND SUMMARY

Measurement in Physics Physics is based on measurement of physical quantities. Certain physical quantities have been chosen as **base quantities** (such as length, time, and mass); each has been defined in terms of a **standard** and given a **unit** of measure (such as meter, second, and kilogram). Other physical quantities are defined in terms of the base quantities and their standards and units.

SI Units The unit system emphasized in this book is the International System of Units (SI). The three physical quantities displayed in Table 1-1 are used in the early chapters. Standards, which must be both accessible and invariable, have been established for these base quantities by international agreement. These standards are used in all physical measurement, for both the base quantities and the quantities derived from them. Scientific notation and the prefixes of Table 1-2 are used to simplify measurement notation.

Changing Units Conversion of units may be performed by using *chain-link conversions* in which the original data are multiplied successively by conversion factors written as unity and the units are manipulated like algebraic quantities until only the desired units remain.

Length The meter is defined as the distance traveled by light during a precisely specified time interval.

Time The second is defined in terms of the oscillations of light emitted by an atomic (cesium-133) source. Accurate time

signals are sent worldwide by radio signals keyed to atomic clocks in standardizing laboratories.

Mass The kilogram is defined in terms of a platinum–iridium standard mass kept near Paris. For measurements on an atomic scale, the atomic mass unit, defined in terms of the atom carbon-12, is usually used.

Errors The uncertainty in measurement of fundamental quantity using an instrument is known as error. Every calculated physical quantity (derived physical quantity), which depends upon the measured fundamental quantity will contain error.

Dimensions Dimensional formula of a given physical quantity is the expression which shows how and which of the base quantities represent the dimensions of that physical quantity. An equation obtained by equating a physical quantity with its dimensional formula is called the dimensional equation of the physical quantity.

PROBLEMS

1. A *gry* is an old English measure for length, defined as 1/10 of a line, where *line* is another old English measure for length, defined as 1/12 inch. A common measure for length in the publishing business is a *point,* defined as 1/72 inch. What is an area of 0.75 gry^2 in points squared (points^2)?
2. How many m/s are there in 1.0 mi/h?
3. Spacing in this book was generally done in units of points and picas: 12 points = 1 pica, and 6 picas = 1 inch. If a figure was misplaced in the page proofs by 0.70 cm, what was the misplacement in (a) picas and (b) points?
4. The height of a motion picture film's frame is 35.0 cm. If 24.0 frames go by in 1.0 s, calculate the total number of frames required to show a 2.0 h long motion picture.
5. Assume the legal limit of speed is 70.0 mi/h. If driving day and night without stopping for 1.00 year, what is the maximum number of miles one can drive?
6. A boy measures the thickness of a human hair by looking at it through a microscope of magnification 100×. After 25 observations, the boy finds that the average width of the hair in the field of view of the microscope is 3.8 mm. What is the estimate on the thickness of hair?
7. A cubical object has an edge length of 1.00 cm. If a cubical box contained a mole of cubical objects, find its edge length (one mole = 6.02×10^{23} units).
8. There exists a claim that if allowed to run for 100.0 years, two cesium clocks, free from any disturbance, may differ by only about 0.020 s. Using that discrepancy, find the uncertainty in a cesium clock measuring a time interval of 1.0 s.
9. The age of the universe is approximately 10^{10} years and mankind has existed for about 10^6 years. If the age of the universe were "1.0 day," how many "seconds" would mankind have existed?
10. Three digital clocks *A, B,* and *C* run at different rates and do not have simultaneous readings of zero. Figure 1-9 shows simultaneous readings on pairs of the clocks for four occasions. (At the earliest occasion, for example, *B* reads 25.0 s and *C* reads 92.0 s.) If two events are 600 s apart on clock *A,* how far apart are they on (a) clock *B* and (b) clock *C*? (c) When clock *A* reads 400 s, what does clock *B* read? (d) When clock *C* reads 15.0 s, what does clock *B* read? (Assume negative readings for prezero times.)

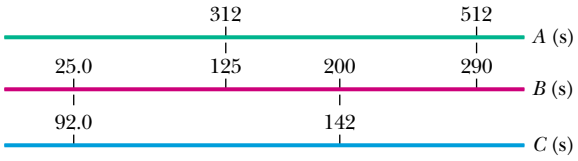


Figure 1-9 Problem 10.

11. A lecture period (50 min) is close to 1 microcentury. (a) How long is a microcentury in minutes? (b) Using

$$\text{percentage difference} = \left(\frac{\text{actual} - \text{approximation}}{\text{actual}}\right)100,$$

find the percentage difference from the approximation.

12. Time standards are now based on atomic clocks. A promising second standard is based on *pulsars,* which are rotating neutron stars (highly compact stars consisting only of neutrons). Some rotate at a rate that is highly stable, sending out a radio beacon that sweeps briefly across Earth once with each rotation, like a lighthouse beacon. Pulsar PSR 1937 + 21 is an example; it rotates once every 1.557 806 448 872 75 ±3 ms, where the trailing ±3 indicates the uncertainty in the last decimal place (it does *not* mean ±3 ms). (a) How many rotations does PSR 1937 + 21 make in 8.00 days? (b) How much time does the pulsar take to rotate exactly one million times and (c) what is the associated uncertainty?
13. Five clocks are being tested in a laboratory. Exactly at noon, as determined by the WWV time signal, on successive days of a week the clocks read as in Table 1-6. Rank the five clocks according to their relative value as good timekeepers, best to worst. Justify your choice.

Table 1-6 Problem 13

Clock	Sun.	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
A	12:36:40	12:36:56	12:37:12	12:37:27	12:37:44	12:37:59	12:38:14
B	11:59:59	12:00:02	11:59:57	12:00:07	12:00:02	11:59:56	12:00:03
C	15:50:45	15:51:43	15:52:41	15:53:39	15:54:37	15:55:35	15:56:33
D	12:03:59	12:02:52	12:01:45	12:00:38	11:59:31	11:58:24	11:57:17
E	12:03:59	12:02:49	12:01:54	12:01:52	12:01:32	12:01:22	12:01:12

14. Because Earth's rotation is gradually slowing, the length of each day increases: The day at the end of 1.0 century is 1.0 ms longer than the day at the start of the century. In 30 centuries, what is the total of the daily increases in time?

15. Gold, which has a density of 19.32 g/cm^3, is the most ductile metal and can be pressed into a thin leaf or drawn out into a longfiber. (a) If a sample of gold with a mass of 29.34 g is pressed into a leaf of 1.000 μm thickness, what is the area of the leaf? (b) If, instead, the gold is drawn out into a cylindrical fiber of radius 2.500 μm, what is the length of the fiber?

16. (a) Using the known values of Avogadro's number and the atomic mass of sodium, find the average mass density of a sodium atom assuming its radius to be about 1.90 Å. (b) The density of sodium in its crystalline phase is 970 kg/m^3. Why do the two densities differ? (Avogadro's number, that is, the number of atoms or molecules in one mole of a substance, is 6.023×10^{23}.)

17. A grocer's balance shows the mass of an object as 2.500 kg. Two gold pieces of masses 21.15 g and 21.17 g are added to the box. What is (a) the total mass in the box and (b) the difference in the masses of the gold pieces to the correct number of significant figures?

18. Einstein's mass–energy equation relates mass m to energy E as $E = mc^2$, where c is speed of light in vacuum. The energy at nuclear level is usually measured in MeV, where 1 MeV = $1.602\ 18 \times 10^{-13}$ J; the masses are measured in unified atomic mass unit (u), where 1 u = $1.660\ 54 \times 10^{-27}$ kg. Prove that the energy equivalent of 1 u is 931.5 MeV.

19. Water is poured into a container that has a small leak. The mass m of the water is given as a function of time t by $m = 5.00t^{0.8} - 3.00t + 20.00$, with $t \geq 0$, m in grams, and t in seconds. (a) At what time is the water mass greatest, and (b) what is that greatest mass? In kilograms per minute, what is the rate of mass change at (c) $t = 3.00$ s and (d) $t = 5.00$ s?

20. A vertical container with base area measuring 14.0 cm by 17.0 cm is being filled with identical pieces of candy, each with a volume of 50.0 mm^3 and a mass of 0.0200 g. Assume that the volume of the empty spaces between the candies is negligible. If the height of the candies in the container increases at the rate of 0.250 cm/s, at what rate (kilograms per minute) does the mass of the candies in the container increase?

PRACTICE QUESTIONS

Single Correct Choice Type

1. The pitch of a screw gauge is 1 mm with 100 divisions on the circular scale. For a given wire, the linear scale reads 2 mm and 64th divisions on the circular scale coincides with the reference line. If the length of the wire is 3.5 cm, find the volume in cm^3?
   (a) 19.0 cm^3 **(b)** 0.19 cm^3
   (c) 76.0 cm^3 **(d)** 0.76 cm^3

2. The formula $W = (F + 2Ma)v^n$, where W is the work done, F is the force, M is the mass, a is the acceleration and v is the velocity can be made dimensionally correct for
   (a) $n = 0$ **(b)** $n = 1$
   (c) $n = -1$ **(d)** no value of n

3. To determine the Young's modulus of a wire, the formula is $Y = F/A \cdot L/\Delta l$; where L is the length, A is the area of cross-section of the wire, Δl is the change in length of the wire when stretched with a force F. The conversion factor to change it from CGS to MKS system is
   (a) 1 **(b)** 10
   (c) 0.1 **(d)** 0.01

4. The measurement of a physical quantity is given by $x = A^2B/C^{1/3}D^3$. The percentage errors introduced in the measurements of the quantities A, B, C and D, are 2%, 2%, 4% and 5%, respectively. Then, which of the following quantity contribute to the minimum amount of percentage of error in the measurement of x?
   (a) A **(b)** B
   (c) C **(d)** D

5. If frequency (F), velocity (V) and density (D) are considered as fundamental units, the dimensional formula for momentum will be
   (a) DVF^2 **(b)** DV^2F^{-1}
   (c) $D^2V^2F^2$ **(d)** DV^4F^{-3}

6. If pressure can be expressed as
$$P = \frac{b}{a}\sqrt{1 + \frac{k\theta t^3}{ma}}$$
   where k is the Boltzmann's constant, θ is the temperature, t is the time and a and b are constants, then dimensional formula of b is equal to the dimensional formula of
   (a) linear momentum. **(b)** force.
   (c) angular momentum. **(d)** torque.

7. During a short interval of time the speed v in m/s of an automobile is given by $v = at^2 + bt^3$, where the time t is in seconds. The units of a and b are, respectively,
   (a) m s^2; m s^4 **(b)** s^3/m; s^4/m
   (c) m/s^2; m/s^3 **(d)** m/s^3; m/s^4

8. You may not know integration but by using dimensional analysis you can check on some results. In the integral $\int dx/(2ax - x^2)^{1/2} = a^n \sin^{-1}(x/a - 1)$, the value of n is
   (a) 1 **(b)** -1
   (c) 0 **(d)** $\frac{1}{2}$

9. The volume of liquid flowing per second is called the volume flow rate Q and has the dimensions of [L]^3/[T]. The flow rate of a liquid through a hypodermic needle

during an injection can be estimated with the following equation:

$$Q = \frac{\pi R^n (P_2 - P_1)}{8\eta L}$$

The length and radius of the needle are L and R, respectively, both of which have the dimension [L]. The pressures at opposite ends of the needle are P_2 and P_1, both of which have the dimensions of [M]/{[L] [T]^2}. The symbol η represents the viscosity of the liquid and has the dimensions of [M]/{[L][T]}. The symbol π stands for pi and, like the number 8 and the exponent n, has no dimensions. Using dimensional analysis, determine the value of n in the expression for Q.

(a) 1 **(b)** 3
(c) 2 **(d)** 4

10. A quantity X is given by $\varepsilon_0 L \Delta V/\Delta t$, where ε_0 is the permittivity of free space, L is the length, ΔV is the potential difference and Δt is the time interval. The dimensional formula for X is the same as that of

(a) resistance. **(b)** charge.
(c) voltage. **(d)** current.

11. The main scale of Vernier calipers reads in millimeter. Its Vernier is divided into 10 divisions, which coincide with 9 divisions of the main scale. When there is nothing between its jaws, the fifth division of its Vernier scale coincides with a division on the main scale. Also, zero of the Vernier scale lies on the right side of the zero of the main scale. When a coin is tightly gripped by its jaws to measure the diameter, the zero of the Vernier scale is observed to be slightly left to the 3.4 cm and the third Vernier scale division coincides with a main scale division. The diameter of the coin is

(a) 3.27 cm **(b)** 3.28 cm
(c) 3.33 cm **(d)** 3.23 cm

12. At the end of a year, a motor car company announces that sales of a pickup are down by 43% for the year. If sales continue to decrease by 43% in each succeeding year, how long will it take for sales to decrease to zero?

(a) 1 year **(b)** 2 years
(c) 3 years **(d)** More than 5 years

13. A body travels uniformly a distance of (10.2 ± 0.4) m in a time interval (6.0 ± 0.2) s. The speed of the particle is best expressed as (in m/s^1).

(a) 1.7 ± 0.12 **(b)** 1.7 ± 0.01
(c) 1.7 ± 0.07 **(d)** 1.7 ± 0.00

14. In two systems of relations among velocity, acceleration and force are, respectively, $v_2 = \alpha^2/\beta\, v_1$, $a_2 = \alpha\beta a_1$ and $F_2 = F_1/\alpha\beta$. If α and β are constants, then relations among mass, length and time in two systems are

(a) $M_2 = \frac{\alpha}{\beta} M_1,\ L_2 = \frac{\alpha^2}{\beta^2} L_1,\ T_2 = \frac{\alpha^3 T_1}{\beta}$

(b) $M_2 = \frac{1}{\alpha^2\beta^2} M_1,\ L_2 = \frac{\alpha^3}{\beta^3} L_1,\ T_2 = T_1 \frac{\alpha}{\beta^2}$

(c) $M_2 = \frac{\alpha^3}{\beta^3} M_1,\ L_2 = \frac{\alpha^2}{\beta^2} L_1,\ T_2 = \frac{\alpha}{\beta} T_1$

(d) $M_2 = \frac{\alpha^2}{\beta^2} M_1,\ L_2 = \frac{\alpha}{\beta^2} L_1,\ T_2 = \frac{\alpha^3}{\beta^3} T_1$

15. Metric time is defined so that one day equals 10 hours; one hour equals 100 minutes; and one minute equals 100 seconds. One metric second equals how many normal seconds?

(a) 0.60 **(b)** 0.864
(c) 1.00 **(d)** 1.16

16. The equation of the stationary wave is $y = 2a \sin(2\pi ct/\lambda) \cos(2\pi x/\lambda)$, which of the following statements is wrong?

(a) The unit of ct is same as that of λ.
(b) The unit of x is same as that of λ.
(c) The unit of $2\pi c/\lambda$ is same as that of $2\pi x/\lambda t$.
(d) The unit of c/λ is same as that of x/λ.

17. A spring is hanging down from the ceiling, and an object of mass m is attached to the free end. The object is pulled down, thereby stretching the spring, and then released. The object oscillates up and down, and the time T required for one complete up-and-down oscillation is given by the equation $T = 2\pi\sqrt{m/k}$, where k is known as the spring constant. What must be the dimension of k for this equation to be dimensionally correct?

(a) $\frac{[M]}{[T]}$ **(b)** $\frac{[T]}{[M]}$

(c) $\frac{[M]}{[T]^2}$ **(d)** $\frac{[T]}{[M]^2}$

18. A small steel ball of radius r is allowed to fall under gravity through a column of a viscous liquid of coefficient of viscosity. After some time, the velocity of the ball attains a constant value known as terminal velocity v_T. The terminal velocity depends on (i) the mass of the ball, (ii) η, (iii) r and (iv) acceleration due to gravity g. Which of the following relations is dimensionally correct?

(a) $v_T \propto \frac{mg}{\eta r}$ **(b)** $v_T \propto \frac{\eta r}{mg}$

(c) $v_T \propto \eta r m g$ **(d)** $v_T \propto \frac{mgr}{\eta}$

19. The term $(1/2)\ \rho v^2$ occurs in Bernoulli's equation, with ρ being the density of a fluid and v its speed. The dimensions of this term are

(a) $[M^{-1} L^5 T^2]$ **(b)** $[MLT^2]$
(c) $[ML^{-1}T^{-2}]$ **(d)** $[M^{-1}L^9T^{-2}]$

20. The variables x, v, and a have the dimensions of [L], [L]/[T], and [L]/[T]^2, respectively. These variables are related by an equation that has the form $v^n = 2ax$, where n is an integer constant (1, 2, 3, etc.) without dimensions. What must be the value of n, so that both sides of the equation have the same dimensions? Explain your reasoning.

(a) 1 **(b)** 3
(c) 2 **(d)** 4

21. The SI standard of length is based on:

(a) the distance from the north pole to the equator along a meridian passing through Paris.
(b) wavelength of light emitted by Hg^{198}.
(c) wavelength of light emitted by Kr^{86}.
(d) the speed of light.

22. With the usual notations, the following equation $S_t - u + 1/2a(2t-1)$ is
(a) only numerically correct.
(b) only dimensionally correct.
(c) both numerically and dimensionally correct.
(d) Neither numerically nor dimensionally correct.

23. If the time period (T) of vibration of a liquid drop depends on surface tension (S), radius (r) of the drop and density (ρ) of the liquid, then the expression of T is
(a) $T = K\sqrt{\rho r^3/S}$ **(b)** $T = K\sqrt{\rho^{1/2} r^3/S}$
(c) $T = K\sqrt{\rho r^3/S^{1/2}}$ **(d)** None of these

24. The relative density of material of a body is found by weighing it first in air and then in water. If the weight in air is (5.00 ± 0.05) N and weight in water is (4.00 ± 0.05) N. Then, the relative density along with the maximum permissible percentage error is
(a) $5.0 \pm 11\%$ **(b)** $5.0 \pm 1\%$
(c) $5.0 \pm 6\%$ **(d)** $1.25 \pm 5\%$

25. A physical quantity $P = B^2l^2/m$ where B is the magnetic induction, l is the length and m is the mass. The dimensions of P is
(a) MLT^{-3} **(b)** $ML^2T^{-4}I^{-2}$
(c) $M^2L^2T^{-4}I$ **(d)** $MLT^{-2}I^{-2}$

26. If each frame of a motion picture film is 35 cm high, and 24 frames go by in a second, estimate how many frames are needed to show a two hour long movie.
(a) 1400 **(b)** 25 000
(c) 50 000 **(d)** 170 000

27. The famous Stefan's law of radiation states that the rate of emission of thermal radiation per unit by a black body is proportional to area and fourth power of its absolute temperature that is $Q = \sigma AT^4$ where A = area, T = temperature and σ is a universal constant. In the 'energy–length–time temperature' (E–L–T–K) system the dimension of σ is
(a) $E^2T^2L^{-2}K^{-2}$ **(b)** $E^{-1}T^{-2}L^{-2}K^{-1}$
(c) $ET^{-1}L^{-3}K^{-4}$ **(d)** $ET^{-1}L^{-2}K^{-4}$

28. If the velocity of light c, Planck's constant h and time t are taken as basis of fundamental units, then the dimension of force will change to
(a) $hc^{-1}t^{-2}$ **(b)** $hc^{-1}t^2$
(c) $hc^{-1}t^{-1}$ **(d)** $h^{-1}c^{-1}t^{-2}$

29. If the acceleration due to gravity be taken as the unit of acceleration and the velocity generated in a falling body in one second as the unit of velocity then
(a) the new unit of length is g meter.
(b) the new unit of length is 1 meter.
(c) the new unit of length is g^2 meter.
(d) the new unit of time is $1/g$ second.

30. The largest mass (m) that can be moved by a flowing river depends on velocity (v), density (ρ) of river water and acceleration due to gravity (g). The correct relation is
(a) $m \propto \dfrac{\rho^2 v^4}{g^2}$ **(b)** $m \propto \dfrac{\rho v^6}{g^2}$
(c) $m \propto \dfrac{\rho v^4}{g^3}$ **(d)** $m \propto \dfrac{\rho v^6}{g^3}$

31. If velocity v, acceleration A and force F are chosen as fundamental quantities, then the dimensional formula of angular momentum in terms of v, A and F would be
(a) $FA^{-1}v$ **(b)** Fv^3A^{-2}
(c) Fv^2A^{-1} **(d)** $F^2v^2A^{-1}$

More than One Correct Choice Type

32. Which of the following pairs have different dimensions?
(a) Frequency and angular velocity
(b) Tension and surface tension
(c) Density and energy density
(d) Linear momentum and angular momentum

33. L, C and R represent physical quantities of inductance, capacitance and resistance, respectively. The combination which has the dimensions of frequency is
(a) $1/RC$ **(b)** R/L
(c) $\sqrt{LC}$ **(d)** C/L

34. Let $[\varepsilon_0]$ denote the dimensional formula of the permittivity of the vacuum and $[\mu_0]$ that of the permeability of the vacuum. If M is the mass, L is the length, T is the time and I is the electric current, then
(a) $[\varepsilon_0] = M^{-1}L^3T^2I$
(b) $[\varepsilon_0] = M^{-1}L^{-3}T^4I^2$
(c) $[\mu_0] = MLT^{-2}I^{-2}$
(d) $[\mu_0] = ML^2T^{-1}I$

35. The velocity, acceleration and force in two systems of units are related as under:
(i) $v' = \dfrac{\alpha^2}{\beta} v$
(ii) $a' = (\alpha\beta)a$
(iii) $F' = \left(\dfrac{1}{\alpha\beta}\right)F$

All the primed symbols belong to one system and unprimed ones belong to the other system. Here α and β are dimensionless constants. Which of the following is/are correct?
(a) Length standards of the two systems are related by:

$$L' = \left(\frac{\alpha^3}{\beta^3}\right)L$$

(b) Mass standards of the two systems are related by:

$$m' = \left(\frac{1}{\alpha^2\beta^2}\right)m$$

(c) Time standards of the two systems are related by:

$$T' = \left(\frac{\alpha}{\beta^2}\right)T$$

(d) Momentum standards of the two systems are related by:

$$P' = \left(\frac{1}{\beta^3}\right)P$$

Linked Comprehension

***Paragraph for Questions 36 and 37*:** Astronomical distances are so large compared to terrestrial ones that much larger units of length are used for easy comprehension of the relative distances of astronomical objects. An astronomical unit (AU) is equal to the average distance from Earth to the Sun, 1.50×10^8 km. A parsec (pc) is the distance at which 1 AU would subtend an angle of 1 second of arc. A light year (ly) is the distance that light, traveling through a vacuum with a speed of 3.00×10^5 km/s, would cover in 1 year.

36. What is the distance from Earth to the Sun in parsecs?

(a) 4.85×10^{-6} pc
(b) 5.85×10^{-6} pc
(c) 4.85×10^{-5} pc
(d) 3.85×10^{-6} pc

37. Express a light – year and a parsec in kilometers.

(a) 9.48×10^{11} km; 4.08×10^{13} km
(b) 9.48×10^{12} km; 3.08×10^{13} km
(c) 9.48×10^{12} km; 3.08×10^{14} km
(d) 8.48×10^{12} km; 4.08×10^{13} km

Matrix-Match

38. Match Column I (Event) with Column II (order of the time interval for happening of the event) and select the correct combination from the options given below.

Column I		Column II	
(i)	Rotation period of Earth	**(p)**	10^5 s
(ii)	Revolution period of Earth	**(q)**	10^7 s
(iii)	Period of a light wave	**(r)**	10^{-15} s
(iv)	Period of a sound wave	**(s)**	10^{-3} s

(a) (i) → (p), (ii) → (q), (iii) → (r), (iv) → (s)
(b) (i) → (q), (ii) → (p), (iii) → (s), (iv) → (r)
(c) (i) → (p), (ii) → (q), (iv) → (s), (iv) → (r)
(d) (i) → (q), (ii) → (p), (v) → (r), (iv) → (s)

39. Match the physical quantities in Column I with the units given in Column II.

Column I	Column II
(a) GM_eM_s G – universal gravitational constant, M_e – mass of the Earth, M_s – mass of the Sun.	**(p)** (volt)(coulomb)(meter)
(b) $\frac{3RT}{M}$ R – universal gas constant, T – absolute temperature, M – molar mass.	**(q)** (kilogram)(meter)^3(second)$^{-2}$
(c) $\frac{F^2}{q^2B^2}$ F – force, q – charge, B – magnetic field.	**(r)** (meter)^2(second)$^{-2}$
(d) $\frac{GM_e}{R_e}$ G – universal gravitational constant, M_e – mass of the Earth, R_e – radius of the Earth.	**(s)** (farad)(volt)^2(kg)$^{-1}$

Directions for Questions 40 and 41: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

40. The physical quantities, their formulae and dimensions expressed in terms of fundamental quantities are given in Column I, Column II and Column III, respectively.

Column I		Column II		Column III	
(I)	Angular momentum	**(i)**	Force × Distance	**(J)**	$[M^1L^2T^{-2}]$
(II)	Stefan's constant	**(ii)**	Moment of Inertia × Angular Velocity	**(K)**	$[M^1L^1T^{-1}]$
(III)	Planck's constant	**(iii)**	$\frac{\text{Energy}}{\text{Frequency}}$	**(L)**	$[M^1L^0T^{-3}K^{-4}]$
(IV)	Torque	**(iv)**	$\frac{\left(\frac{\text{Energy}}{\text{Area} \times \text{Time}}\right)}{(\text{Temperature})^4}$	**(M)**	$[M^1L^2T^{-1}]$

(1) Which of the options correctly represents the physical quantity that remains conserved in planetary motion?

(a) (III) (i) (K) **(b)** (I) (ii) (M)
(c) (II) (iv) (J) **(d)** (IV) (iii) (L)

(2) Which one of the following options correctly represents the physical quantity with the same dimensions as that of angular momentum?

(a) (I) (ii) (J) **(b)** (IV) (iii) (L)
(c) (III) (iii) (M) **(d)** (II) (iv) (M)

(3) Which one of the following options represents correct combination for relation between power radiated and temperature?

(a) (IV) (i) (M)
(b) (III) (iii) (K)
(c) (I) (ii) (J)
(d) (II) (iv) (L)

41. Suppose two students are trying to make a new measurement system so that they can use it like a code measurement. Instead of taking 1 kg, 1 m and 1 s, as the basic unit they took unit of mass as α kg, the unit of length as β m and the unit of time as γ s. They called the new system

as ACME. Column I represents the units of the physical quantities, Column II represents the ACME units and Column III represents the CGS units of the physical quantities.

Column I	Column II	Column III
(I) 1 N in new system	**(i)** $[\alpha^{-1}\beta^{-2}\gamma^{2}]$	**(J)** g cm^2 s^{-3}
(II) 1 J in new system	**(ii)** $[\alpha^{2}\beta^{2}\gamma^{-3}]$	**(K)** dyne cm^{-2}
(III) 1 Pa in new system	**(iii)** $[\alpha^{-1}\beta^{1}\gamma^{2}]$	**(L)** dyne
(IV) 1 W in new system	**(iv)** $[\alpha^{-1}\beta^{-1}\gamma^{2}]$	**(M)** erg

(1) Which of the following options is a correct representation of physical quantity measured by Torricelli?
(a) (II) (i) (J) **(b)** (IV) (ii) (M)
(c) (I) (iv) (L) **(d)** (III) (iii) (K)

(2) Which of the following represents the correct combination for tension?
(a) (I) (iv) (L) **(b)** (III) (ii) (K)
(c) (IV) (iii) (M) **(d)** (II) (i) (J)

(3) Which of the following represents the correct combination for a physical quantity known as rate of energy?
(a) (III) (iii) (L) **(b)** (IV) (ii) (J)
(c) (II) (i) (M) **(d)** (I) (iv) (K)

Integer Type

42. The resistance R is given by relation $R = V/I$. If potential difference $V = (100 \pm 5)$ V and current $I = (10 \pm 0.2)$ A, calculate the percentage error in R.

43. Estimate the wavelength at which plasma reflection will occur for a metal having the density of electrons $N \approx 4 \times 10^{27}$ m^{-3}. Taking $\varepsilon_0 = 10^{-11}$ and m $\approx 10^{-30}$, where these quantities are in proper SI units.

44. The time period of oscillation of a simple pendulum is given by $T = 2\pi\sqrt{l/g}$. The length of the pendulum is measured as $1 = 10 \pm 0.1$ cm and the time period as $T = 0.2 \pm 0.02$ s. Determine percentage error in the value of g.

ANSWER KEY

Problems

1. 0.27 point^2 **2.** 0.45 m/s **3.** (a) 1.7 picas; (b) 20 points **4.** 1.7×10^5 frames
5. 6.13×10^5 miles **6.** 0.038 mm **7.** 8.4×10^2 km **8.** 6.3×10^{-12} s
9. 8.64 s $\approx$ 9.0 s **10.** (a) 495 s; (b) 141 s; (c) 198 s; (d) –245 s **11.** (a) 52.6 min; (b) 4.9%
12. (a) 4.44×10^8 rotations; (b) 155780644887275 s; (c) $\pm 3 \times 10^{-11}$ s
13. important criterion is the consistency of the daily variation, not its magnitude. The clocks in order of ranking from best to worst – C, D, A, B, E.
14. 1.64×10^4 s $\approx$ 4.6 hours **15.** (a) 1.519 m^2; (b) 77.34 km **16.** 1.3×10^3 kg/m^3
17. (a) 2.542 kg; (b) 0.02 **19.** (a) 4.21 s; (b) 23.2 g; (c) 1.27×10^{-2}; (c) -6.05×10^{-3} kg/min
20. 1.43 kg/min

Practice Questions

Single Correct Choice Type

1. (b) **2.** (d) **3.** (c) **4.** (c) **5.** (d)
6. (a) **7.** (d) **8.** (c) **9.** (d) **10.** (d)
11. (b) **12.** (d) **13.** (a) **14.** (b) **15.** (b)
16. (d) **17.** (c) **18.** (a) **19.** (c) **20.** (c)
21. (d) **22.** (c) **23.** (a) **24.** (a) **25.** (b)
26. (d) **27.** (d) **28.** (a) **29.** (a) **30.** (d)
31. (b)

More than One Correct Choice Type

32. (b), (c), (d) **33.** (a), (b) **34.** (b), (c) **35.** (a), (b), (c), (d)

Linked Comprehension

36. (a) **37.** (b)

Matrix-Match

38. (a)

39. (a) → (p), (q); (b) → (r), (s); (c) → (r), (s); (d) → (r), (s)

40. (1) → (b); (2) → (c); (3) → (d)

41. (1) → (b); (2) → (a); (3) → (b)

Integer Type

42. 7 **43.** 600 **44.** 5

CHAPTER

2

Motion Along a Straight Line

2.1 | WHAT IS PHYSICS?

One purpose of physics is to study the motion of objects—how fast they move, for example, and how far they move in a given amount of time. NASCAR engineers are fanatical about this aspect of physics as they determine the performance of their cars before and during a race. Geologists use this physics to measure tectonic-plate motion as they attempt to predict earthquakes. Medical researchers need this physics to map the blood flow through a patient when diagnosing a partially closed artery, and motorists use it to determine how they might slow sufficiently when their radar detector sounds a warning. There are countless other examples. In this chapter, we study the basic physics of motion where the object (race car, tectonic plate, blood cell, or any other object) moves along a single axis. Such motion is called *one-dimensional motion*.

2.2 | MOTION

The world, and everything in it, moves. Even seemingly stationary things, such as a roadway, move with Earth's rotation, Earth's orbit around the Sun, the Sun's orbit around the center of the Milky Way galaxy, and that galaxy's migration relative to other galaxies. The classification and comparison of motions (called **kinematics**) is often challenging. What exactly do you measure, and how do you compare?

Before we attempt an answer, we shall examine some general properties of motion that is restricted in three ways.

1. The motion is along a straight line only. The line may be vertical, horizontal, or slanted, but it must be straight.
2. Forces (pushes and pulls) cause motion but will not be discussed until Chapter 5. In this chapter we discuss only the motion itself and changes in the motion. Does the moving object speed up, slow down, stop, or reverse direction? If the motion does change, how is time involved in the change?

Contents

2.1 What is Physics?
2.2 Motion
2.3 Position and Displacement
2.4 Average Velocity and Average Speed
2.5 Instantaneous Velocity and Speed
2.6 Acceleration
2.7 Constant Acceleration: A Special Case
2.8 Free-Fall Acceleration
2.9 Graphical Integration in Motion Analysis

3. The moving object is either a **particle** (by which we mean a point-like object such as an electron) or an object that moves like a particle (such that every portion moves in the same direction and at the same rate). A stiff pig slipping down a straight playground slide might be considered to be moving like a particle; however, a tumbling tumbleweed would not.

2.3 | POSITION AND DISPLACEMENT

Key Concepts

- The position x of a particle on an x axis locates the particle with respect to the origin, or zero point, of the axis.
- The position is either positive or negative, according to which side of the origin the particle is on, or zero if the particle is at the origin. The positive direction on an axis is the direction of increasing positive numbers; the opposite direction is the negative direction on the axis.
- The displacement Δx of a particle is the change in its position:

$$\Delta x = x_2 - x_1.$$

- Displacement is a vector quantity. It is positive if the particle has moved in the positive direction of the x axis and negative if the particle has moved in the negative direction.

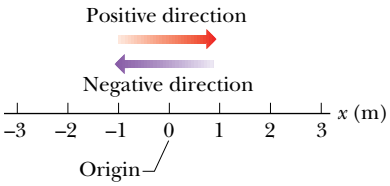


Figure 2-1 Position is determined on an axis that is marked in units of length (here meters) and that extends indefinitely in opposite directions. The axis name, here x, is always on the positive side of the origin.

To locate an object means to find its position relative to some reference point, often the **origin** (or zero point) of an axis such as the x axis in Fig 2-1. The **positive direction** of the axis is in the direction of increasing numbers (coordinates), which is to the right in Fig. 2-1. The opposite is the **negative direction**.

For example, a particle might be located at $x = 5$ m, which means it is 5 m in the positive direction from the origin. If it were at $x = -5$ m, it would be just as far from the origin but in the opposite direction. On the axis, a coordinate of –5 m is less than a coordinate of –1 m, and both coordinates are less than a coordinate of +5 m. A plus sign for a coordinate need not be shown, but a minus sign must always be shown.

A change from position x_1 to position x_2 is called a **displacement** Δx, where

$$\Delta x = x_2 - x_1. \tag{2-1}$$

(The symbol Δ, the Greek uppercase delta, represents a change in a quantity, and it means the final value of that quantity minus the initial value.) When numbers are inserted for the position values x_1 and x_2 in Eq. 2-1, a displacement in the positive direction (to the right in Fig. 2-1) always comes out positive, and a displacement in the opposite direction (left in the figure) always comes out negative. For example, if the particle moves from $x_1 = 5$ m to $x_2 = 12$ m, then the displacement is $\Delta x = (12\text{ m}) - (5\text{ m}) = +7$ m. The positive result indicates that the motion is in the positive direction. If, instead, the particle moves from $x_1 = 5$ m to $x_2 = 1$ m, then $\Delta x = (1\text{ m}) - (5\text{ m}) = -4$ m. The negative result indicates that the motion is in the negative direction.

The actual number of meters covered for a trip is irrelevant; displacement involves only the original and final positions. For example, if the particle moves from $x = 5$ m out to $x = 200$ m and then back to $x = 5$ m, the displacement from start to finish is $\Delta x = (5\text{ m}) - (5\text{ m}) = 0$.

Signs. A plus sign for a displacement need not be shown, but a minus sign must always be shown. If we ignore the sign (and thus the direction) of a displacement, we are left with the **magnitude** (or absolute value) of the displacement. For example, a displacement of $\Delta x = -4$ m has a magnitude of 4 m.

Displacement is an example of a **vector quantity**, which is a quantity that has both a direction and a magnitude. We explore vectors more fully in Chapter 3, but here all we need is the idea that displacement has two features: (1) Its *magnitude* is the distance (such as the number of meters) between the original and final positions. (2) Its *direction*, from an original position to a final position, can be represented by a plus sign or a minus sign if the motion is along a single axis.

Here is the first of many checkpoints where you can check your understanding with a bit of reasoning.

CHECKPOINT 1

Here are three pairs of initial and final positions, respectively, along an x axis. Which pairs give a negative displacement: (a) –3 m, +5 m; (b) –3 m, –7 m; (c) 7 m, –3 m?

2.4 | AVERAGE VELOCITY AND AVERAGE SPEED

Key Concepts

- When a particle has moved from position x_1 to position x_2 during a time interval $\Delta t = t_2 - t_1$, its average velocity during that interval is

$$v_{avg} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - x_1}.$$

- The algebraic sign of v_{avg} indicates the direction of motion (v_{avg} is a vector quantity). Average velocity does not depend on the actual distance a particle moves, but instead depends on its original and final positions.
- On a graph of x versus t, the average velocity for a time interval Δt is the slope of the straight line connecting the points on the curve that represent the two ends of the interval.
- The average speed s_{avg} of a particle during a time interval Δt depends on the total distance the particle moves in that time interval:

$$s_{avg} = \frac{\text{total distance}}{\Delta t}.$$

A compact way to describe position is with a graph of position x plotted as a function of time t–a graph of $x(t)$. (The notation $x(t)$ represents a function x of t, not the product x times t.) As a simple example, Fig. 2-2 shows the position function $x(t)$ for a stationary armadillo (which we treat as a particle) over a 7 s time interval. The animal's position stays at $x = -2$ m.

Figure 2-3 is more interesting, because it involves motion. The armadillo is apparently first noticed at $t = 0$ when it is at the position $x = -5$ m. It moves toward $x = 0$, passes through that point at $t = 3$ s, and then moves on

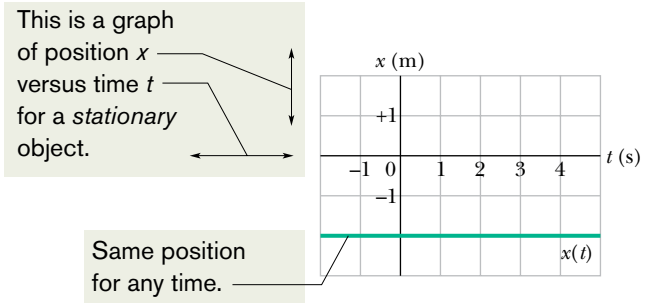


Figure 2-2 The graph of $x(t)$ for an armadillo that is stationary at $x = -2$ m. The value of x is –2 m for all times t.

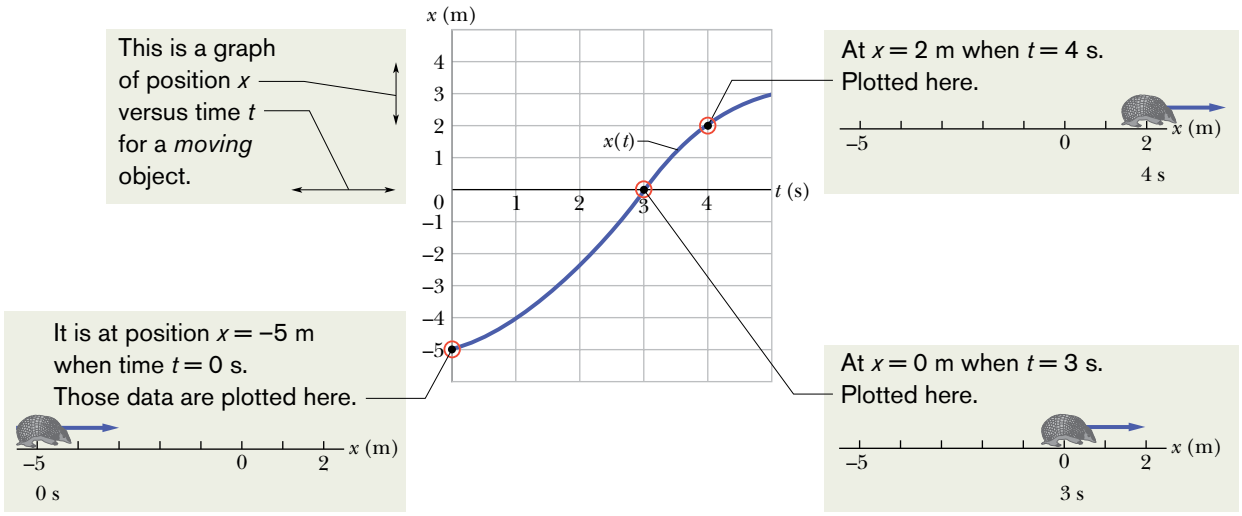


Figure 2-3 The graph of $x(t)$ for a moving armadillo. The path associated with the graph is also shown, at three times.

to increasingly larger positive values of x. Figure 2-3 also depicts the straight-line motion of the armadillo (at three times) and is something like what you would see. The graph in Fig. 2-3 is more abstract, but it reveals how fast the armadillo moves.

Actually, several quantities are associated with the phrase "how fast." One of them is the **average velocity** v_{avg}, which is the ratio of the displacement Δx that occurs during a particular time interval Δt to that interval:

$$v_{avg} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}. \tag{2-2}$$

The notation means that the position is x_1 at time t_1 and then x_2 at time t_2. A common unit for v_{avg} is the meter per second (m/s). You may see other units in the problems, but they are always in the form of length/time.

Graphs. On a graph of x versus t, v_{avg} is the **slope** of the straight line that connects two particular points on the $x(t)$ curve: one is the point that corresponds to x_2 and t_2, and the other is the point that corresponds to x_1 and t_1. Like displacement, v_{avg} has both magnitude and direction (it is another vector quantity). Its magnitude is the magnitude of the line's slope. A positive v_{avg} (and slope) tells us that the line slants upward to the right; a negative v_{avg} (and slope) tells us that the line slants downward to the right. The average velocity v_{avg} always has the same sign as the displacement Δx because Δt in Eq. 2-2 is always positive.

Figure 2-4 shows how to find v_{avg} in Fig. 2-3 for the time interval $t = 1$ s to $t = 4$ s. We draw the straight line that connects the point on the position curve at the beginning of the interval and the point on the curve at the end of the interval. Then we find the slope $\Delta x/\Delta t$ of the straight line. For the given time interval, the average velocity is

$$v_{avg} = \frac{6\text{ m}}{3\text{ s}} = 2\text{ m/s}.$$

Average speed s_{avg} is a different way of describing "how fast" a particle moves. Whereas the average velocity involves the particle's displacement Δx, the average speed involves the total distance covered (for example, the number of meters moved), independent of direction; that is,

$$s_{avg} = \frac{\text{total distance}}{\Delta t}. \tag{2-3}$$

Because average speed does *not* include direction, it lacks any algebraic sign. Sometimes s_{avg} is the same (except for the absence of a sign) as v_{avg}. However, the two can be quite different.

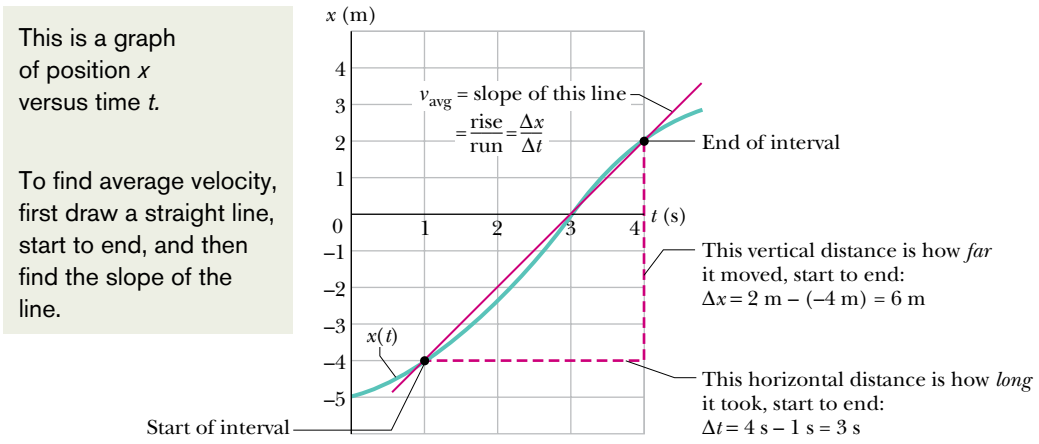

This is a graph of position x versus time t.
To find average velocity, first draw a straight line, start to end, and then find the slope of the line.
x (m)
t (s)
v_avg = slope of this line
= rise/run = Δx/Δt
End of interval
This vertical distance is how far it moved, start to end:
Δx = 2 m – (–4 m) = 6 m
This horizontal distance is how long it took, start to end:
Δt = 4 s – 1 s = 3 s
x(t)
Start of interval

Figure 2-4 Calculation of the average velocity between $t = 1$ s and $t = 4$ s as the slope of the line that connects the points on the $x(t)$ curve representing those times. The swirling icon indicates that a figure is available in *WileyPLUS* as an animation with voiceover.

SAMPLE PROBLEM 2.01

Average velocity, beat-up pickup truck

You drive a beat-up pickup truck along a straight road for 8.4 km at 70 km/h, at which point the truck runs out of gasoline and stops. Over the next 30 min, you walk another 2.0 km farther along the road to a gasoline station.

(a) What is your overall displacement from the beginning of your drive to your arrival at the station?

KEY IDEA

Assume, for convenience, that you move in the positive direction of an x axis, from a first position of $x_1 = 0$ to a second position of x_2 at the station. That second position must be at $x_2 = 8.4\text{ km} + 2.0\text{ km} = 10.4\text{ km}$. Then your displacement Δx along the x axis is the second position minus the first position.

Calculation: From Eq. 2-1, we have

$$\Delta x = x_2 - x_1 = 10.4\text{ km} - 0 = 10.4\text{ km}. \qquad \text{(Answer)}$$

Thus, your overall displacement is 10.4 km in the positive direction of the x axis.

(b) What is the time interval Δt from the beginning of your drive to your arrival at the station?

KEY IDEA

We already know the walking time interval Δt_{wlk} (= 0.50 h), but we lack the driving time interval Δt_{dr}. However, we know that for the drive the displacement Δx_{dr} is 8.4 km and the average velocity $v_{avg,dr}$ is 70 km/h. Thus, this average velocity is the ratio of the displacement for the drive to the time interval for the drive.

Calculations: We first write

$$v_{avg,dr} = \frac{\Delta x_{dr}}{\Delta t_{dr}}.$$

Rearranging and substituting data then give us

$$\Delta t_{dr} = \frac{\Delta x_{dr}}{v_{avg,dr}} = \frac{8.4\text{ km}}{70\text{ km/h}} = 0.12\text{ h}.$$

So,

$$\Delta t = \Delta t_{dr} + \Delta t_{wlk} = 0.12\text{ h} + 0.50\text{ h} = 0.62\text{ h}. \qquad \text{(Answer)}$$

(c) What is your average velocity v_{avg} from the beginning of your drive to your arrival at the station? Find it both numerically and graphically.

KEY IDEA

From Eq. 2-2 we know that v_{avg} *for the entire trip* is the ratio of the displacement of 10.4 km *for the entire trip* to the time interval of 0.62 h *for the entire trip*.

Calculation: Here we find

$$v_{avg} = \frac{\Delta x}{\Delta t} = \frac{10.4\text{ km}}{0.62\text{ h}} = 16.8\text{ km/h} \approx 17\text{ km/h}. \qquad \text{(Answer)}$$

To find v_{avg} graphically, first we graph the function $x(t)$ as shown in Fig. 2-5, where the beginning and arrival points on the graph are the origin and the point labeled as "Station." Your average velocity is the slope of the straight line connecting those points; that is, v_{avg} is the ratio of the *rise* ($\Delta x = 10.4$ km) to the *run* ($\Delta t = 0.62$ h), which gives us $v_{avg} = 16.8$ km/h.

(d) Suppose that to pump the gasoline, pay for it, and walk back to the truck takes you another 45 min. What is your average speed from the beginning of your drive to your return to the truck with the gasoline?

KEY IDEA

Your average speed is the ratio of the total distance you move to the total time interval you take to make that move.

Calculation: The total distance is 8.4 km + 2.0 km + 2.0 km = 12.4 km. The total time interval is 0.12 h + 0.50 h + 0.75 h = 1.37 h. Thus, Eq. 2-3 gives us

$$s_{avg} = \frac{12.4\text{ km}}{1.37\text{ h}} = 9.1\text{ km/h}. \qquad \text{(Answer)}$$

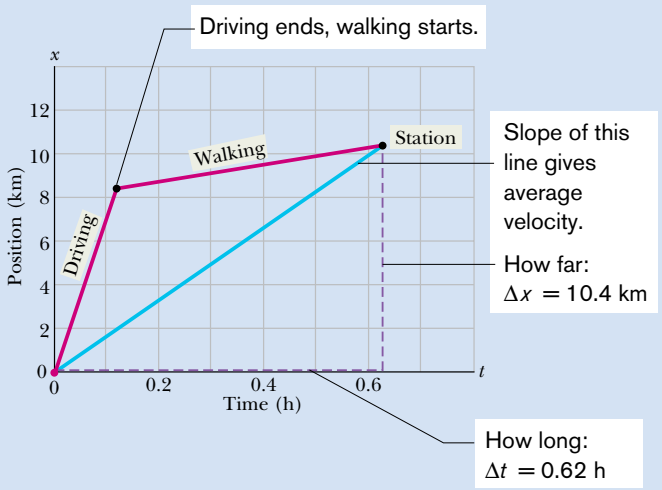


Figure 2-5 The lines marked "Driving" and "Walking" are the position–time plots for the driving and walking stages. (The plot for the walking stage assumes a constant rate of walking.) The slope of the straight line joining the origin and the point labelled "Station" is the average velocity for the trip, from the beginning to the station.

PROBLEM-SOLVING TACTICS

Tactic 1: Do You Understand the Problem? The common difficulty is simply not understanding the problem. The best test of understanding is this: Can *you* explain the problem?

Write down the given data, with units, using the symbols of the chapter. (In Sample Problem 2.01, the given data allow you to find your net displacement Δx in part (a) and the corresponding time interval Δt in part (b).) Identify the unknown and its symbol. (In the sample problem, the unknown in part (c) is your average velocity v_{avg}.) Then find the connection between the unknown and the data. (The connection is provided by Eq. 2-2, the definition of average velocity.)

Tactic 2: Are the Units OK? Be sure to use a consistent set of units when putting numbers into the equations. In Sample Problem 2.01, the logical units in terms of the given data are kilometers for distances, hours for time intervals, and kilometers per hour for velocities. You may sometimes need to convert units.

Tactic 3: Is Your Answer Reasonable? Does your answer make sense, or is it far too large or far too small? Is the sign correct? Are the units appropriate? In part (c) of Sample Problem 2.01, for example, the correct answer is 17 km/h. If you find 0.00017 km/h, –17 km/h, 17 km/s, or 17000 km/h, you should realize at once that you have done something wrong. The error may lie in your method, in your algebra, or in your keystroking of numbers on a calculator.

Tactic 4: Reading a Graph Figures 2-2, 2-3*a*, 2-4, and 2-5 are graphs you should be able to read easily. In each graph, the variable on the horizontal axis is the time t, with the direction of increasing time to the right. In each, the variable on the vertical axis is the position x of the moving particle with respect to the origin, with the positive direction of x upward. Always note the units (seconds or minutes; meters or kilometers) in which the variables are expressed.

2.5 | INSTANTANEOUS VELOCITY AND SPEED

Key Concepts

- The instantaneous velocity (or simply velocity) v of a moving particle is
$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt},$$
where $\Delta x = x_2 - x_1$ and $\Delta t = t_2 - t_1$.
- The instantaneous velocity (at a particular time) may be found as the slope (at that particular time) of the graph of x versus t.
- Speed is the magnitude of instantaneous velocity.

You have now seen two ways to describe how fast something moves: average velocity and average speed, both of which are measured over a time interval Δt. However, the phrase "how fast" more commonly refers to how fast a particle is moving at a given instant—its **instantaneous velocity** (or simply **velocity**) v.

The velocity at any instant is obtained from the average velocity by shrinking the time interval Δt closer and closer to 0. As Δt dwindles, the average velocity approaches a limiting value, which is the velocity at that instant:

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}. \tag{2-4}$$

Note that v is the rate at which position x is changing with time at a given instant; that is, v is the derivative of x with respect to t. Also note that v at any instant is the slope of the position–time curve at the point representing that instant. Velocity is another vector quantity and thus has an associated direction.

Speed is the magnitude of velocity; that is, speed is velocity that has been stripped of any indication of direction, either in words or via an algebraic sign. (*Caution:* Speed and average speed can be quite different.) A velocity of +5 m/s and one of –5 m/s both have an associated speed of 5 m/s. The speedometer in a car measures speed, not velocity (it cannot determine the direction).

CHECKPOINT 2

The following equations give the position $x(t)$ of a particle in four situations (in each equation, x is in meters, t is in seconds, and $t > 0$): (1) $x = 3t - 2$; (2) $x = -4t^2 - 2$; (3) $x = 2/t^2$; and (4) $x = -2$. (a) In which situation is the velocity v of the particle constant? (b) In which is v in the negative x direction?

SAMPLE PROBLEM 2.02

Velocity and slope of *x* versus *t*, elevator cab

Figure 2-6*a* is an $x(t)$ plot for an elevator cab that is initially stationary, then moves upward (which we take to be the positive direction of x), and then stops. Plot $v(t)$.

KEY IDEA

We can find the velocity at any time from the slope of the $x(t)$ curve at that time.

Calculations: The slope of $x(t)$, and so also the velocity, is zero in the intervals from 0 to 1 s and from 9 s on, so then the cab is stationary. During the interval bc, the slope is constant and nonzero, so then the cab moves with constant velocity. We calculate the slope of $x(t)$ then as

$$\frac{\Delta x}{\Delta t} = v = \frac{24\text{ m} - 4.0\text{ m}}{8.0\text{ s} - 3.0\text{ s}}$$
$$= +4.0\text{ m/s} \qquad (2\text{-}5)$$

The plus sign indicates that the cab is moving in the positive x direction. These intervals (where $v = 0$ and $v = 4$ m/s) are plotted in Fig. 2-6*b*. In addition, as the cab

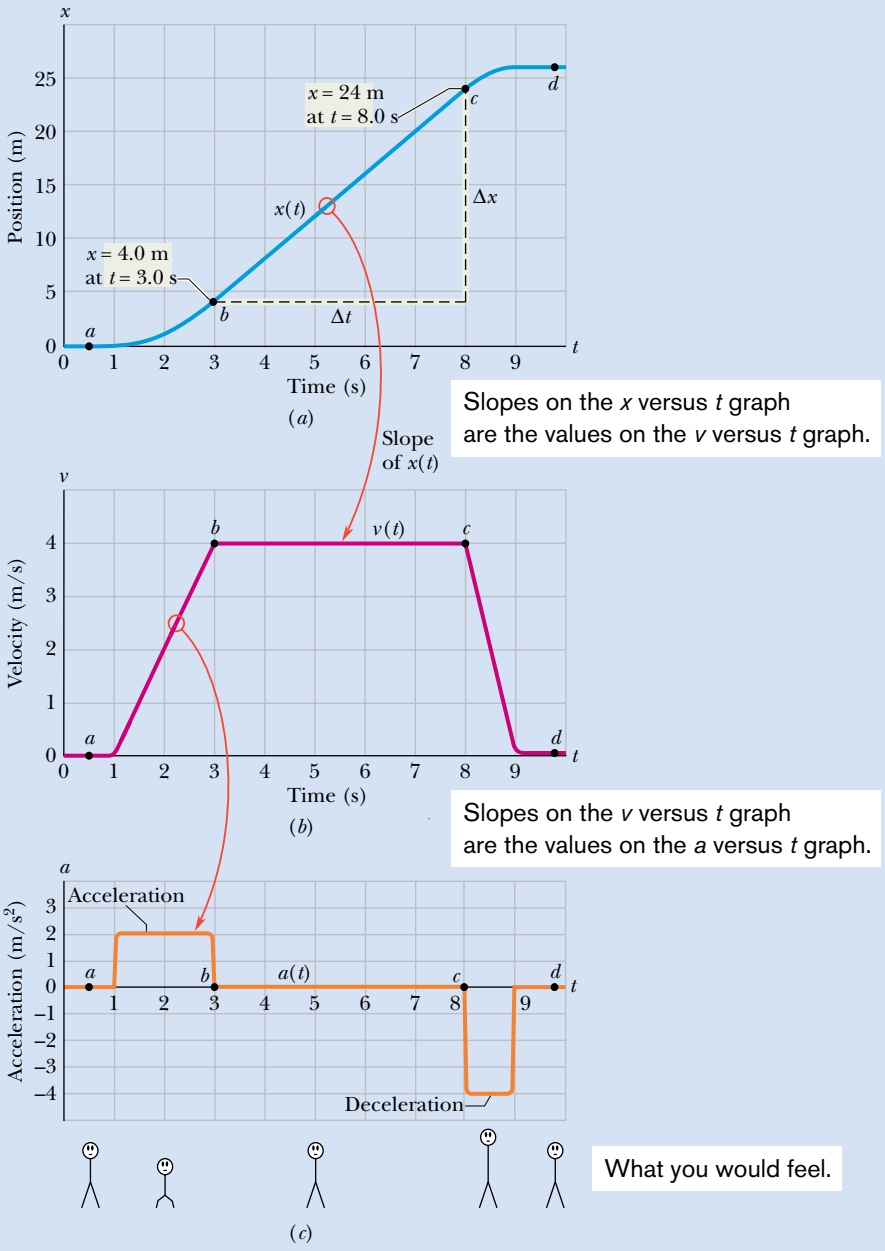


Figure 2-6 (*a*) The $x(t)$ curve for an elevator cab that moves upward along an x axis. (*b*) The $v(t)$ curve for the cab. Note that it is the derivative of the $x(t)$ curve ($v = dx/dt$). (*c*) The $a(t)$ curve for the cab. It is the derivative of the $v(t)$ curve ($a = dv/dt$). The stick figures along the bottom suggest how a passenger's body might feel during the accelerations.

initially begins to move and then later slows to a stop, v varies as indicated in the intervals 1 s to 3 s and 8 s to 9 s. Thus, Fig. 2-6*b* is the required plot. (Figure 2-6*c* is considered in Section 2.6.)

Given a $v(t)$ graph such as Fig. 2-6*b*, we could "work backward" to produce the shape of the associated $x(t)$ graph (Fig. 2-6*a*). However, we would not know the actual values for x at various times, because the $v(t)$ graph indicates only *changes* in x. To find such a change in x during any interval, we must, in the language of calculus, calculate the area "under the curve" on the $v(t)$ graph for that interval. For example, during the interval 3 s to 8 s in which the cab has a velocity of 4.0 m/s, the change in x is

$$\Delta x = (4.0 \text{ m/s})(8.0 \text{ s} - 3.0 \text{ s}) = +20 \text{ m}. \qquad (2\text{-}6)$$

(This area is positive because the $v(t)$ curve is above the t axis.) Figure 2-6*a* shows that x does indeed increase by 20 m in that interval. However, Fig. 2-6*b* does not tell us the *values* of x at the beginning and end of the interval. For that, we need additional information, such as the value of x at some instant.

SAMPLE PROBLEM 2.03

Average and instantaneous velocity

The position of a particle moving along the x axis is given in centimeters by $x = 9.75 + 1.50t^3$, where t is in seconds. Calculate (a) the average velocity during the time interval $t = 2.00$ s to $t = 3.00$ s; (b) the instantaneous velocity at $t = 2.00$ s; (c) the instantaneous velocity at $t = 3.00$ s; (d) the instantaneous velocity at $t = 2.50$ s; and (e) the instantaneous velocity when the particle is midway between its positions at $t = 2.00$ s and $t = 3.00$ s. (f) Graph x versus t and indicate your answers graphically

KEY IDEA

We use Eq. 2-2 for average velocity and Eq. 2-4 for instantaneous velocity, and work with distances in centimeters and times in seconds.

Calculations: (a) We plug into the given equation for x for $t = 2.00$ s and $t = 3.00$ s and obtain $x_2 = 21.75$ cm and $x_3 = 50.25$ cm, respectively. The average velocity during the time interval $2.00 \le t \le 3.00$ s is

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t} = \frac{50.25 \text{ cm} - 21.75 \text{ cm}}{3.00 \text{ s} - 2.00 \text{ s}}$$

which yields $v_{\text{avg}} = 28.5$ cm/s.

(b) The instantaneous velocity is $v = \frac{dx}{dt} = 4.5t^2$, which, at time $t = 2.00$ s, yields $v = (4.5)(2.00)^2 = 18.0$ cm/s.

(c) At $t = 3.00$ s, the instantaneous velocity is

$$v = (4.5)(3.00)^2 = 40.5 \text{ cm/s}.$$

(d) At $t = 2.50$ s, the instantaneous velocity is

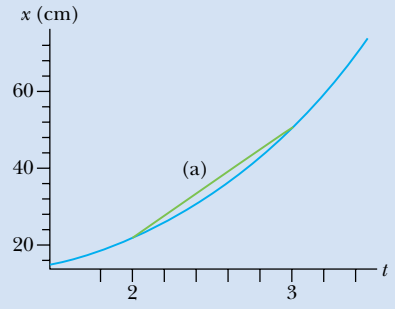
$$v = (4.5)(2.50)^2 = 28.1 \text{ cm/s}.$$

(e) Let t_m stand for the moment when the particle is midway between x_2 and x_3 (that is, when the particle is at $x_m = (x_2 + x_3)/2 = 36$ cm). Therefore,

$$x_m = 9.75 + 1.5t_m^3 \Rightarrow t_m = 2.596$$

in seconds. Thus, the instantaneous speed at this time is $v = 4.5(2.596)^2 = 30.3$ cm/s.

(f) The answer to part (a) is given by the slope of the straight line between $t = 2$ and $t = 3$ in this x-vs-t plot. The answers to parts (b), (c), (d), and (e) correspond to the slopes of tangent lines (not shown but easily imagined) to the curve at the appropriate points.



2.6 | ACCELERATION

Key Concepts

- Average acceleration is the ratio of a change in velocity Δv to the time interval Δt in which the change occurs:

$$a_{avg} = \frac{\Delta v}{\Delta t}.$$

  The algebraic sign indicates the direction of a_{avg}.
- Instantaneous acceleration (or simply acceleration) a is the first time derivative of velocity $v(t)$ and the second time derivative of position $x(t)$:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

- On a graph of v versus t, the acceleration a at any time t is the slope of the curve at the point that represents t.

When a particle's velocity changes, the particle is said to undergo **acceleration** (or to accelerate). For motion along an axis, the **average acceleration** a_{avg} over a time interval Δt is

$$a_{avg} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t}, \tag{2-7}$$

where the particle has velocity v_1 at time t_1 and then velocity v_2 at time t_2. The **instantaneous acceleration** (or simply **acceleration**) is

$$a = \frac{dv}{dt}. \tag{2-8}$$

In words, the acceleration of a particle at any instant is the rate at which its velocity is changing at that instant. Graphically, the acceleration at any point is the slope of the curve of $v(t)$ at that point. We can combine Eq. 2-8 with Eq. 2-4 to write

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}. \tag{2-9}$$

In words, the acceleration of a particle at any instant is the second derivative of its position $x(t)$ with respect to time.

A common unit of acceleration is the meter per second per second: m/(s · s) or m/s^2. Other units are in the form of length/(time · time) or length/time^2. Acceleration has both magnitude and direction (it is yet another vector quantity). Its algebraic sign represents its direction on an axis just as for displacement and velocity; that is, acceleration with a positive value is in the positive direction of an axis, and acceleration with a negative value is in the negative direction.

Figure 2-6 gives plots of the position, velocity, and acceleration of an elevator moving up a shaft. Compare the $a(t)$ curve with the $v(t)$ curve—each point on the $a(t)$ curve shows the derivative (slope) of the $v(t)$ curve at the corresponding time. When v is constant (at either 0 or 4 m/s), the derivative is zero and so also is the acceleration. When the cab first begins to move, the $v(t)$ curve has a positive derivative (the slope is positive), which means that $a(t)$ is positive. When the cab slows to a stop, the derivative and slope of the $v(t)$ curve are negative; that is, $a(t)$ is negative.

Next compare the slopes of the $v(t)$ curve during the two acceleration periods. The slope associated with the cab's slowing down (commonly called "deceleration") is steeper because the cab stops in half the time it took to get up to speed. The steeper slope means that the magnitude of the deceleration is larger than that of the acceleration, as indicated in Fig. 2-6*c*.

Sensations. The sensations you would feel while riding in the cab of Fig. 2-6 are indicated by the sketched figures at the bottom. When the cab first accelerates, you feel as though you are pressed downward; when later the cab is braked to a stop, you seem to be stretched upward. In between, you feel nothing special. In other words, your body reacts to accelerations (it is an accelerometer) but not to velocities (it is not a speedometer). When you are in a car traveling at 90 km/h or an airplane traveling at 900 km/h, you have no bodily awareness of the motion. However, if the car or plane quickly changes velocity, you may become keenly aware of the change, perhaps even frightened by it. Part of the thrill of an amusement park ride is due to the quick changes of velocity that you undergo (you pay for

Courtesy U.S. Air Force

Figure 2-7 Colonel J. P. Stapp in a rocket sled as it is brought up to high speed (acceleration out of the page) and then very rapidly braked (acceleration into the page).

the accelerations, not for the speed). A more extreme example is shown in the photographs of Fig. 2-7, which were taken while a rocket sled was rapidly accelerated along a track and then rapidly braked to a stop.

***g* Units.** Large accelerations are sometimes expressed in terms of g units, with

$$1g = 9.8 \text{ m/s}^2 \ (g \text{ unit}). \tag{2-10}$$

(As we shall discuss in Section 2.8, g is the magnitude of the acceleration of a falling object near Earth's surface.) On a roller coaster, you may experience brief accelerations up to $3g$, which is $(3)(9.8 \text{ m/s}^2)$, or about 29 m/s^2, more than enough to justify the cost of the ride.

Signs. In common language, the sign of an acceleration has a non-scientific meaning: positive acceleration means that the speed of an object is increasing, and negative acceleration means that the speed is decreasing (the object is decelerating). In this book, however, the sign of an acceleration indicates a direction, not whether an object's speed is increasing or decreasing. For example, if a car with an initial velocity $v = -25$ m/s is braked to a stop in 5.0 s, then $a_{avg} = +5.0 \text{ m/s}^2$. The acceleration is *positive*, but the car's speed has decreased. The reason is the difference in signs: the direction of the acceleration is opposite that of the velocity.

Here then is the proper way to interpret the signs:

If the signs of the velocity and acceleration of a particle are the same, the speed of the particle increases. If the signs are opposite, the speed decreases.

CHECKPOINT 3

A wombat moves along an x axis. What is the sign of its acceleration if it is moving (a) in the positive direction with increasing speed, (b) in the positive direction with decreasing speed, (c) in the negative direction with increasing speed, and (d) in the negative direction with decreasing speed?

SAMPLE PROBLEM 2.04

Acceleration and *dv*/*dt*

A particle's position on the x axis of Fig. 2-1 is given by

$$x = 4 - 27t + t^3,$$

with x in meters and t in seconds.

(a) Because position x depends on time t, the particle must be moving. Find the particle's velocity function $v(t)$ and acceleration function $a(t)$.

KEY IDEAS

(1) To get the velocity function $v(t)$, we differentiate the position function $x(t)$ with respect to time. (2) To get the acceleration function $a(t)$, we differentiate the velocity function $v(t)$ with respect to time.

Calculations: Differentiating the position function, we find

$$v = -27 + 3t^2, \qquad \text{(Answer)}$$

with v in meters per second. Differentiating the velocity function then gives us

$$a = +6t, \qquad \text{(Answer)}$$

with a in meters per second squared.

(b) Is there ever a time when $v = 0$?

Calculation: Setting $v(t) = 0$ yields

$$0 = -27 + 3t^2,$$

which has the solution

$$t = \pm 3 \text{ s}. \qquad \text{(Answer)}$$

Thus, the velocity is zero both 3 s before and 3 s after the clock reads 0.

(c) Describe the particle's motion for $t \geq 0$.

Reasoning: We need to examine the expressions for $x(t)$, $v(t)$, and $a(t)$.

At $t = 0$, the particle is at $x(0) = +4$ m and is moving with a velocity of $v(0) = -27$ m/s—that is, in the negative direction of the x axis. Its acceleration is $a(0) = 0$ because just then the particle's velocity is not changing (Fig. 2-8*a*).

For $0 < t < 3$ s, the particle still has a negative velocity, so it continues to move in the negative direction. However, its acceleration is no longer 0 but is increasing and positive. Because the signs of the velocity and the acceleration are opposite, the particle must be slowing (Fig. 2-8*b*).

Indeed, we already know that it stops momentarily at $t = 3$ s. Just then the particle is as far to the left of the origin in Fig. 2-1 as it will ever get. Substituting $t = 3$ s into the expression for $x(t)$, we find that the particle's position just then is $x = -50$ m (Fig. 2-8*c*). Its acceleration is still positive.

For $t > 3$ s, the particle moves to the right on the axis. Its acceleration remains positive and grows progressively larger in magnitude. The velocity is now positive, and it too grows progressively larger in magnitude (Fig. 2-8*d*).

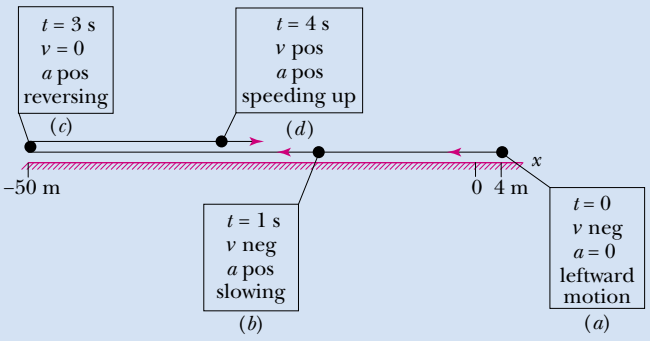


Figure 2-8 Four stages of the particle's motion.

2.7 | CONSTANT ACCELERATION: A SPECIAL CASE

Key Concept

- The following five equations describe the motion of a particle with constant acceleration:

$$v = v_0 + at, \qquad x - x_0 = v_0 t + \frac{1}{2}at^2,$$

$$v^2 = v_0^2 + 2a(x - x_0), \qquad x - x_0 = \frac{1}{2}(v_0 + v)t, \qquad x - x_0 = vt - \frac{1}{2}at^2,$$

These are *not* valid when the acceleration is not constant.

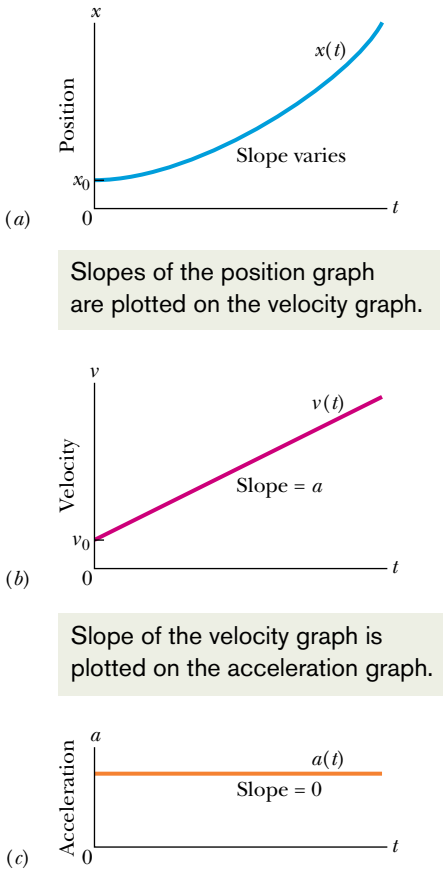


Figure 2-9 (*a*) The position $x(t)$ of a particle moving with constant acceleration. (*b*) Its velocity $v(t)$, given at each point by the slope of the curve of $x(t)$. (*c*) Its (constant) acceleration, equal to the (constant) slope of the curve of $v(t)$.

In many types of motion, the acceleration is either constant or approximately so. For example, you might accelerate a car at an approximately constant rate when a traffic light turns from red to green. Then graphs of your position, velocity, and acceleration would resemble those in Fig. 2-9. (Note that $a(t)$ in Fig. 2-9*c* is constant, which requires that $v(t)$ in Fig. 2-9*b* have a constant slope.) Later when you brake the car to a stop, the acceleration (or deceleration in common language) might also be approximately constant.

Such cases are so common that a special set of equations has been derived for dealing with them. One approach to the derivation of these equations is given in this section. A second approach is given in the next section. Throughout both sections and later when you work on the homework problems, keep in mind that *these equations are valid only for constant acceleration (or situations in which you can approximate the acceleration as being constant)*.

First Basic Equation. When the acceleration is constant, the average acceleration and instantaneous acceleration are equal and we can write Eq. 2-7, with some changes in notation, as

$$a = a_{avg} = \frac{v - v_0}{t - 0}.$$

Here v_0 is the velocity at time $t = 0$ and v is the velocity at any later time t. We can recast this equation as

$$v = v_0 + at. \tag{2-11}$$

As a check, note that this equation reduces to $v = v_0$ for $t = 0$, as it must. As a further check, take the derivative of Eq. 2-11. Doing so yields $dv/dt = a$, which is the definition of a. Figure 2-9*b* shows a plot of Eq. 2-11, the $v(t)$ function; the function is linear and thus the plot is a straight line.

Second Basic Equation. In a similar manner, we can rewrite Eq. 2-2 (with a few changes in notation) as

$$v_{avg} = \frac{x - x_0}{t - 0}$$

and then as

$$x = x_0 + v_{avg}t, \tag{2-12}$$

in which x_0 is the position of the particle at $t = 0$ and v_{avg} is the average velocity between $t = 0$ and a later time t.

For the linear velocity function in Eq. 2-11, the *average* velocity over any time interval (say, from $t = 0$ to a later time t) is the average of the velocity at the beginning of the interval ($= v_0$) and the velocity at the end of the interval ($= v$). For the interval from $t = 0$ to the later time t then, the average velocity is

$$v_{avg} = \frac{1}{2}(v_0 + v). \tag{2-13}$$

Substituting the right side of Eq. 2-11 for v yields, after a little rearrangement,

$$v_{avg} = v_0 + \frac{1}{2}at. \tag{2-14}$$

Finally, substituting Eq. 2-14 into Eq. 2-12 yields

$$x - x_0 = v_0 t + \frac{1}{2}at^2. \tag{2-15}$$

As a check, note that putting $t = 0$ yields $x = x_0$, as it must. As a further check, taking the derivative of Eq. 2-15 yields Eq. 2-11, again as it must. Figure 2-9*a* shows a plot of Eq. 2-15; the function is quadratic and thus the plot is curved.

Three Other Equations. Equations 2-11 and 2-15 are the *basic equations for constant acceleration*; they can be used to solve any constant acceleration problem in this book. However, we can derive other equations that might prove useful in certain specific situations. First, note that as many as five quantities can possibly be involved in any problem about constant acceleration—namely, $x - x_0$, v, t, a, and v_0. Usually, one of these quantities is *not* involved in the problem, *either as a given or as an unknown*. We are then presented with three of the remaining quantities and asked to find the fourth.

Equations 2-11 and 2-15 each contain four of these quantities, but not the same four. In Eq. 2-11, the "missing ingredient" is the displacement $x - x_0$. In Eq. 2-15, it is the velocity v. These two equations can also be combined in three ways to yield three additional equations, each of which involves a different "missing variable." First, we can eliminate t to obtain

$$v^2 = v_0^2 + 2a(x - x_0). \tag{2-16}$$

This equation is useful if we do not know t and are not required to find it. Second, we can eliminate the acceleration a between Eqs. 2-11 and 2-15 to produce an equation in which a does not appear:

$$x - x_0 = \frac{1}{2}(v_0 + v)t. \tag{2-17}$$

Finally, we can eliminate v_0, obtaining

$$x - x_0 = vt - \frac{1}{2}at^2. \tag{2-18}$$

Note the subtle difference between this equation and Eq. 2-15. One involves the initial velocity v_0; the other involves the velocity v at time t.

Table 2-1 lists the basic constant acceleration equations (Eqs. 2-11 and 2-15) as well as the specialized equations that we have derived. To solve a simple constant acceleration problem, you can usually use an equation from this list (*if* you have the list with you). Choose an equation for which the only unknown variable is the variable requested in the problem. A simpler plan is to remember only Eqs. 2-11 and 2-15, and then solve them as simultaneous equations whenever needed.

Table 2-1 Equations for Motion with Constant Acceleration[a]

Equation Number	Equation	Missing Quantity
2-11	$v = v_0 + at$	$x - x_0$
2-15	$x - x_0 = v_0 t + \frac{1}{2}at^2$	v
2-16	$v^2 = v_0^2 + 2a(x - x_0)$	t
2-17	$x - x_0 = \frac{1}{2}(v_0 + v)t$	a
2-18	$x - x_0 = vt - \frac{1}{2}at^2$	v_0

[a]Make sure that the acceleration is indeed constant before using the equations in this table.

CHECKPOINT 4

The following equations give the position $x(t)$ of a particle in four situations: (1) $x = 3t - 4$; (2) $x = -5t^3 + 4t^2 + 6$; (3) $x = 2/t^2 - 4/t$; (4) $x = 5t^2 - 3$. To which of these situations do the equations of Table 2-1 apply?

SAMPLE PROBLEM 2.05

Distance and time for motion under constant acceleration

An electric vehicle starts from rest and accelerates at a rate of 2.0 m/s² in a straight line until it reaches a speed of 20 m/s. The vehicle then slows at a constant rate of 1.0 m/s² until it stops. (a) How much time elapses from start to stop? (b) How far does the vehicle travel from start to stop?

KEY IDEA

We separate the motion into two parts, and take the direction of motion to be positive. In part 1, the vehicle accelerates from rest to its highest speed; we are given $v_0 = 0$; $v = 20$ m/s and $a = 2.0$ m/s². In part 2, the vehicle decelerates from its highest speed to a halt; we are given $v_0 = 20$ m/s; $v = 0$ and $a = -1.0$ m/s² (negative because the acceleration vector points opposite to the direction of motion).

Calculations: (a) From Table 2-1, we find t_1 (the duration of part 1) from $v = v_0 + at$. In this way, $20 = 0 + 2.0t_1$ yields $t_1 = 10$ s. We obtain the duration t_2 of part 2 from the same equation. Thus, $0 = 20 + (-1.0)t_2$ leads to $t_2 = 20$ s, and the total is $t = t_1 + t_2 = 30$ s.

(b) For part 1, taking $x_0 = 0$, we use the equation $v^2 = v_0^2 + 2a(x - x_0)$ from Table 2-1 and find

$$x = \frac{v^2 - v_0^2}{2a} = \frac{(20\text{ m/s})^2 - (0)^2}{2(2.0\text{ m/s}^2)} = 100\text{ m}.$$

This position is then the *initial* position for part 2, so that when the same equation is used in part 2 we obtain

$$x - 100\text{ m} = \frac{v^2 - v_0^2}{2a} = \frac{(0)^2 - (20\text{ m/s})^2}{2(-1.0\text{ m/s}^2)}.$$

Thus, the final position is $x = 300$ m. That this is also the total distance traveled should be evident (the vehicle did not reverse its direction of motion).

SAMPLE PROBLEM 2.06

Speed, distance and time

(a) If the maximum acceleration that is tolerable for passengers in a subway train is 1.34 m/s² and subway stations are located 806 m apart, what is the maximum speed a subway train can attain between stations? (b) What is the travel time between stations? (c) If a subway train stops for 20 s at each station, what is the maximum average speed of the train, from one start-up to the next? (d) Graph x, v, and a versus t for the interval from one start-up to the next.

KEY IDEA

We assume the train accelerates from rest ($v_0 = 0$ and $x_0 = 0$) at $a_1 = +1.34$ m/s² until the midway point and then decelerates at $a_1 = -1.34$ m/s² until it comes to a stop ($v_2 = 0$) at the next station. The velocity at the midpoint is v_1, which occurs at $x_1 = 806/2 = 403$ m.

Calculations: (a) Equation 2-16 leads to

$$v_1^2 = v_0^2 + 2a_1x_1 \Rightarrow v_1 = \sqrt{2(1.34\text{ m/s}^2)(403\text{ m})} = 32.9\text{ m/s}.$$

(b) The time t_1 for the accelerating stage is (using Eq. 2-15)

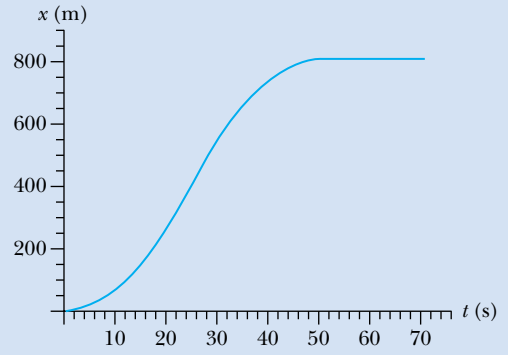
$$x_1 = v_0t_1 + \frac{1}{2}a_1t_1^2 \Rightarrow t_1 = \sqrt{\frac{2(403\text{ m})}{1.34\text{ m/s}^2}} = 24.53\text{ s}.$$

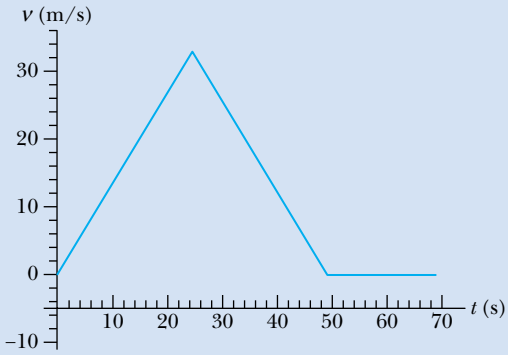
Since the time interval for the decelerating stage turns out to be the same, we double this result and obtain $t = 49.1$ s for the travel time between stations.

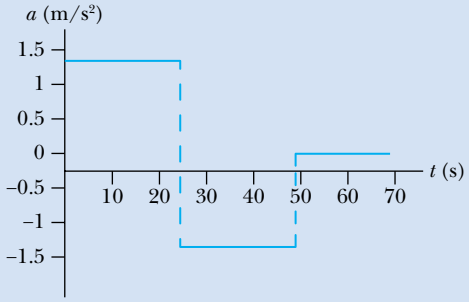
(c) With a "dead time" of 20 s, we have $T = t + 20 = 69.1$ s for the total time between start-ups. Thus, Eq. 2-2 gives

$$v_{avg} = \frac{806\text{ m}}{69.1\text{ s}} = 11.7\text{ m/s}.$$

(d) The graphs for x, v and a as a function of t are shown below. The third graph, $a(t)$, consists of three horizontal "steps"—one at 1.34 m/s² during $0 < t < 24.53$ s, and the next at –1.34 m/s² during 24.53 s $< t <$ 49.1 s and the last at zero during the "dead time" 49.1 s $< t <$ 69.1 s).







SAMPLE PROBLEM 2.07

Drag race of car and motorcycle

A popular web video shows a jet airplane, a car, and a motorcycle racing from rest along a runway (Fig. 2-10). Initially the motorcycle takes the lead, but then the jet takes the lead, and finally the car blows past the motorcycle. Here let's focus on the car and motorcycle and assign some reasonable values to the motion. The motorcycle first takes the lead because its (constant) acceleration $a_m = 8.40\ \text{m/s}^2$ is greater than the car's (constant) acceleration $a_c = 5.60\ \text{m/s}^2$, but it soon loses to the car because it reaches its greatest speed $v_m = 58.8$ m/s before the car reaches its greatest speed $v_c = 106$ m/s. How long does the car take to reach the motorcycle?

KEY IDEAS

We can apply the equations of constant acceleration to both vehicles, but for the motorcycle we must consider the motion in two stages: (1) First it travels through distance x_{m1} with zero initial velocity and acceleration $a_m = 8.40\ \text{m/s}^2$, reaching speed $v_m = 58.8$ m/s. (2) Then it travels through distance x_{m2} with constant velocity $v_m = 58.8$ m/s and zero acceleration (that, too, is a constant acceleration). (Note that we symbolized the distances even though we do not know their values. Symbolizing unknown quantities is often helpful in solving physics problems.)

Calculations: So that we can draw figures and do calculations, let's assume that the vehicles race along the positive direction of an x axis, starting from $x = 0$ at time $t = 0$. (We can choose any initial numbers because we are looking for the elapsed time, not a particular time in, say, the afternoon, but let's stick with these easy numbers.) We want the car to pass the motorcycle, but what does that mean mathematically?

It means that at some time t, the side-by-side vehicles are at the same coordinate: x_c for the car and the sum $x_{m1} + x_{m2}$ for the motorcycle. We can write this statement mathematically as

$$x_c = x_{m1} = x_{m2}. \tag{2-19}$$

Now let's fill out both sides of Eq. 2-19, left side first. To reach the passing point at x_c, the car accelerates from rest. From Eq. 2-15 ($x - x_0 = v_0 t + \frac{1}{2}at^2$), with x_0 and $v_0 = 0$, we have

$$x_c = \frac{1}{2}a_c t^2 \tag{2-20}$$

To write an expression for x_{m1} for the motorcycle, we first find the time t_m it takes to reach its maximum speed v_m, using Eq. 2-11 ($v = v_0 + at$). Substituting $v_0 = 0$, $v = v_m = 58.8$ m/s, and $a = a_m = 8.40\ \text{m/s}^2$, that time is

$$t_m = \frac{v_m}{a_m} = \frac{58.8\ \text{m/s}}{8.40\ \text{m/s}^2} = 7.00\ \text{s}. \tag{2-21}$$

To get the distance x_{m1} traveled by the motorcycle during the first stage, we again use Eq. 2-15 with $x_0 = 0$ and $v_0 = 0$, but we also substitute from Eq. 2-21 for the time. We find

$$x_{m1} = \frac{1}{2}a_m t_m^2 = \frac{1}{2}a_m\left(\frac{v_m}{a_m}\right)^2 = \frac{1}{2}\frac{v_m^2}{a_m}. \tag{2-22}$$

For the remaining time of $t - t_m$, the motorcycle travels at its maximum speed with zero acceleration. To get the distance, we use Eq. 2-15 for this second stage of the motion, but now the initial velocity is $v_0 = v_m$ (the speed at the end of the first stage) and the acceleration is $a = 0$. So, the distance traveled during the second stage is

$$x_{m2} = v_m(t - t_m) = v_m(t - 7.00\ \text{s}). \tag{2-23}$$

To finish the calculation, we substitute Eqs. 2-20, 2-22, and 2-23 into Eq. 2-19, obtaining

$$\frac{1}{2}a_c t^2 = \frac{1}{2}\frac{v_m^2}{a_m} + v_m(t - 7.00\ \text{s}) \tag{2-24}$$

This is a quadratic equation. Substituting in the given data, we solve the equation (by using the usual quadratic-equation formula or a polynomial solver on a calculator), finding $t = 4.44$ s and $t = 16.6$ s.

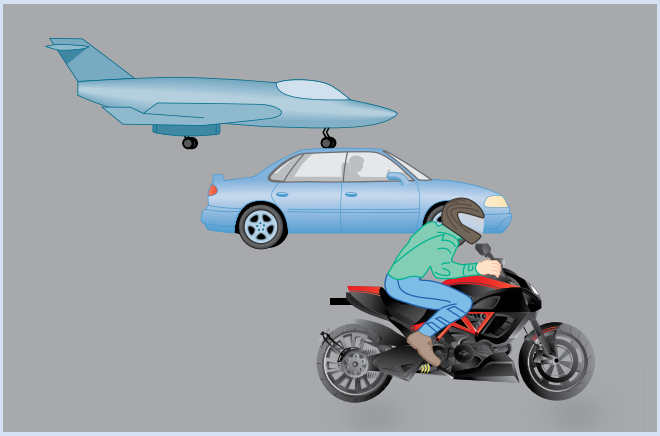


Figure 2-10 A jet airplane, a car, and a motorcycle just after accelerating from rest.

But what do we do with two answers? Does the car pass the motorcycle twice? No, of course not, as we can see in the video. So, one of the answers is mathematically correct but not physically meaningful. Because we know that the car passes the motorcycle *after* the motorcycle reaches its maximum speed at $t = 7.00$ s, we discard the solution with $t < 7.00$ s as being the unphysical answer and conclude that the passing occurs at

$$t = 16.6 \text{ s.} \qquad \text{(Answer)}$$

Figure 2-11 is a graph of the position versus time for the two vehicles, with the passing point marked. Notice that at $t = 7.00$ s the plot for the motorcycle switches from being curved (because the speed had been increasing) to being straight (because the speed is thereafter constant).

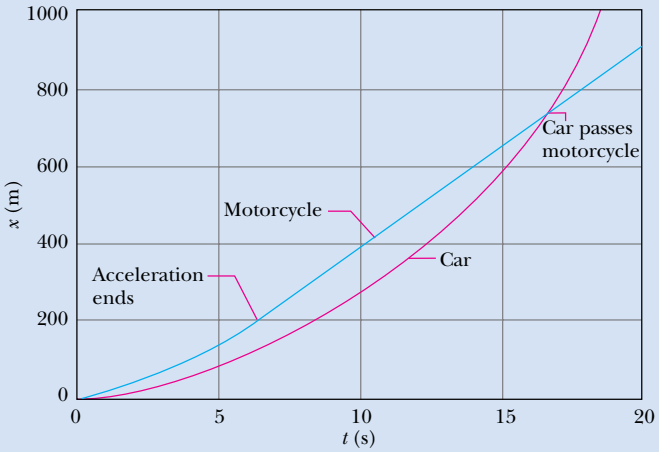


Figure 2-11 Graph of position versus time for car and motorcycle.

Another Look at Constant Acceleration*

The first two equations in Table 2-1 are the basic equations from which the others are derived. Those two can be obtained by integration of the acceleration with the condition that a is constant. To find Eq. 2-11, we rewrite the definition of acceleration (Eq. 2-8) as

$$dv = a\,dt.$$

We next write the *indefinite integral* (or *antiderivative*) of both sides:

$$\int dv = \int a\,dt.$$

Since acceleration a is a constant, it can be taken outside the integration. We obtain

$$\int dv = a\int dt$$

or

$$v = at + C. \qquad (2\text{-}25)$$

To evaluate the constant of integration C, we let $t = 0$, at which time $v = v_0$. Substituting these values into Eq. 2-25 (which must hold for all values of t, including $t = 0$) yields

$$v_0 = (a)(0) + C = C.$$

Substituting this into Eq. 2-25 gives us Eq. 2-11.

To derive Eq. 2-15, we rewrite the definition of velocity (Eq. 2-4) as

$$dx = v\,dt$$

and then take the indefinite integral of both sides to obtain

$$\int dx = \int v dt.$$

Next, we substitute for v with Eq. 2-11:

$$\int dx = \int (v_0 + at)dt.$$

Since v_0 is a constant, as is the acceleration a, this can be rewritten as

$$\int dx = v_0\int dt + a\int t\,dt.$$

Integration now yields

$$x = v_0 t + \frac{1}{2}at^2 + C', \qquad (2\text{-}26)$$

where C' is another constant of integration. At time $t = 0$, we have $x = x_0$. Substituting these values in Eq. 2-26 yields $x_0 = C'$. Replacing C' with x_0 in Eq. 2-26 gives us Eq. 2-15.

*For introduction to elementary calculus (differential and integral), refer to Appendix at the end of chapter.

2.8 | FREE-FALL ACCELERATION

Key Concept

- An important example of straight-line motion with constant acceleration is that of an object rising or falling freely near Earth's surface. The constant acceleration equations describe this motion, but we make two changes in notation: (1) we refer the motion to the vertical y axis with $+y$ vertically up; (2) we replace a with $-g$, where g is the magnitude of the free-fall acceleration. Near Earth's surface,

$$g = 9.8 \text{ m/s}^2 = 32 \text{ ft/s}^2.$$

If you tossed an object either up or down and could somehow eliminate the effects of air on its flight, you would find that the object accelerates downward at a certain constant rate. That rate is called the **free-fall acceleration**, and its magnitude is represented by g. The acceleration is independent of the object's characteristics, such as mass, density, or shape; it is the same for all objects.

Two examples of free-fall acceleration are shown in Fig. 2-12, which is a series of stroboscopic photos of a feather and an apple. As these objects fall, they accelerate downward—both at the same rate g. Thus, their speeds increase at the same rate, and they fall together.

The value of g varies slightly with latitude and with elevation. At sea level in Earth's midlatitudes the value is 9.8 m/s^2 (or 32 ft/s^2), which is what you should use as an exact number for the problems in this book unless otherwise noted.

The equations of motion in Table 2-1 for constant acceleration also apply to free fall near Earth's surface; that is, they apply to an object in vertical flight, either up or down, when the effects of the air can be neglected. However, note that for free fall: (1) The directions of motion are now along a vertical y axis instead of the x axis, with the positive direction of y upward. (This is important for later chapters when combined horizontal and vertical motions are examined.) (2) The free-fall acceleration is negative—that is, downward on the y axis, toward Earth's center—and so it has the value $-g$ in the equations.

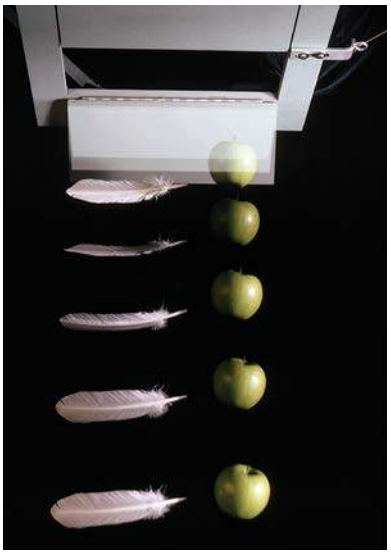


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Figure 2-12 A feather and an apple free fall in vacuum at the same magnitude of acceleration g. The acceleration increases the distance between successive images. In the absence of air, the feather and apple fall together.

The free-fall acceleration near Earth's surface is $a = -g = -9.8$ m/s^2, and the *magnitude* of the acceleration is $g = 9.8$ m/s^2. Do not substitute -9.8 m/s^2 for g.

Suppose you toss a tomato directly upward with an initial (positive) velocity v_0 and then catch it when it returns to the release level. During its *free-fall flight* (from just after its release to just before it is caught), the equations of Table 2-1 apply to its motion. The acceleration is always $a = -g = -9.8$ m/s^2, negative and thus downward. The velocity, however, changes, as indicated by Eqs. 2-11 and 2-16: during the ascent, the magnitude of the positive velocity decreases, until it momentarily becomes zero. Because the tomato has then stopped, it is at its maximum height. During the descent, the magnitude of the (now negative) velocity increases.

CHECKPOINT 5

(a) If you toss a ball straight up, what is the sign of the ball's displacement for the ascent, from the release point to the highest point? (b) What is it for the descent, from the highest point back to the release point? (c) What is the ball's acceleration at its highest point?

SAMPLE PROBLEM 2.08

Speed and time under free-fall acceleration

A hot-air balloon is ascending at the rate of 12 m/s and is 80 m above the ground when a package is dropped over the side. (a) How long does the package take to reach the ground? (b) With what speed does it hit the ground?

KEY IDEA

We neglect air resistance, which justifies setting $a = -g = -9.8\ \text{m/s}^2$ (taking *down* as the $-y$ direction) for the duration of the motion. We can use Table 2-1 (with Δy replacing Δx) because this is constant acceleration motion. We are placing the coordinate origin on the ground. We note that the initial velocity of the package is the same as the velocity of the balloon, $v_0 = +12$ m/s and that its initial coordinate is $y_0 = +80$ m.

Calculations: (a) We solve $y = y_0 + v_0 t - 1/2gt^2$ for time, with $y = 0$, using the quadratic formula (choosing the positive root to yield a positive value for t).

$$t = \frac{v_0 + \sqrt{v_0^2 + 2gy_0}}{g} = \frac{12 + \sqrt{12^2 + 2(9.8)(80)}}{9.8} = 5.4\ \text{s}$$

(b) If we wish to avoid using the result from part (a), we could use Eq. 2-16, but if that is not a concern, then a variety of formulas from Table 2-1 can be used. For instance, Eq. 2-11 leads to

$$v = v_0 - gt = 12 - (9.8)(5.4) = -41\ \text{m/s}.$$

Its final *speed* is 41 m/s.

SAMPLE PROBLEM 2.09

Time for full up-down flight, baseball toss

In Fig. 2-13, a pitcher tosses a baseball up along a y axis, with an initial speed of 12 m/s.

(a) How long does the ball take to reach its maximum height?

KEY IDEAS

(1) Once the ball leaves the pitcher and before it returns to his hand, its acceleration is the free-fall acceleration $a = -g$. Because this is constant, Table 2-1 applies to the motion. (2) The velocity v at the maximum height must be 0.

Calculation: Knowing v, a, and the initial velocity $v_0 = 12$ m/s, and seeking t, we solve Eq. 2-11, which contains those four variables. This yields

$$t = \frac{v - v_0}{a} = \frac{0 - 12\ \text{m/s}}{-9.8\ \text{m/s}^2} = 1.2\ \text{s}. \qquad \text{(Answer)}$$

(b) What is the ball's maximum height above its release point?

Calculation: We can take the ball's release point to be $y_0 = 0$. We can then write Eq. 2-16 in y notation, set $y - y_0 = y$ and $v = 0$ (at the maximum height), and solve for y. We get

$$y = \frac{v^2 - v_0^2}{2a} = \frac{0 - (12\ \text{m/s})^2}{2(-9.8\ \text{m/s}^2)} = 7.3\ \text{m}. \qquad \text{(Answer)}$$

(c) How long does the ball take to reach a point 5.0 m above its release point?

Calculations: We know v_0, $a = -g$, and displacement $y - y_0 = 5.0$ m, and we want t, so we choose Eq. 2-15. Rewriting it for y and setting $y_0 = 0$ give us

$$y = v_0 t - \frac{1}{2}gt^2,$$

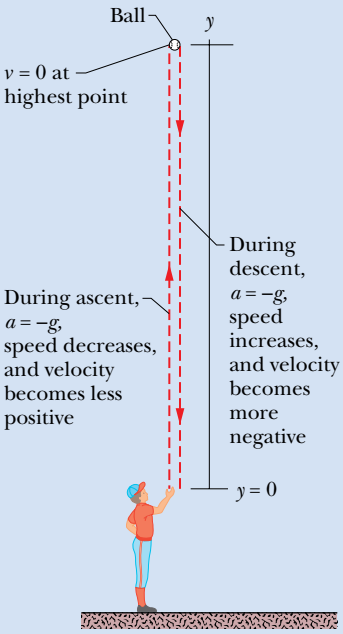

Ball
y
v = 0 at highest point
During descent, a = −g, speed increases, and velocity becomes more negative
During ascent, a = −g, speed decreases, and velocity becomes less positive
y = 0

Figure 2-13 A pitcher tosses a baseball straight up into the air. The equations of free fall apply for rising as well as for falling objects, provided any effects from the air can be neglected.

or $$5.0\text{ m} = (12\text{ m/s})t - \left(\frac{1}{2}\right)(9.8\text{ m/s}^2)t^2.$$

If we temporarily omit the units (having noted that they are consistent), we can rewrite this as

$$4.9t^2 - 12t + 5.0 = 0.$$

Solving this quadratic equation for t yields

$$t = 0.53\text{ s} \quad \text{and} \quad t = 1.9\text{ s.} \qquad \text{(Answer)}$$

There are two such times! This is not really surprising because the ball passes twice through $y = 5.0$ m, once on the way up and once on the way down.

PROBLEM-SOLVING TACTICS

Tactic 6: Meanings of Minus Signs In Sample Problems 2.08 and 2.09, we established a vertical axis (the y axis) and we chose—quite arbitrarily—its upward direction to be positive. We then chose the origin of the y axis (that is, the $y = 0$ position) to suit the problem. In Sample Problem 2.08, the origin was at the top of the falls, and in Sample Problem 2.09 it was at the pitcher's hand. A negative value of y then means that the body is below the chosen origin. A negative velocity means that the body is moving in the negative direction of the y axis—that is, downward. This is true no matter where the body is located.

We take the acceleration to be negative (-9.8 m/s^2) in all problems dealing with falling bodies. A negative acceleration means that, as time goes on, the velocity of the body becomes either less positive or more negative. This is true no matter where the body is located and no matter how fast or in what direction it is moving. In Sample Problem 2.08, the acceleration of the ball is negative (downward) throughout its flight, whether the ball is rising or falling.

Tactic 7: Unexpected Answers Mathematics often generates answers that you might not have thought of as possibilities, as in Sample Problem 2.09(c). If you get more answers than you expect, do not automatically discard the ones that do not seem to fit. Examine them carefully for physical meaning. If time is your variable, even a negative value can mean something; negative time simply refers to time before $t = 0$, the (arbitrary) time at which you decided to start your stopwatch.

2.9 | GRAPHICAL INTEGRATION IN MOTION ANALYSIS

Key Concepts

- On a graph of acceleration a versus time t, the change in the velocity is given by

$$v_1 - v_0 = \int_{t_0}^{t_1} a\,dt.$$

The integral amounts to finding an area on the graph:

$$\int_{t_0}^{t_1} a\,dt = \begin{pmatrix}\text{area between acceleration curve}\\ \text{and time axis, from } t_0 \text{ to } t_1\end{pmatrix}.$$

- On a graph of velocity v versus time t, the change in the position is given by

$$x_1 - x_0 = \int_{t_0}^{t_1} v\,dt,$$

where the integral can be taken from the graph as

$$\int_{t_0}^{t_1} v\,dt = \begin{pmatrix}\text{area between velocity curve}\\ \text{and time axis, from } t_0 \text{ to } t_1\end{pmatrix}.$$

Integrating Acceleration. When we have a graph of an object's acceleration a versus time t, we can integrate on the graph to find the velocity at any given time. Because a is defined as $a = dv/dt$, the Fundamental Theorem of Calculus tells us that

$$v_1 - v_0 = \int_{t_0}^{t_1} a\,dt. \qquad (2\text{-}27)$$

The right side of the equation is a definite integral (it gives a numerical result rather than a function), v_0 is the velocity at time t_0, and v_1 is the velocity at later time t_1. The definite integral can be evaluated from an $a(t)$ graph, such as in Fig. 2-14*a*. In particular,

$$\int_{t_0}^{t_1} a\,dt = \begin{pmatrix}\text{area between acceleration curve}\\ \text{and time axis, from } t_0 \text{ to } t_1\end{pmatrix}. \tag{2-28}$$

If a unit of acceleration is 1 m/s^2 and a unit of time is 1 s, then the corresponding unit of area on the graph is

$$(1\ \text{m/s}^2)(1\ \text{s}) = 1\ \text{m/s},$$

which is (properly) a unit of velocity. When the acceleration curve is above the time axis, the area is positive; when the curve is below the time axis, the area is negative.

Integrating Velocity. Similarly, because velocity v is defined in terms of the position x as $v = dx/dt$, then

$$x_1 - x_0 = \int_{t_0}^{t_1} v\,dt, \tag{2-29}$$

where x_0 is the position at time t_0 and x_1 is the position at time t_1. The definite integral on the right side of Eq. 2-29 can be evaluated from a $v(t)$ graph, like that shown in Fig. 2-14*b*. In particular,

$$\int_{t_0}^{t_1} v\,dt = \begin{pmatrix}\text{area between velocity curve}\\ \text{and time axis, from } t_0 \text{ to } t_1\end{pmatrix}. \tag{2-30}$$

If the unit of velocity is 1 m/s and the unit of time is 1 s, then the corresponding unit of area on the graph is

$$(1\ \text{m/s})(1\ \text{s}) = 1\ \text{m},$$

which is (properly) a unit of position and displacement. Whether this area is positive or negative is determined as described for the $a(t)$ curve of Fig. 2-14*a*.

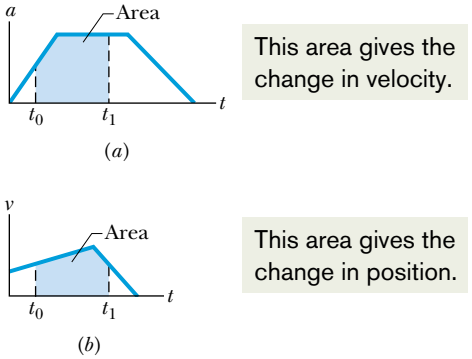


Figure 2-14 The area between a plotted curve and the horizontal time axis, from time t_0 to time t_1, is indicated for (*a*) a graph of acceleration a versus t and (*b*) a graph of velocity v versus t.

SAMPLE PROBLEM 2.10

Graphical integration *a* versus *t*, whiplash injury

"Whiplash injury" commonly occurs in a rear-end collision where a front car is hit from behind by a second car. In the 1970s, researchers concluded that the injury was due to the occupant's head being whipped back over the top of the seat as the car was slammed forward. As a result of this finding, head restraints were built into cars, yet neck injuries in rearend collisions continued to occur.

In a recent test to study neck injury in rear-end collisions, a volunteer was strapped to a seat that was then moved abruptly to simulate a collision by a rear car moving at 10.5 km/h. Figure 2-15*a* gives the accelerations of the volunteer's torso and head during the collision, which began at time $t = 0$. The torso acceleration was delayed by 40 ms because during that time interval the seat back had to compress against the volunteer. The head acceleration was delayed by an additional 70 ms. What was the torso speed when the head began to accelerate?

KEY IDEA

We can calculate the torso speed at any time by finding an area on the torso $a(t)$ graph.

Calculations: We know that the initial torso speed is $v_0 = 0$ at time $t_0 = 0$, at the start of the "collision." We

want the torso speed v_1 at time $t_1 = 110$ ms, which is when the head begins to accelerate.

Combining Eqs. 2-27 and 2-28, we can write

$$v_1 - v_0 = \begin{pmatrix}\text{area between acceleration curve} \\ \text{and time axis, from } t_0 \text{ to } t_1\end{pmatrix}. \quad (2\text{-}31)$$

For convenience, let us separate the area into three regions (Fig. 2-15*b*). From 0 to 40 ms, region *A* has no area:

$$\text{area}_A = 0.$$

From 40 ms to 100 ms, region *B* has the shape of a triangle, with area

$$\text{area}_B = \frac{1}{2}(0.060\text{ s})(50\text{ m/s}^2) = 1.5\text{ m/s}.$$

From 100 ms to 110 ms, region *C* has the shape of a rectangle, with area

$$\text{area}_C = (0.010\text{ s})(50\text{ m/s}^2) = 0.50\text{ m/s}.$$

Substituting these values and $v_0 = 0$ into Eq. 2-31 gives us

$$v_1 - 0 = 0 + 1.5\text{ m/s} + 0.50\text{ m/s},$$

or $$v_1 = 2.0\text{ m/s} = 7.2\text{ km/h}. \quad \text{(Answer)}$$

Reasoning: When the head is just starting to move forward, the torso already has a speed of 7.2 km/h. Researchers argue that it is this difference in speeds during the early stage of a rear-end collision that injures the neck. The backward whipping of the head happens later and could, especially if there is no head restraint, increase the injury.

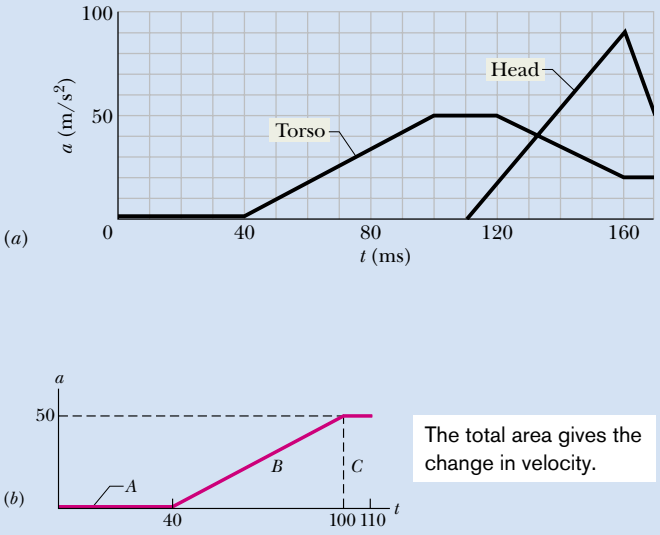


Figure 2-15 (*a*) The $a(t)$ curve of the torso and head of a volunteer in a simulation of a rear-end collision. (*b*) Breaking up the region between the plotted curve and the time axis to calculate the area.

REVIEW AND SUMMARY

Position The *position* x of a particle on an x axis locates the particle with respect to the **origin**, or zero point, of the axis. The position is either positive or negative, according to which side of the origin the particle is on, or zero if the particle is at the origin. The **positive direction** on an axis is the direction of increasing positive numbers; the opposite direction is the **negative direction** on the axis.

Displacement The *displacement* Δx of a particle is the change in its position:

$$\Delta x = x_2 - x_1. \quad (2\text{-}1)$$

Displacement is a vector quantity. It is positive if the particle has moved in the positive direction of the x axis and negative if the particle has moved in the negative direction.

Average Velocity When a particle has moved from position x_1 to position x_2 during a time interval $\Delta t = t_2 - t_1$, its *average velocity* during that interval is

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}. \quad (2\text{-}2)$$

The algebraic sign of v_{avg} indicates the direction of motion (v_{avg} is a vector quantity). Average velocity does not depend on the actual distance a particle moves, but instead depends on its original and final positions.

On a graph of x versus t, the average velocity for a time interval Δt is the slope of the straight line connecting the points on the curve that represent the two ends of the interval.

Average Speed The *average speed* s_{avg} of a particle during a time interval Δt depends on the total distance the particle moves in that time interval:

$$s_{\text{avg}} = \frac{\text{total distance}}{\Delta t}. \quad (2\text{-}3)$$

Instantaneous Velocity The *instantaneous velocity* (or simply **velocity**) v of a moving particle is

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}, \tag{2-4}$$

where Δx and Δt are defined by Eq. 2-2. The instantaneous velocity (at a particular time) may be found as the slope (at that particular time) of the graph of x versus t. **Speed** is the magnitude of instantaneous velocity.

Average Acceleration *Average acceleration* is the ratio of a change in velocity Δv to the time interval Δt in which the change occurs:

$$a_{avg} = \frac{\Delta v}{\Delta t}. \tag{2-7}$$

The algebraic sign indicates the direction of a_{avg}.

Instantaneous Acceleration *Instantaneous acceleration* (or simply **acceleration**) a is the first time derivative of velocity $v(t)$ and the second time derivative of position $x(t)$:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}. \tag{2-8, 2-9}$$

On a graph of v versus t, the acceleration a at any time t is the slope of the curve at the point that represents t.

Constant Acceleration The five equations in Table 2-1 describe the motion of a particle with constant acceleration:

$$v = v_0 + at, \tag{2-11}$$

$$x - x_0 = v_0 t + \frac{1}{2}at^2, \tag{2-15}$$

$$v^2 = v_0^2 + 2a(x - x_0), \tag{2-16}$$

$$x - x_0 = \frac{1}{2}(v_0 + v)t, \tag{2-17}$$

$$x - x_0 = vt - \frac{1}{2}at^2. \tag{2-18}$$

These are *not* valid when the acceleration is not constant.

Free-Fall Acceleration An important example of straightline motion with constant acceleration is that of an object rising or falling freely near Earth's surface. The constant acceleration equations describe this motion, but we make two changes in notation: (1) we refer the motion to the vertical y axis with $+y$ vertically *up*; (2) we replace a with $-g$, where g is the magnitude of the free-fall acceleration. Near Earth's surface, $g = 9.8\ m/s^2$ ($= 32\ ft/s^2$).

PROBLEMS

1. In 25 min, a man ran 2.40 km on a treadmill facing due east. Relative to the gym, what were his (a) displacement and (b) average velocity during this time interval?
2. Compute your average velocity in the following two cases: (a) You walk 73.2 m at a speed of 1.22 m/s and then run 73.2 m at a speed of 2.85 m/s along a straight track. (b) You walk for 1.00 min at a speed of 1.22 m/s and then run for 1.00 min at 3.05 m/s along a straight track. (c) Graph x versus t for both cases and indicate how the average velocity is found on the graph.
3. Rachel walks on a straight road from her home to a gymnasium 2.80 km away with a speed of 6.00 km/h. As soon as she reaches the gymnasium, she immediately turns and walks back home with a speed of 7.70 km/h as she finds the gymnasium closed. What are the (a) magnitude of average velocity and (b) average speed of Rachel over the interval of time 0.00–35.0 min?
4. A car travels up a hill at a constant speed of 35 km/h and returns down the hill at a constant speed of 60 km/h. Calculate the average speed for the round trip.
5. The position of an object moving along an x axis is given by $x = 3t - 4t^2 + t^3$, where x is in meters and t in seconds. Find the position of the object at the following values of t: (a) 1 s, (b) 2 s, (c) 3 s, and (d) 4 s. (e) What is the object's displacement between $t = 0$ and $t = 4$ s? (f) What is its average velocity for the time interval from $t = 2$ s to $t = 4$ s? (g) Graph x versus t for $0 \le t \le 4$ s and indicate how the answer for (f) can be found on the graph.
6. A pigeon flies at 36 km/h to and fro between two cars moving toward each other on a straight road, starting from the first car when the car separation is 40 km. The first car has a speed of 16 km/h and the second one has a speed of 25 km/h. By the time the cars meet head on, what are the (a) total distance and (b) net displacement flown by the pigeon?
7. *Panic escape*. Figure 2-16 shows a general situation in which a stream of people attempt to escape through an exit door that turns out to be locked. The people move toward the door at speed $v_s = 3.50$ m/s, are each $d = 0.25$ m in depth, and are separated by $L = 1.75$ m. The arrangement in Fig. 2-16 occurs at time $t = 0$. (a) At what average rate does the layer of people at the door increase? (b) At what time does the layer's depth reach 5.0 m? (The answers reveal how quickly such a situation becomes dangerous.)

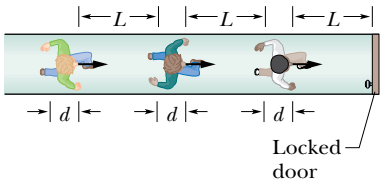


Figure 2-16 Problem 7.

8. To set a speed record in a measured (straight-line) distance d, a race car must be driven first in one direction (in time t_1) and then in the opposite direction (in time t_2). (a) To eliminate the effects of the wind and obtain the car's speed v_c in a windless situation, should we find the average of d/t_1 and d/t_2 (method 1) or should we divide d by the average of t_1 and t_2? (b) What is the fractional difference in the two methods when a steady wind blows along the car's route and the ratio of the wind speed v_w to the car's speed v_c is 0.0240?

9. A pickup vehicle is moving with a speed of 15.00 m/s on a straight road. A scooterist wishes to overtake the pickup vehicle in 150.0 s. If the pickup vehicle is at an initial distance of 1.500 km from the scooterist, with what constant speed should the scooterist chase the pickup vehicle?

10. *Traffic shock wave.* An abrupt slowdown in concentrated traffic can travel as a pulse, termed a *shock wave,* along the line of cars, either downstream (in the traffic direction) or upstream, or it can be stationary. Figure 2-17 shows a uniformly spaced line of cars moving at speed $v = 25.0$ m/s toward a uniformly spaced line of slow cars moving at speed $v_s = 5.00$ m/s. Assume that each faster car adds length $L = 12.0$ m (car length plus buffer zone) to the line of slow cars when it joins the line, and assume it slows abruptly at the last instant. (a) For what separation distance d between the faster cars does the shock wave remain stationary? If the separation is twice that amount, what are the (b) speed and (c) direction (upstream or downstream) of the shock wave?

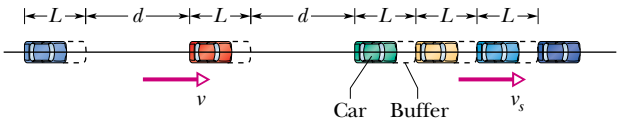


Figure 2-17 Problem 10.

11. The displacement of a particle moving along an x axis is given by $x = 18t + 5.0t^2$, where x is in meters and t is in seconds. Calculate (a) the instantaneous velocity at $t = 2.0$ s and (b) the average velocity between $t = 2.0$ s and $t = 3.0$ s.

12. The position function $x(t)$ of a particle moving along an x axis is $x = 4.0 - 6.0t^2$, with x in meters and t in seconds. (a) At what time and (b) where does the particle (momentarily) stop? At what (c) negative time and (d) positive time does the particle pass through the origin? (e) Graph x versus t for the range −5 s to +5 s. (f) To shift the curve rightward on the graph, should we include the term $+20t$ or the term $-20t$ in $x(t)$? (g) Does that inclusion increase or decrease the value of x at which the particle momentarily stops?

13. The position of a particle moving along an x axis is given by $x = 12t^2 - 2t^3$, where x is in meters and t is in seconds. Determine (a) the position, (b) the velocity, and (c) the acceleration of the particle at $t = 3.5$ s. (d) What is the maximum positive coordinate reached by the particle and (e) at what time is it reached? (f) What is the maximum positive velocity reached by the particle and (g) at what time is it reached? (h) What is the acceleration of the particle at the instant the particle is not moving (other than at $t = 0$)? (i) Determine the average velocity of the particle between $t = 0$ and $t = 3$ s.

14. At a certain time a particle had a speed of 18 m/s in the positive x direction, and 2.4 s later its speed was 30 m/s in the opposite direction. What is the average acceleration of the particle during this 2.4 s interval?

15. (a) If the position of a particle is given by $x = 25t - 6.0t^3$, where x is in meters and t is in seconds, when, if ever, is the particle's velocity v zero? (b) When is its acceleration a zero? For what time range (positive or negative) is a (c) negative and (d) positive? (e) Graph $x(t)$, $v(t)$, and $a(t)$.

16. Along a straight road, a car moving with a speed of 130 km/h is brought to a stop in a distance of 210 m. (a) Find the magnitude of the deceleration of the car (assumed uniform). (b) How long does it take for the car to stop?

17. A body starting from rest moves with constant acceleration. What is the ratio of distance covered by the body during the fifth second of time to that covered in the first 5.00 s?

18. A particle confined to motion along an x axis moves with constant acceleration from $x = 2.0$ m to $x = 8.0$ m during a 2.5 s time interval. The velocity of the particle at $x = 8.0$ m is 2.8 m/s. What is the constant acceleration during this time interval?

19. A muon (an elementary particle) enters a region with a speed of 6.00×10^6 m/s and then is slowed at the rate of 1.25×10^{14} m/s^2. (a) How far does the muon take to stop? (b) Graph x versus t and v versus t for the muon.

20. An electron, starting from rest and moving with a constant acceleration, travels 2.00 cm in 5.00 ms. What is the magnitude of this acceleration?

21. On a dry road, a car with good tires may be able to brake with a constant deceleration of 4.92 m/s^2. (a) How long does such a car, initially traveling at 27.2 m/s, take to stop? (b) How far does it travel in this time? (c) Graph x versus t and v versus t for the deceleration.

22. A certain elevator cab has a total run of 190 m and a maximum speed of 305 m/min, and it accelerates from rest and then back to rest at 1.22 m/s^2. (a) How far does the cab move while accelerating to full speed from rest? (b) How long does it take to make the nonstop 190 m run, starting and ending at rest?

23. The brakes on your car can slow you at a rate of 5.2 m/s^2. (a) If you are going 146 km/h and suddenly see a state trooper, what is the minimum time in which you can get your car under the 90 km/h speed limit? (The answer reveals the futility of braking to keep your high speed from being detected with a radar or laser gun.) (b) Graph x versus t and v versus t for such a slowing.

24. A rocket, initially at rest, is fired vertically with an upward acceleration of 10.0 m/s^2. At an altitude of 0.500 km, the engine of the rocket cuts off. What is the maximum altitude it achieves?

25. A world's land speed record was set by Colonel John P. Stapp when in March 1954 he rode a rocket-propelled sled that moved along a track at 1020 km/h. He and the sled were brought to a stop in 1.4 s. (See Fig. 2-7.) In terms of g, what acceleration did he experience while stopping?

26. A stone is thrown from the top of a building with an initial velocity of 20 m/s downward. The top of the building is 60 m above the ground. How much time elapses between the instant of release and the instant of impact with the ground?

27. In Fig. 2-18, a red car and a green car, identical except for the color, move toward each other in adjacent lanes and parallel to an x axis. At time $t = 0$, the red car is at $x_r = 0$ and the green car is at $x_g = 220$ m. If the red car has a constant velocity of 20 km/h, the cars pass each other at $x = 44.5$ m, and if it has a constant velocity of 40 km/h, they pass each other at $x = 77.9$ m. What are (a) the initial velocity and (b) the constant acceleration of the green car?

Figure 2-18 Problem 27.

28. In a particle accelerator, an electron enters a region in which it accelerates uniformly in a straight line from a speed of 4.00×10^5 m/s to a speed of 6.00×10^7 m/s in a distance of 3.00 cm. For what time interval does the electron accelerate?

29. A car moves along an x axis through a distance of 900 m, starting at rest (at $x = 0$) and ending at rest (at $x = 900$ m). Through the first $\frac{1}{4}$ of that distance, its acceleration is +2.75 m/s^2. Through the rest of that distance, its acceleration is –0.750 m/s^2. What are (a) its travel time through the 900 m and (b) its maximum speed? (c) Graph position x, velocity v, and acceleration a versus time t for the trip.

30. Figure 2-19 depicts the motion of a particle moving along an x axis with a constant acceleration. The figure's vertical scaling is set by $x_s = 6.0$ m. What are the (a) magnitude and (b) direction of the particle's acceleration?

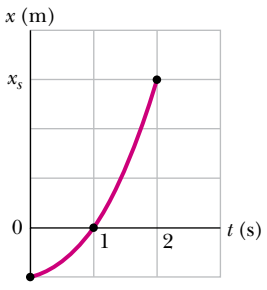


Figure 2-19 Problem 30.

31. A car moving at a constant velocity of 46 m/s passes a traffic cop who is readily sitting on his motorcycle. After a reaction time of 1.0 s, the cop begins to chase the speeding car with a constant acceleration of 4.0 m/s^2. How much time does the cop then need to overtake the speeding car?

32. You are driving toward a traffic signal when it turns yellow. Your speed is the legal speed limit of $v_0 = 55$ km/h; your best deceleration rate has the magnitude $a = 5.18$ m/s^2. Your best reaction time to begin braking is $T = 0.75$ s. To avoid having the front of your car enter the intersection after the light turns red, should you brake to a stop or continue to move at 55 km/h if the distance to the intersection and the duration of the yellow light are (a) 40 m and 2.8 s, and (b) 32 m and 1.8 s? Give an answer of brake, continue, either (if either strategy works), or neither (if neither strategy works and the yellow duration is inappropriate).

33. On a defense aircraft carrier, a military jet lands at a speed of 64 m/s. (a) Assuming the acceleration to be constant, what is the acceleration of the jet if it stops in 3.0 s due to the arresting cable that snags it? (b) If the jet first touches at position $x_i = 0$, what is its final position along an x axis lying under its landing path?

34. You are arguing over a cell phone while trailing an unmarked police car by 25 m; both your car and the police car are traveling at 120 km/hr. Your argument diverts your attention from the police car for 2.0 s (long enough for you to look at the phone and yell, "I won't do that!"). At the beginning of that 2.0 s, the police officer begins braking suddenly at 5.0 m/s^2. (a) What is the separation between the two cars when your attention finally returns? Suppose that you take another 0.40 s to realize your danger and begin braking. (b) If you too brake at 5.0 m/s^2, what is your speed when you hit the police car?

35. When a high-speed passenger train traveling at 161 km/h rounds a bend, the engineer is shocked to see that a locomotive has improperly entered onto the track from a siding and is a distance $D = 676$ m ahead (Fig. 2-20). The locomotive is moving at 29.0 km/h. The engineer of the high-speed train immediately applies the brakes. (a) What must be the magnitude of the resulting constant deceleration if a collision is to be just avoided? (b) Assume that the engineer is at $x = 0$ when, at $t = 0$, he first spots the locomotive. Sketch $x(t)$ curves for the locomotive and high-speed train for the cases in which a collision is just avoided and is not quite avoided.

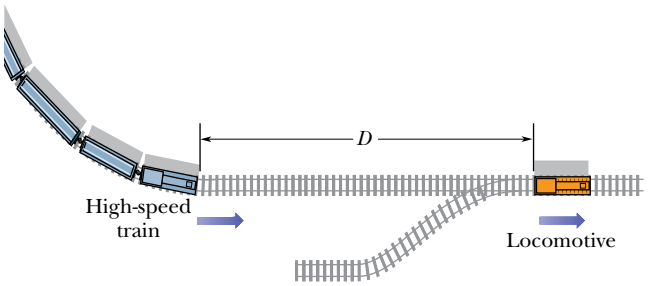


Figure 2-20 Problem 35.

36. A man releases a stone at the top edge of a tower. During the last second of its travel, the stone falls through a distance of $(9/25)H$, where H is the tower's height. Find H.

37. Raindrops fall 1800 m from a cloud to the ground. (a) If they were not slowed by air resistance, how fast would the drops be moving when they struck the ground? (b) Would it be safe to walk outside during a rainstorm?

38. A hot-air balloon is ascending at a rate of 14 m/s at a height of 98 m above the ground when a packet is dropped from it. (a) With what speed does the packet reach the ground, and (b) how much time does the fall take?

39. A hoodlum throws a stone vertically downward with an initial speed of 15.0 m/s from the roof of a building, 30.0 m above the ground. (a) How long does it take the stone to reach the ground? (b) What is the speed of the stone at impact?

40. A melon is dropped from a height of 39.2 m. After it crosses through half that distance, the acceleration due to gravity is reduced to zero by air drag. With what velocity does the melon hit the ground?

41. A key falls from a bridge that is 45 m above the water. It falls directly into a model boat, moving with constant velocity, that is 12 m from the point of impact when the key is released. What is the speed of the boat?

42. A stone is dropped into a river from a bridge 53.6 m above the water. Another stone is thrown vertically down 1.00 s after the first is dropped. The stones strike the water at the same time. (a) What is the initial speed of the second stone? (b) Plot velocity versus time on a graph for each stone, taking zero time as the instant the first stone is released.

43. Figure 2-21 shows the speed v versus height y of a ball tossed directly upward, along a y axis. Distance d is 0.40 m. The speed at height y_A is v_A. The speed at height y_B is $\frac{1}{3}v_A$. What is speed v_A?

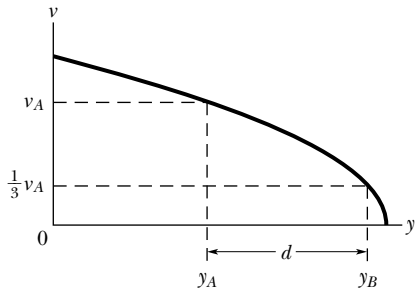


Figure 2-21 Problem 43.

44. To test the quality of a tennis ball, you drop it onto the floor from a height of 4.00 m. It rebounds to a height of 2.00 m. If the ball is in contact with the floor for 12.0 ms, (a) what is the magnitude of its average acceleration during that contact and (b) is the average acceleration up or down?

45. Water drips from the nozzle of a shower onto the floor 200 cm below. The drops fall at regular (equal) intervals of time, the first drop striking the floor at the instant the fourth drop begins to fall. When the first drop strikes the floor, how far below the nozzle are the (a) second and (b) third drops?

46. A rock is thrown vertically upward from ground level at time $t = 0$. At $t = 1.5$ s it passes the top of a tall tower, and 1.0 s later it reaches its maximum height. What is the height of the tower?

47. A drowsy cat spots a flowerpot that sails first up and then down past an open window. The pot is in view for a total of 0.50 s, and the top-to-bottom height of the window is 2.00 m. How high above the window top does the flowerpot go?

48. A rock is thrown downward from an unknown height above the ground with an initial speed of 10 m/s. It strikes the ground 3.0 s later. Determine the initial height of the rock above the ground.

49. In a forward punch in karate, the fist begins at rest at the waist and is brought rapidly forward until the arm is fully extended. The speed $v(t)$ of the fist is given in Fig. 2-22 for someone skilled in karate. The vertical scaling is set by $v_s = 8.0$ m/s. How far has the fist moved at (a) time $t = 50$ ms and (b) when the speed of the fist is maximum?

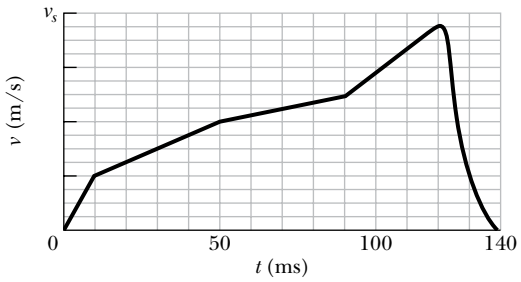


Figure 2-22 Problem 49.

50. When a soccer ball is kicked toward a player and the player deflects the ball by "heading" it, the acceleration of the head during the collision can be significant. Figure 2-23 gives the measured acceleration $a(t)$ of a soccer player's head for a bare head and a helmeted head, starting from rest. The scaling on the vertical axis is set by $a_s = 200$ m/s^2. At time $t = 7.0$ ms, what is the difference in the speed acquired by the bare head and the speed acquired by the helmeted head?

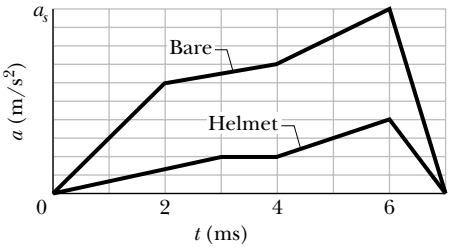


Figure 2-23 Problem 50.

51. A salamander of the genus *Hydromantes* captures prey by launching its tongue as a projectile: The skeletal part of the tongue is shot forward, unfolding the rest of the tongue, until the outer portion lands on the prey, sticking to it. Figure 2-24 shows the acceleration magnitude a versus time t for the acceleration phase of the launch in a typical situation. The indicated accelerations are $a_2 = 400$ m/s^2

and $a_1 = 100\ \text{m/s}^2$. What is the outward speed of the tongue at the end of the acceleration phase?

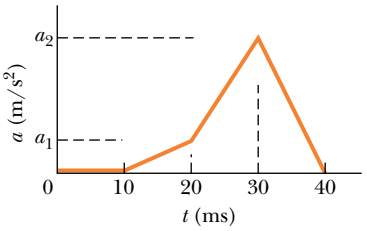


Figure 2-24 Problem 51.

52. How far does the runner whose velocity–time graph is shown in Fig. 2-25 travel in 16 s? The figure's vertical scaling is set by $v_s = 8.0$ m/s.

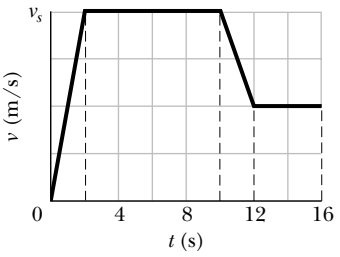


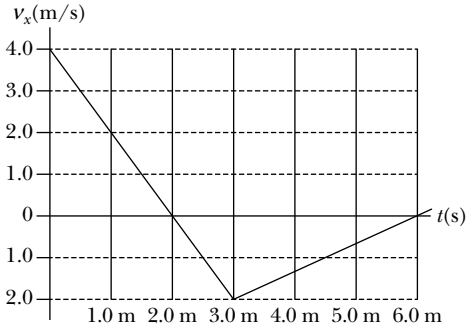
Figure 2-25 Problem 52.

53. Two particles move along an x axis. The position of particle 1 is given by $x = 6.00t^2 + 3.00t + 2.00$ (in meters and seconds); the acceleration of particle 2 is given by $a = -8.00t$ (in meters per second squared and seconds) and, at $t = 0$, its velocity is 15 m/s. When the velocities of the particles match, what is their velocity?

PRACTICE QUESTIONS

Single Correct Choice Type

1. v_x is the velocity of a particle moving along the x-axis as shown in the figure. If $x = 2.0$ m at $t = 1.0$ s, what is the position of the particle at $t = 6.0$ s?

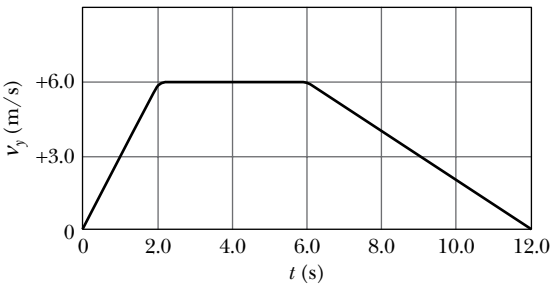


(a) −2.0 m **(b)** +2.0 m
(c) +1.0 m **(d)** −1.0 m

2. What is the magnitude of the average acceleration of a skier who, starting from rest, reaches a speed of 8.0 m/s when going down a slope for 5.0 s?
(a) 0.85 m/s² **(b)** 1.1 m/s²
(c) 1.6 m/s² **(d)** 1.9 m/s²

3. During the first 18 minutes of a 1.0 hour trip, a car has an average speed of 11 m/s. What must the average speed of the car be during the last 42 minutes of the trip be if the car is to have an average speed of 21 m/s for the entire trip?
(a) 21 m/s **(b)** 25 m/s
(c) 23 m/s **(d)** 27 m/s

4. A helicopter is lifting off from the ground and is moving vertically upward. The graph shows its vertical velocity v_y versus time. How high is the helicopter after 12.0 s have elapsed?



(a) 6.0 m **(b)** 18 m
(c) 48 m **(d)** 12 m

5. A target is made of two plates, one of wood and the other of iron. The thickness of the wooden plate is 4 cm and that of iron plate is 2 cm. A bullet fired goes through the wood first and then penetrates 1 cm into iron. A similar bullet fired with the same velocity from opposite direction goes through iron first and then penetrates 2 cm into wood. If a_1 and a_2 be the retardations offered to the bullet by wood and iron plates, respectively, then
(a) $a_1 = 2a_2$ **(b)** $a_2 = 2a_1$
(c) $a_1 = a_2$ **(d)** Data insufficient

6. At time $t = 0$ s, an object is observed at $x = 0$ m; and its position along the x axis follows this expression: $x = -3t + t^3$, where the units for distance and time are meter and second, respectively. What is the object's displacement Δx between $t = 1.0$ s and $t = 3.0$ s?
(a) +20 m **(b)** +10 m
(c) −20 m **(d)** +2 m

7. A ball is thrown upward from the top of a 25.0 m tall building. The ball's initial speed is 12.0 m/s. At the same instant, a person is running on the ground at a distance of 31.0 m from the building. What must be the average speed of the person if he is to catch the ball at the bottom of the building?

(a) 8.18 m/s **(b)** 0.122 m/s
(c) 7.20 m/s **(d)** 0.139 m/s

8. An 18 year old runner can complete a 10.0 km course with an average speed of 4.39 m/s. A 50 year old runner can cover the same distance with an average speed of 4.27 m/s. How much later (in seconds) should the younger runner start in order to finish the course at the same time as the older runner?
(a) 12 s **(b)** 48 s
(c) 64 s **(d)** 24 s

9. A body is dropped from a height of 39.2 m. After it crosses half distance, the acceleration due to gravity ceases to act. The body will hit the ground with velocity (Take $g = 10$ m/s^2)
(a) 19.6 m/s **(b)** 20 m/s
(c) 1.96 m/s **(d)** 196 m/s

10. A drag racing car starts from rest at $t = 0$ and moves along a straight line with velocity given by $v = bt^2$, where b is a constant. The expression for the distance traveled by this car from its position at $t = 0$ is:
(a) bt^3 **(b)** $bt^3/3$
(c) $4bt^2$ **(d)** $3bt^2$

11. The velocity of a diver just before hitting the water is –10.1 m/s, where the minus sign indicates that her motion is directly downward. What is her displacement during the last 1.20 s of the dive?
(a) –5.06 m **(b)** –7.06 m
(c) –12.1 m **(d)** –4.27 m

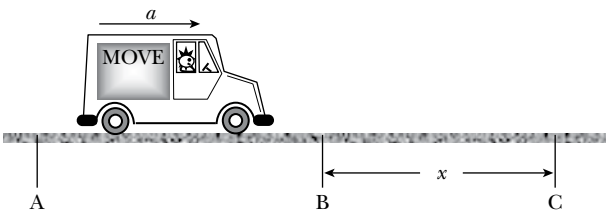
12. The velocity at the midway point of a ball able to reach a height y when thrown with velocity v_i at the origin is
(a) $v_i/2$ **(b)** $\sqrt{v_i 2gy}$
(c) $\sqrt{v_i^2/2}$ **(d)** $\sqrt{v_i^2/2gy}$

13. A car is initially travelling at 50.0 km/h. The brakes are applied and the car stops over a distance of 35 m. What was magnitude of the car's acceleration while it was braking?
(a) 2.8 m/s^2 **(b)** 5.4 m/s^2
(c) 36 m/s^2 **(d)** 71 m/s^2

14. The three-toed sloth is the slowest moving land mammal. On the ground, the sloth moves at an average speed of 0.037 m/s, considerably slower than the giant tortoise, which walks at 0.076 m/s. After 12 minutes of walking, how much further would the tortoise have gone relative to the sloth?
(a) 11 m **(b)** 22 m
(c) 14 m **(d)** 28 m

15. Two cars travel along a level highway. It is observed that the distance between the cars is increasing. Which one of the following statements concerning this situation is necessarily true?
(a) The velocity of each car is increasing.
(b) At least one of the cars has non-zero acceleration.
(c) The trailing car has the smaller acceleration.
(d) Both cars could be accelerating at the same rate.

16. A truck accelerates from rest at point A with constant acceleration of magnitude a and subsequently, passes points B and C as shown in the figure.



The distance between points B and C is x, and the time required for the truck to travel from B to C is t. Which expression determines the average speed of the truck between the points B and C?
(a) $v^2 = 2ax$ **(b)** $v = xt$
(c) $v = \dfrac{x}{t}$ **(d)** $v = \dfrac{1}{2}at^2$

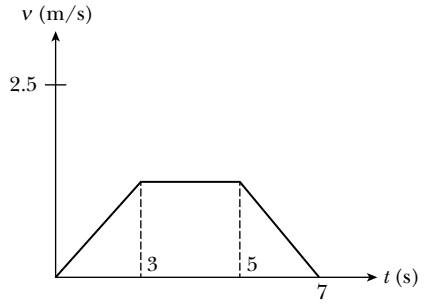
17. In reaching her destination, a backpacker walks with an average velocity of 1.34 m/s, due west. This average velocity results because she hikes for 6.44 km with an average velocity of 2.68 m/s, due west, turns around, and hikes with an average velocity of 0.447 m/s, due east. How far east did she walk?
(a) 3.5 km **(b)** 1.8 km
(c) 2.4 km **(d)** 0.81 km

18. Water drops fall at regular intervals from a roof. At an instant when a drop is about to leave the roof, the separations between 3 successive drops below the roof are in the ratio
(a) 1:2:3 **(b)** 1:4:9
(c) 1:3:5 **(d)** 1:5:13

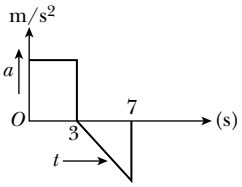
19. A golfer rides in a golf cart at an average speed of 3.10 m/s for 28.0 s. She then gets out of the cart and starts walking at an average speed of 1.30 m/s. For how long (in seconds) must she walk if her average speed for the entire trip, riding and walking, is 1.80 m/s?
(a) 73 s **(b)** 31 s
(c) 57 s **(d)** 44 s

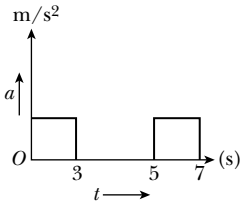
20. A train approaches a small town with a constant velocity of +28.6 m/s. The operator applies the brake, reducing the train's velocity to +11.4 m/s. If the average acceleration of the train during braking is –1.35 m/s^2, for what elapsed time does the operator apply the brake?
(a) 8.44 s **(b)** 3.38 s
(c) 12.7 s **(d)** 5.92 s

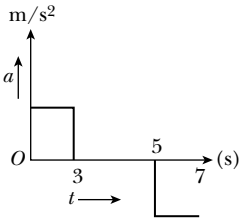
21. The velocity—time graph of a body is given in figure below.

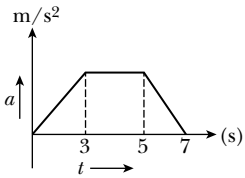


The acceleration–time graph of the motion of the body is

(a) 

(b) 

(c) 

(d) 

22. A hot-air balloon is rising upward with a constant speed of 2.50 m/s. When the balloon is 3.00 m above the ground, the balloonist accidentally drops a compass over the side of the balloon. How much time elapses before the compass hits the ground?

(a) 2.43 s **(b)** 0.568 s
(c) 1.08 s **(d)** 0.410 s

23. Starting from rest, a particle confined to move along a straight line is accelerated at a rate of 5.0 m/s². Which one of the following statements accurately describes the motion of this particle?

(a) The particle travels 5.0 m during each second.
(b) The particle travels 5.0 m only during the first second.
(c) The acceleration of the particle increases by 5.0 m/s² during each second.
(d) The speed of the particle increases by 5.0 m/s during each second.

24. An automobile starts from rest and accelerates to a final velocity in two stages along a straight road. Each stage occupies the same amount of time. In stage 1, the magnitude of the car's acceleration is 3.0 m/s². The magnitude of the car's velocity at the end of stage 2 is 2.5 times greater than it is at the end of stage 1. Find the magnitude of the acceleration in stage 2.

(a) 4.5 m/s² **(b)** 3.0 m/s²
(c) 3.8 m/s² **(d)** 2.2 m/s²

25. A point moving along the x-direction starts from rest at $x = 0$ and comes to rest at $x = 1$ after 1 s. Its acceleration at any point is denoted by α. Which of the following is not correct?

(a) α must change sign during the motion.
(b) $|\alpha| \geq 4$ units at some or all points during the motion.
(c) It is not possible to specify an upper limit for $|\alpha|$ from the given data.
(d) $|\alpha|$ cannot be less than 1/2 during the motion.

26. From the top of a cliff, a person uses a slingshot to fire a pebble straight downward, which is the negative direction. The initial speed of the pebble is 9.0 m/s. What is the acceleration (magnitude and direction) of the pebble during the downward motion?

(a) zero m/s² **(b)** 9.8 m/s², downward
(c) 9.8 m/s², upward **(d)** 1.1 m/s², upward

27. A particle has a velocity u towards east at $t = 0$. Its acceleration is towards west and is constant. Let x_A and x_B be the magnitude of displacements in the first 10 seconds and the next 10 seconds.

(a) $x_A < x_B$
(b) $x_A = x_B$
(c) $x_A > x_B$
(d) the information is insufficient to decide the relation of x_A with x_B

28. Two motorcycles are travelling due east with different velocities. However, four seconds later, they have the same velocity. During this four-second interval, motorcycle A has an average acceleration of 2.0 m/s² due east, while motorcycle B has an average acceleration of 4.0 m/s² due east. By how much did the speeds differ at the beginning of the four-second interval, and which motorcycle was moving faster?

(a) –2.0 m/s **(b)** –6.0 m/s
(c) +8.0 m/s **(d)** +1.0 m/s

29. A bullet is fired through a board, 14.0 cm thick, with its line of motion perpendicular to the face of the board. If it enters with a speed of 450 m/s and emerges with a speed of 220 m/s, what is the bullet's acceleration as it passes through the board?

(a) –500 km/s² **(b)** –550 km/s²
(c) –360 km/s² **(d)** –520 km/s²

30. Ball A is dropped from rest from a window. At the same instant, ball B is thrown downward; and ball C is thrown upward from the same window. Which statement concerning the balls after their release is necessarily true if air resistance is neglected?

(a) At some instant after it is thrown, the acceleration of ball C is zero.
(b) All three balls strike the ground at the same time.
(c) All three balls have the same velocity at any instant.
(d) All three balls have the same acceleration at any instant.

31. A football player, starting from rest at the line of scrimmage, accelerates along a straight line for a time of 1.5 s. Then, during a negligible amount of time, he changes the magnitude of his acceleration to a value of 1.1 m/s². With this acceleration, he continues in the same direction for another 1.2 s, until he reaches a speed of 3.4 m/s. What is the value of his acceleration (assumed to be constant) during the initial 1.5 s period?

(a) 0.91 m/s² **(b)** 1.1 m/s²
(c) 1.4 m/s² **(d)** 2.2 m/s²

32. Water drips from rest from a leaf that is 20 meters above the ground. Neglecting air resistance, what is the speed of each water drop when it hits the ground?

(a) 30 m/s **(b)** 40 m/s
(c) 20 m/s **(d)** 15 m/s

33. A proton moving along the x axis has an initial velocity of 4.0×10^6 m/s and a constant acceleration of 6.0×10^{12} m/s². What is the velocity of the proton after it has traveled a distance of 80 cm?

(a) 5.1×10^6 m/s **(b)** 6.3×10^6 m/s
(c) 4.8×10^6 m/s **(d)** 3.9×10^6 m/s

34. A sprinter accelerates from rest to a top speed with an acceleration whose magnitude is 3.80 m/s². After achieving top speed, he runs the remainder of the race without speeding up or slowing down. The total race is fifty meters long. If the total race is run in 7.88 s, how far does he run during the acceleration phase?
(a) 6.85 m **(b)** 9.03 m
(c) 7.62 m **(d)** 13.6 m

35. A boy on a skate board skates off a horizontal bench at a velocity of 10 m/s. One tenth of a second after he leaves the bench, to two significant figures, the magnitudes of his velocity and acceleration are:
(a) 10 m/s; 9.8 m/s² **(b)** 9.0 m/s; 9.8 m/s²
(c) 9.0 m/s; 9.0 m/s² **(d)** 1.0 m/s; 9.0 m/s²

36. At the beginning of a basketball game, a referee tosses the ball straight up with a speed of 4.6 m/s. A player cannot touch the ball until after it reaches its maximum height and begins to fall down. What is the minimum time that a player must wait before touching the ball?
(a) 0.24 s **(b)** 0.66 s
(c) 0.47 s **(d)** 0.94 s

37. A car starts from rest and accelerates at a constant rate in a straight line. In the first second, the car covers a distance of 2.0 meters. How much additional distance will the car cover during the second second of its motion?
(a) 2.0 m **(b)** 6.0 m
(c) 4.0 m **(d)** 8.0 m

38. A pellet gun is fired straight downward from the edge of a cliff that is 15 m above the ground. The pellet strikes the ground with a speed of 27 m/s. How far above the cliff edge would the pellet have gone had the gun been fired straight upward?
(a) 4.5 m **(b)** 22 m
(c) 15 m **(d)** 29 m

39. An automobile travelling along a straight road increases its speed from 30.0 m/s to 50.0 m/s in a distance of 180 m. If the acceleration is constant, how much time elapses while the auto moves this distance?
(a) 6.00 s **(b)** 4.50 s
(c) 3.60 s **(d)** 4.00 s

40. A landing airplane makes contact with the runway with a speed of 78.0 m/s and moves toward the south. After 18.5 seconds, the airplane comes to rest. What is the average acceleration of the airplane during the landing?
(a) 2.11 m/s², north **(b)** 4.22 m/s², north
(c) 2.11 m/s², south **(d)** 4.22 m/s², south

41. While standing on a bridge 15.0 m above the ground, you drop a stone from rest. When the stone has fallen 3.20 m, you throw a second stone straight down. What initial velocity must you give the second stone if they are both to reach the ground at the same instant? Take the downward direction to be the negative direction.
(a) –4.9 m/s **(b)** –9.8 m/s
(c) –8.4 m/s **(d)** –11 m/s

42. An object starts from rest at the origin and moves along the x axis with a constant acceleration of 4 m/s². Its average velocity as it goes from $x = 2$ m to $x = 8$ m is:
(a) 2 m/s **(b)** 3 m/s
(c) 5 m/s **(d)** 6 m/s

43. A rock is dropped from rest from a height h above the ground. It falls and hits the ground with a speed of 11 m/s. From what height should the rock be dropped so that its speed on hitting the ground is 22 m/s? Neglect air resistance.
(a) 1.4h **(b)** 3.0h
(c) 2.0h **(d)** 4.0h

44. A car is traveling to the left, which is the negative direction. The direction of travel remains the same throughout this problem. The car's initial speed is 27.0 m/s, and during a 5.0 s interval, it changes to a final speed of 29.0 m/s. Find the acceleration (magnitude and algebraic sign) and state whether or not the car is decelerating.
(a) –11 m/s² **(b)** –5.8 m/s²
(c) –0.40 m/s² **(d)** +0.40 m/s²

45. A drag racer, starting from rest, speeds up for 402 m with an acceleration of +17.0 m/s². A parachute then opens, slowing the car down with an acceleration of –6.10 m/s². How fast is the racer moving 3.50×10^2 m after the parachute opens?
(a) 96.9 m/s **(b)** 65.3 m/s
(c) 82.9 m/s **(d)** 20.1 m/s

46. A baseball is hit straight up and is caught by the catcher 2.0 s later, at the same height at which it left the bat. The maximum height of the ball during this interval is:
(a) 4.9 m **(b)** 7.4 m
(c) 12.4 m **(d)** 19.6 m

47. An object is thrown vertically upward with a certain initial velocity in a world where the acceleration due to gravity is 19.6 m/s². The height to which it rises is _____ that to which the object would rise if thrown upward with the same initial velocity on the Earth. Neglect air resistance.
(a) one fourth **(b)** half
(c) twice **(d)** four times

48. A car is stopped at a red traffic light. When the light turns to green, the car has a constant acceleration and crosses the 9.10 m intersection in 2.47 s. What is the magnitude of the car's acceleration?
(a) 1.77 m/s² **(b)** 3.60 m/s²
(c) 2.98 m/s² **(d)** 7.36 m/s²

49. A hot air balloon is ascending straight up at a constant speed of 7.0 m/s. When the balloon is 12.0 m above the ground, a gun fires a pellet straight up from ground level with an initial speed of 30.0 m/s. Along the paths of the balloon and the pellet, there are two places where each of them has the same altitude at the same time. How far above ground level are these places?
(a) 13 m, 37 m **(b)** 16 m, 41 m
(c) 17 m, 44 m **(d)** 14 m, 28 m

50. The minimum takeoff speed for a certain airplane is 75 m/s. What minimum acceleration is required if the plane must leave a runway of length 950 m? Assume the plane starts from rest at one end of the runway.
(a) 1.5 m/s² **(b)** 4.5 m/s²
(c) 3.0 m/s² **(d)** 6.0 m/s²

51. An elevator is moving upward with a speed of 11 m/s. Three seconds later, the elevator is still moving upward, but its speed has been reduced to 5.0 m/s. What is the average acceleration of the elevator during the 3.0 s interval?

(a) 2.0 m/s^2, upward **(b)** 5.3 m/s^2, upward
(c) 2.0 m/s^2, downward **(d)** 5.3 m/s^2, downward

52. A particle moving along the x axis has a position given by $x = 54t - 2.0t^3$ m. At the time $t = 3.0$ s, the speed of the particle is zero. Which statement is correct?

(a) The particle remains at rest after $t = 3.0$ s.
(b) The particle no longer accelerates after $t = 3.0$ s.
(c) The particle can be found at positions $x < 0$ m only when $t < 0$ s.
(d) None of the above is correct.

53. A pitcher delivers a fast ball with a velocity of 43 m/s to the south. The batter hits the ball and gives it a velocity of 51 m/s to the north. What was the average acceleration of the ball during the 1.0 ms when it was in contact with the bat?

(a) 4.3×10^4 m/s^2, south
(b) 9.4×10^4 m/s^2, north
(c) 5.1×10^4 m/s^2, north
(d) 2.2×10^3 m/s^2, south

54. A car traveling along a road begins accelerating with a constant acceleration of 1.5 m/s^2 in the direction of motion. After travelling 392 m at this acceleration, its speed is 35 m/s. Determine the speed of the car when it began accelerating.

(a) 1.5 m/s **(b)** 34 m/s
(c) 7.0 m/s **(d)** 49 m/s

More than One Correct Choice Type

55. Choose the correct statements:

(a) When the total area of the acceleration–time graph is negative, it always means that the final velocity of the particle is negative.
(b) When the total area of the velocity–time graph is negative, it always means that the final displacement of the particle is negative.
(c) When the total area of the velocity–time graph is negative, it may happen that the particle returns to its original position.
(d) When the total area of the acceleration–time graph is negative, it may happen that the final velocity of the particle is zero.

56. Two bodies of masses m_1 and m_2 are dropped from heights h_1 and h_2, respectively. They reach the ground after time t_1 and t_2 and strike the ground with v_1 and v_2, respectively. Choose the correct relations from the following:

(a) $\dfrac{t_1}{t_2} = \sqrt{\dfrac{h_1}{h_2}}$ **(b)** $\dfrac{t_1}{t_2} = \sqrt{\dfrac{h_2}{h_1}}$

(c) $\dfrac{v_1}{v_2} = \sqrt{\dfrac{h_1}{h_2}}$ **(d)** $\dfrac{v_1}{v_2} = \dfrac{h_2}{h_1}$

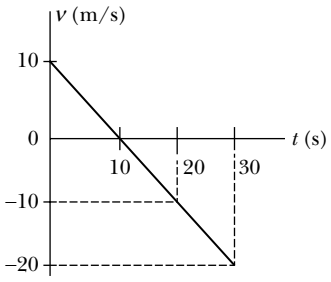
57. A particle of mass m moves on the x-axis as follows: it starts from rest at $t = 0$ from the point $x = 0$, and comes to rest at $t = 1$ at the point $x = 1$. No other information is available about its motion at intermediate times ($0 < t < 1$). If α denote the instantaneous acceleration of the particle, then

(a) α cannot remain positive for all t in the interval $0 \le t \le 1$.
(b) $|\alpha|$ cannot exceed 2 at any point in its path.
(c) $|\alpha|$ must be ≥ 4 at some point or points in its path.
(d) α must change sign during the motion, but no other assertion can be made with the information given.

58. Mark the correct statements for a particle going on a straight line:

(a) If the velocity and acceleration have opposite sign, the object is slowing down.
(b) If the position and velocity have opposite sign, the particle is moving towards the origin.
(c) If the velocity is zero at an instant, the acceleration should also be zero at that instant.
(d) If the velocity is zero for a time interval, the acceleration is zero at any instant within the time interval.

59. The acceleration a of a particle depends on displacements covered in a time t as $a = S + 5$. It is given that initially $S = 0$ m and $v = 5$ m/s. Then



(a) $v = S + 5$
(b) $v = \sqrt{S + 5}$
(c) $t = \log_e\left(\dfrac{S+5}{S}\right)$
(d) $t = \log_e\left(\dfrac{S+5}{5}\right)$

60. Pick the correct statements:

(a) Average speed of a particle in a given time is never less than the magnitude of the average velocity.
(b) It is possible to have a situation in which

$$\left|\frac{d\vec{v}}{dt}\right| \neq 0 \text{ but } \frac{d}{dt}|\vec{v}| = 0$$

(c) The average velocity of a particle is zero in a time interval. It is possible than the instantaneous velocity is never zero in the interval.
(d) The average velocity of a particle moving on a straight line is zero in a time interval. It is possible that the instantaneous velocity is never zero in the interval. (Infinite accelerations are not allowed)

Linked Comprehension

***Paragraph for Questions 61–64*:** A racecar, traveling at constant speed, makes one lap around a circular track of radius r in a time t.

Note: The circumference of a circle is given by $C = 2\pi r$.

61. When the car has traveled halfway around the track, what is the magnitude of its *displacement* from the starting point?

(a) r **(b)** $2r$
(c) πr **(d)** $2\pi r$

62. What is the average speed of the car for one complete lap?

(a) $\frac{r}{t}$ **(b)** $\frac{\pi r}{t}$
(c) $\frac{2r}{t}$ **(d)** $\frac{2\pi r}{t}$

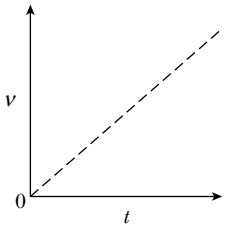
63. Determine the magnitude of the average velocity of the car for one complete lap.

(a) $\frac{r}{t}$ m/s **(b)** $\frac{\pi r}{t}$ m/s
(c) zero m/s **(d)** $\frac{2r}{t}$

64. Which one of the following statements concerning this car is true?

(a) The displacement of the car does not change with time.
(b) The instantaneous velocity of the car is constant.
(c) The average speed of the car is the same over any time interval.
(d) The average velocity of the car is the same over any time interval.

***Paragraph for Questions 65–68*:** The figure shows the speed as a function of time for an object in free fall near the surface of the earth. The object was dropped from rest in a long evacuated cylinder.



65. Which one of the following statements best explains why the graph goes through the origin?

(a) The object was in a vacuum.
(b) All v vs. t curves pass through the origin.
(c) The object was dropped from rest.
(d) The velocity of the object was constant.

66. What is the numerical value of the slope of the line?

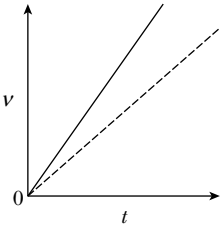
(a) 1.0 m/s^2 **(b)** 9.8 m/s^2
(c) 2.0 m/s^2 **(d)** 7.7 m/s^2

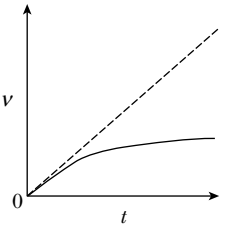
67. What is the speed of the object 3.0 seconds after it is dropped?

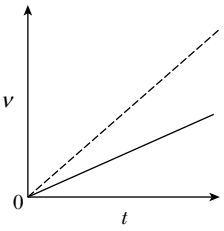
(a) 3.0 m/s **(b)** 7.7 m/s
(c) 9.8 m/s **(d)** 29 m/s

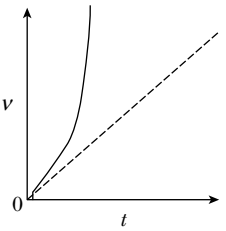
68. If the same object were released in air, the magnitude of its acceleration would begin at the free-fall value, but it would decrease continuously to zero as the object continued to fall. For which one of the choices given does the solid line best represent the speed of the object as a function of time when it is dropped from rest in air?

Note: The dashed line shows the free-fall under vacuum graph for comparison.

(a)


(b)


(c)


(d)


***Paragraph for Questions 69–72*:** A rock, dropped from rest near the surface of an atmosphere-free planet, attains a speed of 20.0 m/s after falling 8.0 m.

69. What is the magnitude of the acceleration due to gravity on the surface of this planet?

(a) 0.40 m/s^2 **(b)** 2.5 m/s^2
(c) 1.3 m/s^2 **(d)** 25 m/s^2

70. How long did it take the object to fall the 8.0 m mentioned?

(a) 0.40 s **(b)** 1.3 s
(c) 0.80 s **(d)** 2.5 s

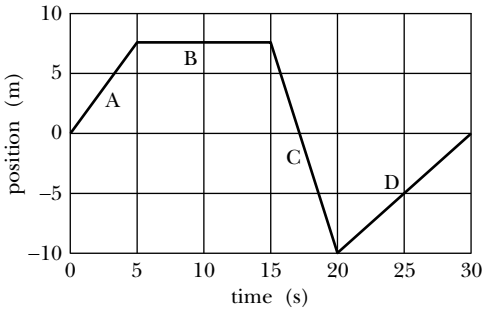
71. How long would it take the object, falling from rest, to fall 16 m on this planet?

(a) 0.8 s **(b)** 2.5 s
(c) 1.1 s **(d)** 3.5 s

72. Determine the speed of the object after falling from rest through 16 m on this planet.

(a) 28 m/s **(b)** 32 m/s
(c) 56 m/s **(d)** 64 m/s

***Paragraph for Questions 73–75*:** An object is moving along a straight line in the positive x direction. The graph shows its position from the starting point as a function of time. Various segments of the graph are identified by the letters A, B, C, and D.



73. Which segment(s) of the graph represent(s) a *constant velocity* of +1.0 m/s?

(a) A **(b)** C
(c) B **(d)** D

74. What was the instantaneous velocity of the object at the end of the eighth second?

(a) +7.5 m/s **(b)** –0.94 m/s
(c) 0 m/s **(d)** +0.94 m/s

75. During which interval(s) did the object move in the negative x direction?

(a) Only during interval B.
(b) During both intervals C and D.
(c) Only during interval C.
(d) Only during interval D.

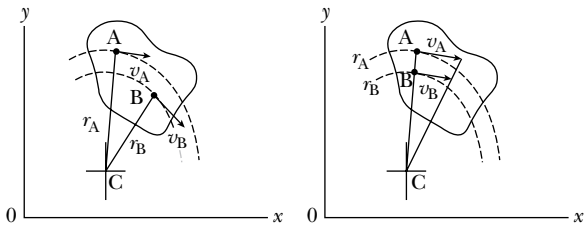
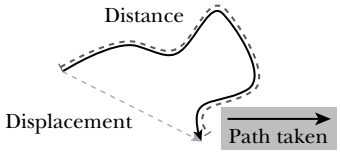
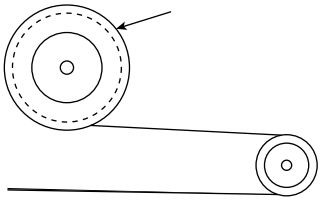
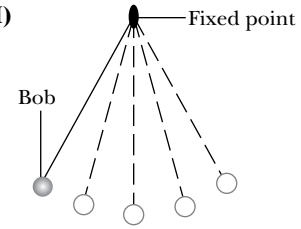
Matrix-Match

76. A particle of mass 10 kg is moving in a straight line. If its displacement, x with time t is given by $x = (at^3 + bt + c)$ m, the force acting on it at the end of 4 s is 240 N. Its displacement at $t = 0$ is 110 m and its velocity at $t = 0$ is –2 m/s. Here, values of a, b, c and acceleration are in their proper SI units:

Column I	Column II
(a) Value of c is	**(p)** –2
(b) Value of b is	**(q)** 1
(c) Value of a is	**(r)** 24
(d) Acceleration at $t = 4$ s is	**(s)** –10

Directions for Questions 77 and 78: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

77. In the given table, Columns I and II give statements regarding different types of motion of particle along straight line and Column III shows figures depicting the motion.

Column I	Column II	Column III
(I) When it moves in such a way that the linear distance covered by each particle is	**(i)** repeats its motion about a fixed point	**(J)** 
(II) A particle covering	**(ii)** the same during the motion	**(K)** 
(III) The kind of movement when an object	**(iii)** undergoes the same angular displacement about a particular axis of rotation	**(L)** 
(IV) Every particle of the body	**(iv)** equal displacements in equal intervals of time	**(M)** 

(1) Which combination of statements is characteristic of oscillatory motion?

(a) (I) (ii) (J) **(b)** (III) (i) (M)
(c) (II) (i) (M) **(d)** (I) (iii) (J)

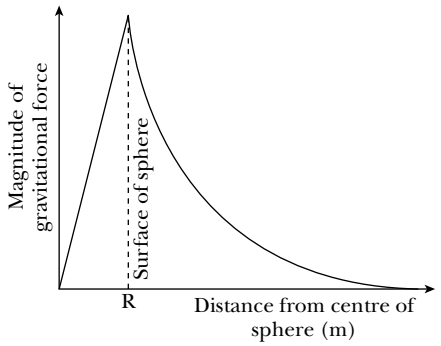
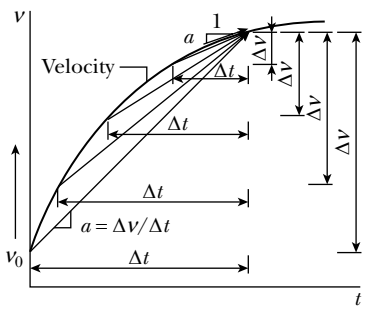
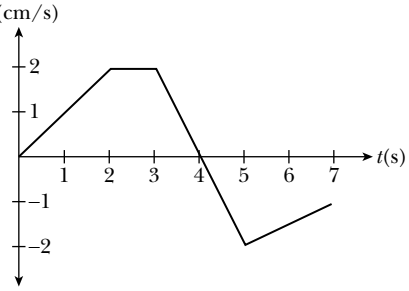
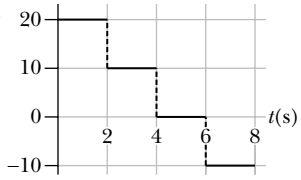
(2) Which combination of statements is characteristic of a translatory motion?

(a) (I) (iii) (K) **(b)** (IV) (iii) (L)
(c) (II) (iii) (L) **(d)** (I) (ii) (J)

(3) Which combination of statements is characteristic of rotatory motion?

(a) (III) (i) (K) **(b)** (I) (i) (L)
(c) (IV) (iii) (L) **(d)** (II) (iii) (M)

78. In the given table, Columns I and II give statements regarding the velocity and acceleration of particle in different types of motion along a straight line and Column III shows figures depicting the motion.

Column I	Column II	Column III
(I) Ratio of the total change in velocity to the	**(i)** velocity at any point of time	**(J)** 
(II) The rate of change of	**(ii)** total time taken in which this velocity change takes place.	**(K)** 
(III) Acceleration on an object caused	**(iii)** with respect to time	**(L)** 
(IV) The rate of change of angular velocity	**(iv)** by gravity, irrespective of the mass or composition of the body	**(M)** 

(1) Which combination of statements explains about instantaneous acceleration?

(a) (I) (iii) (L) **(b)** (IV) (i) (M)
(c) (II) (iv) (K) **(d)** (II) (i) (K)

(2) Which combination of statements explains about angular acceleration?

(a) (III) (ii) (L) **(b)** (IV) (iii) (M)
(c) (II) (iii) (K) **(d)** (I) (i) (M)

(3) Which combination of statements explains about gravitational acceleration?

(a) (III) (iv) (J)
(b) (I) (i) (K)
(c) (III) (iii) (K)
(d) (I) (ii) (L)

Integer Type

79. A particle is dropped from a height h and at the same instant another particle is projected vertically up from the ground. They meet when the upper one has descended a height $h/3$. Find the ratio of their velocities at this instant.

80. A key falls from a bridge that is 45 m above the water. It falls directly into a model boat, moving with constant velocity, that is, 12 m from the point of impact when the key is released. What is the speed of the boat?

81. A bullet loses 1/20 of its velocity in passing through a plank. What is the least number of planks required to stop the bullet?

ANSWER KEY

Checkpoints

1. b and c
2. (check the derivative dx/dt) (a) 1 and 4; (b) 2 and 3
3. (a) plus; (b) minus; (c) minus; (d) plus
4. 1 and 4 ($a = d^2x/dt^2$ must be constant)
5. (a) plus (upward displacement on y axis); (b) minus (downward displacement on y axis); (c) $a = -g = -9.8$ m/s^2

Problems

1. (a) zero; (b) 0.0
2. (a) 1.71 m/s; (b) 2.14 m/s
3. (a) 3.26 km/h; (b) 6.34 km/h
4. 44 km/h
5. (a) 0; (b) –2 m; (c) 0 m; (d) 12 m; (e) 12 m; (f) 7 m/s
6. (a) ≈35 km; (b) ≈16 km
7. (a) 0.50 m/s; (b) 10 s
8. (a) method 1 gives the car's speed v_c a in windless situation; (b) 5.76×10^{-4}
9. 25.00 m/s
10. (a) 48.0 m; (b) 2.50 m/s; (c) direction of the shock wave is downstream
11. (a) 38 m/s; (b) 43 m/s
12. (a) 0; (b) 4.0 m; (c) and (d) ±0.82 s; (f) add $20t$; (g) increase
13. (a) 61 m; (b) 11 m/s; (c) –18 m/s^2; (d) $x = 64$ m; (e) 4.0 s; (f) 24 m/s; (g) 2.0 s; (h) –24 m/s^2; (i) 18 m/s
14. –20 m/s^2
15. (a) ±1.2 s; (b) $t = 0$; (c) $t > 0$; (d) $t < 0$
16. (a) 3.10 m/s^2; (b) 11.64 s ≈ 11.6 s
17. 0.36
18. 0.32 m/s^2
19. (a) 0.144 m
20. (1.6×10^3) m/s^2
21. (a) 5.53 s; (b) 75.2 m
22. (a) ≈10.6 m; (b) ≈41.5 s
23. (a) 3.0 s
24. 1.01 km
25. 202.4 m/s^2 (or $21g$)
26. +2.0 s
27. –8.7 m/s; (b) –3.3 m/s^2
28. 9.93×10^{-10} s = 0.993 ns
29. (a) 51.2 s; (b) 35.2 m/s
30. (a) 4.0 m/s^2; (b) +x direction
31. 23.96 ≈ 24 s
32. (a) continue; (b) stop
33. (a) 21 m/s^2; (b) 96 m
34. (a) 15.0 m; (b) ≈28.9 m/s, or 104 km/h
35. (a) –0.994 m/s^2
36. 122.5 m
37. (a) 188 m/s; (b) no
38. (a) –46 m/s; (b) 6.1 m/s
39. (a) 1.38 s; (b) 28.5 m/s
40. –19.6 m/s
41. 4.0 m/s
42. (a) 11.9 m/s
43. ≈3.0 m/s
44. (a) 1.26×10^3 m/s; (b) upward
45. (a) 89 cm; (b) 22 cm
46. 26 m
47. 2.34 m
48. 74 m
49. (a) 0.13 m; (b) 0.50 m
50. 0.56 m/s
51. 5.0 m/s
52. 100 m
53. 12.5 m/s

Practice Questions

Single Correct Choice Type

1. (d)
2. (c)
3. (b)
4. (c)
5. (b)
6. (a)
7. (a)
8. (c)
9. (a)
10. (b)
11. (a)
12. (c)
13. (a)
14. (d)
15. (d)
16. (c)
17. (d)
18. (b)
19. (a)
20. (c)
21. (c)
22. (c)
23. (d)
24. (a)
25. (d)
26. (b)
27. (d)
28. (c)
29. (b)
30. (d)

31. (c) **32.** (c) **33.** (a) **34.** (a) **35.** (a)
36. (c) **37.** (b) **38.** (b) **39.** (b) **40.** (b)
41. (d) **42.** (d) **43.** (d) **44.** (c) **45.** (a)
46. (a) **47.** (b) **48.** (c) **49.** (b) **50.** (c)
51. (c) **52.** (d) **53.** (b) **54.** (c)

More than One Correct Choice Type

55. (c), (d) **56.** (a), (c) **57.** (a), (c) **58.** (a), (b), (d) **59.** (a), (d)
60. (a), (b), (c)

Linked Comprehension

61. (b) **62.** (d) **63.** (c) **64.** (c) **65.** (c)
66. (b) **67.** (d) **68.** (b) **69.** (d) **70.** (c)
71. (c) **72.** (a) **73.** (d) **74.** (c) **75.** (c)

Matrix-Match

76. (a) → (s); (b) → (p); (c) → (q); (d) → (r)
77. (1) → (b); (2) → (d); (3) → (c)
78. (1) → (d); (2) → (b); (3) → (a)

Integer Type

79. 2 **80.** 4 **81.** 11

APPENDIX | ELEMENTS OF CALCULUS

Differential Calculus

Calculus is a branch of mathematics which uses operators to extract information from functions. The derivative of a function is defined in terms of a limit, but in practice shortcuts are used to calculate the derivatives of the basic types of functions. When the basic types of functions are combined by addition, subtraction, multiplication, division and/or composition, chain rules are used to determine the derivatives of the combinations.

Branch of Mathematics	Mathematical Elements Used
Arithmetic Numbers	$(0, 1, 101.7, \pi, e, \ldots)$
Algebra Variables	$(x, y, z, t, L, A, \ldots)$
Analysis Functions	$(f(x), v(t), a(t), \ldots)$
Calculus Operators	(derivative, integral, Laplacian, …)

There is a very large number of operators, but for introductory physics only the derivative and integral are needed. There is also a very large number of functions, but the only ones typically needed are constant, polynomial, trigonometric, exponential and logarithmic. These can also be combined by addition, subtraction, multiplication, division and/or composition.

The **derivative** of a function is defined in terms of a limit.

$$\frac{df(x)}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \tag{A-1}$$

Unless a new function is encountered, the limit definition of the derivative is never used. Instead, the limits are determined once for each class of functions, and the resulting expressions, which are called shortcuts, are used to calculate them.

Constant: If $f(x) = a$, then $f'(x) = 0$. (A-2)

Polynomial: If $f(x) = ax^n$, then $f'(x) = nax^{n-1}$. (A-3)

Trigonometric: If $f(x) = \cos x$, then $f'(x) = -\sin x$ and if $f(x) = \sin x$, then $f'(x) = \cos x$. (A-4)

Exponential: If $f(x) = e^x$, then $f'(x) = e^x$. (A-5)

Logarithmic: If $f(x) = \ln x$, where $x > 0$, then $f'(x) = x - 1$. (A-6)

When the basic types of functions are combined by addition, subtraction, multiplication, division and/or composition, chain rules are used to determine the derivatives of the combinations.

Constant Factor: If $f(x) = cg\ (x)$, then $f'(x) = cg'(x)$. (A-7)

Addition/Subtraction: If $f(x) = g(x) \pm h(x)$, then $f'(x) = g'(x) \pm h'(x)$. (A-8)

Multiplication: If $f(x) = g(x)h(x)$, then $f'(x) = g'(x)h(x) + g(x)h'(x)$. (A-9)

Composition: If $f(x) = g(h(x))$, then $f'(x) = g(h(x))h'(x)$. (A-10)

The division chain rule is given by

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{(h(x))^2}. \tag{A-11}$$

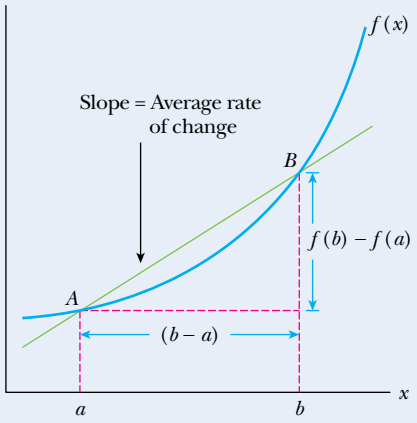


Figure A-1 Visualizing the average rate of change of f between a and b.

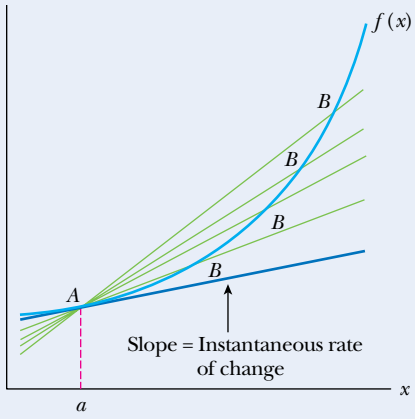


Figure A-2 Visualizing the instantaneous rate of change of f at a.

Visualizing Derivative: Slope of The Graph and Slope of the Tangent Line

Figure A-1 shows the average rate of change of a function represented by the slope of the secant line joining points A and B. The derivative is found by taking the average rate of change over smaller and smaller intervals. In Figure A-2, as point B moves toward point A, the secant line becomes the tangent line at point A. Thus, the derivative is represented by the slope of the tangent line to the graph at the point.

To determine the slope of the graph of a function at a point, calculate the derivative of the function and substitute the value of the variable. To determine the minimum and/or maximum values of a function, take its derivative, set it equal to zero, solve for the variable by factoring and using the Zero-Product Rule, and substitute the value or values of the variable into the function.

Derivatives of some common functions are tabulated as follows:

$\dfrac{d(au)}{dx} = a\dfrac{du}{dx}$	$\dfrac{du}{dt} = \dfrac{du}{dx}\cdot\dfrac{dx}{dt}$
$\dfrac{d(uv)}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$	$\dfrac{d(u/v)}{dx} = \dfrac{1}{v^2}\dfrac{du}{dx} - u\dfrac{du}{dx}$
$\dfrac{du}{dv} = \dfrac{du/dx}{dv/dx}$	
$\dfrac{d(\sin x)}{dx} = \cos x$	$\dfrac{d(\cos x)}{dx} = -\sin x$
$\dfrac{d(\tan x)}{dx} = \sec^2 x$	$\dfrac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$
$\dfrac{d(\sec x)}{dx} = \tan x \sec x$	$\dfrac{d(\operatorname{cosec}^2 x)}{dx} = -\cot x \operatorname{cosec} x$
$\dfrac{d(u)^n}{dx} = nu^{n-1}\dfrac{du}{dx}$	$\dfrac{d}{du}(\ln u) = \dfrac{1}{u}$
$\dfrac{d}{du}(e^u) = e^u$	

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Velocity

If a particle in rectilinear motion moves along an s-axis so that its position coordinate function of the elapsed time t is

$$s = f(t) \tag{A-12}$$

then f is called the **position function of the particle**; the graph of (A-12) is the position versus time curve. The **average velocity** of the particle over a time interval $[t_0, t_0 + h]$, $h > 0$, is defined to be

$$v_{\text{avg}} = \frac{\text{change in position}}{\text{time elapsed}} = \frac{f(t_0 + h) - f(t_0)}{h} \tag{A-13}$$

We can define the **instantaneous velocity** v_{inst} of the particle at time t_0 to be the limit as $h \to 0$ of the average velocities v_{avg} in Eq. (A-13):

$$v_{\text{inst}} = \lim_{h \to 0} v_{\text{avg}} = \lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h} \tag{A-14}$$

For a particle in rectilinear motion, average velocity describes its behavior over an *interval* of time. We are also interested in the particle's "instantaneous velocity," its speed and direction at a specific *instant* $t = t_0$ in time. Average velocities over small time intervals between $t = t_0$ and $t = t_0 + h$ can be viewed as approximations to this instantaneous velocity. If these average velocities have a limit as h approaches 0, then we can take that limit to be the instantaneous velocity of the particle at time t_0.

Integral Calculus

The **indefinite integral** of a function can also be defined in terms of a limit, but for our purposes a more useful definition relates it to the derivative. The indefinite integral is the inverse of the derivative, and vice versa. That is to say, the derivative of the indefinite integral of a function is equal to the function itself and the indefinite integral of the derivative of a function is equal to the function itself, within a constant called the **constant of integration**.

The symbol for the derivative is d, which is an abbreviation for difference, and the symbol for the indefinite integral is $\int$, which looks much like an s, which is an abbreviation for sum. Using these symbols, the inverse relationships are:

$$\frac{d}{dx}\int f(x)dx = f(x) \tag{A-15}$$

$$\int \frac{df(x)}{dx}dx = f(x) + C \tag{A-16}$$

This pair of equations is called the **fundamental theorem of calculus** and provides an intuitive way of calculating the indefinite integrals of many functions. Since the indefinite integral is the inverse of the derivative, it's often called the **antiderivative**.

Unless a new function is encountered, the limit definition of the indefinite integral is never used. Instead, the limits are determined once for each class of functions, and the resulting expressions, which are called shortcuts, are used to calculate them.

Constant: If $f(x) = a$, then $\int f(x)dx = ax + C$. (A-17)

Polynomial: If $f(x) = ax^n$, then $\int f(x)dx = ax^{n+1}/(n+1) + C$. (A-18)

Trigonometric: $\int \cos x dx = \sin x + C$ and $\int \sin x dx = -\cos x + C$. (A-19)

Exponential: If $f(x) = e^x$, then $\int f(x)dx = e^x + C$. (A-20)

Logarithm: If $f(x) = \ln x$, then $\int f(x)dx = x(\ln x - 1) + C$. (A-21)

You should verify these shortcuts are correct by applying the derivative to the indefinite integrals of each function. If the shortcut is correct, then the derivative of the indefinite integral will be equal to the original function. Also, notice the indefinite integral of e^x is itself. This is another very important property of this very special function.

Two applications of the integral used in physics are determining area on the graph of a function and the average value of a function.

The **area on the graph of a function** is defined as the area enclosed by the graph of the function, the horizontal axis, and two vertical lines. (See Figure A-3) If the area is above the horizontal axis, then it is positive and if it is below, then it is negative. If the two vertical lines are located in the same place, then the area is zero.

One way to determine this area is to calculate the **definite integral** of the function. This integral is the same as the indefinite integral except it results in a specific value rather than a generalized function. For this kind of integral, the constant of integration is omitted, the two values of the variable defining the vertical lines are substituted and the results are subtracted.

The two values defining the vertical lines are called **limits of integration**. The larger value is called the **upper limit** and the smaller is called the **lower limit**. The result using the lower limit is subtracted from the one using the upper limit. The subset of the domain on the horizontal axis between the two vertical lines is called the **interval**.

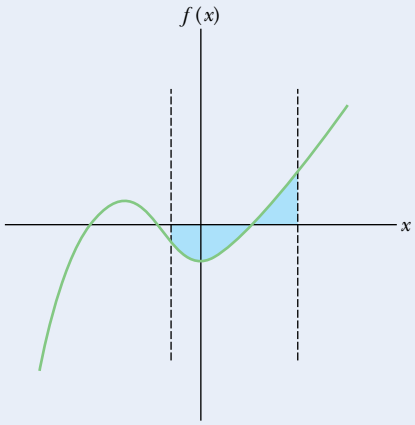

Figure A-3 Area enclosed by the graph of a function.

To determine the **average value of a function** over some interval, calculate the definite integral over the interval and divide by the length of the interval.

The average value of the function can be thought of as the average height of the graph, measured with respect to the horizontal axis. This method finds the average height by calculating the enclosed area and dividing by the length of the interval. Since area is length multiplied by height, dividing the area by the length of the interval gives the average height.

Visualizing Distance on the Velocity Graph

Suppose a car is moving with increasing velocity and suppose we measure the car's velocity every two seconds, obtaining the two-second data in Table A-1.

Table A-1 Velocity of Car in Every Two Seconds

Time (s)	0	2	4	6	8	10
Velocity (ft/s)	20	30	38	44	48	50

We can represent both upper and lower estimates on a graph of the velocity against time. The velocity can be graphed by plotting these data and drawing a smooth curve through the data points (See Figure A-4).

We use the fact that for a rectangle, Area = Height × Width. The area of the first dark rectangle is $20 \times 2 = 40$, the lower estimate of the distance moved during the first two seconds. The area of the second dark rectangle is $30 \times 2 = 60$, the lower estimate for the distance moved in the next two seconds. The total area of the dark rectangles represents the lower estimate for the total distance moved during the ten seconds.

If the dark and light rectangles are considered together, the first area is $30 \times 2 = 60$, the upper estimate for the distance moved in the first two seconds. The second area is $38 \times 2 = 76$, the upper estimate for the next two seconds. Continuing this calculation suggests that the upper estimate for the total distance is represented by the sum of the areas of the dark and light rectangles. Therefore, the area of the light rectangles alone represents the difference between the two estimates.

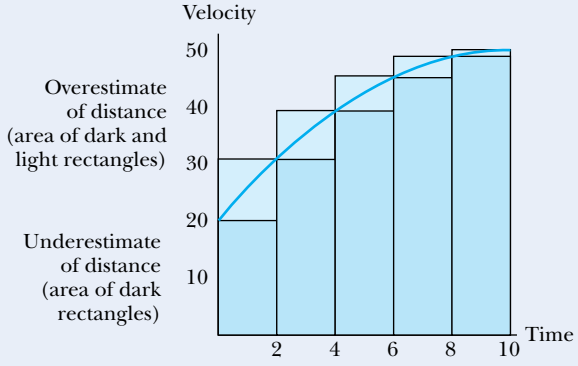

Figure A-4 Shaded area estimates distance traveled. Velocity measured every 2 seconds.

Figure A-5 shows a graph of the one-second data. The area of the dark rectangles again represents the lower estimate, and the area of the dark and light rectangles together represents the upper estimate. The total area of the light rectangles is smaller in Figure A-5 than in Figure A-4, so the underestimate and overestimate are closer for the one-second data than for the two-second data.

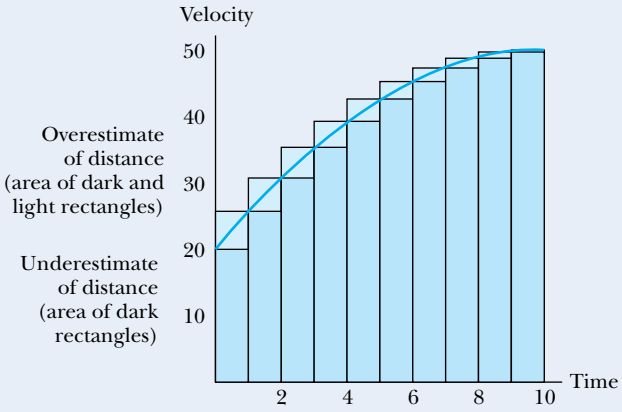

Figure A-5 Shaded area estimates distance traveled. Velocity measured every second.

CHAPTER

3

Vectors

Contents

3.1 What is Physics?
3.2 Vectors and Scalars
3.3 Vector Addition
3.4 Components of Vectors
3.5 Unit Vectors
3.6 Adding Vectors by Components
3.7 Multiplying Vectors

3.1 | WHAT IS PHYSICS?

Physics deals with a great many quantities that have both size and direction, and it needs a special mathematical language—the language of vectors—to describe those quantities. This language is also used in engineering, the other sciences, and even in common speech. If you have ever given directions such as "Go five blocks down this street and then hang a left," you have used the language of vectors. In fact, navigation of any sort is based on vectors, but physics and engineering also need vectors in special ways to explain phenomena involving rotation and magnetic forces, which we get to in later chapters. In this chapter, we focus on the basic language of vectors.

3.2 | VECTORS AND SCALARS

Key Concept

- Scalars, such as temperature, have magnitude only. They are specified by a number with a unit (10°C) and obey the rules of arithmetic and ordinary algebra. Vectors, such as displacement, have both magnitude and direction (5 m, north) and obey the rules of vector algebra.

A particle moving along a straight line can move in only two directions. We can take its motion to be positive in one of these directions and negative in the other. For a particle moving in three dimensions, however, a plus sign or minus sign is no longer enough to indicate a direction. Instead, we must use a *vector*.

A **vector** has magnitude as well as direction, and vectors follow certain (vector) rules of combination, which we examine in this chapter. A **vector quantity** is a quantity that has both a magnitude and a direction and thus can be represented with a vector. Some physical quantities that are vector quantities are displacement, velocity, and acceleration. You will see many

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more throughout this book, so learning the rules of vector combination now will help you greatly in later chapters.

Not all physical quantities involve a direction. Temperature, pressure, energy, mass, and time, for example, do not "point" in the spatial sense. We call such quantities **scalars**, and we deal with them by the rules of ordinary algebra. A single value, with a sign (as in a temperature of –40°F), specifies a scalar.

The simplest vector quantity is displacement, or change of position. A vector that represents a displacement is called, reasonably, a **displacement vector**. (Similarly, we have velocity vectors and acceleration vectors.) If a particle changes its position by moving from A to B in Fig. 3-1a, we say that it undergoes a displacement from A to B, which we represent with an arrow pointing from A to B. The arrow specifies the vector graphically. To distinguish vector symbols from other kinds of arrows in this book, we use the outline of a triangle as the arrowhead.

In Fig. 3-1a, the arrows from A to B, from A' to B', and from A'' to B'' have the same magnitude and direction. Thus, they specify identical displacement vectors and represent the same *change of position* for the particle. A vector can be shifted without changing its value *if* its length and direction are not changed.

The displacement vector tells us nothing about the actual path that the particle takes. In Fig. 3-1b, for example, all three paths connecting points A and B correspond to the same displacement vector, that of Fig. 3-1a. Displacement vectors represent only the overall effect of the motion, not the motion itself.

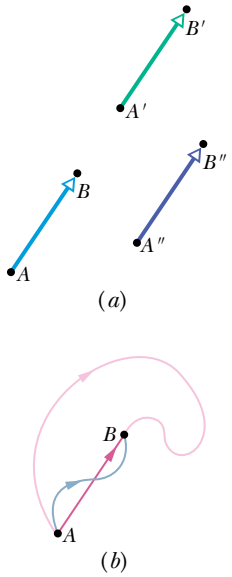

Figure 3-1 (a) All three arrows have the same magnitude and direction and thus represent the same displacement. (b) All three paths connecting the two points correspond to the same displacement vector.

3.3 | VECTOR ADDITION

Key Concept

- Two vectors $\vec{a}$ and $\vec{b}$ may be added geometrically by drawing them to a common scale and placing them head to tail. The vector connecting the tail of the first to the head of the second is the vector sum $\vec{s}$. To subtract $\vec{b}$ from $\vec{a}$, reverse the direction of $\vec{b}$ to get $-\vec{b}$; then add $-\vec{b}$ to $\vec{a}$. Vector addition is commutative and obeys the associative law.

Suppose that, as in the vector diagram of Fig. 3-2a, a particle moves from A to B and then later from B to C. We can represent its overall displacement (no matter what its actual path) with two successive displacement vectors, AB and BC. The *net* displacement of these two displacements is a single displacement from A to C. We call AC the **vector sum** (or **resultant**) of the vectors AB and BC. This sum is not the usual algebraic sum.

In Fig. 3-2b, we redraw the vectors of Fig. 3-2a and relabel them in the way that we shall use from now on, namely, with an arrow over an italic symbol, as in $\vec{a}$. If we want to indicate only the magnitude of the vector (a quantity that lacks a sign or direction), we shall use the italic symbol, as in a, b, and s. (You can use just a handwritten symbol.) A symbol with an overhead arrow always implies both properties of a vector, magnitude and direction.

We can represent the relation among the three vectors in Fig. 3-2b with the *vector equation*

$$\vec{s} = \vec{a} + \vec{b}, \tag{3-1}$$

which says that the vector $\vec{s}$ is the vector sum of vectors $\vec{a}$ and $\vec{b}$. The symbol + in Eq. 3-1 and the words "sum" and "add" have different meanings for vectors than they do in the usual algebra because they involve both magnitude *and* direction.

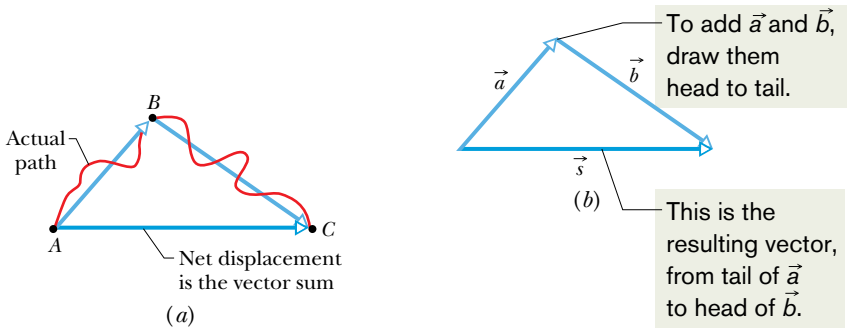

Figure 3-2 (*a*) *AC* is the vector sum of the vectors *AB* and *BC*. (*b*) The same vectors relabeled.

Figure 3-2 suggests a procedure for adding two-dimensional vectors $\vec{a}$ and $\vec{b}$ geometrically. (1) On paper, sketch vector $\vec{a}$ to some convenient scale and at the proper angle. (2) Sketch vector $\vec{b}$ to the same scale, with its tail at the head of vector $\vec{a}$, again at the proper angle. (3) The vector sum $\vec{s}$ is the vector that extends from the tail of $\vec{a}$ to the head of $\vec{b}$.

Properties. Vector addition, defined in this way, has two important properties. First, the order of addition does not matter. Adding $\vec{a}$ to $\vec{b}$ gives the same result as adding $\vec{b}$ to $\vec{a}$ (Fig. 3-3); that is,

$$\vec{a}+\vec{b}=\vec{b}+\vec{a} \quad \text{(commutative law)}. \tag{3-2}$$

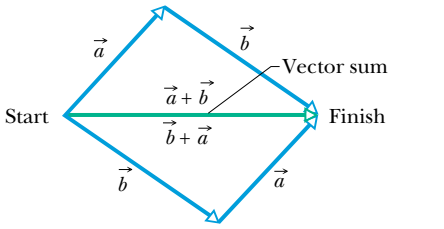

Figure 3-3 The two vectors $\vec{a}$ and $\vec{b}$ can be added in either order; see Eq. 3-2.

Second, when there are more than two vectors, we can group them in any order as we add them. Thus, if we want to add vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, we can add $\vec{a}$ and $\vec{b}$ first and then add their vector sum to $\vec{c}$. We can also add $\vec{b}$ and $\vec{c}$ first and then add *that* sum to $\vec{a}$. We get the same result either way, as shown in Fig. 3-4. That is,

$$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c}) \quad \text{(associative law)}. \tag{3-3}$$

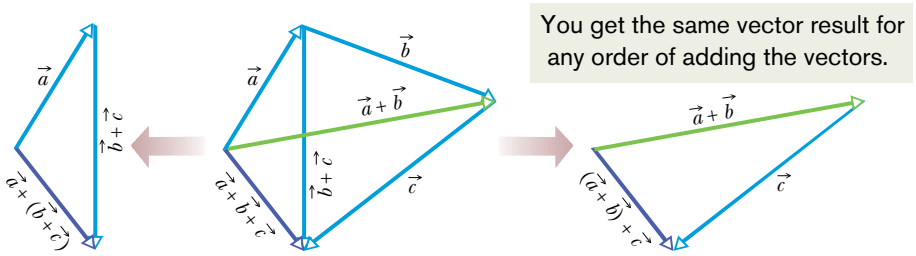

Figure 3-4 The three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ can be grouped in any way as they are added; see Eq. 3-3.

The vector $-\vec{b}$ is a vector with the same magnitude as $\vec{b}$ but the opposite direction (see Fig. 3-5). Adding the two vectors in Fig. 3-5 would yield

$$\vec{b}+(-\vec{b})=0.$$

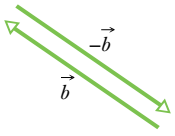

Figure 3-5 The vectors $\vec{b}$ and $-\vec{b}$ have the same magnitude and opposite directions.

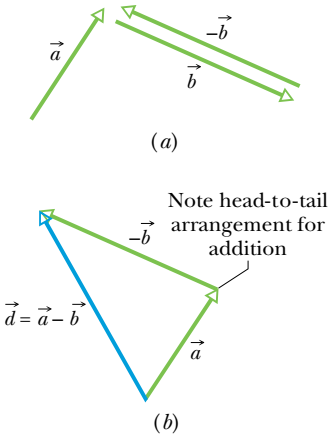

Figure 3-6 (*a*) Vectors $\vec{a}$, $\vec{b}$, and $-\vec{b}$. (*b*) To subtract vector $\vec{b}$ from vector $\vec{a}$, add vector $-\vec{b}$ to vector $\vec{a}$.

Thus, adding $-\vec{b}$ has the effect of subtracting $\vec{b}$. We use this property to define the difference between two vectors: let $\vec{d} = \vec{a} - \vec{b}$. Then

$$\vec{d} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b}) \quad \text{(vector subtraction);} \tag{3-4}$$

that is, we find the difference vector $\vec{d}$ by adding the vector $-\vec{b}$ to the vector $\vec{a}$. Figure 3-6 shows how this is done geometrically.

As in the usual algebra, we can move a term that includes a vector symbol from one side of a vector equation to the other, but we must change its sign. For example, if we are given Eq. 3-4 and need to solve for $\vec{a}$, we can rearrange the equation as

$$\vec{d} + \vec{b} = \vec{a} \quad \text{or} \quad \vec{a} = \vec{d} + \vec{b}.$$

Remember that, although we have used displacement vectors here, the rules for addition and subtraction hold for vectors of all kinds, whether they represent velocities, accelerations, or any other vector quantity. However, we can add only vectors of the same kind. For example, we can add two displacements, or two velocities, but adding a displacement and a velocity makes no sense. In the arithmetic of scalars, that would be like trying to add 21 s and 12 m.

CHECKPOINT 1

The magnitudes of displacements $\vec{a}$ and $\vec{b}$ are 3 m and 4 m, respectively, and $\vec{c} = \vec{a} + \vec{b}$. Considering various orientations of $\vec{a}$ and $\vec{b}$, what are (a) the maximum possible magnitude for $\vec{c}$ and (b) the minimum possible magnitude?

SAMPLE PROBLEM 3.01

Adding vectors in a drawing, orienteering

In an orienteering class, you have the goal of moving as far (straight-line distance) from base camp as possible by making three straight-line moves. You may use the following displacements in any order: (a) $\vec{a}$, 2.0 km due east (directly toward the east); (b) $\vec{b}$, 2.0 km 30° north of east (at an angle of 30° toward the north from due east); (c) $\vec{c}$, 1.0 km due west. Alternatively, you may substitute either $-\vec{b}$ for $\vec{b}$ or $-\vec{c}$ for $\vec{c}$. What is the greatest distance you can be from base camp at the end of the third displacement? (We are not concerned about the direction.)

Reasoning: Using a convenient scale, we draw vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $-\vec{b}$, and $-\vec{c}$ as in Fig. 3-7*a*. We then mentally slide the vectors over the page, connecting three of them at a time in head-to-tail arrangements to find their vector sum $\vec{d}$. The tail of the first vector represents base camp. The head of the third vector represents the point at which you stop. The vector sum $\vec{d}$ extends from the tail of the first vector to the head of the third vector. Its magnitude d is your distance from base camp. Our goal here is to maximize that base-camp distance.

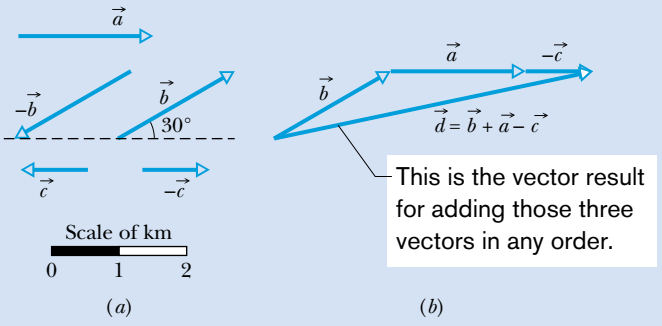

Figure 3-7 (*a*) Displacement vectors; three are to be used. (*b*) Your distance from base camp is greatest if you undergo displacements $\vec{a}$, $\vec{b}$, and $-\vec{c}$, in any order.

We find that distance d is greatest for a head-to-tail arrangement of vectors $\vec{a}$, $\vec{b}$, and $-\vec{c}$. They can be in any order, because their vector sum is the same for any order. (Recall from Eq. 3-2 that vectors commute.) The order shown in Fig. 3-7*b* is for the vector sum

$$\vec{d} = \vec{b} + \vec{a} + (-\vec{c}).$$

Using the scale given in Fig. 3-7*a*, we measure the length d of this vector sum, finding

$$d = 4.8 \text{ km}. \qquad \text{(Answer)}$$

3.4 | COMPONENTS OF VECTORS

Key Concept

- The (scalar) components a_x and a_y of any two-dimensional vector $\vec{a}$ along the coordinate axes are found by dropping perpendicular lines from the ends of $\vec{a}$ onto the coordinate axes. The components are given by

$$a_x = a\cos\theta \quad \text{and} \quad a_y = a\sin\theta,$$

where θ is the angle between the positive direction of the x axis and the direction of $\vec{a}$. The algebraic sign of a component indicates its direction along the associated axis. Given its components, we can find the magnitude and orientation of the vector $\vec{a}$ with

$$a = \sqrt{a_x^2 + a_y^2} \quad \text{and} \quad \tan\theta = \frac{a_y}{a_x}.$$

Adding vectors geometrically can be tedious. A neater and easier technique involves algebra but requires that the vectors be placed on a rectangular coordinate system. The x and y axes are usually drawn in the plane of the page, as shown in Fig. 3-8*a*. The z axis comes directly out of the page at the origin; we ignore it for now and deal only with two-dimensional vectors.

A **component** of a vector is the projection of the vector on an axis. In Fig. 3-8*a*, for example, a_x is the component of vector $\vec{a}$ on (or along) the x axis and a_y is the component along the y axis. To find the projection of a vector along an axis, we draw perpendicular lines from the two ends of the vector to the axis, as shown. The projection of a vector on an x axis is its x *component*, and similarly the projection on the y axis is the y *component*. The process of finding the components of a vector is called **resolving the vector**.

A component of a vector has the same direction (along an axis) as the vector. In Fig. 3-8, a_x and a_y are both positive because $\vec{a}$ extends in the positive direction of both axes. (Note the small arrowheads on the components, to indicate their direction.) If we were to reverse vector $\vec{a}$, then both components would be negative and their arrowheads would point toward negative x and y. Resolving vector $\vec{b}$ in Fig. 3-9 yields a positive component b_x and a negative component b_y.

In general, a vector has three components, although for the case of Fig. 3-8*a* the component along the z axis is zero. As Figs. 3-8*a* and *b* show, if you shift a vector without changing its direction, its components do not change.

Finding the Components. We can find the components of $\vec{a}$ in Fig. 3-8*a* geometrically from the right triangle there:

$$a_x = a\cos\theta \quad \text{and} \quad a_y = a\sin\theta, \tag{3-5}$$

where θ is the angle that the vector $\vec{a}$ makes with the positive direction of the x axis, and a is the magnitude of $\vec{a}$. Figure 3-8*c* shows that $\vec{a}$ and its x and y components form a right triangle. It also shows how we can reconstruct a vector

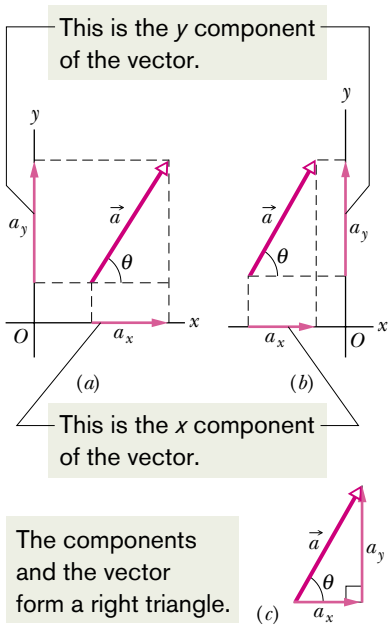

Figure 3-8 (*a*) The components a_x and a_y of vector $\vec{a}$. (*b*) The components are unchanged if the vector is shifted, as long as the magnitude and orientation are maintained. (*c*) The components form the legs of a right triangle whose hypotenuse is the magnitude of the vector.

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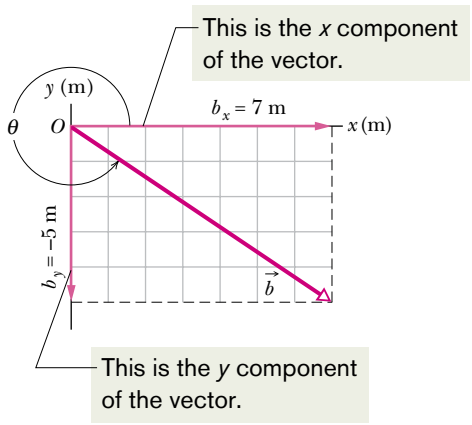


Figure 3-9 The component of $\vec{b}$ on the x axis is positive, and that on the y axis is negative.

from its components: we arrange those components *head to tail*. Then we complete a right triangle with the vector forming the hypotenuse, from the tail of one component to the head of the other component.

Once a vector has been resolved into its components along a set of axes, the components themselves can be used in place of the vector. For example, $\vec{a}$ in Fig. 3-8*a* is given (completely determined) by a and θ. It can also be given by its components a_x and a_y. Both pairs of values contain the same information. If we know a vector in *component notation* (a_x and a_y) and want it in *magnitude-angle notation* (a and θ), we can use the equations

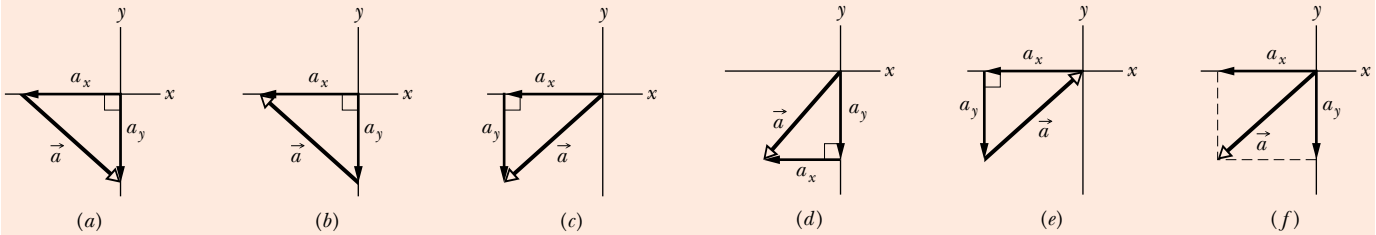
$$a = \sqrt{a_x^2 + a_y^2} \quad \text{and} \quad \tan\theta = \frac{a_y}{a_x} \tag{3-6}$$

to transform it.

In the more general three-dimensional case, we need a magnitude and two angles (say, a, θ, and ϕ) or three components (a_x, a_y, and a_z) to specify a vector.

CHECKPOINT 2

In the figure,which of the indicated methods for combining the x and y components of vector $\vec{a}$ are proper to determine that vector?



SAMPLE PROBLEM 3.02

Finding components, airplane flight

A small airplane leaves an airport on an overcast day and is later sighted 215 km away, in a direction making an angle of 22° east of due north. This means that the direction is not due north (directly toward the north) but is rotated 22° toward the east from due north. How far east and north is the airplane from the airport when sighted?

KEY IDEA

We are given the magnitude (215 km) and the angle (22° east of due north) of a vector and need to find the components of the vector.

Calculations: We draw an xy coordinate system with the positive direction of x due east and that of y due north (Fig. 3-10). For convenience, the origin is placed

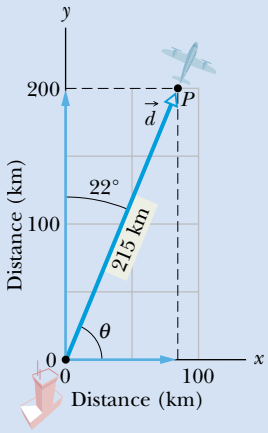


Figure 3-10 A plane takes off from an airport at the origin and is later sighted at *P*.

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at the airport. (We do not have to do this. We could shift and misalign the coordinate system but, given a choice, why make the problem more difficult?) The airplane's displacement $\vec{d}$ points from the origin to where the airplane is sighted.

To find the components of $\vec{d}$, we use Eq. 3-5 with $\theta = 68°$ $(= 90° - 22°)$:

$$d_x = d\cos\theta = (215\text{ km})(\cos 68°) = 81\text{ km} \quad \text{(Answer)}$$

$$d_y = d\sin\theta = (215\text{ km})(\sin 68°) = 199\text{ km} \approx 2.0 \times 10^2\text{ km}. \quad \text{(Answer)}$$

Thus, the airplane is 81 km east and 2.0×10^2 km north of the airport.

PROBLEM-SOLVING TACTICS

Angles, trig functions, and inverse trig functions

Tactic 1: Angles—Degrees and Radians Angles that are measured relative to the positive direction of the x axis are positive if they are measured in the counterclockwise direction and negative if measured clockwise. For example, 210° and –150° are the same angle.

Angles may be measured in degrees or radians (rad). To relate the two measures, recall that a full circle is 360° and 2π rad. To convert, say, 40° to radians, write

$$40° \frac{2\pi\text{ rad}}{360°} = 0.70\text{ rad}$$

Tactic 2: Trig Functions You need to know the definitions of the common trigonometric functions—sine, cosine, and tangent—because they are part of the language of science and engineering. They are given in Fig. 3-11 in a form that does not depend on how the triangle is labeled.

You should also be able to sketch how the trig functions vary with angle, as in Fig. 3-12, in order to be able to judge whether a calculator result is reasonable. Even knowing the signs of the functions in the various quadrants can be of help.

$$\sin\theta = \frac{\text{leg opposite }\theta}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{leg adjacent to }\theta}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{leg opposite }\theta}{\text{leg adjacent to }\theta}$$

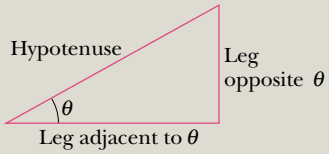

Figure 3-11 A triangle used to define the trigonometric functions. See also Appendix E.

Tactic 3: Inverse Trig Functions When the inverse trig functions $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ are taken on a calculator, you must consider the reasonableness of the answer you get, because there is usually another possible answer that the calculator does not give. The range of operation for a calculator in taking each inverse trig function is indicated in Fig. 3-12. As an example, $\sin^{-1} 0.5$ has associated angles of 30° (which is displayed by the calculator, since 30° falls within its range of operation) and 150°. To see both values, draw a horizontal line through 0.5 in Fig. 3-12*a* and note where it cuts the sine curve. How do you distinguish a correct answer? It is the one that seems more reasonable for the given situation.

Tactic 4: Measuring Vector Angles The equations for $\cos\theta$ and $\sin\theta$ in Eq. 3-5 and for $\tan\theta$ in Eq. 3-6 are valid only if the angle is measured from the positive direction of the x axis. If it is measured relative to some other direction, then the trig functions in Eq. 3-5 may have to be interchanged and the ratio in Eq. 3-6 may have to be inverted. A safer method is to convert the angle to one measured from the positive direction of the x axis.

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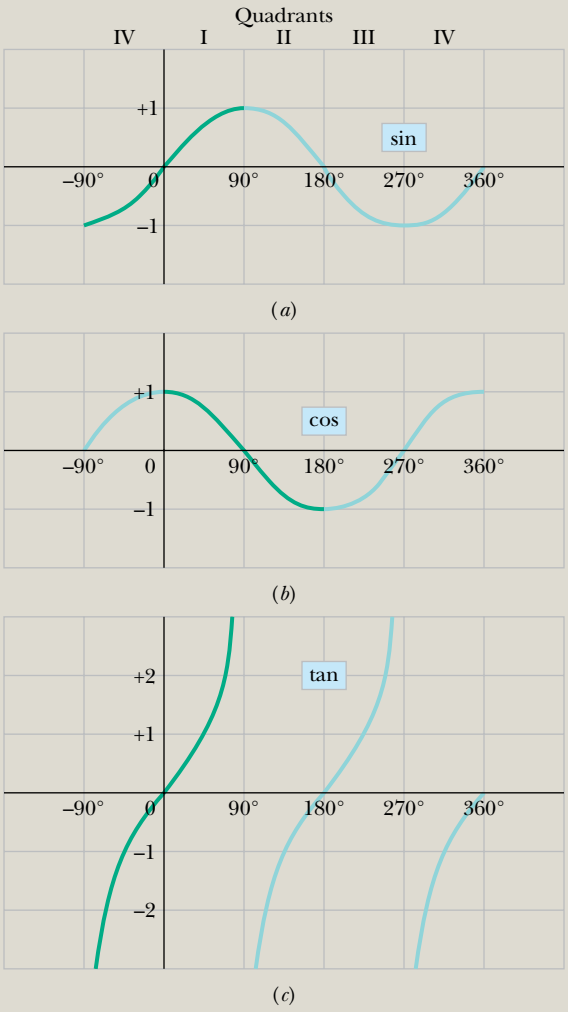

Figure 3-12 Three useful curves to remember. A calculator's range of operation for taking *inverse* trig functions is indicated by the darker portions of the colored curves.

3.5 | UNIT VECTORS

Key Concept

- Unit vectors $\hat{\mathrm{i}}$, $\hat{\mathrm{j}}$, and $\hat{\mathrm{k}}$ have magnitudes of unity and are directed in the positive directions of the x, y, and z axes, respectively, in a right-handed coordinate system. We can write a vector $\vec{a}$ in terms of unit vectors as

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}},$$

in which $a_x\hat{\mathrm{i}}$, $a_y\hat{\mathrm{j}}$, and $a_z\hat{\mathrm{k}}$ are the vector components of $\vec{a}$ and a_x, a_y, and a_z are its scalar components.

A **unit vector** is a vector that has a magnitude of exactly 1 and points in a particular direction. It lacks both dimension and unit. Its sole purpose is to point—that is, to specify a direction. The unit vectors in the positive directions of the x, y, and z axes are labeled $\hat{\mathrm{i}}$, $\hat{\mathrm{j}}$, and $\hat{\mathrm{k}}$, where the hat $\hat{}$ is used instead of an overhead arrow as for other vectors

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(Fig. 3-13). The arrangement of axes in Fig. 3-13 is said to be a **right-handed coordinate system**. The system remains right-handed if it is rotated rigidly. We use such coordinate systems exclusively in this book.

Unit vectors are very useful for expressing other vectors; for example, we can express $\vec{a}$ and $\vec{b}$ of Figs. 3-7 and 3-8 as

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} \tag{3-7}$$

and

$$\vec{b} = b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}}. \tag{3-8}$$

The unit vectors point along axes.

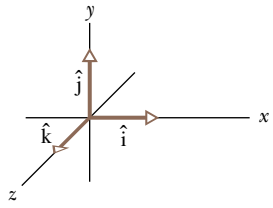
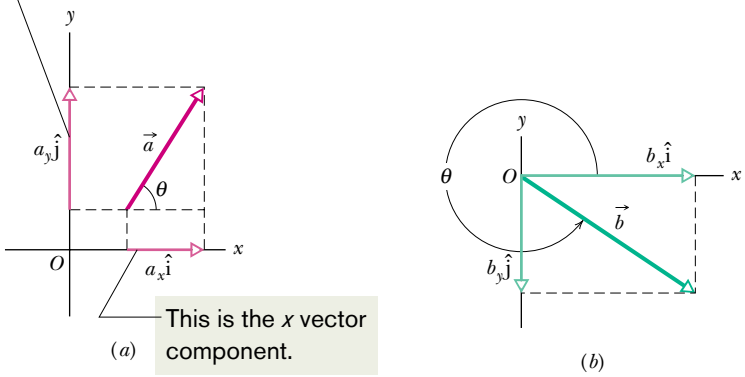


Figure 3-13 Unit vectors $\hat{\mathrm{i}}$, $\hat{\mathrm{j}}$, and $\hat{\mathrm{k}}$ define the directions of a right-handed coordinate system.

These two equations are illustrated in Fig. 3-14. The quantities $a_x\hat{\mathrm{i}}$ and $a_y\hat{\mathrm{j}}$ are vectors, called the **vector components** of $\vec{a}$. The quantities a_x and a_y are scalars, called the **scalar components** of $\vec{a}$ (or, as before, simply its **components**).

This is the *y* vector component.



This is the *x* vector component.

Figure 3-14 (*a*) The vector components of vector $\vec{a}$. (*b*) The vector components of vector $\vec{b}$.

Representing Position Vector in Terms of Unit Vectors

A vector used to define the position of a point in space is called position vector. The position of a point P in a two-dimensional coordinate system can be represented by a position vector with the tail at the origin of the coordinate system and the head at the point P itself. Let us draw a vector $\vec{a}$ in the plane with its initial point O at the origin and terminal point P with coordinates (x, y), as shown in Fig. 3-15.

Then

$$\overrightarrow{OP} = \vec{a}$$

$$\overrightarrow{OP} = \overrightarrow{ON} + \overrightarrow{NP} = x\hat{\mathrm{i}} + y\hat{\mathrm{j}}$$

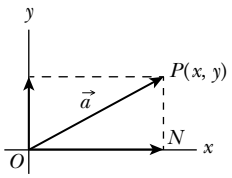


Figure 3-15 Representation of position vector a point P in two-dimensional coordinates.

Therefore,

$$\vec{a} = x\hat{\mathrm{i}} + y\hat{\mathrm{j}}$$

The magnitude of the vector can be calculated as

$$|\vec{a}| = \sqrt{x^2 + y^2}$$

Similarly, in three-dimensional coordinate system, the vector $\vec{a}$ is drawn with the initial point O at the origin and the terminal point P with coordinates (x, y, z) shown in Fig. 3-16. Unit vectors a_x, a_y and a_z are required for specifying the vector in (x, y, z) coordinate system as.

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Then,

$$\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{a} = a\cos\alpha\,\hat{i} + a\cos\beta\,\hat{j} + a\cos\gamma\,\hat{k}$$

where $a_x = a\cos\alpha;\ a_y = a\cos\beta$ and $a_z = a\cos\gamma.$

The magnitude of the vector can be calculated as

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

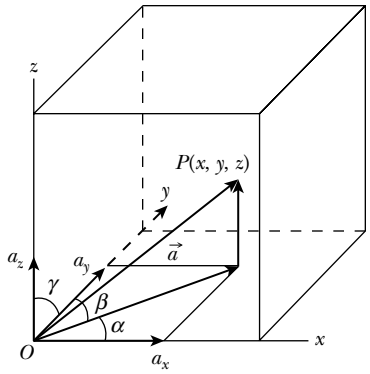

Figure 3-16 Representation of position vector OP in three-dimensional coordinate system.

Representing Displacement Vector in Terms of Unit Vectors

The change in position vector of a particle is known as displacement vector. Consider a particle in two-dimensional coordinates, initially situated at point $P_1(x_1, y_1)$ which changes its position to point $P_2\,(x_2, y_2)$, as shown in Fig. 3-17. If $\vec{a}_1$ and $\vec{a}_2$ are the position vectors for the two points with respect to origin O, then the position of point P_1 with respect to P_2 is defined in terms of displacement vector $\vec{P}_{12}$ and is the shortest distance between position P_1 and P_2 of the particle.

$$\overrightarrow{OP}_1 = \vec{a}_1 = x_1\hat{i} + y_1\hat{j}$$
$$\overrightarrow{OP}_2 = \vec{a}_2 = x_2\hat{i} + y_2\hat{j}$$
$$\vec{P}_{12} = \vec{a}_2 - \vec{a}_1 = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}$$

Thus, displacement vector is expressed in terms of initial position of point in space (x_1, y_1) and final position of point (x_2, y_2) and is independent of the choice of the origin.

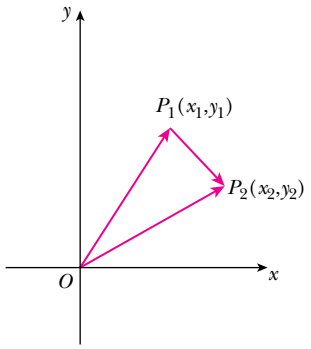

Figure 3-17 Two-dimensional representation of position vector $\vec{P}_{12}$.

3.6 | ADDING VECTORS BY COMPONENTS

Key Concept

- To add vectors in component form, we use the rules

$$r_x = a_x + b_x \quad r_y = a_y + b_y \quad r_z = a_z + b_z.$$

Here $\vec{a}$ and $\vec{b}$ are the vectors to be added, and $\vec{r}$ is the vector sum. Note that we add components axis by axis.

We can add vectors geometrically on a sketch or directly on a vector-capable calculator. A third way is to combine their components axis by axis.

To start, consider the statement

$$\vec{r} = \vec{a} + \vec{b}, \tag{3-9}$$

which says that the vector $\vec{r}$ is the same as the vector $(\vec{a} + \vec{b})$. Thus, each component of $\vec{r}$ must be the same as the corresponding component of $(\vec{a} + \vec{b})$:

$$r_x = a_x + b_x \tag{3-10}$$
$$r_y = a_y + b_y \tag{3-11}$$
$$r_z = a_z + b_z \tag{3-12}$$

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In other words, two vectors must be equal if their corresponding components are equal. Equations 3-9 to 3-12 tell us that to add vectors $\vec{a}$ and $\vec{b}$, we must (1) resolve the vectors into their scalar components; (2) combine these scalar components, axis by axis, to get the components of the sum $\vec{r}$; and (3) combine the components $\vec{r}$ of to get $\vec{r}$ itself. We have a choice in step 3. We can express $\vec{r}$ in unit-vector notation or in magnitude-angle notation.

This procedure for adding vectors by components also applies to vector subtractions. Recall that a subtraction such as $\vec{d} = \vec{a} - \vec{b}$ can be rewritten as an addition $\vec{d} = \vec{a} + (-\vec{b})$. To subtract, we add $\vec{a}$ and $-\vec{b}$ by components, to get

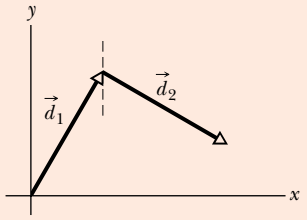
$$d_x = a_x - b_x, \quad d_y = a_y - b_y, \quad \text{and} \quad d_z = a_z - b_z,$$

where

$$\vec{d} = d_x\hat{\mathrm{i}} + d_y\hat{\mathrm{j}} + d_z\hat{\mathrm{k}}. \tag{3-13}$$

CHECKPOINT 3

(a) In the figure here, what are the signs of the x components of $\vec{d}_1$ and $\vec{d}_2$? (b) What are the signs of the y components of $\vec{d}_1$ and $\vec{d}_2$? (c) What are the signs of the x and y components of $\vec{d}_1 + \vec{d}_2$?



Vectors and the Laws of Physics

So far, in every figure that includes a coordinate system, the x and y axes are parallel to the edges of the book page. Thus, when a vector $\vec{a}$ is included, its components a_x and a_y are also parallel to the edges (as in Fig. 3-18*a*). The only reason for that orientation of the axes is that it looks "proper"; there is no deeper reason. We could, instead, rotate the axes (but not the vector $\vec{a}$) through an angle ϕ as in Fig. 3-18*b*, in which case the components would have new values, call them a'_x and a'_y. Since there are an infinite number of choices of ϕ, there are an infinite number of different pairs of components for $\vec{a}$.

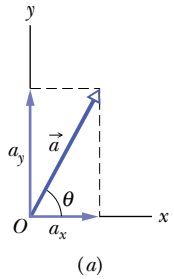
Which then is the "right" pair of components? The answer is that they are all equally valid because each pair (with its axes) just gives us a different way of describing the same vector $\vec{a}$; all produce the same magnitude and direction for the vector. In Fig. 3-18, we have

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{a'^2_x + a'^2_y} \tag{3-14}$$

and

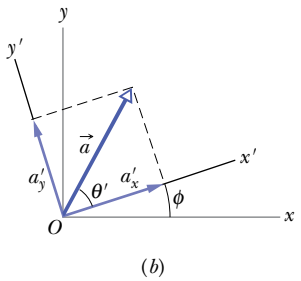
$$\theta = \theta' + \phi. \tag{3-15}$$

The point is that we have great freedom in choosing a coordinate system, because the relations among vectors do not depend on the location of the origin or on the orientation of the axes. This is also true of the relations of physics; they are all independent of the choice of coordinate system. Add to that the simplicity and richness of the language of vectors and you can see why the laws of physics are almost always presented in that language: one equation, like Eq. 3-9, can represent three (or even more) relations, like Eqs. 3-10, 3-11, and 3-12.



(*a*)

Rotating the axes changes the components but not the vector.



(*b*)

Figure 3-18 (*a*) The vector $\vec{a}$ and its components. (*b*) The same vector, with the axes of the coordinate system rotated through an angle ϕ.

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SAMPLE PROBLEM 3.03

Searching through a hedge maze

A hedge maze is a maze formed by tall rows of hedge. After entering, you search for the center point and then for the exit. Figure 3-19*a* shows the entrance to such a maze and the first two choices we make at the junctions we encounter in moving from point *i* to point *c*. We undergo three displacements as indicated in the overhead view of Fig. 3-19*b*:

$$d_1 = 6.00\text{ m} \quad \theta_1 = 40°$$
$$d_2 = 8.00\text{ m} \quad \theta_2 = 30°$$
$$d_3 = 5.00\text{ m} \quad \theta_3 = 0°,$$

where the last segment is parallel to the superimposed *x* axis. When we reach point *c*, what are the magnitude and angle of our net displacement $\vec{d}_{net}$ from point *i*?

KEY IDEAS

(1) To find the net displacement $\vec{d}_{net}$, we need to sum the three individual displacement vectors:

$$\vec{d}_{net} = \vec{d}_1 + \vec{d}_2 + \vec{d}_3.$$

(2) To do this, we first evaluate this sum for the *x* components alone,

$$d_{net,x} = d_{1x} + d_{2x} + d_{3x}, \tag{3-16}$$

and then the *y* components alone,

$$d_{net,y} = d_{1y} + d_{2y} + d_{3y}. \tag{3-17}$$

(3) Finally,we construct $\vec{d}_{net}$ from its *x* and *y* components.

Calculations: To evaluate Eqs. 3-16 and 3-17, we find the *x* and *y* components of each displacement. As an example, the components for the first displacement are shown in Fig. 3-19*c*. We draw similar diagrams for the other two displacements and then we apply the *x* part of Eq. 3-5 to each displacement, using angles relative to the positive direction of the *x* axis:

$$d_{1x} = (6.00\text{ m})\cos 40° = 4.60\text{ m}$$
$$d_{2x} = (8.00\text{ m})\cos(-60°) = 4.00\text{ m}$$
$$d_{3x} = (5.00\text{ m})\cos 0° = 5.00\text{ m}.$$

Equation 3-16 then gives us

$$d_{net,x} = +4.60\text{ m} + 4.00\text{ m} + 5.00\text{ m}$$
$$= 13.60\text{ m}$$

Similarly, to evaluate Eq. 3-17, we apply the *y* part of Eq. 3-5 to each displacement:

$$d_{1y} = (6.00\text{ m})\sin 40° = 3.86\text{ m}$$
$$d_{2y} = (8.00\text{ m})\sin(-60°) = -6.93\text{ m}$$
$$d_{3y} = (5.00\text{ m})\sin 0° = 0\text{ m}.$$

Equation 3-17 then gives us

$$d_{net,y} = +3.86\text{ m} - 6.93\text{ m} + 0\text{ m}$$
$$= -3.07\text{ m}.$$

Next we use these components of $\vec{d}_{net}$ to construct the vector as shown in Fig. 3-19*d*: the components are in a head-to-tail arrangement and form the legs of a right triangle, and the vector forms the hypotenuse. We find the magnitude and angle of $\vec{d}_{net}$ with Eq. 3-6. The magnitude is

$$d_{net} = \sqrt{d_{net,x}^2 + d_{net,y}^2} \tag{3-18}$$
$$= \sqrt{(13.60\text{ m})^2 + (-3.07\text{ m})^2} = 13.9\text{ m}. \quad \text{(Answer)}$$

To find the angle (measured from the positive direction of *x*), we take an inverse tangent:

$$\theta = \tan^{-1}\left(\frac{d_{net,y}}{d_{net,y}}\right) \tag{3-19}$$
$$= \tan^{-1}\left(\frac{-3.07\text{ m}}{13.60\text{ m}}\right) = -12.7°. \quad \text{(Answer)}$$

The angle is negative because it is measured clockwise from positive *x*. We must always be alert when we take an inverse tangent on a calculator. The answer it displays is mathematically correct but it may not be the correct answer for the physical situation. In those cases, we have to add 180° to the displayed answer, to reverse the vector. To check, we always need to draw the vector and its components as we did in Fig. 3-16*d*. In our physical situation, the figure shows us that $\theta = -12.7°$ is a reasonable answer, whereas $-12.7° + 180° = 167°$ is clearly not.

We can see all this on the graph of tangent versus angle in Fig. 3-12*c*. In our maze problem, the argument of the inverse tangent is –3.07/13.60, or –0.226. On the graph draw a horizontal line through that value on the vertical axis. The line cuts through the darker plotted branch at –12.7° and also through the lighter branch at 167°.

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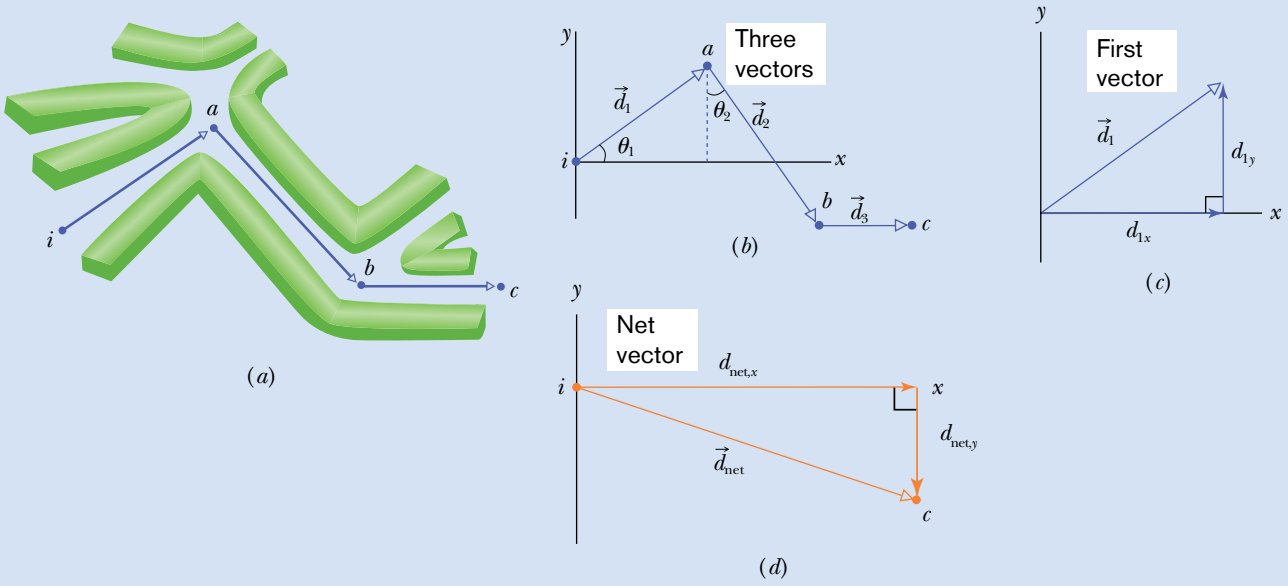

Figure 3-19 (*a*) Three displacements through a hedge maze. (*b*) The displacement vectors. (*c*) The first displacement vector and its components. (*d*) The net displacement vector and its components.

SAMPLE PROBLEM 3.04

Adding vectors, unit-vector components

Figure 3-20*a* shows the following three vectors:

$$\vec{a} = (4.2\text{ m})\hat{i} - (1.5\text{ m})\hat{j},$$

$$\vec{b} = (-1.6\text{ m})\hat{i} + (2.9\text{ m})\hat{j},$$

and $$\vec{c} = (-3.7\text{ m})\hat{j}.$$

What is their vector sum $\vec{r}$ which is also shown?

KEY IDEA

We can add the three vectors by components, axis by axis, and then combine the components to write the vector sum $\vec{r}$.

Calculations: For the x axis, we add the x components of $\vec{a}$, $\vec{b}$, and $\vec{c}$, to get the x component of the vector sum $\vec{r}$:

$$r_x = a_x + b_x + c_x$$
$$= 4.2\text{ m} - 1.6\text{ m} + 0 = 2.6\text{ m}.$$

Similarly, for the y axis,

$$r_y = a_y + b_y + c_y$$
$$= -1.5\text{ m} + 2.9\text{ m} - 3.7\text{ m} = -2.3\text{ m}.$$

We then combine these components of $\vec{r}$ to write the vector in unit-vector notation:

$$\vec{r} = (2.6\text{ m})\hat{i} - (2.3\text{ m})\hat{j}, \qquad \text{(Answer)}$$

where $(2.6\text{ m})\hat{i}$ is the vector component of $\vec{r}$ along the x axis and $-(2.6\text{ m})\hat{j}$ is that along the y axis. Figure 3-20*b* shows one way to arrange these vector components to form $\vec{r}$. (Can you sketch the other way?)

We can also answer the question by giving the magnitude and an angle for $\vec{r}$. From Eq. 3-6, the magnitude is

$$r = \sqrt{(2.6\text{ m})^2 + (-2.3\text{ m})^2} \approx 3.5\text{ m} \qquad \text{(Answer)}$$

and the angle (measured from the $+x$ direction) is

$$\theta = \tan^{-1}\left(\frac{-2.3\text{ m}}{2.6\text{ m}}\right) = -41°, \qquad \text{(Answer)}$$

where the minus sign means clockwise.

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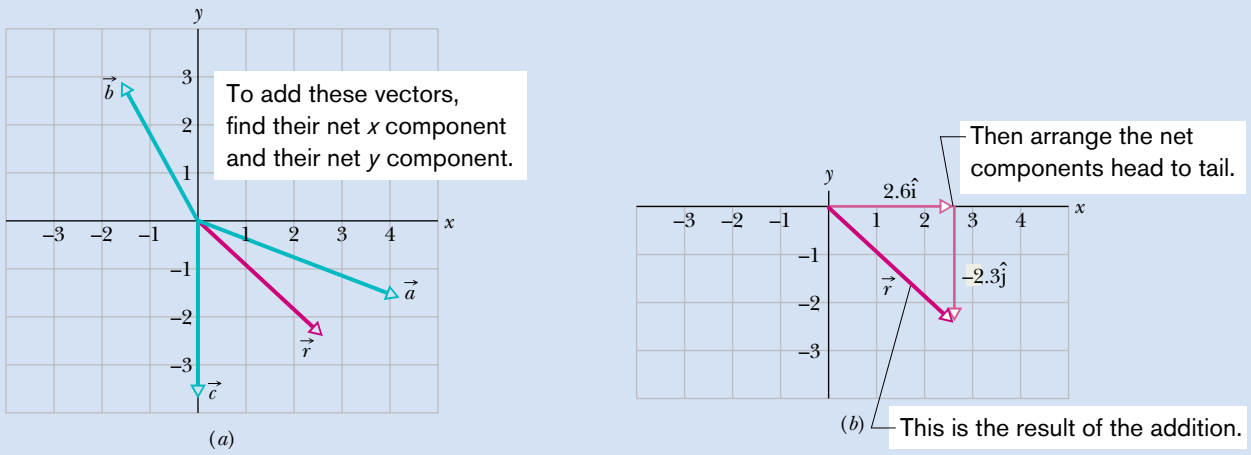
To add these vectors, find their net x component and their net y component.
Then arrange the net components head to tail.
2.6î
–2.3ĵ
This is the result of the addition.
(a)
(b)

Figure 3-20 Vector $\vec{r}$ is the vector sum of the other three vectors.

SAMPLE PROBLEM 3.05

Adding vectors, unit-vector components

For the vectors $\vec{a} = (3.0\text{ m})\hat{i} + (3.0\text{ m})\hat{j}$ and $\vec{b} = (5.0\text{ m})\hat{i} + (-2.0\text{ m})\hat{j}$, give $\vec{a} + \vec{b}$ in (a) unit-vector notation, and as (b) a magnitude and (c) an angle (relative to $\hat{i}$). Now give $\vec{b} - \vec{a}$ in (d) unit-vector notation, as (e) a magnitude and (f) an angle.

Calculations:

(a) $$\vec{a} + \vec{b} = (3.0\hat{i} + 4.0\hat{j})\text{ m} + (5.0\hat{i} - 2.0\hat{j})\text{ m}$$
$$= (8.0\text{ m})\hat{i} + (2.0\text{ m})\hat{j}.$$

(b) The magnitude of $\vec{a} + \vec{b}$ is

$$|\vec{a} + \vec{b}| = \sqrt{(8.0\text{ m})^2 + (2.0\text{ m})^2} = 8.2\text{ m}.$$

(c) The angle between this vector and the $+x$ axis is $\tan^{-1}[(2.0\text{ m})/(8.0\text{ m})] = 14°$.

(d) $$\vec{b} - \vec{a} = (5.0\hat{i} - 2.0\hat{j})\text{ m} - (3.0\hat{i} + 4.0\hat{j})\text{ m}$$
$$= (2.0\text{ m})\hat{i} - (6.0\text{ m})\hat{j}.$$

(e) The magnitude of the difference vector $\vec{b} - \vec{a}$ is

$$|\vec{b} - \vec{a}| = \sqrt{(2.0\text{ m})^2 + (-6.0\text{ m})^2} = 6.3\text{ m}.$$

(f) The angle between this vector and the $+x$ axis is $\tan^{-1}[(-6.0\text{ m})/(2.0\text{ m})] = -72°$. The vector is 72° *clockwise* from the axis defined by $\hat{i}$.

SAMPLE PROBLEM 3.06

Adding vectors, unit-vector components

Two beetles run across flat sand, starting at the same point. Beetle 1 runs 0.50 m due east, then 0.80 m at 30° north of due east. Beetle 2 also makes two runs; the first is 1.6 m at 40° east of due north. What must be (a) the magnitude and (b) the direction of its second run if it is to end up at the new location of beetle 1?

KEY IDEAS

Let $\vec{A}$ represent the first part of Beetle 1's trip (0.50 m east or 0.5 $\hat{i}$) and $\vec{C}$ represent the first part of Beetle 2's trip intended voyage (1.6 m at 50° north of east). For their respective second parts: $\vec{B}$ is 0.80 m at 30° north of east

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and $\vec{D}$ is the unknown. The final position of Beetle 1 is

$$\vec{A}+\vec{B}=(0.5\text{ m})\hat{i}+(0.8\text{ m})(\cos 30^\circ\,\hat{i}+\sin 30^\circ\,\hat{j})$$
$$=(1.19\text{ m})\,\hat{i}+(0.40\text{ m})\,\hat{j}.$$

The equation relating these is $\vec{A}+\vec{B}=\vec{C}+\vec{D}$, where

$$\vec{C}=(1.60\text{ m})(\cos 50.0^\circ\hat{i}+\sin 50.0^\circ\hat{j})=(1.03\text{ m})\hat{i}+(1.23\text{ m})\hat{j}$$

Calculations:

(a) We find $\vec{D}=\vec{A}+\vec{B}-\vec{C}=(0.16\text{ m})\hat{i}+(-0.83\text{ m})\hat{j}$, and the magnitude is $D=0.84$ m.

(b) The angle is $\tan^{-1}(-0.83/0.16)=-79^\circ$ which is interpreted to mean 79° south of east (or 11° east of south).

SAMPLE PROBLEM 3.07

Components of vectors on rotation of coordinate axis

In Fig. 3-21, a vector $\vec{a}$ with a magnitude of 17.0 m is directed at angle $\theta = 56.0^\circ$ counterclockwise from the $+x$ axis. What are the components(a) a_x and (b) a_y of the vector? A second coordinate system is inclined by angle $\theta' = 18.0^\circ$ with respect to the first. What are the components (c) a'_x and (d) a'_y in this primed coordinate system?

Calculations:

(a) With $a = 17.0$ m and $\theta = 56.0^\circ$ we find $a_x = a\cos\theta = 9.51$ m.

(b) Similarly, $a_y = a\sin\theta = 14.1$ m.

(c) The angle relative to the new coordinate system is $\theta' = (56.0^\circ - 18.0^\circ) = 38.0^\circ$. Thus, $a'_x = a\cos\theta' = 13.4$ m.

(d) Similarly, $a'_y = a\sin\theta' = 10.5$ m.

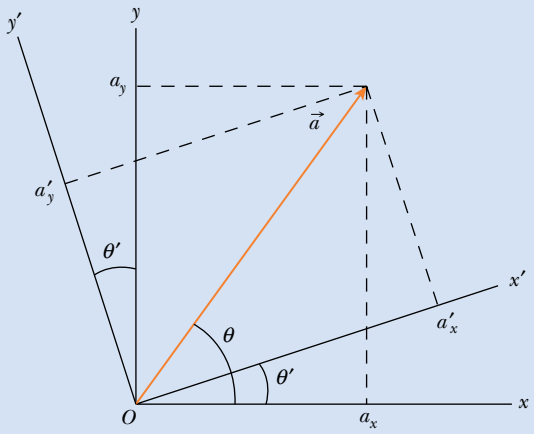

Figure 3-21 The vector $\vec{a}$ and its components and the same vector with the axes of coordinate system rotated through an angle.

3.7 | MULTIPLYING VECTORS*

Key Concepts

- The product of a scalar s and a vector $\vec{v}$ is a new vector whose magnitude is sv and whose direction is the same as that of $\vec{v}$ if s is positive, and opposite that of $\vec{v}$ if s is negative. To divide $\vec{v}$ by s, multiply $\vec{v}$ by $1/s$.
- The scalar (or dot) product of two vectors $\vec{a}$ and $\vec{b}$ is written $\vec{a}\cdot\vec{b}$ and is the *scalar* quantity given by

$$\vec{a}\cdot\vec{b}=ab\cos\phi,$$

in which ϕ is the angle between the directions of $\vec{a}$ and $\vec{b}$. A scalar product is the product of the magnitude of one vector and the scalar component of the second vector along the direction of the first vector. In unit-vector notation,

$$\vec{a}\cdot\vec{b}=(a_x\hat{i}+a_y\hat{j}+a_z\hat{k})\cdot(b_x\hat{i}+b_y\hat{j}+b_z\hat{k}),$$

which may be expanded according to the distributive law. Note that $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$.

- The vector (or cross) product of two vectors $\vec{a}$ and $\vec{b}$ is written $\vec{a}\times\vec{b}$ and is a *vector* $\vec{c}$ whose magnitude c is given by

$$c=ab\sin\phi,$$

*This material will not be used until later (Chapter 8 for scalar products and Chapter 10 for vector products), and so you may wish to postpone reading it.

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in which ϕ is the smaller of the angles between the directions of $\vec{a}$ and $\vec{b}$. The direction of $\vec{c}$ is perpendicular to the plane defined by $\vec{a}$ and $\vec{b}$ and is given by a right-hand rule, as shown in Fig. 3-23. Note $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$. In unit-vector notation,

$$\vec{a}\times\vec{b} = (a_x\hat{i} + a_y\hat{j} + a_z\hat{k})\cdot(b_x\hat{i} + b_y\hat{j} + b_z\hat{k}),$$

which we may expand with the distributive law.

- In nested products, where one product is buried inside another, follow the normal algebraic procedure by starting with the innermost product and working outward.

There are three ways in which vectors can be multiplied, but none is exactly like the usual algebraic multiplication.

1. **Multiplying a Vector by a Scalar:** If we multiply a vector $\vec{a}$ by a scalar s, we get a new vector. Its magnitude is the product of the magnitude of $\vec{a}$ and the absolute value of s. Its direction is the direction of $\vec{a}$ if s is positive but the opposite direction if s is negative. To divide $\vec{a}$ by s, we multiply $\vec{a}$ by $1/s$.
2. **Multiplying a Vector by a Vector:** There are two ways to multiply a vector by a vector: one way produces a scalar (called the *scalar product*), and the other produces a new vector (called the *vector product*).

The Scalar Product

The **scalar product** of the vectors $\vec{a}$ and $\vec{b}$ in Fig. 3-22*a* is written as $\vec{a}\cdot\vec{b}$ and defined to be

$$\vec{a}\cdot\vec{b} = ab\cos\phi, \tag{3-20}$$

where a is the magnitude of $\vec{a}$, b is the magnitude of $\vec{b}$, and ϕ is the angle between $\vec{a}$ and $\vec{b}$ (or, more properly, between the directions of $\vec{a}$ and $\vec{b}$). There are actually two such angles: ϕ and $360° - \phi$. Either can be used in Eq. 3-20, because their cosines are the same.

When two vectors are perpendicular, $\theta = \pi/2$, therefore the scalar product is zero. For any other angle between the two vectors

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

Note that there are only scalars on the right side of Eq. 3-20 (including the value of $\cos\phi$). Thus $\vec{a}\cdot\vec{b}$ on the left side represents a *scalar* quantity. Because of the notation, $\vec{a}\cdot\vec{b}$ is also known as the **dot product** and is spoken as "a dot b."

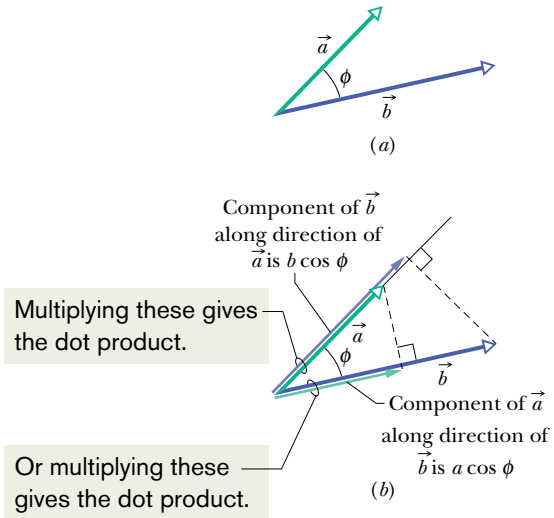

Figure 3-22 (*a*) Two vectors $\vec{a}$ and $\vec{b}$, with an angle ϕ between them. (*b*) Each vector has a component along the direction of the other vector.

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A dot product can be regarded as the product of two quantities: (1) the magnitude of one of the vectors and (2) the scalar component of the second vector along the direction of the first vector. For example, in Fig. 3-22*b*, $\vec{a}$ has a scalar component $a\cos\phi$ along the direction of $\vec{b}$; note that a perpendicular dropped from the head of $\vec{a}$ onto $\vec{b}$ determines that component. Similarly, $\vec{b}$ has a scalar component $b\cos\phi$ along the direction of $\vec{a}$.

If the angle ϕ between two vectors is 0°, the component of one vector along the other is maximum, and so also is the dot product of the vectors. If, instead, ϕ is 90°, the component of one vector along the other is zero, and so is the dot product.

Equation 3-20 can be rewritten as follows to emphasize the components:

$$\vec{a}\cdot\vec{b} = (a\cos\phi)(b) = (a)(b\cos\phi). \tag{3-21}$$

The commutative law applies to a scalar product, so we can write

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}.$$

When two vectors are in unit-vector notation, we write their dot product as

$$\vec{a}\cdot\vec{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}})\cdot(b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}), \tag{3-22}$$

which we can expand according to the distributive law: Each vector component of the first vector is to be dotted with each vector component of the second vector. By doing so, we can show that

$$\vec{a}\cdot\vec{b} = a_xb_x + a_yb_y + a_zb_z. \tag{3-23}$$

CHECKPOINT 4

Vectors $\vec{C}$ and $\vec{D}$ have magnitudes of 3 units and 4 units, respectively. What is the angle between the directions of $\vec{C}$ and $\vec{D}$ if $\vec{C}\cdot\vec{D}$ equals (a) zero, (b) 12 units, and (c) –12 units?

The Vector Product

The **vector product** of $\vec{a}$ and $\vec{b}$, written $\vec{a}\times\vec{b}$, produces a third vector $\vec{c}$ whose magnitude is

$$c = ab\sin\phi, \tag{3-24}$$

where ϕ is the *smaller* of the two angles between $\vec{a}$ and $\vec{b}$. (You must use the smaller of the two angles between the vectors because $\sin\phi$ and $\sin(360° - \phi)$ differ in algebraic sign.) Because of the notation, $\vec{a}\times\vec{b}$ is also known as the **cross product,** and in speech it is "a cross b."

If $\vec{a}$ and $\vec{b}$ are parallel or antiparallel, $\vec{a}\times\vec{b} = 0$. The magnitude of $\vec{a}\times\vec{b}$, which can be written as $|\vec{a}\times\vec{b}|$, is maximum when $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

The direction of $\vec{c}$ is perpendicular to the plane that contains $\vec{a}$ and $\vec{b}$. Figure 3-23*a* shows how to determine the direction of $\vec{c} = \vec{a}\times\vec{b}$ with what is known as a **right-hand rule**. Place the vectors $\vec{a}$ and $\vec{b}$ tail to tail without altering their orientations, and imagine a line that is perpendicular to their plane where they meet. Pretend to place your *right* hand around that line in such a way that your fingers would sweep $\vec{a}$ into $\vec{b}$ through the smaller angle between them. Your outstretched thumb points in the direction of $\vec{c}$.

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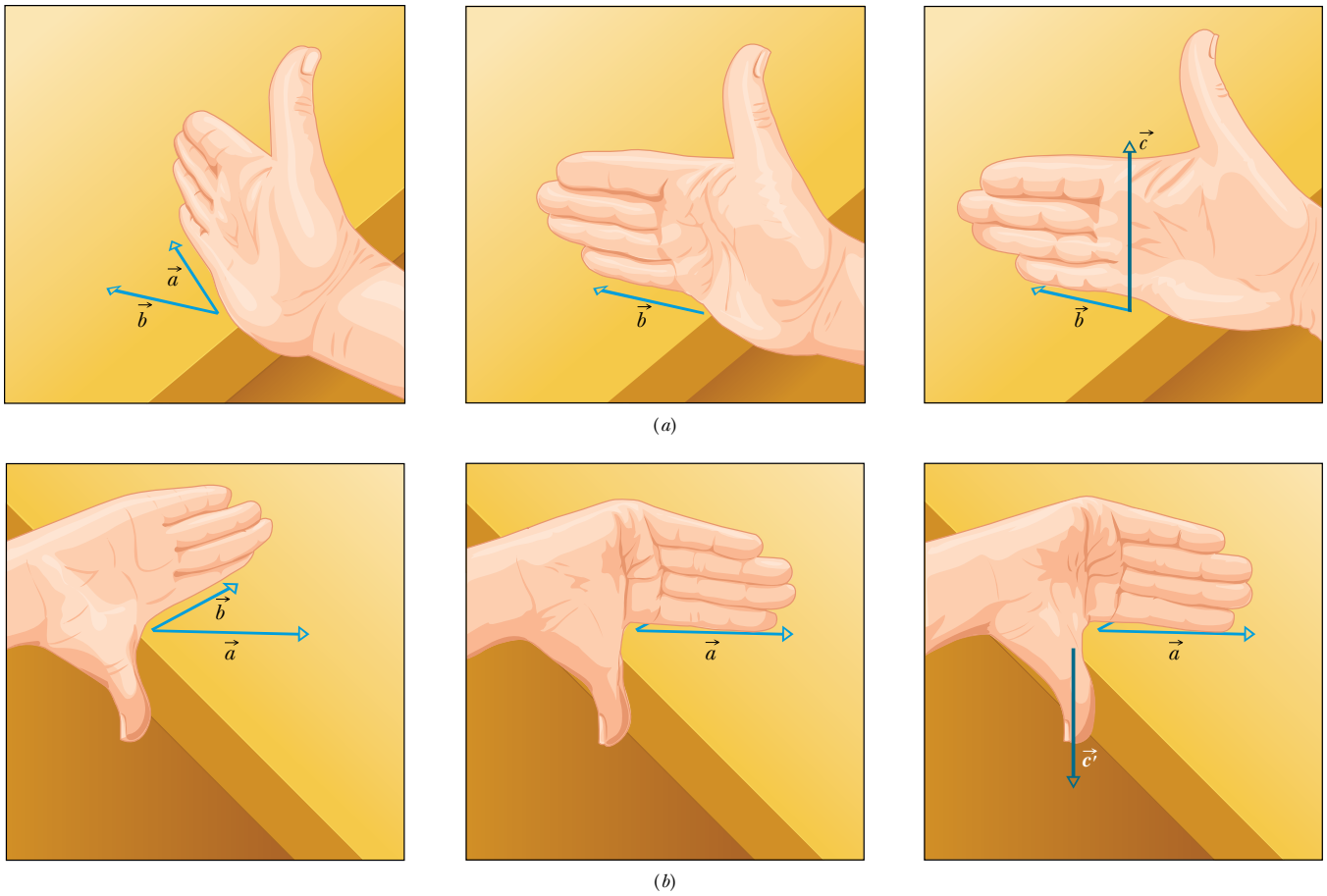

Figure 3-23 Illustration of the right-hand rule for vector products. (*a*) Sweep vector $\vec{a}$ into vector $\vec{b}$ with the fingers of your right hand. Your outstretched thumb shows the direction of vector $\vec{c} = \vec{a} \times \vec{b}$. (*b*) Showing that $\vec{b} \times \vec{a}$ is the reverse of $\vec{a} \times \vec{b}$.

The order of the vector multiplication is important. In Fig. 3-23*b*, we are determining the direction of $\vec{c}' = \vec{b} \times \vec{a}$, so the fingers are placed to sweep $\vec{b}$ into $\vec{a}$ through the smaller angle. The thumb ends up in the opposite direction from previously, and so it must be that $\vec{c}' = -\vec{c}$; that is,

$$\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b}). \tag{3-25}$$

In other words, the commutative law does not apply to a vector product.

In unit-vector notation, we write

$$\vec{a} \times \vec{b} = (a_x \hat{\mathrm{i}} + a_y \hat{\mathrm{j}} + a_z \hat{\mathrm{k}}) \times (b_x \hat{\mathrm{i}} + b_y \hat{\mathrm{j}} + b_z \hat{\mathrm{k}}), \tag{3-26}$$

which can be expanded according to the distributive law; that is, each component of the first vector is to be crossed with each component of the second vector. The cross products of unit vectors are given in Appendix E (see "Products of Vectors"). For example, in the expansion of Eq. 3-26, we have

$$a_x \hat{\mathrm{i}} \times b_x \hat{\mathrm{i}} = a_x b_x (\hat{\mathrm{i}} \times \hat{\mathrm{i}}) = 0,$$

because the two unit vectors $\hat{\mathrm{i}}$ and $\hat{\mathrm{i}}$ are parallel and thus have a zero cross product. Similarly, we have

$$a_x \hat{\mathrm{i}} \times b_y \hat{\mathrm{j}} = a_x b_y (\hat{\mathrm{i}} \times \hat{\mathrm{j}}) = a_x b_y \hat{\mathrm{k}}.$$

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In the last step we used Eq. 3-24 to evaluate the magnitude of $\hat{i}\times\hat{j}$ as unity. (These vectors $\hat{i}$ and $\hat{j}$ each have a magnitude of unity, and the angle between them is 90°.) Also, we used the right-hand rule to get the direction of $\hat{i}\times\hat{j}$ as being in the positive direction of the z axis (thus in the direction of $\hat{k}$).

Continuing to expand Eq. 3-26, you can show that

$$\vec{a}\times\vec{b}=(a_yb_z-b_ya_z)\hat{i}+(a_zb_x-b_za_x)\hat{j}+(a_xb_y-b_xa_y)\hat{k}. \tag{3-27}$$

A determinant (Appendix E) can also be used.

To check whether any xyz coordinate system is a right-handed coordinate system, use the right-hand rule for the cross product $\hat{i}\times\hat{j}=\hat{k}$ with that system. If your fingers sweep $\hat{i}$ (positive direction of x) into $\hat{j}$ (positive direction of y) with the outstretched thumb pointing in the positive direction of z (not the negative direction), then the system is right-handed.

CHECKPOINT 5

Vectors $\vec{C}$ and $\vec{D}$ have magnitudes of 3 units and 4 units, respectively. What is the angle between the directions of $\vec{C}$ and $\vec{D}$ if the magnitude of the vector product $\vec{C}\times\vec{D}$ is (a) zero and (b) 12 units?

SAMPLE PROBLEM 3.08

Angle between two vectors using dot products

What is the angle ϕ between $\vec{a}=3.0\hat{i}-4.0\hat{j}$ and $\vec{b}=-2.0\hat{i}+3.0\hat{k}$? (*Caution:* Do not bypass any step, you will learn more about scalar products you use all these steps.)

KEY IDEA

The angle between the directions of two vectors is included in the definition of their scalar product (Eq. 3-20):

$$\vec{a}\cdot\vec{b}=ab\cos\phi. \tag{3-28}$$

Calculations: In Eq. 3-28, a is the magnitude of $\vec{a}$, or

$$a=\sqrt{3.0^2+(-4.0)^2}=5.00, \tag{3-29}$$

and b is the magnitude of $\vec{b}$, or

$$b=\sqrt{(-2.0)^2+3.0^2}=3.61. \tag{3-30}$$

We can separately evaluate the left side of Eq. 3-28 by writing the vectors in unit-vector notation and using the distributive law:

$$\begin{aligned}\vec{a}\cdot\vec{b}&=(3.0\hat{i}-4.0\hat{j})\cdot(-2.0\hat{i}+3.0\hat{k})\\&=(3.0\hat{i})\cdot(-2.0\hat{i})+(3.0\hat{i})\cdot(3.0\hat{k})\\&\quad+(-4.0\hat{j})\cdot(-2.0\hat{i})+(-4.0\hat{j})\cdot(3.0\hat{k}).\end{aligned}$$

We next apply Eq. 3-20 to each term in this last expression. The angle between the unit vectors in the first term ($\hat{i}$ and $\hat{i}$) is 0°, and in the other terms it is 90°. We then have

$$\begin{aligned}\vec{a}\cdot\vec{b}&=-(6.0)(1)+(9.0)(0)+(8.0)(0)-(12)(0)\\&=-6.0.\end{aligned}$$

Substituting this result and the results of Eqs. 3-29 and 3-30 into Eq. 3-28 yields

$$-6.0=(5.00)(3.61)\cos\phi,$$

so $$\phi=\cos^{-1}\frac{-6.0}{(5.00)(3.61)}=109°\approx 110°. \quad \text{(Answer)}$$

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SAMPLE PROBLEM 3.09

Cross product, right-hand rule

In Fig. 3-24, vector $\vec{a}$ lies in the xy plane, has a magnitude of 18 units, and points in a direction 250° from the positive direction of the x axis. Also, vector $\vec{b}$ has a magnitude of 12 units and points in the positive direction of the z axis. What is the vector product $\vec{c} = \vec{a} \times \vec{b}$?

KEY IDEA

When we have two vectors in magnitude-angle notation, we find the magnitude of their cross product with Eq. 3-24 and the direction of their cross product with the right-hand rule of Fig. 3-23.

Calculations: For the magnitude we write

$$c = ab \sin \phi = (18)(12)(\sin 90°) = 216. \qquad \text{(Answer)}$$

To determine the direction in Fig. 3-24, imagine placing the fingers of your right hand around a line perpendicular to the plane of $\vec{a}$ and $\vec{b}$ (the line on which $\vec{c}$ is shown) such that your fingers sweep $\vec{a}$ into $\vec{b}$. Your outstretched thumb then gives the direction of $\vec{c}$. Thus, as shown in the figure, $\vec{c}$ lies in the xy plane. Because its direction is perpendicular to the direction of $\vec{a}$ (a cross product always gives a perpendicular vector), it is at an angle of

$$250° - 90° = 160° \qquad \text{(Answer)}$$

from the positive direction of the x axis.

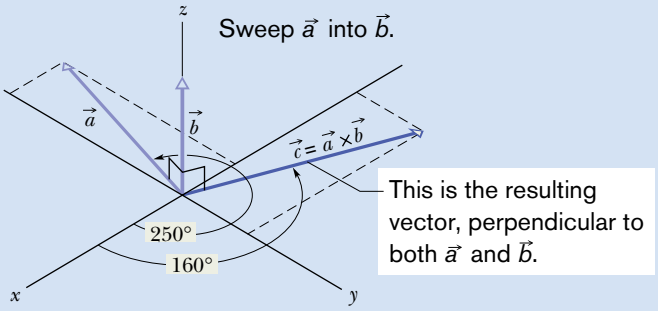

Figure 3-24 Vector $\vec{c}$ (in the xy plane) is the vector (or cross) product of vectors $\vec{a}$ and $\vec{b}$.

SAMPLE PROBLEM 3.10

Cross product, unit-vector notation

If $\vec{a} = 3\hat{i} - 4\hat{j}$ and $\vec{b} = -2\hat{i} + 3\hat{k}$, what is $\vec{c} = \vec{a} \times \vec{b}$?

KEY IDEA

When two vectors are in unit-vector notation, we can find their cross product by using the distributive law.

Calculations: Here we write

$$\vec{c} = (3\hat{i} - 4\hat{j}) \times (-2\hat{i} + 3\hat{k})$$
$$= 3\hat{i} \times (-2\hat{i}) + 3\hat{i} \times 3\hat{k} + (-4\hat{j}) \times (-2\hat{i}) + (-4\hat{j}) \times 3\hat{k}.$$

We next evaluate each term with Eq. 3-24, finding the direction with the right-hand rule. For the first term here, the angle ϕ between the two vectors being crossed is 0. For the other terms, ϕ is 90°. We find

$$\vec{c} = -6(0) + 9(-\hat{j}) + 8(-\hat{k}) - 12\hat{i}$$
$$= -12\hat{i} - 9\hat{j} - 8\hat{k}. \qquad \text{(Answer)}$$

This vector $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, a fact you can check by showing that $\vec{c} \cdot \vec{a} = 0$ and $\vec{c} \cdot \vec{b} = 0$; that is, there is no component of $\vec{c}$ along the direction of either $\vec{a}$ or $\vec{b}$.

In general: A cross product gives a perpendicular vector, two perpendicular vectors have a zero dot product, and two vectors along the same axis have a zero cross product.

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REVIEW AND SUMMARY

Scalars and Vectors *Scalars,* such as temperature, have magnitude only. They are specified by a number with a unit (10°C) and obey the rules of arithmetic and ordinary algebra. *Vectors,* such as displacement, have both magnitude and direction (5 m, north) and obey the rules of vector algebra.

Adding Vectors Geometrically Two vectors $\vec{a}$ and $\vec{b}$ may be added geometrically by drawing them to a common scale and placing them head to tail. The vector connecting the tail of the first to the head of the second is the vector sum $\vec{s}$. To subtract $\vec{b}$ from $\vec{a}$, reverse the direction of $\vec{b}$ to get $-\vec{b}$; then add $-\vec{b}$ to $\vec{a}$. Vector addition is commutative

$$\vec{a}+\vec{b}=\vec{b}+\vec{a} \tag{3-2}$$

and obeys the associative law

$$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c}). \tag{3-3}$$

Components of a Vector The (scalar) *components* a_x and a_y of any two-dimensional vector $\vec{a}$ along the coordinate axes are found by dropping perpendicular lines from the ends of $\vec{a}$ onto the coordinate axes. The components are given by

$$a_x = a\cos\theta \quad \text{and} \quad a_y = a\sin\theta, \tag{3-5}$$

where θ is the angle between the positive direction of the x axis and the direction of $\vec{a}$. The algebraic sign of a component indicates its direction along the associated axis. Given its components, we can find the magnitude and orientation (direction) of the vector $\vec{a}$ by using

$$a=\sqrt{a_x^2+a_y^2} \quad \text{and} \quad \tan\theta=\frac{a_y}{a_x} \tag{3-6}$$

Unit-Vector Notation *Unit vectors* $\hat{\mathrm{i}}, \hat{\mathrm{j}}$, and $\hat{\mathrm{k}}$ have magnitudes of unity and are directed in the positive directions of the x, y, and z axes, respectively, in a right-handed coordinate system (as defined by the vector products of the unit vectors). We can write a vector $\vec{a}$ in terms of unit vectors as

$$\vec{a}=a_x\hat{\mathrm{i}}+a_y\hat{\mathrm{j}}+a_z\hat{\mathrm{k}}, \tag{3-7}$$

in which $a_x\hat{\mathrm{i}}, a_y\hat{\mathrm{j}}$, and $a_z\hat{\mathrm{k}}$ are the **vector components** of $\vec{a}$ and a_x, a_y, and a_z are its **scalar components**.

Adding Vectors in Component Form To add vectors in component form, we use the rules

$$r_x=a_x+b_x \quad r_y=a_y+b_y \quad r_z=a_z+b_z. \tag{3-10 to 3-12}$$

Here $\vec{a}$ and $\vec{b}$ are the vectors to be added, and $\vec{r}$ is the vector sum. Note that we add components axis by axis. We can then express the sum in unit-vector notation or magnitude-angle notation.

Product of a Scalar and a Vector The product of a scalar s and a vector $\vec{v}$ is a new vector whose magnitude is sv and whose direction is the same as that of $\vec{v}$ if s is positive, and opposite that of $\vec{v}$ if s is negative. (The negative sign reverses the vector.) To divide $\vec{v}$ by s, multiply $\vec{v}$ by $1/s$.

The Scalar Product The **scalar** (or **dot**) **product** of two vectors $\vec{a}$ and $\vec{b}$ is written $\vec{a}\cdot\vec{b}$ and is the *scalar* quantity given by

$$\vec{a}\cdot\vec{b}=ab\cos\phi, \tag{3-20}$$

in which ϕ is the angle between the directions of $\vec{a}$ and $\vec{b}$. A scalar product is the product of the magnitude of one vector and the scalar component of the second vector along the direction of the first vector. Note that $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$, which means that the scalar product obeys the commutative law.

In unit-vector notation,

$$\vec{a}\cdot\vec{b}=(a_x\hat{\mathrm{i}}+a_y\hat{\mathrm{j}}+a_z\hat{\mathrm{k}})\cdot(b_x\hat{\mathrm{i}}+b_y\hat{\mathrm{j}}+b_z\hat{\mathrm{k}}), \tag{3-22}$$

which may be expanded according to the distributive law.

The Vector Product The **vector** (or **cross**) **product** of two vectors $\vec{a}$ and $\vec{b}$ is written $\vec{a}\times\vec{b}$ and is a *vector* $\vec{c}$ whose magnitude c is given by

$$c=ab\sin\phi, \tag{3-24}$$

in which ϕ is the smaller of the angles between the directions of $\vec{a}$ and $\vec{b}$. The direction of $\vec{c}$ is perpendicular to the plane defined by $\vec{a}$ and $\vec{b}$ and is given by a right-hand rule, as shown in Fig. 3-23. Note that $\vec{a}\times\vec{b}=-(\vec{b}\times\vec{a})$, which means that the vector product does not obey the commutative law.

In unit-vector notation,

$$\vec{a}\times\vec{b}=(a_x\hat{\mathrm{i}}+a_y\hat{\mathrm{j}}+a_z\hat{\mathrm{k}})\times(b_x\hat{\mathrm{i}}+b_y\hat{\mathrm{j}}+b_z\hat{\mathrm{k}}), \tag{3-26}$$

which we may expand with the distributive law.

PROBLEMS

1. If the x component of a vector $\vec{a}$, in the xy plane, is half as large as the magnitude of the vector, find the tangent of the angle between the vector and the x axis.
2. A displacement vector $\vec{r}$ in the xy plane is 12 m long and directed at angle $\theta = 30°$ in Fig. 3-25. Determine (a) the x component and (b) the y component of the vector.

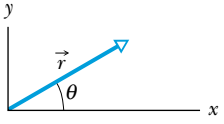

Figure 3-25 Problem 2.

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3. A vector has a component of 15 m in the $+x$ direction, a component of 15 m in the $+y$ direction, and a component of 10 m in the $+z$ direction. What is the magnitude of this vector?

4. A golfer takes three putts to get the ball into the hole. The first putt displaces the ball 3.66 m north, the second 1.83 m southeast, and the third 0.91 m southwest. What are (a) the magnitude and (b) the direction of the displacement needed to get the ball into the hole on the first putt?

5. Consider two displacements, one of magnitude 3 m and another of magnitude 4 m. Show how the displacement vectors may be combined to get a resultant displacement of magnitude (a) 7 m, (b) 1 m, and (c) 5 m.

6. Consider two vectors $\vec{a} = (5.0)\hat{i} - (4.0)\hat{j} + (2.0)\hat{k}$ and $\vec{b} = (-2.0m)\hat{i} + (2.0m)\hat{j} + (5.0m)\hat{k}$, where m is a scalar. Find (a) $\vec{a} + \vec{b}$, (b) $\vec{a} - \vec{b}$, and (c) a third vector $\vec{c}$ such that $\vec{a} - \vec{b} + \vec{c} = 0$.

7. Find the (a) x, (b) y, and (c) z components of the sum $\vec{r}$ of the displacements $\vec{c}$ and $\vec{d}$ whose components in meters are $c_x = 7.4, c_y = -3.8, c_z = -6.1; d_x = 4.4, d_y = -2.0, d_z = 3.3$.

8. (a) In unit-vector notation, what is the sum $\vec{a} + \vec{b}$ if $\vec{a} = (4.0\text{ m})\hat{i} + (3.0\text{ m})\hat{j}$ and $\vec{b} = (-13.0\text{ m})\hat{i} + (7.0\text{ m})\hat{j}$? What are the (b) magnitude and (c) direction of $\vec{a} + \vec{b}$?

9. A car is driven east for a distance of 40 km, then north for 30 km, and then in a direction 30° east of north for 25 km. Sketch the vector diagram and determine (a) the magnitude and (b) the angle of the car's total displacement from its starting point.

10. An object moves 1.00 m in a straight-line displacement $\vec{a}$, changes direction, moves another 1.00 m in straight-line displacement $\vec{b}$, and then ends up 1.0 m from the starting point. (a) Through what angle did it turn? (b) What is the magnitude of $\vec{a} - \vec{b}$?

11. The two vectors $\vec{a}$ and $\vec{b}$ in Fig. 3-26 have equal magnitudes of 10.0 m and the angles are $\theta_1 = 30°$ and $\theta_2 = 105°$. Find the (a) x and (b) y components of their vector sum $\vec{r}$, (c) the magnitude of $\vec{r}$, and (d) the angle $\vec{r}$ makes with the positive direction of the x axis.

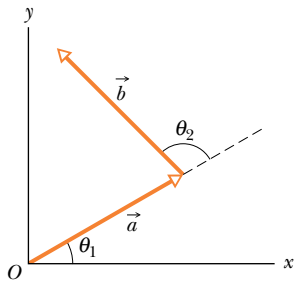

Figure 3-26 Problem 11.

12. For the displacement vectors $\vec{a} = (3.0\text{ m})\hat{i} + (4.0\text{ m})\hat{j}$ and $\vec{b} = (5.0\text{ m})\hat{i} + (-2.0\text{ m})\hat{j}$, give $\vec{a} + \vec{b}$ in (a) unit-vector notation, and as (b) a magnitude and (c) an angle (relative to $\hat{i}$). Now give $\vec{a} - \vec{b}$ in (d) unit-vector notation, and as (e) a magnitude and (f) an angle.

13. Three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ each have a magnitude of 50 m and lie in an xy plane. Their directions relative to the positive direction of the x axis are 30°, 195°, and 315°, respectively. What are (a) the magnitude and (b) the angle of the vector $\vec{a} + \vec{b} + \vec{c}$, and (c) the magnitude and (d) the angle of $\vec{a} - \vec{b} - \vec{c}$? What are the (e) magnitude and (f) angle of a fourth vector $\vec{d}$ such that $(\vec{a} + \vec{b}) - (\vec{c} + \vec{d}) = 0$?

14. In the sum $\vec{A} + \vec{B} + \vec{C}$, vector $\vec{A}$ has a magnitude of 12.0 m and is angled 40.0° counterclockwise from the $+x$ direction, and vector $\vec{C}$ has a magnitude of 16.0 m and is angled 20.0° counterclockwise from the $-x$ direction. What are (a) the magnitude and (b) the angle (relative to $+x$) of $\vec{B}$?

15. In a game of lawn chess, where pieces are moved between the centers of squares that are each 1.00 m on edge, a knight is moved in the following way: (1) two squares forward, one square rightward; (2) two squares leftward, one square forward; (3) two squares forward, one square leftward. What are (a) the magnitude and (b) the angle (relative to "forward") of the knight's overall displacement for the series of three moves?

16. An explorer is caught in a whiteout (in which the snowfall is so thick that the ground cannot be distinguished from the sky) while returning to base camp. He was supposed to travel due north for 4.8 km, but when the snow clears, he discovers that he actually traveled 7.8 km at 50° north of due east. (a) How far and (b) in what direction must he now travel to reach base camp?

17. An ant, crazed by the Sun on a hot Texas afternoon, darts over an xy plane scratched in the dirt. The x and y components of four consecutive darts are the following, all in centimeters: (30.0, 40.0), (b_x, −70.0), (−20.0, c_y), (−80.0, −70.0). The overall displacement of the four darts has the xy components (−140, −20.0). What are (a) b_x and (b) c_y? What are the (c) magnitude and (d) angle (relative to the positive direction of the x axis) of the overall displacement?

18. If $\vec{b} = (3.0)\hat{i} + (4.0)\hat{j}$ and $\vec{a} = \hat{i} + \hat{j}$, what is the vector having the same magnitude as that of $\vec{b}$ and parallel to $\vec{a}$?

19. Vector $\vec{A}$, which is directed along an x axis, is to be added to vector $\vec{B}$, which has a magnitude of 6.0 m. The sum is a third vector that is directed along the y axis, with a magnitude that is 3.0 times that of $\vec{A}$. What is that magnitude of $\vec{A}$?

20. Starting from an oasis, a camel walks 25 km in a direction 30° south of west and then walks 30 km toward the north to a second oasis. What is the direction from the first oasis to the second oasis?

21. What is the sum of the following four vectors in (a) unit-vector notation, and as (b) a magnitude and (c) an angle?

$\vec{A} = (2.00\text{ m})\hat{i} + (3.00\text{ m})\hat{j}$ $\qquad$ $\vec{B}$: 4.00 m, at +65.0°

$\vec{C} = (-4.00\text{ m})\hat{i} + (-6.00\text{ m})\hat{j}$ $\qquad$ $\vec{D}$: 5.00m, at −235°

22. If $\vec{d}_1 + \vec{d}_2 = 5\vec{d}_3$, $\vec{d}_1 - \vec{d}_2 = 3\vec{d}_3$, and $\vec{d}_3 = 2\hat{i} + 4\hat{j}$, then what are, in unit-vector notation, (a) $\vec{d}_1$ and (b) $\vec{d}_2$?

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23. Typical backyard ants often create a network of chemical trails for guidance. Extending outward from the nest, a trail branches (*bifurcates*) repeatedly, with 60° between the branches. If a roaming ant chances upon a trail, it can tell the way to the nest at any branch point: If it is moving away from the nest, it has two choices of path requiring a small turn in its travel direction, either 30° leftward or 30° rightward. If it is moving toward the nest, it has only one such choice. Figure 3-27 shows a typical ant trail, with lettered straight sections of 2.0 cm length and symmetric bifurcation of 60°. Path v is parallel to the y axis. What are the (a) magnitude and (b) angle (relative to the positive direction of the superimposed x axis) of an ant's displacement from the nest (find it in the figure) if the ant enters the trail at point A? What are the (c) magnitude and (d) angle if it enters at point B?

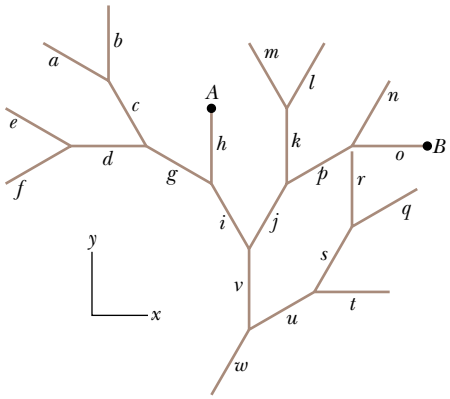

Figure 3-27 Problem 23.

24. Here are two vectors:

$$\vec{a} = (4.0\text{ m})\hat{i} - (3.0\text{ m})\hat{j} \quad \text{and} \quad \vec{b} = (6.0\text{ m})\hat{i} + (8.0\text{ m})\hat{j}.$$

What are (a) the magnitude and (b) the angle (relative to $\hat{i}$) of $\vec{a}$? What are (c) the magnitude and (d) the angle of $\vec{b}$? What are (e) the magnitude and (f) the angle $\vec{a}+\vec{b}$; of (g) the magnitude and (h) the angle of $\vec{a}-\vec{b}$; and (i) the magnitude and (j) the angle of $\vec{a}-\vec{b}$? (k) What is the angle between the directions of $\vec{a}-\vec{b}$ and $\vec{a}-\vec{b}$?

25. For the vectors in Fig. 3-28, with $a = 4$, $b = 3$, and $c = 5$, what are (a) the magnitude and (b) the direction of $\vec{a}\times\vec{b}$, (c) the magnitude and (d) the direction of $\vec{a}\times\vec{c}$, and (e) the magnitude and (f) the direction of $\vec{b}\times\vec{c}$? (The z axis is not shown.)

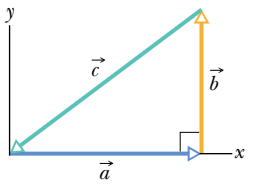

Figure 3-28 Problems 25.

26. In Fig. 3-29, a cube of edge length a sits with one corner at the origin of an xyz coordinate system. A *body diagonal* is a line that extends from one corner to another through the center. In unit-vector notation, what is the body diagonal that extends from the corner at (a) coordinates (0, 0, 0), (b) coordinates (a, 0, 0), (c) coordinates (0, a, 0), and (d) coordinates (a, a, 0)? (e) Determine the angles that the body diagonals make with the adjacent edges. (f) Determine the length of the body diagonals in terms of a.

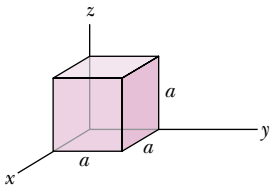

Figure 3-29 Problem 26.

27. Two vectors are presented as $\vec{a} = 3.0\hat{i} + 5.0\hat{j}$ and $\vec{b} = 2.0\hat{i} + 4.0\hat{j}$. Find (a) $\vec{a}\times\vec{b}$, (b) $\vec{a}\cdot\vec{b}$, (c) $(\vec{a}+\vec{b})\cdot\vec{b}$, and (d) the component of $\vec{a}$ along the direction of $\vec{b}$. (*Hint:* For (d), consider Eq. 3-20 and Fig. 3-22.)

28. Two vectors $\vec{p}$ and $\vec{q}$ lie in the xy plane. Their magnitudes are 3.50 and 6.30 units, respectively, and their directions are 220° and 75.0°, respectively, as measured counterclockwise from the positive x axis. What are the values of (a) $\vec{p}\times\vec{q}$ and (b) $\vec{p}\cdot\vec{q}$?

29. Consider two vectors $\vec{p}_1 = 4\hat{i} - 3\hat{j} + 5\hat{k}$ and $\vec{p}_2 = -6\hat{i} + 3\hat{j} - 2\hat{k}$. What is $(\vec{p}_1 + \vec{p}_2)\cdot(\vec{p}_1 \times 5\vec{p}_2)$?

30. Three vectors are given by $\vec{a} = 3.0\hat{i} + 3.0\hat{j} - 2.0\hat{k}$, $\vec{b} = -1.0\hat{i} - 4.0\hat{j} + 2.0\hat{k}$, and $\vec{c} = 2.0\hat{i} + 2.0\hat{j} + 1.0\hat{k}$. Find (a) $\vec{a}\cdot(\vec{b}\times\vec{c})$, (b) $\vec{a}\cdot(\vec{b}+\vec{c})$, and (c) $\vec{a}\times(\vec{b}+\vec{c})$.

31. For the following three vectors, what is $3\vec{C}\cdot(2\vec{A}\times\vec{B})$?

$$\vec{A} = 2.00\hat{i} + 3.00\hat{j} - 4.00\hat{k}$$

$$\vec{B} = -3.00\hat{i} + 4.00\hat{j} + 2.00\hat{k} \qquad \vec{C} = 7.00\hat{i} - 8.00\hat{j}$$

32. Vector $\vec{A}$ has a magnitude of 6.00 units, vector $\vec{B}$ has a magnitude of 7.00 units, and $\vec{A}\cdot\vec{B}$ has a value of 14.0. What is the angle between the directions of $\vec{A}$ and $\vec{B}$?

33. Displacement $\vec{d}_1$ is in the yz plane 63.0° from the positive direction of the y axis, has a positive z component, and has a magnitude of 4.80 m. Displacement $\vec{d}_2$ is in the xz plane 30.0° from the positive direction of the x axis, has a positive z component, and has magnitude 1.40 m. What are (a) $\vec{d}_1\cdot\vec{d}_2$, (b) $\vec{d}_1\times\vec{d}_2$, and (c) the angle between $\vec{d}_1$ and $\vec{d}_2$?

34. The three vectors in Fig. 3-30 have magnitudes a = 3.00 m, b = 4.00 m, and c = 10.0 m and angle $\theta = 30.0°$. What are (a) the x component and (b) the y component of $\vec{a}$; (c) the x component and (d) the y component of $\vec{b}$; and (e) the x component and (f) the y component of $\vec{c}$? If $\vec{c} = p\vec{a} + q\vec{b}$, what are the values of (g) p and (h) q?

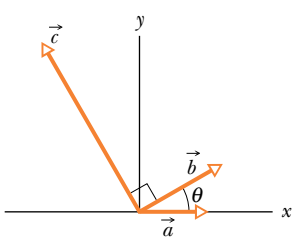

Figure 3-30 Problem 34.

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PRACTICE QUESTIONS

Single Correct Choice Type

1. Which one of the following statements is true concerning scalar quantities?
- **(a)** Scalar quantities must be represented by base units.
- **(b)** Scalar quantities have both magnitude and direction.
- **(c)** Scalar quantities can be added to vector quantities using rules of trigonometry.
- **(d)** Scalar quantities can be added to other scalar quantities using rules of ordinary addition.

2. Which one of the following quantities is a vector quantity?
- **(a)** the age of the earth
- **(b)** the mass of a freight train
- **(c)** the Earth's pull on your body
- **(d)** the temperature of hot cup of coffee

3. Which one of the following statements concerning vectors and scalars is *false*?
- **(a)** In calculations, the vector components of a vector may be used in place of the vector itself.
- **(b)** It is possible to use vector components that are not perpendicular.
- **(c)** A scalar component may be either positive or negative.
- **(d)** A vector that is zero may have components other than zero.

4. Twelve coplanar forces (all of equal magnitude) maintain a body in equilibrium, then the angle between any two adjacent forces is
- **(a)** 15° **(b)** 30°
- **(c)** 45° **(d)** 60°

5. If vectors $\vec{A} = \hat{i} + 2\hat{j} + 4\hat{k}$ and $\vec{B} = 5\hat{i}$ represent the two sides of a triangle, then the third side of the triangle can have length equal to
- **(a)** 6 **(b)** $\sqrt{56}$
- **(c)** both (a) and (b) **(d)** none of the above

6. Mark the correct statement.
- **(a)** $|\vec{a} + \vec{b}| \geq |\vec{a}| + |\vec{b}|$ **(b)** $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$
- **(c)** $|\vec{a} - \vec{b}| \geq |\vec{a}| + |\vec{b}|$ **(d)** All of the above

7. Two vectors A and B, are added together to form the vector $C = A + B$. The relationship between the magnitudes of these vectors is given by: $C_x = A \cos 30° + B$ and $C_y = -A \sin 30°$. Which statement best describes the orientation of these vectors?
- **(a)** A points in the negative x direction while B points in the positive y direction.
- **(b)** A points in the negative y direction while B points in the positive x direction.
- **(c)** A points 30° below the positive x axis while B points in the positive x direction.
- **(d)** A points 30° above the positive x axis while B points in the positive x direction.

8. Three vectors A, B, and C add together to yield zero: $A + B + C = 0$. The vectors A and C point in *opposite* directions and their magnitudes are related by the expression: $A = 2C$. Which one of the following conclusions is correct?
- **(a)** A and B have equal magnitudes and point in opposite directions.
- **(b)** B and C have equal magnitudes and point in the same direction.
- **(c)** B and C have equal magnitudes and point in opposite directions.
- **(d)** A and B point in the same direction, but A has twice the magnitude of B.

9. If vector $\vec{C}$ is added to vector $\vec{D}$, the results is a third vector that is perpendicular to $\vec{D}$ and has a magnitude equal to $3\vec{D}$. What is the ratio of the magnitude of $\vec{C}$ to that of $\vec{D}$?
- **(a)** 1.8 **(b)** 2.2
- **(c)** 3.2 **(d)** 1.3

10. Given that $\vec{A} + 2\vec{B} = x_1\hat{i} + y_1\hat{j}$ and $2\vec{A} - \vec{B} = x_2\hat{i} + y_2\hat{j}$, what is $\vec{A}$?
- **(a)** $\vec{A} = \frac{1}{5}(x_1 + 2x_2)\hat{i} + \frac{1}{5}(y_1 + 2y_2)\hat{j}$
- **(b)** $\vec{A} = \frac{1}{5}(x_1 + 2x_2)\hat{i} - \frac{1}{5}(y_1 + 2y_2)\hat{j}$
- **(c)** $\vec{A} = \frac{1}{5}(x_1 + 4x_2)\hat{i} + \frac{1}{5}(y_1 + 2y_2)\hat{j}$
- **(d)** $\vec{A} = \frac{1}{5}(x_1 + 4x_2)\hat{i} + \frac{1}{5}(y_1 + 4y_2)\hat{j}$

11. A vector $\vec{A}$ is rotated by a small angle $\Delta\theta$ radians ($\Delta\theta << 1$) to get a new vector $\vec{B}$. In that case $|\vec{B} - \vec{A}|$ is
- **(a)** 0 **(b)** $|\vec{A}|\left(1 - \frac{\Delta\theta^2}{2}\right)$
- **(c)** $|\vec{A}|\Delta\theta$ **(d)** $|\vec{B}|\Delta\theta - |\vec{A}|$

12. The vector $\vec{A}$ has components +5 and +7 along the x-axes and y-axes, respectively. Along a set of axes rotated 90 degrees counterclockwise relative to the original axes, the vector's components are
- **(a)** −7; −5 **(b)** 7; −5
- **(c)** −7; 5 **(d)** 7; 5

13. In a two-dimensional motion of a particle, the particle moves from point A of position vector $\vec{r}_1$, to point B of position vector $\vec{r}_2$ (see figure). If the magnitudes of these vectors are, respectively, $r_1 = 3$ and $r_2 = 4$ and the angles

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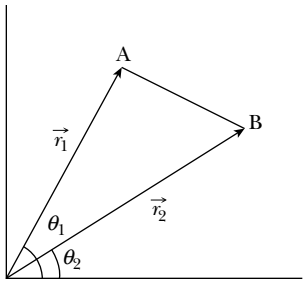
they make with the x axis are $\theta_1 = 75°$ and $\theta_2 = 15°$, respectively, then find the magnitude of the displacement vector.



(a) 15 **(b)** $\sqrt{13}$
(c) 17 **(d)** $\sqrt{15}$

14. Each of two vectors $\vec{D}_1$ and $\vec{D}_2$ lies along a coordinate axis in the x–y plane. Each vector has its tail at the origin, and the dot product of the two vectors is $\vec{D}_1 \cdot \vec{D}_2 = -\left|\vec{D}_1\right| \cdot \left|\vec{D}_2\right|$. Which is possibility is correct?

(a) $\vec{D}_1$ and $\vec{D}_2$ both lie along the positive x-axis.

(b) $\vec{D}_1$ lies along the positive x-axis $\vec{D}_2$ lies along the negative x-axis.

(c) $\vec{D}_1$ and $\vec{D}_2$ both lie along the positive y-axis.

(d) $\vec{D}_1$ lies along the negative x-axis $\vec{D}_2$ lies along the negative y-axis.

15. Town A lies 20 km north of town B. Town C lies 13 km west of town A. A small plane flies directly from town B to town C. What is the displacement of the plane?
(a) 33 km, 33° north of west
(b) 19 km, 33° north of west
(c) 24 km, 57° north of west
(d) 31 km, 57° north of west

16. A force, F_1, of magnitude 2.0 N and directed due east is exerted on an object. A second force exerted on the object is $F_2 = 2.0$ N, due north. What is the magnitude and direction of a third force, F_3, which must be exerted on the object so that the resultant force is zero?
(a) 1.4 N, 45° north of east
(b) 1.4 N, 45° south of west
(c) 2.8 N, 45° north of east
(d) 2.8 N, 45° south of west

17. A race car will make one lap around a circular track of radius R. When the car has traveled *halfway* around the track, what is the magnitude of the car's displacement from the starting point?
(a) $2R$ **(b)** R
(c) πR **(d)** $2\pi R$

18. An escaped convict runs 1.70 km due east of the prison. He then runs due north to a friend's house. If the magnitude of the convict's total displacement vector is 2.50 km, what is the direction of his total displacement vector with respect to due east?
(a) 43° south of east **(b)** 47° north of east
(c) 56° north of east **(d)** 34° south of east

19. Consider the following four force vectors:

$\vec{F}_1 = 50.0$ N, due east, $\vec{F}_2 = 10.0$ N, due east,

$\vec{F}_3 = 40.0$ N, due west, $\vec{F}_4 = 30.0$ N, due west

Which two vectors add together to give a resultant with smallest magnitude? In the option below, the two vectors are given followed by a magnitude and direction.

(a) $\vec{F}_1 + \vec{F}_3$, 10 N, due east

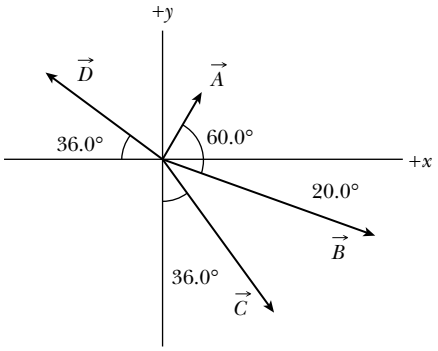
(b) $\vec{F}_1 + \vec{F}_3$, 10 N, due west

(c) $\vec{F}_1 + \vec{F}_4$, 20 N, due east

(d) $\vec{F}_1 + \vec{F}_4$, 20 N, due west

20. Two bicyclists, starting at the same place, are riding toward the same campground by two different routes. One cyclist rides 1080 m due east and then turns due north and travels another 1430 m before reaching the campground. The second cyclist starts out by heading due north for 1950 m and then turns and heads directly toward the campground. At the turning point, how far is the second cyclist from the campground?
(a) 1200 m **(b)** 1700 m
(c) 1100 m **(d)** 1600 m

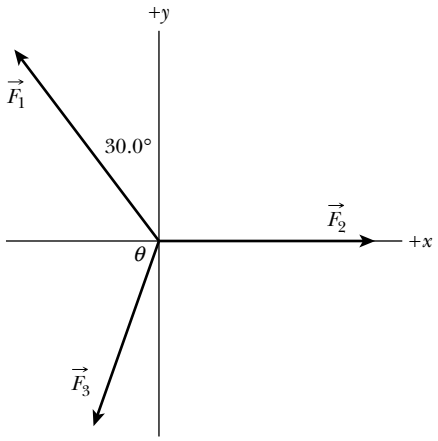
21. Use the component method of vector addition to find the components of the resultant of the four displacements shown in the figure. The magnitudes of the displacements are: $A = 2.25$ cm, $B = 6.35$ cm, $C = 5.47$ cm, and $D = 4.19$ cm.



	x component	*y* component
(a)	2.19 cm	–6.92 cm
(b)	3.71 cm	–1.09 cm
(c)	5.45 cm	–2.82 cm
(d)	6.93 cm	–2.19 cm

22. Three forces are applied to an object, as indicated in the drawing. Force $\vec{F}_1$ has a magnitude of 21.0 newton (21.0 N) and is directed 30.0° to the left of the $+y$ axis. Force $\vec{F}_2$

has a magnitude of 15.0 N and points along the $+x$ axis. What must be the magnitude and direction (specified by the angle θ in the drawing) of the third force $\vec{F}_3$ such that the vector sum of the three forces is 0 N?



(a) 16.9 N, 81°
(b) 20.4 N, 75°
(c) 22.3 N, 79°
(d) 18.7 N, 76°

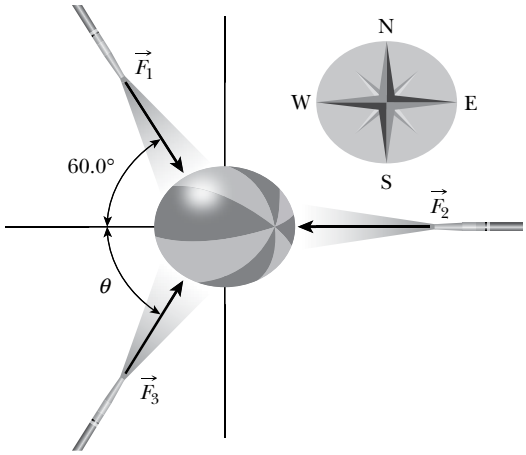
23. Displacement vector $\vec{A}$ points due east and has a magnitude of 2.00 km. Displacement vector $\vec{C}$ points due north and has a magnitude of 3.75 km. Displacement vector $\vec{A}$ points due west and has a magnitude of 2.50 km. Displacement vector $\vec{D}$ points due south and has a magnitude of 3.00 km. Find the magnitude and direction (relative to due west) of the resultant vector $\vec{A}+\vec{B}+\vec{C}+\vec{D}$.

(a) 0.90 km, 56° north of west
(b) 0.50 km, 34° south of west
(c) 0.75 km, 48° north of west
(d) 0.25 km, 52° south of west
(e) 1.25 km, 44° north of west

24. A jogger travels a route that has two parts. The first is a displacement $\vec{A}$ of 2.50 km due south, and the second involves a displacement $\vec{B}$ that points due east. Suppose that $\vec{A}-\vec{B}$ had a magnitude of 3.75 km. What then would be the magnitude of $\vec{B}$, and what is the direction of $\vec{A}-\vec{B}$ relative to due south?

(a) 4.5 km, 63° east of south
(b) 4.5 km, 63° west of south
(c) 3.7 km, 56° west of south
(d) 2.8 km, 48° west of south

25. At a picnic, there is a contest in which hoses are used to shoot water at a beach ball from three directions. As a result, three forces act on the ball, $\vec{F}_1, \vec{F}_2$ and $\vec{F}_3$ (see the drawing). The magnitudes of $\vec{F}_1$ and $\vec{F}_2$ are $F_1 = 50.0$ N and $F_2 = 90.0$ N. Using the graphical technique, determine the angle such that the resultant force acting on the ball is zero.



(a) 21°
(b) 26°
(c) 34°
(d) 39°

More than One Correct Choice Type

26. The x-component of the resultant of several vectors
(a) is equal to the sum of the x-components of the vectors.
(b) may be equal to the sum of the magnitude of the vectors.
(c) may be smaller than the sum of the magnitude of the vectors.
(d) may be greater than the sum of the magnitude of the vectors.

27. If $\vec{A} = 2\hat{i}+\hat{j}+\hat{k}$ and $\vec{B} = \hat{i}+\hat{j}+\hat{k}$ are two vectors, then the unit vector is

(a) perpendicular to $\vec{A}$ is $\left(\dfrac{-\hat{j}+\hat{k}}{\sqrt{2}}\right)$.

(b) parallel to $\vec{A}$ is $\left(\dfrac{2\hat{i}+\hat{j}+\hat{k}}{\sqrt{6}}\right)$.

(c) perpendicular to $\vec{A}$ is $\left(\dfrac{-\hat{j}+\hat{k}}{\sqrt{2}}\right)$.

(d) parallel to $\vec{A}$ is $\left(\dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}\right)$.

Linked Comprehension

Paragraph for Questions 28 and 29: Two vectors, A and B, are added together to form the vector $C = A + B$. The relationship between the magnitudes of these vectors is given by:

$$C_x = 0$$

$$C_y = A \sin 60° + B \sin 30°$$

A_x and A_y point in the positive x and y directions, respectively.

28. Which one of the following statements best describes the orientation of vectors A and B?

(a) A and B point in opposite directions.
(b) A points 60° above the positive x axis while B points 30° above the negative x axis.
(c) A points 60° above the negative x axis while B points 30° above the positive x axis.
(d) A points 60° below the positive x axis while B points 30° above the positive y axis.

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29. How does the magnitude of A compare with that of B?
(a) $A = B$ **(b)** $A = 1.7B$
(c) $A = 0.4B$ **(d)** $A = 0.5B$

Paragraph for Questions 30–32: The table gives the x and y components of two vectors A and B.

Vector	x component	y component
A	+15 units	+10 units
B	+15 units	–10 units

30. Which one of the following statements concerning these vectors is true?
(a) The vector $A–B$ has no x component.
(b) The two vectors have different magnitudes.
(c) A makes a 56° angle with the positive x axis.
(d) B makes a 34° angle with the positive y axis.

31. Determine the magnitude of the vector sum, $A + B$.
(a) 5 units **(b)** 15 units
(c) 20 units **(d)** 30 units

32. Determine the magnitude of the vector difference, $A - B$.
(a) 5 units **(b)** 15 units
(c) 20 units **(d)** 30 units

Paragraph for Questions 33 and 34: A boat radioed a distress call to a Coast Guard station. At the time of the call, a vector A from the station to the boat had a magnitude of 45.0 km and was directed 15.0° east of north. A vector from the station to the point where the boat was later found is $B = 30.0$ km, 15.0° north of east.

33. What are the components of the vector from the point where the distress call was made to the point where the boat was found? In other words, what are the components of vector $C = B - A$?

x component	y component
(a) 35.7 km, west	17.4 km, north
(b) 40.6 km, east	51.2 km, south
(c) 17.3 km, west	51.2 km, south
(d) 17.3 km, east	35.7 km, south

34. How far did the boat travel from the point where the distress call was made to the point where the boat was found? In other words, what is the magnitude of vector C?
(a) 65.3 km **(b)** 39.7 km
(c) 26.5 km **(d)** 54.0 km

Matrix-Match

Directions for Question 35: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

35. If $\vec{A} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{B} = \hat{i} + \hat{j} + \hat{k}$ are the two vectors, match the following answer the questions by appropriately matching the information given in the three columns of the following table:

Column I	Column II	Column III
(I) Perpendicular	**(i)** $\vec{A}$	**(J)** $\left(\frac{\hat{j}+\hat{k}}{2\sqrt{2}}\right)$
(II) Parallel	**(ii)** $\vec{B}$	**(K)** $\left(\frac{2\hat{i}+\hat{j}+\hat{k}}{\sqrt{6}}\right)$
(III) Collinear	**(iii)** $\vec{A}$ and $\vec{B}$	**(L)** $\left(\frac{-\hat{j}+\hat{k}}{3\sqrt{2}}\right)$
(IV) Orthogonal	**(iv)** $\vec{A}\times\vec{B}$	**(M)** $\left(\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}\right)$

(1) The correct alternative for a collinear vector to $\vec{A}$ is
(a) (III) (iv) (L) **(b)** (II) (ii) (K)
(c) (IV) (i) (J) **(d)** (I) (iii) (M)

(2) Which option represents a vector orthogonal to $\vec{A}\times\vec{B}$?
(a) (IV) (ii) (M) **(b)** (II) (iii) (J)
(c) (I) (i) (L) **(d)** (III) (iv) (K)

(3) Which vector represents the direction of $\vec{A}\times\vec{B}$?
(a) (II) (iv) (K) **(b)** (I) (ii) (J)
(c) (IV) (i) (M) **(d)** (I) (iii) (L)

Integer Type

36. If $\vec{B}$ is added to $\vec{C} = 3.0\hat{i} + 4.0\hat{j}$, the result is a vector in the positive direction of the y axis, with a magnitude equal to that of $\vec{C}$. What is the magnitude of $\vec{B}$?

37. Vector $\vec{A}$, which is directed along an x axis, is to be added to vector $\vec{B}$, which has a magnitude of 7.0 m. The sum is a third vector that is directed along the y axis, with a magnitude that is 3.0 times that of $\vec{A}$. What is that magnitude of $\vec{A}$?

ANSWER KEY

Checkpoints

1. (a) 7 m ($\vec{a}$ and $\vec{b}$ are in same direction); (b) 1 m ($\vec{a}$ and $\vec{b}$ are in opposite directions)
2. c, d, f (components must be head to tail; must extend from tail of one component to head of the other)
3. (a) +, +; (b) +, –; (c) +, + (draw vector from tail of $\vec{d}_1$ to head of $\vec{d}_2$)

4. (a) 90°; (b) 0° (vectors are parallel—same direction); (c) 180° (vectors are antiparallel—opposite directions)
5. (a) 0° or 180°; (b) 90°

Problems

1. ± 1.73
2. (a) 10 m; (b) 6.0 m
3. ≈23
4. (a) 1.8 m; (b) 69° north of due east
5. (a) parallel; (b) antiparallel; (c) perpendicular
6. (a) $(5.0-2.0\text{ m})\hat{i}+(-4.0+2.0\text{ m})\hat{j}+(2.0+5.0\text{ m})\hat{k}$;
   (b) $(5.0+2.0\text{ m})\hat{i}+(-4.0-2.0\text{ m})\hat{j}+(2.0-5.0\text{ m})\hat{k}$;
   (c) $(-5.0-2.0)\hat{i}+(4.0+2.0)\hat{j}+(2.0+5.0)\hat{k}$
7. (a) 12 m; (b) –5.8 m; (c) –2.8 m
8. (a) $(-9.0\text{ m})\hat{i}+(10\text{ m})\hat{j}$; (b) 13 m; (c) 132°
9. (a) 74 km; (b) 45° north of east
10. (a) 120°; (b) 1.73 m
11. (a) 1.59 m; (b) 12.1 m; (c) 12.2 m; (d) 82.5°
12. (a) $(8.0\text{ m})\hat{i}+(2.0\text{ m})\hat{j}$; (b) 8.2 m; (c) 14°; (d) $(2.0\text{ m})\hat{i}-(6.0\text{ m})\hat{j}$; (e) 6.3 m; (f) –72°
13. (a) 38 m; (b) –37.5°; (c) 130 m; (d) 1.2°; (e) 62 m; (f) 130°
14. (a) 27.6 m; (b) 209° (or –151°)
15. 5.39 m at 21.8 left of forward
16. (a) 5.1 km; (b) 13.3°, south of due west
17. (a) –70.0 cm; (b) 80.0 cm; (c) 141 cm; (d) –172°
18. $(5/\sqrt{2})(\hat{i}-\hat{j})$
19. 1.9 m
20. 51° west of due north
21. (a) $\vec{R}=(-3.18\text{ m})\hat{i}+(4.72\text{ m})\hat{j}$; (b) 5.69 m; (c) –56.0° (with $-x$ axis) or $(5.69\angle 124°)$
22. (a) $\vec{d}_1=4\vec{d}_3=8\hat{i}+16\hat{j}$; (b) $\vec{d}_2=\vec{d}_3=2\hat{i}+4\hat{j}$
23. (a) 7.5 cm; (b) 90°; (c) 8.6 cm; (d) 48°
24. (a) 5.0 m; (b) –37° (clockwise); (c) 10 m; (d) 53°; (e) 11 m; (f) 27°; (g) 11 m; (h) 80°; (i) 11 m; (j) 260°; (k) 180°
25. (a) 12; (b) $+z$ direction; (c) 12; (d) – direction; (e) 12; (f) $+z$ direction
26. (a) $a\hat{i}+a\hat{j}+a\hat{k}$; (b) $-a\hat{i}+a\hat{j}+a\hat{k}$; (c) $a\hat{i}-a\hat{j}+a\hat{k}$; (d) $-a\hat{i}-a\hat{j}+a\hat{k}$; (e) 54.7°; (f) $a\sqrt{3}$
27. (a) $2.0\,\hat{k}$; (b) 26; (c) 46; (d) 5.8
28. (a) $-12.6\,\hat{k}$; (b) –18.1
29. 0
30. (a) –21; (b) –9; (c) $5\hat{i}-11\hat{j}-9\hat{k}$
31. 540
32. 70.5°
33. (a) 2.99 m^2; (b) $(1.53\hat{i}+5.19\hat{j}-2.64\hat{k})$m^2; (c) 63.6°
34. (a) 3.00 m; (b) 0; (c) 3.46 m; (d) 2.00 m; (e) –5.00 m; (f) 8.66 m; (g) –6.67; (h) 4.33

Practice Questions

Single Correct Choice Type

1. (d)	2. (c)	3. (d)	4. (b)	5. (c)
6. (b)	7. (c)	8. (b)	9. (c)	10. (a)
11. (c)	12. (b)	13. (b)	14. (b)	15. (c)
16. (d)	17. (a)	18. (b)	19. (a)	20. (a)
21. (d)	22. (d)	23. (a)	24. (d)	25. (c)

More than One Correct Choice Type

26. (a), (b), (c)
27. (a), (b), (c)

Linked Comprehension

28. (b)	29. (b)	30. (a)	31. (d)	32. (c)
33. (d)	34. (b)			

Matrix-Match

35. (1) → (b); (2) → (c); (3) → (d)

Integer Type

36. 3.2
37. 2.2

CHAPTER

4

Motion in Two and Three Dimensions

4.1 | WHAT IS PHYSICS?

In this chapter we continue looking at the aspect of physics that analyzes motion, but now the motion can be in two or three dimensions. For example, medical researchers and aeronautical engineers might concentrate on the physics of the two- and three-dimensional turns taken by fighter pilots in dogfights because a modern high-performance jet can take a tight turn so quickly that the pilot immediately loses consciousness. A sports engineer might focus on the physics of basketball. For example, in a *free throw* (where a player gets an uncontested shot at the basket from about 4.3 m), a player might employ the *overhand push shot,* in which the ball is pushed away from about shoulder height and then released. Or the player might use an *underhand loop shot,* in which the ball is brought upward from about the belt-line level and released. The first technique is the overwhelming choice among professional players, but the legendary Rick Barry set the record for free-throw shooting with the underhand technique.

Motion in three dimensions is not easy to understand. For example, you are probably good at driving a car along a freeway (one-dimensional motion) but would probably have a difficult time in landing an airplane on a runway (threedimensional motion) without a lot of training.

In our study of two- and three-dimensional motion, we start with position and displacement.

4.2 | POSITION AND DISPLACEMENT

Key Concepts

- The location of a particle relative to the origin of a coordinate system is given by a position vector $\vec{r}$, which in unitvector notation is

$$\vec{r} = x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}}.$$

Here $x\hat{\mathrm{i}}$, $y\hat{\mathrm{j}}$, and $z\hat{\mathrm{k}}$ are the vector components of position vector $\vec{r}$, and x, y, and z are its scalar components (as well as the coordinates of the particle).

Contents

4.1 What is Physics?
4.2 Position and Displacement
4.3 Average Velocity and Instantaneous Velocity
4.4 Average Acceleration and Instantaneous Acceleration
4.5 Projectile Motion
4.6 Relative Motion in One Dimension
4.7 Relative Motion in Two Dimensions

- A position vector is described either by a magnitude and one or two angles for orientation, or by its vector or scalar components.
- If a particle moves so that its position vector changes from $\vec{r}_1$ to $\vec{r}_2$, the particle's displacement $\Delta\vec{r}$ is

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1.$$

The displacement can also be written as

$$\Delta\vec{r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$
$$= \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}.$$

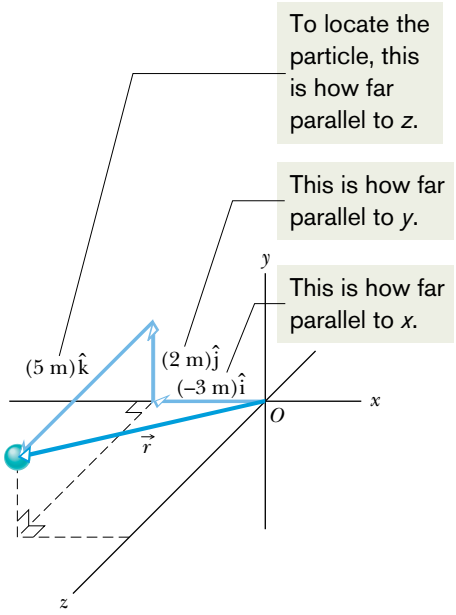

Figure 4-1 The position vector $\vec{r}$ for a particle is the vector sum of its vector components.

One general way of locating a particle (or particle-like object) is with a **position vector** $\vec{r}$, which is a vector that extends from a reference point (usually the origin) to the particle. In the unit-vector notation of Section 3.2, $\vec{r}$ can be written

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, \tag{4-1}$$

where $x\hat{i}$, $y\hat{j}$, and $z\hat{k}$ are the vector components of $\vec{r}$ and the coefficients x, y, and z are its scalar components.

The coefficients x, y, and z give the particle's location along the coordinate axes and relative to the origin; that is, the particle has the rectangular coordinates (x, y, z). For instance, Fig. 4-1 shows a particle with position vector

$$\vec{r} = (-3\text{ m})\hat{i} + (2\text{ m})\hat{j} + (5\text{ m})\hat{k}$$

and rectangular coordinates (–3 m, 2 m, 5 m). Along the x axis the particle is 3 m from the origin, in the $-\hat{i}$ direction. Along the y axis it is 2 m from the origin, in the $+\hat{j}$ direction. Along the z axis it is 5 m from the origin, in the $+\hat{k}$ direction.

As a particle moves, its position vector changes in such a way that the vector always extends to the particle from the reference point (the origin). If the position vector changes—say, from $\vec{r}_1$ to $\vec{r}_2$ during a certain time interval—then the particle's **displacement** $\Delta\vec{r}$ during that time interval is

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1. \tag{4-2}$$

Using the unit-vector notation of Eq. 4-1, we can rewrite this displacement as

$$\Delta\vec{r} = (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$

or as

$$\Delta\vec{r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}, \tag{4-3}$$

where coordinates (x_1, y_1, z_1) correspond to position vector $\vec{r}_1$ and coordinates (x_2, y_2, z_2) correspond to position vector $\vec{r}_2$. We can also rewrite the displacement by substituting Δx for $(x_2 - x_1)$, Δy for $(y_2 - y_1)$, and Δz for $(z_2 - z_1)$:

$$\Delta\vec{r} = \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}. \tag{4-4}$$

SAMPLE PROBLEM 4.01

Two-dimensional position vector, rabbit run

A rabbit runs across a parking lot on which a set of coordinate axes has, strangely enough, been drawn. The coordinates (meters) of the rabbit's position as functions of time t (seconds) are given by

$$x = -0.31t^2 + 7.2t + 28 \tag{4-5}$$

and

$$y = 0.22t^2 - 9.1t + 30. \tag{4-6}$$

(a) At $t = 15$ s, what is the rabbit's position vector $\vec{r}$ in unitvector notation and in magnitude-angle notation?

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KEY IDEA

The x and y coordinates of the rabbit's position, as given by Eqs. 4-5 and 4-6, are the scalar components of the rabbit's position vector $\vec{r}$. Let's evaluate those coordinates at the given time, and then we can use Eq. 3-6 to evaluate the magnitude and orientation of the position vector.

Calculations: We can write

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}. \tag{4-7}$$

(We write $\vec{r}(t)$ rather than $\vec{r}$ because the components are functions of t, and thus $\vec{r}$ is also.)

At $t = 15$ s, the scalar components are

$$x = (-0.31)(15)^2 + (7.2)(15) + 28 = 66 \text{ m}$$

and $$y = (0.22)(15)^2 - (9.1)(15) + 30 = -57 \text{ m},$$

so $$\vec{r} = (66 \text{ m})\hat{i} - (57 \text{ m})\hat{j}, \tag{Answer}$$

which is drawn in Fig. 4-2*a*. To get the magnitude and angle of $\vec{r}$, notice that the components form the legs of a right triangle and r is the hypotenuse. So, we use Eq. 3-6:

$$r = \sqrt{x^2 + y^2} = \sqrt{(66 \text{ m})^2 + (-57 \text{ m})^2}$$
$$= 87 \text{ m}, \tag{Answer}$$

and $$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{-57 \text{ m}}{66 \text{ m}}\right) = -41°. \tag{Answer}$$

Check: Although $\theta = 139°$ has the same tangent as $-41°$, the components of position vector $\vec{r}$ indicate that the desired angle is $139° - 180° = -41°$.

(b) Graph the rabbit's path for $t = 0$ to $t = 25$ s.

Graphing: We have located the rabbit at one instant, but to see its path we need a graph. So we repeat part (a) for several values of t and then plot the results. Figure 4-2*b* shows the plots for six values of t and the path connecting them.

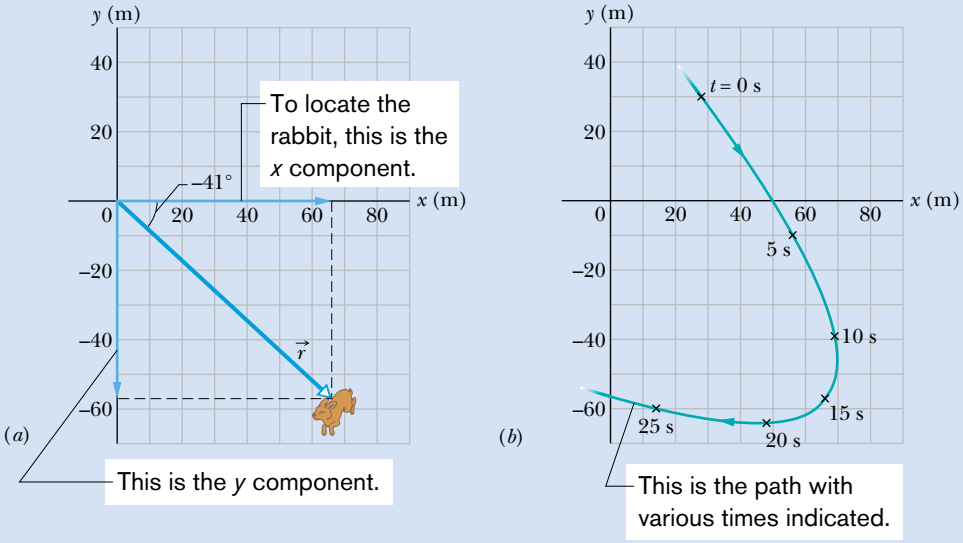

Figure 4-2 (*a*) A rabbit's position vector $\vec{r}$ at time $t = 15$ s. The scalar components of $\vec{r}$ are shown along the axes. (*b*) The rabbit's path and its position at six values of t.

4.3 | AVERAGE VELOCITY AND INSTANTANEOUS VELOCITY

Key Concepts

- If a particle undergoes a displacement $\Delta\vec{r}$ in time interval Δt, its average velocity $\vec{v}_{avg}$ for that time interval is

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t}.$$

- As Δt is shrunk to 0, $\vec{v}_{avg}$ reaches a limit called either the velocity or the instantaneous velocity $\vec{v}$:

$$\vec{v} = \frac{d\vec{r}}{dt},$$

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which can be rewritten in unit-vector notation as

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k},$$

where $v_x = dx/dt$, $v_y = dy/dt$, and $v_z = dz/dt$.

- The instantaneous velocity $\vec{v}$ of a particle is always directed along the tangent to the particle's path at the particle's position.

If a particle moves from one point to another, we might need to know how fast it moves. Just as in Chapter 2, we can define two quantities that deal with "how fast": *average velocity* and *instantaneous velocity*. However, here we must consider these quantities as vectors and use vector notation.

If a particle moves through a displacement $\Delta\vec{r}$ in a time interval Δt, then its **average velocity** $\vec{v}_{avg}$ is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time interval}},$$

or

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t}. \tag{4-8}$$

This tells us that the direction of $\vec{v}_{avg}$ (the vector on the left side of Eq. 4-8) must be the same as that of the displacement $\Delta\vec{r}$ (the vector on the right side). Using Eq. 4-4, we can write Eq. 4-8 in vector components as

$$\vec{v}_{avg} = \frac{\Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j} + \frac{\Delta z}{\Delta t}\hat{k}. \tag{4-9}$$

For example, if a particle moves through displacement $(12\text{ m})\hat{i} + (3.0\text{ m})\hat{k}$ in 2.0 s, then its average velocity during that move is

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{(12\text{ m})\hat{i} + (3.0\text{ m})\hat{k}}{2.0\text{ s}} = (6.0\text{ m/s})\hat{i} + (1.5\text{ m/s})\hat{k}.$$

That is, the average velocity (a vector quantity) has a component of 6.0 m/s along the x axis and a component of 1.5 m/s along the z axis.

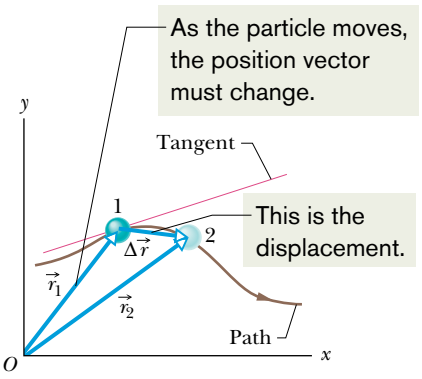

Figure 4-3 The displacement $\Delta\vec{r}$ of a particle during a time interval Δt, from position 1 with position vector $\vec{r}_1$ at time t_1 to position 2 with position vector $\vec{r}_2$ at time t_2. The tangent to the particle's path at position 1 is shown.

When we speak of the **velocity** of a particle, we usually mean the particle's **instantaneous velocity** $\vec{v}$ at some instant. This $\vec{v}$ is the value that $\vec{v}_{avg}$ approaches in the limit as we shrink the time interval Δt to 0 about that instant. Using the language of calculus, we may write $\vec{v}$ as the derivative

$$\vec{v} = \frac{d\vec{r}}{dt}. \tag{4-10}$$

Figure 4-3 shows the path of a particle that is restricted to the xy plane. As the particle travels to the right along the curve, its position vector sweeps to the right. During time interval Δt, the position vector changes from $\vec{r}_1$ to $\vec{r}_2$ and the particle's displacement is $\Delta\vec{r}$.

To find the instantaneous velocity of the particle at, say, instant t_1 (when the particle is at position 1), we shrink interval Δt to 0 about t_1. Three things happen as we do so. (1) Position vector $\vec{r}_2$ in Fig. 4-3 moves toward $\vec{r}_1$ so that $\Delta\vec{r}$ shrinks toward zero. (2) The direction of $\Delta\vec{r}/\Delta t$ (and thus of $\vec{v}_{avg}$) approaches the direction of the line tangent to the particle's path at position 1. (3) The average velocity $\vec{v}_{avg}$ approaches the instantaneous velocity $\vec{v}$ at t_1.

In the limit as $\Delta t \to 0$, we have $\vec{v}_{avg} \to \vec{v}$ and, most important here, $\vec{v}_{avg}$ takes on the direction of the tangent line. Thus, $\vec{v}$ has that direction as well:

The direction of the instantaneous velocity $\vec{v}$ of a particle is always tangent to the particle's path at the particle's position.

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The result is the same in three dimensions: $\vec{v}$ is always tangent to the particle's path.

To write Eq. 4-10 in unit-vector form, we substitute for $\vec{r}$ from Eq. 4-1:

$$\vec{v} = \frac{d}{dt}(x\hat{i} + y\hat{j} + z\hat{k}) = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}.$$

This equation can be simplified somewhat by writing it as

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}, \tag{4-11}$$

where the scalar components of $\vec{v}$ are

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad \text{and} \quad v_z = \frac{dz}{dt}. \tag{4-12}$$

For example, dx/dt is the scalar component of $\vec{v}$ along the x axis. Thus, we can find the scalar components of $\vec{v}$ by differentiating the scalar components of $\vec{r}$.

Figure 4-4 shows a velocity vector $\vec{v}$ and its scalar x and y components. Note that $\vec{v}$ is tangent to the particle's path at the particle's position. *Caution:* When a position vector is drawn, as in Figs. 4-1 through 4-3, it is an arrow that extends from one point (a "here") to another point (a "there"). However, when a velocity vector is drawn, as in Fig. 4-4, it does *not* extend from one point to another. Rather, it shows the instantaneous direction of travel of a particle at the tail, and its length (representing the velocity magnitude) can be drawn to any scale.

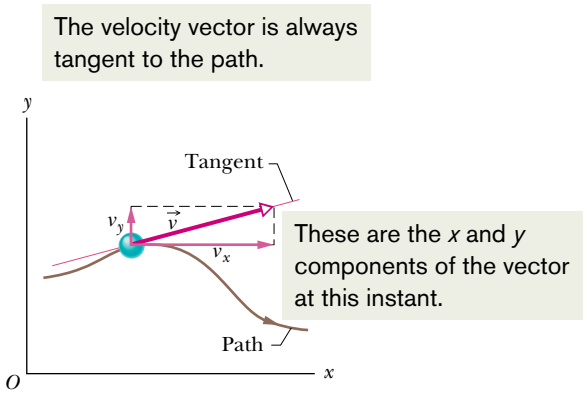
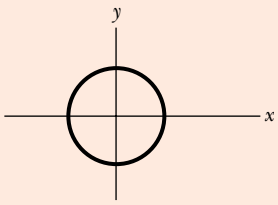


Figure 4-4 The velocity $\vec{v}$ of a particle, along with the scalar components of $\vec{v}$.

CHECKPOINT 1

The figure shows a circular path taken by a particle. If the instantaneous velocity of the particle is $\vec{v} = (2\text{ m/s})\hat{i} - (2\text{ m/s})\hat{j}$, through which quadrant is the particle moving at that instant if it is traveling (a) clockwise and (b) counterclockwise around the circle? For both cases, draw $\vec{v}$ on the figure.



SAMPLE PROBLEM 4.02

Two-dimensional velocity, rabbit run

For the rabbit in the preceding sample problem, find the velocity $\vec{v}$ at time $t = 15$ s.

KEY IDEA

We can find $\vec{v}$ by taking derivatives of the components of the rabbit's position vector.

Calculations: Applying the v_x part of Eq. 4-12 to Eq. 4-5, we find the x component of $\vec{v}$ to be

$$v_x = \frac{dx}{dt} = \frac{d}{dt}(-0.31t^2 + 7.2t + 28)$$
$$= -0.62t + 7.2. \tag{4-13}$$

At $t = 15$ s, this gives $v_x = -2.1$ m/s. Similarly, applying the v_y part of Eq. 4-12 to Eq. 4-6, we find

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$$v_y = \frac{dy}{dt} = \frac{d}{dt}(0.22t^2 - 9.1t + 30)$$
$$= 0.44t - 9.1. \qquad (4\text{-}14)$$

At $t = 15$ s, this gives $v_y = -2.5$ m/s. Equation 4-11 then yields

$$\vec{v} = (-2.1 \text{ m/s})\hat{i} + (-2.5 \text{ m/s})\hat{j}, \qquad \text{(Answer)}$$

which is shown in Fig. 4-5, tangent to the rabbit's path and in the direction the rabbit is running at $t = 15$ s.

To get the magnitude and angle of $\vec{v}$ either we use Eq. 3-6 to write

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(-2.1 \text{ m/s})^2 + (-2.5 \text{ m/s})^2}$$
$$= 3.3 \text{ m/s} \qquad \text{(Answer)}$$

and
$$\theta = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\left(\frac{-2.5 \text{ m/s}}{-2.1 \text{ m/s}}\right)$$
$$= \tan^{-1} 1.19$$
$$= -130°. \qquad \text{(Answer)}$$

Check: Is the angle −130° or −130° + 180° = 50°?

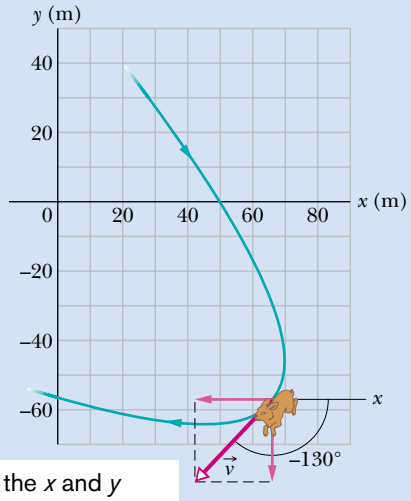

Figure 4-5 The rabbit's velocity $\vec{v}$ at $t = 15$ s.

4.4 | AVERAGE ACCELERATION AND INSTANTANEOUS ACCELERATION

Key Concepts

- If a particle's velocity changes from $\vec{v}_1$ to $\vec{v}_2$ in time interval Δt, its average acceleration during Δt is
$$\vec{a}_{avg} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\Delta \vec{v}}{\Delta t}.$$
- As Δt is shrunk to 0, $\vec{a}_{avg}$ reaches a limiting value called either the acceleration or the instantaneous acceleration $\vec{a}$:
$$\vec{a} = \frac{d\vec{v}}{dt}.$$
- In unit-vector notation,
$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k},$$
where $a_x = dv_x/dt$, $a_y = dv_y/dt$, and $a_z = dv_z/dt$.

When a particle's velocity changes from $\vec{v}_1$ to $\vec{v}_2$ in a time interval Δt, its **average acceleration** $\vec{a}_{avg}$ during Δt is

$$\text{average acceleration} = \frac{\text{change in velocity}}{\text{time interval}},$$

or
$$\vec{a}_{avg} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\Delta \vec{v}}{\Delta t}. \qquad (4\text{-}15)$$

If we shrink Δt to zero about some instant, then in the limit $\vec{a}_{avg}$ approaches the **instantaneous acceleration** (or **acceleration**) $\vec{a}$ at that instant; that is,

$$\vec{a} = \frac{d\vec{v}}{dt}. \qquad (4\text{-}16)$$

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If the velocity changes in *either* magnitude *or* direction (or both), the particle must have an acceleration.

We can write Eq. 4-16 in unit-vector form by substituting Eq. 4-11 for $\vec{v}$ to obtain

$$\vec{a} = \frac{d}{dt}(v_x\hat{i} + v_y\hat{j} + v_z\hat{k})$$
$$= \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k}.$$

We can rewrite this as

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}, \tag{4-17}$$

where the scalar components of $\vec{a}$ are

$$a_x = \frac{dv_x}{dt}, \quad a_y \frac{dv_y}{dt}, \quad \text{and} \quad a_z = \frac{dv_z}{dt}. \tag{4-18}$$

To find the scalar components of $\vec{a}$, we differentiate the scalar components of $\vec{v}$.

Figure 4-6 shows an acceleration vector $\vec{a}$ and its scalar components for a particle moving in two dimensions. *Caution:* When an acceleration vector is drawn, as in Fig. 4-6, it does *not* extend from one position to another. Rather, it shows the direction of acceleration for a particle located at its tail, and its length (representing the acceleration magnitude) can be drawn to any scale.

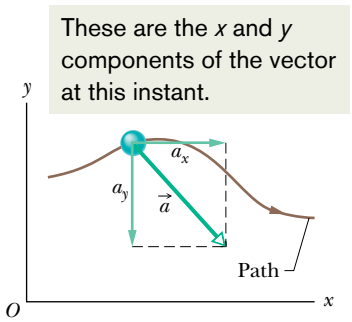

Figure 4-6 The acceleration $\vec{a}$ of a particle and the scalar components of $\vec{a}$.

CHECKPOINT 2

Here are four descriptions of the position (in meters) of a puck as it moves in an *xy* plane:

(1) $x = -3t^2 + 4t - 2$ and $y = 6t^2 - 4t$ (3) $\vec{r} = 2t^2\hat{i} - (4t + 3)\hat{j}$

(2) $x = -3t^3 - 4t$ and $y = -5t^2 + 6$ (4) $\vec{r} = (4t^3 - 2t)\hat{i} + \hat{j}$

Are the *x* and *y* acceleration components constant? Is acceleration $\vec{a}$ constant?

SAMPLE PROBLEM 4.03

Two-dimensional acceleration, rabbit run

For the rabbit in the preceding two sample problems, find the acceleration $\vec{a}$ at time $t = 15$ s.

KEY IDEA

We can find $\vec{a}$ by taking derivatives of the rabbit's velocity components.

Calculations: Applying the a_x part of Eq. 4-18 to Eq. 4-13, we find the *x* component of $\vec{a}$ to be

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}(-0.62t + 7.2) = -0.62 \text{ m/s}^2.$$

Similarly, applying the a_y part of Eq. 4-18 to Eq. 4-14 yields the *y* component as

$$a_y = \frac{dv_y}{dt} = \frac{d}{dt}(0.44t - 9.1) = 0.44 \text{ m/s}^2.$$

We see that the acceleration does not vary with time (it is a constant) because the time variable *t* does not appear in the expression for either acceleration component. Equation 4-17 then yields

$$\vec{a} = (-0.62 \text{ m/s}^2)\hat{i} + (0.44 \text{ m/s}^2)\hat{j}, \qquad \text{(Answer)}$$

which is superimposed on the rabbit's path in Fig. 4-7.

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To get the magnitude and angle of $\vec{a}$, either we use Eq. 3-6. For the magnitude we have

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(-0.62\ \text{m/s}^2)^2 + (0.44\ \text{m/s}^2)^2}$$
$$= 0.76\ \text{m/s}^2. \qquad \text{(Answer)}$$

For the angle we have

$$\theta = \tan^{-1}\frac{a_y}{a_x} = \tan^{-1}\left(\frac{0.44\ \text{m/s}^2}{-0.62\ \text{m/s}^2}\right) = -35°.$$

We know from the components that $\vec{a}$ must be directed to the left and upward. To find the other angle that has the same tangent as −35°, we add 180°:

$$-35° + 180° = 145°. \qquad \text{(Answer)}$$

Learn: This *is* consistent with the components of $\vec{a}$ because it gives a vector that is to the left and upward. Note that $\vec{a}$ has the same magnitude and direction throughout the rabbit's run because the acceleration is constant. That means that we could draw the very same vector at any other point along the rabbit's path (just shift the vector to put its tail at some other point on the path without changing the length or orientation).

Caution: This has been the second sample problem in which we needed to take the derivative of a vector that is written in unit-vector notation. One common error is to neglect the unit vectors themselves, with a result of only a set of numbers and symbols. Keep in mind that a derivative of a vector is always another vector.

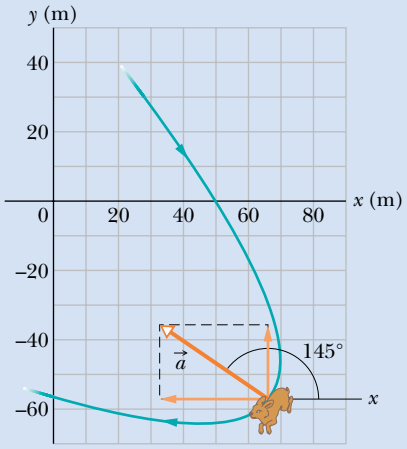

Figure 4-7 The acceleration $\vec{a}$ of the rabbit at $t = 15$ s. The rabbit happens to have this same acceleration at all points on its path.

SAMPLE PROBLEM 4.04

Three-dimensional acceleration motion of a proton

A proton initially has $\vec{v} = 4.0\hat{i} - 2.0\hat{j} + 3.0\hat{k}$ and then 4.0 s later has $\vec{v} = -2.0\hat{i} - 2.0\hat{j} + 5.0\hat{k}$ (in meters per second). For that 4.0 s, what are (a) the proton's average acceleration $\vec{a}_{avg}$ in unit vector notation, (b) the magnitude of $\vec{a}_{avg}$, and (c) the angle between $\vec{a}_{avg}$ and the positive direction of the x axis?

KEY IDEA

We use Eq. 4-15 with $\vec{v}_1$ designating the initial velocity and $\vec{v}_2$ designating the later one.

Calculations: (a) The average acceleration during the $\Delta t = 4$ s interval is

$$\vec{a}_{avg} = \frac{(-2.0\hat{i} - 2.0\hat{j} + 5.0\hat{k})\ \text{m/s} - (4.0\hat{i} - 22\hat{j} + 3.0\hat{k})\ \text{m/s}}{4\ \text{s}}$$
$$= (-1.5\ \text{m/s}^2)\hat{i} + (0.5\ \text{m/s}^2)\hat{k}.$$

(b) The magnitude of $\vec{a}_{avg}$ is

$$\sqrt{(-1.5\ \text{m/s}^2)^2 + (0.5\ \text{m/s}^2)^2} = 1.6\ \text{m/s}^2.$$

(c) Its angle in the xz plane (measured from the $+x$ axis) is one of these possibilities:

$$\tan^{-1}\left(\frac{0.5\ \text{m/s}^2}{-1.5\ \text{m/s}^2}\right) = -18° \text{ or } 162°$$

where we settle on the second choice since the signs of its components imply that it is in the second quadrant.

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4.5 | PROJECTILE MOTION

Key Concepts

- In projectile motion, a particle is launched into the air with a speed v_0 and at an angle θ_0 (as measured from a horizontal x axis). During flight, its horizontal acceleration is zero and its vertical acceleration is $-g$ (downward on a vertical y axis).
- The equations of motion for the particle (while in flight) can be written as

$$x - x_0 = (v_0 \cos\theta_0)t,$$
$$y - y_0 = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2,$$
$$v_y = v_0 \sin\theta_0 - gt,$$
$$v_y^2 = (v_0 \sin\theta_0)^2 - 2g(y - y_0).$$

- The trajectory (path) of a particle in projectile motion is parabolic and is given by

$$y = (\tan\theta_0)x - \frac{gx^2}{2(v_0 \cos\theta_0)^2},$$

if x_0 and y_0 are zero.

- The particle's horizontal range R, which is the horizontal distance from the launch point to the point at which the particle returns to the launch height, is

$$R = \frac{v_0^2}{g}\sin 2\theta_0.$$

We next consider a special case of two-dimensional motion: A particle moves in a vertical plane with some initial velocity $\vec{v}_0$ but its acceleration is always the freefall acceleration $\vec{g}$, which is downward. Such a particle is called a **projectile** (meaning that it is projected or launched), and its motion is called **projectile motion**. A projectile might be a tennis ball (Fig. 4-8) or baseball in flight, but it is not a duck in flight. Many sports involve the study of the projectile motion of a ball. For example, the racquetball player who discovered the Z-shot in the 1970s easily won his games because of the ball's perplexing flight to the rear of the court.

Our goal here is to analyze projectile motion using the tools for two-dimensional motion described in Modules 4-1 through 4-3 and making the assumption that air has no effect on the projectile. Figure 4-9, which we shall analyze soon, shows the path followed by a projectile when the air has no effect. The projectile is launched with an initial velocity $\vec{v}_0$ that can be written as

$$\vec{v}_0 = v_{0x}\hat{i} + v_{0y}\hat{j}. \tag{4-19}$$

The components v_{0x} and v_{0y} can then be found if we know the angle θ_0 between $\vec{v}_0$ and the positive x direction:

$$v_{0x} = v_0 \cos\theta_0 \quad \text{and} \quad v_{0y} = v_0 \sin\theta_0. \tag{4-20}$$

During its two-dimensional motion, the projectile's position vector $\vec{r}$ and velocity vector $\vec{v}$ change continuously, but its acceleration vector $\vec{a}$ is constant and *always* directed vertically downward. The projectile has *no* horizontal acceleration.

Projectile motion, like that in Figs. 4-8 and 4-9, looks complicated, but we have the following simplifying feature (known from experiment):

Richard Megna/Fundamental Photographs

Figure 4-8 A stroboscopic photograph of a yellow tennis ball bouncing off a hard surface. Between impacts, the ball has projectile motion.

In projectile motion, the horizontal motion and the vertical motion are independent of each other; that is, neither motion affects the other.

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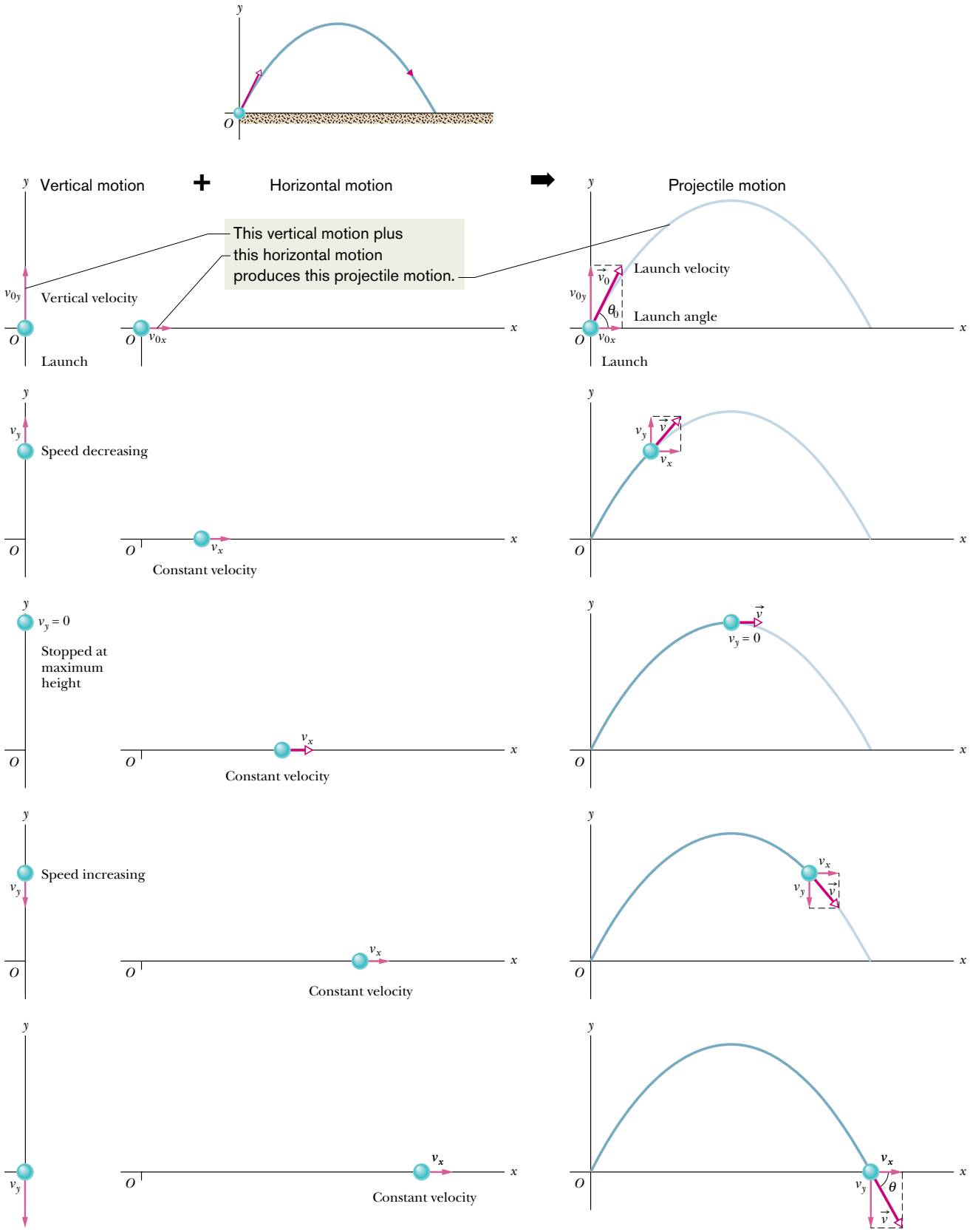

Figure 4-9 The *projectile motion* of an object launched into the air at the origin of a coordinate system and with launch velocity $\vec{v}_0$ at angle θ_0. The motion is a combination of vertical motion (constant acceleration) and horizontal motion (constant velocity), as shown by the velocity components.

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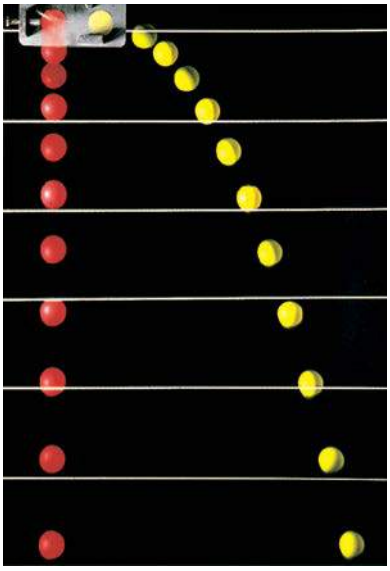
This feature allows us to break up a problem involving two-dimensional motion into two separate and easier one-dimensional problems, one for the horizontal motion (with *zero acceleration*) and one for the vertical motion (with *constant downward acceleration*). Here are two experiments that show that the horizontal motion and the vertical motion are independent.

Two Golf Balls

Figure 4-10 is a stroboscopic photograph of two golf balls, one simply released and the other shot horizontally by a spring. The golf balls have the same vertical motion, both falling through the same vertical distance in the same interval of time. *The fact that one ball is moving horizontally while it is falling has no effect on its vertical motion;* that is, the horizontal and vertical motions are independent of each other.



Richard Megna/Fundamental Photographs

Figure 4-10 One ball is released from rest at the same instant that another ball is shot horizontally to the right. Their vertical motions are identical.

A Great Student Rouser

In Fig. 4-11, a blowgun G using a ball as a projectile is aimed directly at a can suspended from a magnet M. Just as the ball leaves the blowgun, the can is released. If g (the magnitude of the free-fall acceleration) were zero, the ball would follow the straight-line path shown in Fig. 4-11 and the can would float in place after the magnet released it. The ball would certainly hit the can. However, g is *not* zero, but the ball *still* hits the can! As Fig. 4-11 shows, during the time of flight of the ball, both ball and can fall the same distance h from their zero-g locations. The harder the demonstrator blows, the greater is the ball's initial speed, the shorter the flight time, and the smaller the value of h.

The ball and the can fall the same distance h.

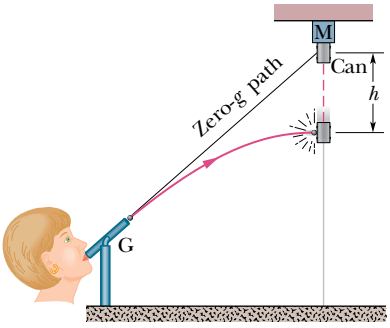


Figure 4-11 The projectile ball always hits the falling can. Each falls a distance h from where it would be were there no free-fall acceleration.

CHECKPOINT 3

At a certain instant, a fly ball has velocity $\vec{v} = 25\hat{i} - 4.9\hat{j}$ (the x axis is horizontal, the y axis is upward, and $\vec{v}$ is in meters per second). Has the ball passed its highest point?

The Horizontal Motion

Now we are ready to analyze projectile motion, horizontally and vertically. We start with the horizontal motion. Because there is *no acceleration* in the horizontal direction, the horizontal component v_x of the projectile's velocity remains unchanged from its initial value v_{0x} throughout the motion, as demonstrated in Fig. 4-12. At any time t, the projectile's horizontal displacement $x - x_0$ from an initial position x_0 is given by Eq. 2-15 with $a = 0$, which we write as

$$x - x_0 = v_{0x}t.$$

Because $v_{0x} = v_0 \cos \theta_0$, this becomes

$$x - x_0 = (v_0 \cos \theta_0)t. \tag{4-21}$$

The Vertical Motion

The vertical motion is the motion we discussed in Section 2.8 for a particle in free fall. Most important is that the acceleration is constant. Thus, the equations of Table 2-1 apply, provided we substitute $-g$ for a and switch to y notation. Then, for example, Eq. 2-15 becomes

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Jamie Budge

Figure 4-12 The vertical component of this skateboarder's velocity is changing but not the horizontal component, which matches the skateboard's velocity. As a result, the skateboard stays underneath him, allowing him to land on it.

$$y - y_0 = v_{0y}t - \frac{1}{2}gt^2$$
$$= (v_0 \sin\theta_0)t - \frac{1}{2}gt^2, \tag{4-22}$$

where the initial vertical velocity component v_{0y} is replaced with the equivalent $v_0 \sin\theta_0$. Similarly, Eqs. 2-11 and 2-16 become

$$v_y = v_0 \sin\theta_0 - gt \tag{4-23}$$

and

$$v_y^2 = (v_0 \sin\theta_0)^2 - 2g(y - y_0). \tag{4-24}$$

As is illustrated in Fig. 4-9 and Eq. 4-23, the vertical velocity component behaves just as for a ball thrown vertically upward. It is directed upward initially, and its magnitude steadily decreases to zero, *which marks the maximum height of the path.* The vertical velocity component then reverses direction, and its magnitude becomes larger with time.

The Equation of the Path

We can find the equation of the projectile's path (its **trajectory**) by eliminating time t between Eqs. 4-21 and 4-22. Solving Eq. 4-21 for t and substituting into Eq. 4-22, we obtain, after a little rearrangement,

$$y = (\tan\theta_0)x - \frac{gx^2}{2(v_0\cos\theta_0)^2} \quad \text{(trajectory)}. \tag{4-25}$$

This is the equation of the path shown in Fig. 4-9. In deriving it, for simplicity we let $x_0 = 0$ and $y_0 = 0$ in Eqs. 4-21 and 4-22, respectively. Because g, θ_0, and v_0 are constants, Eq. 4-25 is of the form $y = ax + bx^2$, in which a and b are constants. This is the equation of a parabola, so the path is *parabolic*.

Equation 4-25 can be rearranged as a quadratic equation in x

$$\left(\frac{g}{2v_0^2\cos^2\theta_0}\right)x^2 - (\tan\theta)x + y = 0$$

Thus, the equation will have two solutions (values of x) for any given value of y. This implies that the projectile can pass through the same height for two different x-coordinates. Further, this equation can be expressed as a quadratic in terms of $\tan\theta$ as follows

$$\left(\frac{g\sec^2\theta_0}{2v_0^2}\right)x^2 - (\tan\theta)x + y = 0$$
$$\left(\frac{gx^2}{2v_0^2}\right)(1+\tan^2\theta_0) - (\tan\theta)x + y = 0$$
$$\left(\frac{gx^2}{2v_0^2}\right)\tan^2\theta_0 - (\tan\theta)x + \left(y + \frac{gx^2}{2v_0^2}\right) = 0$$

This implies that for every value of coordinates x and y, two values of θ are possible or, the target can be hit using two angles of projection.

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The Horizontal Range

The *horizontal range* R of the projectile is the *horizontal* distance the projectile has traveled when it returns to its initial height (the height at which it is launched). To find range R, let us put $x - x_0 = R$ in Eq. 4-21 and $y - y_0 = 0$ in Eq. 4-22, obtaining

$$R = (v_0 \cos \theta_0)t$$

and

$$0 = (v_0 \sin \theta_0)t - \frac{1}{2}gt^2.$$

Eliminating t between these two equations yields

$$R = \frac{2v_0^2}{g} \sin \theta_0 \cos \theta_0.$$

Using the identity $\sin 2\theta_0 = 2 \sin \theta_0 \cos \theta_0$ (see Appendix E), we obtain

$$R = \frac{v_0^2}{g} \sin 2\theta_0. \tag{4-26}$$

This equation does *not* give the horizontal distance traveled by a projectile when the final height is not the launch height. Note that R in Eq. 4-26 has its maximum value when $\sin 2\theta_0 = 1$, which corresponds to $2\theta_0 = 90°$ or $\theta_0 = 45°$.

The horizontal range R is maximum for a launch angle of 45°.

However, when the launch and landing heights differ, as in many sports, a launch angle of 45° does not yield the maximum horizontal distance.

The equation of trajectory in the standard form (Eq. 4-25) can be expressed in terms of horizontal range as

$$y = x \tan \theta - \left[1 - \frac{x}{R}\right] \tag{4-27}$$

Two Angles of Projection for the Same Range

From trigonometry, we know that $\sin 2\theta = \sin (180° - 2\theta)$. Substituting this in Eq. (4-26), we get

$$R = \frac{v^2 \sin(180° - 2\theta)}{g}$$
$$= \frac{v^2 \sin 2(90° - \theta)}{g}.$$

This shows that there are two angles of projection for the same horizontal range, that is, θ and $(90° - \theta)$. The projectile will cover the same horizontal range whether it is thrown at an angle $(90° - \theta)$ with the horizontal, or an angle θ with the vertical. The above concept can be understood with help of the following example. If you kick a football at two different inclinations 30° and 60°, then the horizontal distance covered by the football will be same in both cases; however, in case of 60°, inclination the football will attain greater vertical height than in case of 30° inclination. Figure 4-13 shows that for $\theta = 20°$ and $\theta = 90° - 20° = 70°$, $\theta = 30^0$ and $\theta = 90° - 30° = 60°$, the horizontal distance covered by the projectile is the same. In both the

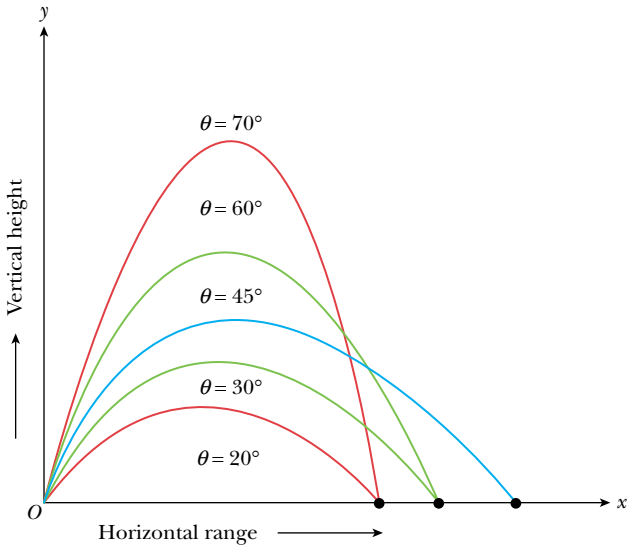

Figure 4-13 Horizontal range for two different angles θ and $(90° - \theta)$.

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cases the vertical height of the projectile if different for different inclination. It can be observed that the maximum horizontal range covered by the projectile can be obtained for an inclination of $\theta = 45°$.

Maximum Height

The maximum vertical height attained by the projectile above the point of projection during its flight is known as **maximum height** of the projectile.

We denote maximum height, also called the vertical range, by h_{max} or H as shown in Fig. 4-14. It is the maximum height to which a projectile can rise above the horizontal plane of projection. To calculate H, we make use of the fact that the velocity $v_y(t)$ of the projectile at the maximum height is zero. If t is the time taken by the projectile to reach maximum height then from the equation

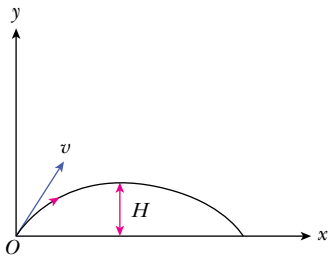

Figure 4-14 Maximum height for the projectile.

$$v_y = v \sin\theta - gt,$$

we have

$$0 = v\sin\theta - gt_1 \quad \text{or} \quad t_1 = \frac{v\sin\theta}{g}.$$

At time t_1, $y(t) = H$, the maximum height

$$H = (v\sin\theta)t_1 - \frac{1}{2}gt_1^2.$$

Substituting for t_1,

$$H = v\sin\theta\left(\frac{v\sin\theta}{g}\right) - \frac{1}{2}g\frac{v^2\sin^2\theta}{g^2}$$

$$= \frac{v^2\sin^2\theta}{g} - \frac{1}{2}\frac{v^2\sin^2\theta}{g}$$

$$H = \frac{v^2\sin^2\theta}{2g}. \tag{4-28}$$

Time of Flight

Time of flight is the total time taken by the projectile to return to the same level from where it was thrown. Since the time of ascent is equal to the time of descent, the time of flight is equal to twice the time taken by the projectile to reach the maximum height. Therefore, time of flight, $T = 2t$ where t is the time of ascent or time taken by the projectile to reach the maximum height. Once again,

$$v_y(t) = 0, a_y = -g, v_y(0) = v\sin\theta.$$

Substituting these values in the equation, $v_y(t) = v_y(0) + a_y t$, we have,

$$0 = v\sin\theta - gt.$$

Therefore,

$$t = \frac{v\sin\theta}{g},$$

and

$$T = 2t = \frac{2v\sin\theta}{g}. \tag{4-29}$$

Projectile Motion on an Inclined Plane

In case of projectile motion on an inclined plane, the points on projection and return are not on the same plane. There are two possibilities in projectile motion on an inclined plane, that is: (i) projectile thrown up the incline,

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so that point of return is higher than the point of projection (Fig. 4-15*a*) and (ii) projectile thrown down the incline where the point of return is lower than the point of projection (Fig. 4-15*b*).

Consider the projectile on an inclined plane making an angle θ with the horizontal as shown in Fig. 4-15*a*. Let the particle be projected at an angle α from the horizontal with an initial velocity v_0 and $\vec{v}$ be its resultant velocity at time t. The motion of the particle can be analyzed in terms of two mutually perpendicular horizontal and vertical directions, wherein coordinate along the incline is (x) and perpendicular to incline is (y). The angle that the velocity of projection makes with the x-axis is $(\theta - \alpha)$. Therefore, the components of the initial velocity and acceleration are:

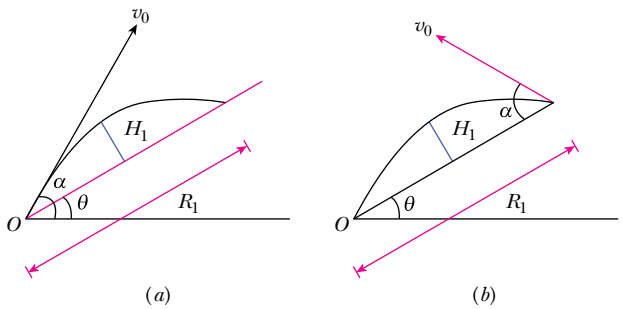

Figure 4-15 Projection of an inclined plane. (*a*) Projectile thrown up the incline. (*b*) Projectile thrown down the incline.

$$\text{Along x-axis: } v_{0x} = v_0 \cos(\alpha - \theta);\ a_x = -g \sin\theta$$

$$\text{Along y-axis: } v_{0y} = v_0 \sin(\alpha - \theta);\ a_y = -g \cos\theta$$

Velocity at any time t is given by

$$\text{Along x-axis: } v_x = v_0 \cos(\alpha - \theta) - g \sin\theta t$$

$$\text{Along y-axis: } v_y = v_0 \sin(\alpha - \theta) - g \cos\theta t$$

Magnitude of the resultant velocity: $v = \sqrt{v^2 + g^2 t^2 - 2vgt \sin\theta}$.

Direction of the velocity: $\tan\alpha = \dfrac{v_y}{v_x} = \dfrac{v\sin\theta - gt}{v\cos\theta} \Rightarrow \alpha = \tan^{-1}\left(\tan\theta - \dfrac{gt}{v\cos\theta}\right)$.

The other parameters can be expressed as follows:

1. **Time of flight:** It is the time in which the projected particle strikes the inclined plane. The total distance covered in y-direction is $y - y_0 = 0$. Therefore, from the first equation of motion, $x - x_0 = v_0 t + \frac{1}{2} 2at^2$, time of flight ($T$) can be determined as

$$0 = v_0 \sin(\alpha - \theta) + \frac{1}{2} g \cos\theta T^2$$

$$T = \frac{2v_0 \sin(\alpha - \theta)}{g\cos\theta}$$

2. **Range along the inclined plane:** We can find range of flight (R) by considering motion in both x and y directions, with the same approach as in normal projectile motion. From Fig. 4-16, we have

$$l = v_0 \cos\alpha$$

$$R = \frac{l}{\cos\theta} = \frac{v_0 \cos\alpha}{\cos\theta} \times \frac{2v_0 \sin(\alpha - \theta)}{g\cos\theta}$$

$$R = \frac{2v_0^2 \sin(\alpha - \theta)\cos\alpha}{g\cos^2\theta}$$

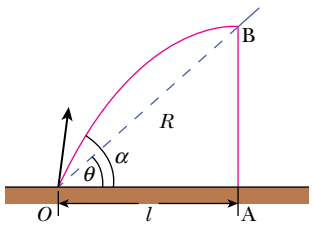

Figure 4-16 Path of projectile along the incline, landing at point B.

3. **Maximum height relative to inclined plane:** It is the maximum height attained by the projected particle relative to the inclined plane. At this height (H), the velocity component in the y-direction, $v_y = 0$. So, from the equation of motion $v^2 = v_0^2 + 2a(x - x_0)$, we have

$$(0)^2 = v_0^2 \sin 2(\alpha - \theta) - 2g\cos\theta H$$

$$H = \frac{v_0^2 \sin 2(\alpha - \theta)}{2g\cos\theta}$$

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When the particle is projected down the inclined plane (Fig. 4-15*b*), the equations for velocity components, acceleration and other parameters are as follows:

1. The components of initial velocity: $v_{0x} = v_0\cos(\theta+\alpha)$; $v_{0y} = v_0\sin(\theta+\alpha)$
2. The components of acceleration: $a_x = g\sin\theta$; $a_y = -g\cos\theta$
3. *e* of flight: $T = \dfrac{2v_0\sin(\theta+\alpha)}{g\cos\theta}$
4. Range of flight: $R = \dfrac{v_0^2\sin 2(\alpha+\theta)}{g\cos^2\theta}$

The Effects of the Air

We have assumed that the air through which the projectile moves has no effect on its motion. However, in many situations, the disagreement between our calculations and the actual motion of the projectile can be large because the air resists (opposes) the motion. Figure 4-17, for example, shows two paths for a fly ball that leaves the bat at an angle of 60° with the horizontal and an initial speed of 44.7 m/s. Path I (the baseball player's fly ball) is a calculated path that approximates normal conditions of play, in air. Path II (the physics professor's fly ball) is the path the ball would follow in a vacuum.

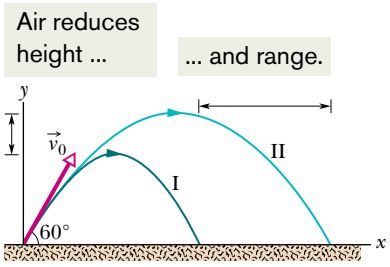

Figure 4-17 (I) The path of a fly ball calculated by taking air resistance into account. (II) The path the ball would follow in a vacuum, calculated by the methods of this chapter. See Table 4-1 for corresponding data. (Based on "The Trajectory of a Fly Ball," by Peter J. Brancazio, *The Physics Teacher,* January 1985.)

Table 4-1 Two Fly Balls[a]

	Path I (Air)	Path II (Vacuum)
Range	98.5 m	177 m
Maximum height	53.0 m	76.8 m
Time of flight	6.6 s	7.9 s

[a]See Fig. 4-17. The launch angle is 60° and the launch speed is 44.7 m/s.

CHECKPOINT 4

A fly ball is hit to the outfield. During its flight (ignore the effects of the air), what happens to its (a) horizontal and (b) vertical components of velocity? What are the (c) horizontal and (d) vertical components of its acceleration during ascent, during descent, and at the topmost point of its flight?

SAMPLE PROBLEM 4.05

Projectile dropped from airplane

In Fig. 4-18, a rescue plane flies at 198 km/h (= 55.0 m/s) and constant height h = 500 m toward a point directly over a victim, where a rescue capsule is to land.

(a) What should be the angle ϕ of the pilot's line of sight to the victim when the capsule release is made?

KEY IDEA

Once released, the capsule is a projectile, so its horizontal and vertical motions can be considered separately (we need not consider the actual curved path of the capsule).

Calculations: In Fig. 4-18, we see that ϕ is given by

$$\phi = \tan^{-1}\frac{x}{h}, \qquad (4\text{-}30)$$

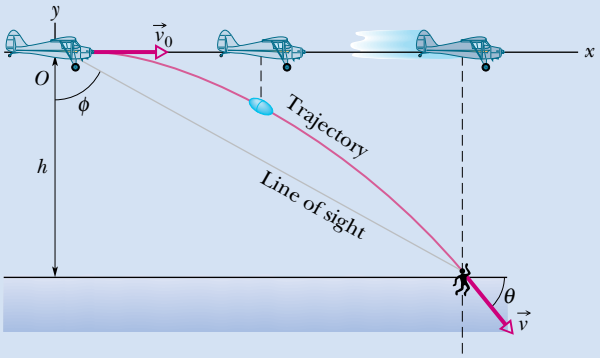

Figure 4-18 A plane drops a rescue capsule while moving at constant velocity in level flight. While falling, the capsule remains under the plane.

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where x is the horizontal coordinate of the victim (and of the capsule when it hits the water) and $h = 500$ m. We should be able to find x with Eq. 4-21:

$$x - x_0 = (v_0 \cos \theta_0)t. \tag{4-31}$$

Here we know that $x_0 = 0$ because the origin is placed at the point of release. Because the capsule is *released* and not shot from the plane, its initial velocity $\vec{v}_0$ is equal to the plane's velocity. Thus, we know also that the initial velocity has magnitude $v_0 = 55.0$ m/s and angle $\theta_0 = 0°$ (measured relative to the positive direction of the x axis). However, we do not know the time t the capsule takes to move from the plane to the victim.

To find t, we next consider the *vertical* motion and specifically Eq. 4-22:

$$x - y_0 = (v_0 \sin \theta_0)t - \frac{1}{2}gt^2. \tag{4-32}$$

Here the vertical displacement $y - y_0$ of the capsule is –500 m (the negative value indicates that the capsule moves *downward*). So,

$$-500 \text{ m} = (55.0 \text{ m/s})(\sin 0°)t - \frac{1}{2}(9.8 \text{ m/s}^2)t^2. \tag{4-33}$$

Solving for t, we find $t = 10.1$ s. Using that value in Eq. 4-31 yields

$$x - 0 = (55.0 \text{ m/s})(\cos 0°)(10.1 \text{ s}), \tag{4-34}$$

or $$x = 555.5 \text{ m}.$$

Then Eq. 4-30 gives us

$$\phi = \tan^{-1}\frac{555.5 \text{ m}}{500 \text{ m}} = 48.0°. \qquad \text{(Answer)}$$

(b) As the capsule reaches the water, what is its velocity $\vec{v}$?

KEY IDEAS

(1) The horizontal and vertical components of the capsule's velocity are independent. (2) Component v_x does not change from its initial value $v_{0x} = v_0 \cos \theta_0$ because there is no horizontal acceleration. (3) Component v_y changes from its initial value $v_{0y} = v_0 \sin \theta_0$ because there is a vertical acceleration.

Calculations: When the capsule reaches the water,

$$v_x = v_0 \cos \theta_0 = (55.0 \text{ m/s})(\cos 0°) = 55.0 \text{ m/s}.$$

Using Eq. 4-23 and the capsule's time of fall $t = 10.1$ s, we also find that when the capsule reaches the water,

$$\begin{aligned} v_y &= v_0 \sin \theta_0 - gt \\ &= (55.0 \text{ m/s})(\sin 0°) - (9.8 \text{ m/s}^2)(10.1 \text{ s}) \\ &= -99.0 \text{ m/s}. \end{aligned}$$

Thus, at the water

$$\vec{v} = (55.0 \text{ m/s})\hat{i} - (99.0 \text{ m/s})\hat{j}. \qquad \text{(Answer)}$$

From Eq. 3-6, the magnitude and the angle of $\vec{v}$ are

$$v = 113 \text{ m/s} \quad \text{and} \quad \theta = -60.9°. \qquad \text{(Answer)}$$

SAMPLE PROBLEM 4.06

Launched into the air from a water slide

One of the most dramatic videos on the web (but entirely fictitious) supposedly shows a man sliding along a long water slide and then being launched into the air to land in a water pool. Let's attach some reasonable numbers to such a flight to calculate the velocity with which the man would have hit the water. Figure 4-19*a* indicates the launch and landing sites and includes a superimposed coordinate system with its origin conveniently located at the launch site. From the video we take the horizontal flight distance as $D = 20.0$ m, the flight time as $t = 2.50$ s, and the launch angle as $\theta_0 = 40.0°$. Find the magnitude of the velocity at launch and at landing.

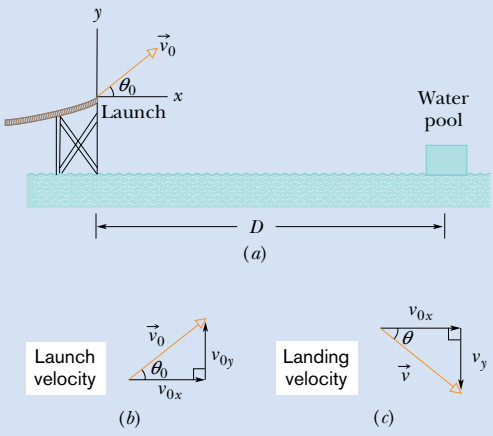

Figure 4-19 (*a*) Launch from a water slide, to land in a water pool. The velocity at (*b*) launch and (*c*) landing.

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KEY IDEAS

(1) For projectile motion, we can apply the equations for constant acceleration along the horizontal and vertical axes *separately*. (2) Throughout the flight, the vertical acceleration is $a_y = -g = -9.8\ \text{m/s}^2$ and the horizontal acceleration is $a_x = 0$.

Calculations: In most projectile problems, the initial challenge is to figure out where to start. There is nothing wrong with trying out various equations, to see if we can somehow get to the velocities. But here is a clue. Because we are going to apply the constant-acceleration equations separately to the x and y motions, we should find the horizontal and vertical components of the velocities at launch and at landing. For each site, we can then combine the velocity components to get the velocity.

Because we know the horizontal displacement $D = 20.0$ m, let's start with the horizontal motion. Since $a_x = 0$, we know that the horizontal velocity component v_x is constant during the flight and thus is always equal to the horizontal component v_{0x} at launch. We can relate that component, the displacement $x - x_0$ and the flight time $t = 2.50$ s with Eq. 2-15:

$$x - x_0 = v_{0x}t + \frac{1}{2}a_x t^2. \tag{4-35}$$

Substituting $a_x = 0$, this becomes Eq. 4-21. With $x - x_0 = D$, we then write

$$20\ \text{m} = v_{0x}(2.0\ \text{s}) + \frac{1}{2}(0)(2.50\ \text{s})^2$$
$$v_{0x} = 8.00\ \text{m/s}.$$

That is a component of the launch velocity, but we need the magnitude of the full vector, as shown in Fig. 4-19*b*, where the components form the legs of a right triangle and the full vector forms the hypotenuse. We can then apply a trig definition to find the magnitude of the full velocity at launch:

$$\cos\theta_0 = \frac{v_{0x}}{v_0},$$

and so

$$v_0 = \frac{v_{0x}}{\cos\theta_0} = \frac{8.00\ \text{m/s}}{\cos 40°}$$
$$= 10.44\ \text{m/s} \approx 10.4\ \text{m/s}. \qquad \text{(Answer)}$$

Now let's go after the magnitude v of the landing velocity. We already know the horizontal component, which does not change from its initial value of 8.00 m/s. To find the vertical component v_y and because we know the elapsed time $t = 2.50$ s and the vertical acceleration $a_y = -9.8\ \text{m/s}^2$, let's rewrite Eq. 2-11 as

$$v_y = v_{0y} + a_y t$$

and then (from Fig. 4-19*b*) as

$$v_y = v_0 \sin\theta_0 + a_y t. \tag{4-36}$$

Substituting $a_y = -g$, this becomes Eq. 4-23. We can then write

$$v_y = (10.44\ \text{m/s})\sin(40.0°) - (9.8\ \text{m/s}^2)(2.50\ \text{s})$$
$$= -17.78\ \text{m/s}.$$

Now that we know both components of the landing velocity, we use Eq. 3-6 to find the velocity magnitude:

$$v = \sqrt{v_x^2 + v_y^2}$$
$$= \sqrt{(8.00\ \text{m/s})^2 + (-17.78\ \text{m/s})^2}$$
$$= 19.49\ \text{m/s} \approx 19.5\ \text{m/s}. \qquad \text{(Answer)}$$

SAMPLE PROBLEM 4.07

Maximum height and horizontal range

A ball is shot from the ground into the air. At the height of 9.1 m, its velocity is $\vec{v} = (7.6\ \text{m/s})\hat{i} + (6.1\ \text{m/s})\hat{j}$, with $\hat{i}$ horizontal and $\hat{j}$ upward. (a) To what maximum height does the ball rise? (b) What total horizontal distance does the ball travel? What are the (c) magnitude and (d) angle (below the horizontal) of the ball's velocity just before it hits the ground?

KEY IDEA

We designate the given velocity $\vec{v} = (7.6\ \text{m/s})\hat{i} + (6.1\ \text{m/s})\hat{j}$ as $\vec{v}_1$, as opposed to the velocity when it reaches the max height $\vec{v}_2$ or the velocity when it returns to the ground $\vec{v}_3$, and take $\vec{v}_0$ as the launch velocity, as usual. The origin is at its launch point on the ground.

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Calculations: (a) Different approaches are available, but since it will be useful (for the rest of the problem) to first find the initial y velocity, that is how we will proceed. Using Eq. 2-16, we have

$$v_{1y}^2 = v_{0y}^2 - 2g\Delta y$$

Substituting values, we have

$$(6.1 \text{ m/s})^2 = v_{0y}^2 - 2(9.8 \text{ m/s}^2)(9.1 \text{ m})$$

which yields $v_{0y} = 14.7$ m/s. Knowing that v_{2y} must equal 0, we use Eq. 2-16 again but now with $\Delta y = h$ for the maximum height:

$$v_{2y}^2 = v_{0y}^2 - 2gh \Rightarrow 0 = (14.7 \text{ m/s})^2 - 2(9.8 \text{ m/s}^2)h$$

which yields $h = 11$ m.

(b) Recalling the derivation of Eq. 4-26, but using v_{0y} for $v_0 \sin\theta_0$ and v_{0x} for $v_0 \cos\theta_0$, we have

$$0 = v_{0y}t - \frac{1}{2}gt^2, \quad R = v_{0x}t$$

which leads to $R = 2v_{0x}v_{0y}/g$. Noting that $v_{0x} = v_{1x} = 7.6$ m/s, we plug in values and obtain

$$R = 2(7.6 \text{ m/s})(14.7 \text{ m/s})/(9.8 \text{ m/s}^2) = 23 \text{ m}.$$

(c) Since $v_{3x} = v_{1x} = 7.6$ m/s and $v_{3y} = -v_{0y} = -14.7$ m/s, we have

$$v_3 = \sqrt{v_{3x}^2 + v_{3y}^2} = \sqrt{(7.6 \text{ m/s})^2 + (-14.7 \text{ m/s})^2} = 17 \text{ m/s}.$$

(d) The angle (measured from horizontal) for $\vec{v}_3$ is one of these possibilities:

$$\tan^{-1}\left(\frac{-14.7 \text{ m}}{7.6 \text{ m}}\right) = -63^\circ \text{ or } 117^\circ$$

where we settle on the first choice (–63°, which is equivalent to 297°) since the signs of its components imply that it is in the fourth quadrant.

SAMPLE PROBLEM 4.08

Projectile motion of a ball on an inclined plane

In Fig. 4-20, a ball is launched with a velocity of magnitude 10.0 m/s, at an angle of 50.0° to the horizontal. The launch point is at the base of a ramp of horizontal length $d_1 = 6.00$ m and height $d_2 = 3.60$ m. A plateau is located at the top of the ramp. (a) Does the ball land on the ramp or the plateau? When it lands, what are the (b) magnitude and (c) angle of its displacement from the launch point?

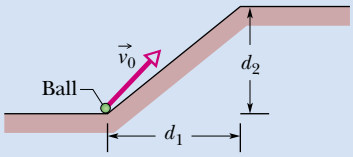

Figure 4-20 A ball launched at the base of a ramp landing on the plateau located at the top of the ramp.

Calculations: (a) Let $m = d_2/d_1 = 0.600$ be the slope of the ramp, so $y = mx$ there. We choose our coordinate origin at the point of launch and use Eq. 4-25. Thus,

$$y = \tan(50.0^\circ)x - \frac{(9.80 \text{ m/s}^2)x^2}{2(10.0 \text{ m/s})^2(\cos 50.0^\circ)^2} = 0.600x$$

which yields $x = 4.99$ m. This is less than d_1 so the ball *does* land on the ramp.

(b) Using the value of x found in part (a), we obtain $y = mx = 2.99$ m. Thus, the Pythagorean theorem yields a displacement magnitude of $\sqrt{x^2 + y^2} = 5.82$ m.

(c) The angle is, of course, the angle of the ramp: $\tan^{-1}(m) = 31.0°$.

SAMPLE PROBLEM 4.09

Two angles of projection

A football kicker can give the ball an initial speed of 25 m/s. What are the (a) least and (b) greatest elevation angles at which he can kick the ball to score a field goal from a point 50 m in front of goalposts whose horizontal bar is 3.44 m above the ground?

KEY IDEA

In this problem a football is given an initial speed and it undergoes projectile motion. We'd like to know the smallest and greatest angles at which a field goal can be scored. We adopt the positive direction choices used

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in the textbook so that equations such as Eq. 4-22 are directly applicable. The coordinate origin is at the point where the ball is kicked. We use x and y to denote the coordinates of the ball at the goalpost, and try to find the kicking angle(s) θ_0 so that $y = 3.44$ m when $x = 50$ m.

Calculations: Writing the kinematic equations for projectile motion:

$$x = v_0 \cos\theta_0, \quad y = v_0 t \sin\theta_0 - \frac{1}{2}gt^2,$$

we see the first equation gives $t = x/v_0 \cos\theta_0$, and when this is substituted into the second the result is

$$y = x\tan\theta_0 - \frac{gx^2}{2v_0^2\cos^2\theta_0}.$$

We can solve the above equation by trial and error: systematically trying values of θ_0 until you find the two that satisfy the equation. A little manipulation, however, will give an algebraic solution: Using the trigonometric identity

$$\frac{1}{\cos^2\theta_0} = 1 + \tan^2\theta_0,$$

we obtain

$$\frac{1}{2}\frac{gx^2}{v_0^2}\tan^2\theta_0 - x\tan\theta_0 + y + \frac{1}{2}\frac{gx^2}{v_0^2} = 0$$

which is a second-order equation for $\tan\theta_0$. To simplify writing the solution, we denote

$$c = \frac{1}{2}gx^2/v_0^2 = \frac{1}{2}(9.80\text{ m/s}^2)(50\text{ m})^2/(25\text{ m/s})^2 = 19.6\text{ m}.$$

Then the second-order equation becomes $c\tan^2\theta_0 - x\tan\theta_0 + y + c = 0$. Using the quadratic formula, we obtain its solution(s).

$$\tan\theta_0 = \frac{x \pm \sqrt{x^2 - 4(y+c)c}}{2c} = \frac{50\text{ m} \pm \sqrt{(50\text{ m})^2 - 4(3.44\text{ m} + 19.6\text{ m})(19.6\text{ m})}}{2(19.6\text{ m})}.$$

The two solutions are given by $\tan\theta_0 = 1.95$ and $\tan\theta_0 = 0.605$. The corresponding (first-quadrant) angles are $\theta_0 = 63°$ and $\theta_0 = 31°$. Thus,

(a) The smallest elevation angle is $\theta_0 = 31°$, and

(b) The greatest elevation angle is $\theta_0 = 63°$.

Therefore, if kicked at any angle between 31° and 63°, the ball will travel above the cross bar on the goalposts.

SAMPLE PROBLEM 4.10

Trajectory of a diver's projectile motion

From the platform edge located 10.0 m above the surface of the water, a high diver pushes off horizontally with a speed of 2.00 m/s. (a) At what horizontal distance from the edge is the diver 0.800 s after pushing off? (b) At what vertical distance above the surface of the water is the diver just then? (c) At what horizontal distance from the edge does the diver strike the water?

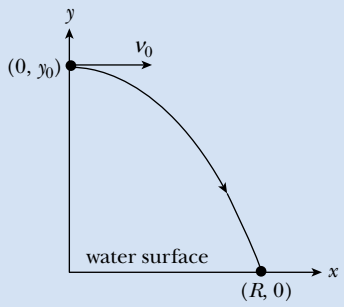
KEY IDEA

The trajectory of the diver is a projectile motion. We are interested in the displacement of the diver at a later time. The initial velocity has no vertical component ($\theta_0 = 0$), but only an x component. Eqs. 4-21 and 4-22 can be simplified to

$$x - x_0 = v_{0x}t$$

$$y - y_0 = v_{0y}t - \frac{1}{2}gt^2 = -\frac{1}{2}gt^2.$$

where $x_0 = 0$, $v_{0x} = v_0 = +2.0$ m/s and $y_0 = +10.0$ m (taking the water surface to be at $y = 0$). The setup of the problem is shown in the figure below.



Calculations: (a) At $t = 0.80$ s, the horizontal distance of the diver from the edge is

$$x = x_0 + v_{0x}t = 0 + (2.0\text{ m/s})(0.80\text{ s}) = 1.60\text{ m}.$$

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(b) Similarly, using the second equation for the vertical motion, we obtain

$$y = y_0 - \frac{1}{2}gt^2 = 10.0\text{ m} - \frac{1}{2}(9.80\text{ m/s}^2)(0.80\text{ s})^2 = 6.86\text{ m}.$$

(c) At the instant the diver strikes the water surface, $y = 0$. Solving for t using the equation $y = y_0 - \frac{1}{2}gt^2 = 0$ leads to

$$t = \sqrt{\frac{2y_0}{g}} = \sqrt{\frac{2(10.0\text{ m})}{9.80\text{ m/s}^2}} = 1.43\text{ s}.$$

During this time, the x-displacement of the diver is $R = x = (2.00\text{ m/s})(1.43\text{ s}) = 2.86\text{ m}$.

Alternately, using Eq. 4-25 with $\theta_0 = 0$, the trajectory of the diver can also be written as

$$y = y_0 - \frac{gx^2}{2v_0^2}.$$

Therefore, Part (c) can also be solved by using this equation:

$$y = y_0 - \frac{gx^2}{2v_0^2} = 0 \Rightarrow x = R = \sqrt{\frac{2v_0^2 y_0}{g}}$$
$$= \sqrt{\frac{2(2.0\text{ m/s})^2(10.0\text{ m})}{9.8\text{ m/s}^2}} = 2.86\text{ m}.$$

4.6 | RELATIVE MOTION IN ONE DIMENSION

Key Concept

- When two frames of reference A and B are moving relative to each other at constant velocity, the velocity of a particle P as measured by an observer in frame A usually differs from that measured from frame B. The two measured velocities are related by

$$\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA},$$

where $\vec{v}_{BA}$ is the velocity of B with respect to A. Both observers measure the same acceleration for the particle:

$$\vec{a}_{PA} = \vec{a}_{PB}.$$

Suppose you see a duck flying north at 30 km/h. To another duck flying alongside, the first duck seems to be stationary. In other words, the velocity of a particle depends on the **reference frame** of whoever is observing or measuring the velocity. For our purposes, a reference frame is the physical object to which we attach our coordinate system. In everyday life, that object is the ground. For example, the speed listed on a speeding ticket is always measured relative to the ground. The speed relative to the police officer would be different if the officer were moving while making the speed measurement.

Suppose that Alex (at the origin of frame A in Fig. 4-21) is parked by the side of a highway, watching car P (the "particle") speed past. Barbara (at the origin of frame B) is driving along the highway at constant speed and is also watching car P. Suppose that they both measure the position of the car at a given moment. From Fig. 4-21 we see that

$$x_{PA} = x_{PB} + x_{BA}. \tag{4-37}$$

The equation is read: "The coordinate x_{PA} of P as measured by A *is equal to* the coordinate x_{PB} of P as measured by B *plus* the coordinate x_{BA} of B as measured by A." Note how this reading is supported by the sequence of the subscripts.

Taking the time derivative of Eq. 4-37, we obtain

$$\frac{d}{dt}(x_{PA}) = \frac{d}{dt}(x_{PB}) + \frac{d}{dt}(x_{BA}).$$

Frame B moves past frame A while both observe P.

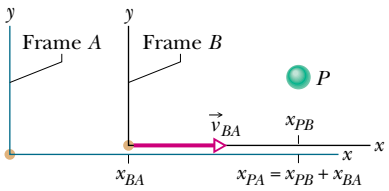

Figure 4-21 Alex (frame A) and Barbara (frame B) watch car P, as both B and P move at different velocities along the common x axis of the two frames. At the instant shown, x_{BA} is the coordinate of B in the A frame. Also, P is at coordinate x_{PB} in the B frame and coordinate $x_{PA} = x_{PB} + x_{BA}$ in the A frame.

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Thus, the velocity components are related by

$$v_{PA} = v_{PB} + v_{BA}. \tag{4-38}$$

This equation is read: "The velocity v_{PA} of P as measured by A *is equal to* the velocity v_{PB} of P as measured by B *plus* the velocity v_{BA} of B as measured by A." The term v_{BA} is the velocity of frame B relative to frame A.

Here we consider only frames that move at constant velocity relative to each other. In our example, this means that Barbara (frame B) drives always at constant velocity v_{BA} relative to Alex (frame A). Car P (the moving particle), however, can change speed and direction (that is, it can accelerate).

To relate an acceleration of P as measured by Barbara and by Alex, we take the time derivative of Eq. 4-38:

$$\frac{d}{dt}(v_{PA}) = \frac{d}{dt}(v_{PB}) + \frac{d}{dt}(v_{BA}).$$

Because v_{BA} is constant, the last term is zero and we have

$$a_{PA} = a_{PB}. \tag{4-39}$$

In other words,

Observers on different frames of reference that move at constant velocity relative to each other will measure the same acceleration for a moving particle.

While travelling on a highway, we have observed that if a car takes over our car in the same direction, we do not find that car moving very fast. On the other hand, if a car crosses us in the opposite direction, we find it moving fast. These observations can be explained based on relative motion in one-direction.

1. Consider two cars, travelling on a straight road in the same direction but at different speeds. Let car C1 travel at a speed of 65 km/h and pass car C2 travelling at 45 km/h. Then the velocity of C1 relative to C2 is

$$v_{C1} - v_{C2} = 65 - 45 = 20 \text{ km/h}.$$

2. Now consider the case when these two cars travel in opposite direction and pass each other. Since the cars are traveling in the opposite direction, the velocity of one car is taken as positive (say +65 km/h for C1) and velocity of the other car is given a negative sign (say –45 km/h for C2). Then, the velocity of C1 relative to C2 is given by

$$v_{C1} - v_{C2} = +65 - (-45) = 110 \text{ km/h}$$

When the velocities are not in the same straight line, the velocity vectors must be added using parallelogram law.

SAMPLE PROBLEM 4.11

Relative motion, one dimensional, Alex and Barbara

In Fig. 4-21, suppose that Barbara's velocity relative to Alex is a constant $v_{BA} = 52$ km/h and car P is moving in the negative direction of the x axis.

(a) If Alex measures a constant $v_{PA} = -78$ km/h for car P, what velocity v_{PB} will Barbara measure?

KEY IDEAS

We can attach a frame of reference A to Alex and a frame of reference B to Barbara. Because the frames move at constant velocity relative to each other along one axis, we can use Eq. 4-41 ($v_{PA} = v_{PB} + v_{BA}$) to relate v_{PB} to v_{PA} and v_{BA}.

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Calculation: We find

$$-78 \text{ km/h} = v_{PB} + 52 \text{ km/h}.$$

Thus, $$v_{PB} = -130 \text{ km/h}. \quad \text{(Answer)}$$

Comment: If car P were connected to Barbara's car by a cord wound on a spool, the cord would be unwinding at a speed of 130 km/h as the two cars separated.

(b) If car P brakes to a stop relative to Alex (and thus relative to the ground) in time $t = 10$ s at constant acceleration, what is its acceleration a_{PA} relative to Alex?

KEY IDEA

To calculate the acceleration of car P *relative to Alex,* we must use the car's velocities *relative to Alex.* Because the acceleration is constant, we can use Eq. 2-11 ($v = v_0 + at$) to relate the acceleration to the initial and final velocities of P.

Calculation: The initial velocity of P relative to Alex is $v_{PA} = -78$ km/h and the final velocity is 0. Thus, the acceleration relative to Alex is

$$a_{PA} = \frac{v - v_0}{t} = \frac{0 - (-78 \text{ km/h})}{10 \text{ s}} \frac{1 \text{ m/s}}{3.6 \text{ km/h}}$$
$$= 2.2 \text{ m/s}^2. \quad \text{(Answer)}$$

(c) What is the acceleration a_{PB} of car P relative to Barbara during the braking?

KEY IDEA

To calculate the acceleration of car P *relative to Barbara,* we must use the car's velocities *relative to Barbara.*

Calculation: We know the initial velocity of P relative to Barbara from part (a) ($v_{PB} = -130$ km/h). The final velocity of P relative to Barbara is –52 km/h (because this is the velocity of the stopped car relative to the moving Barbara). Thus,

$$a_{PA} = \frac{v - v_0}{t} = \frac{-25 \text{ km/h} - (-130 \text{ km/h})}{10 \text{ s}} \frac{1 \text{ m/s}}{3.6 \text{ km/h}}$$
$$= 2.2 \text{ m/s}^2. \quad \text{(Answer)}$$

Comment: We should have foreseen this result: Because Alex and Barbara have a constant relative velocity, they must measure the same acceleration for the car.

4.7 | RELATIVE MOTION IN TWO DIMENSIONS

Key Concept

- When two frames of reference A and B are moving relative to each other at constant velocity, the velocity of a particle P as measured by an observer in frame A usually differs from that measured from frame B. The two measured velocities are related by

$$\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA},$$

where $\vec{v}_{BA}$ is the velocity of B with respect to A. Both observers measure the same acceleration for the particle:

$$\vec{a}_{PA} = \vec{a}_{PB}.$$

Our two observers are again watching a moving particle P from the origins of reference frames A and B, while B moves at a constant velocity $\vec{v}_{BA}$ relative to A. (The corresponding axes of these two frames remain parallel.) Figure 4-22 shows a certain instant during the motion. At that instant, the position vector of the origin of B relative to the origin of A is $\vec{r}_{BA}$. Also, the position vectors of particle P are $\vec{r}_{PA}$ relative to the origin of A and $\vec{r}_{PB}$ relative to the origin of B. From the arrangement of heads and tails of those three position vectors, we can relate the vectors with

$$\vec{r}_{PA} = \vec{r}_{PB} + \vec{r}_{BA}. \quad (4\text{-}40)$$

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By taking the time derivative of this equation, we can relate the velocities $\vec{v}_{PA}$ and $\vec{v}_{PB}$ of particle P relative to our observers:

$$\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}. \tag{4-41}$$

By taking the time derivative of this relation, we can relate the accelerations $\vec{a}_{PA}$ and $\vec{a}_{PB}$ of the particle P relative to our observers. However, note that because $\vec{v}_{BA}$ is constant, its time derivative is zero. Thus, we get

$$\vec{a}_{PA} = \vec{a}_{PB}. \tag{4-42}$$

As for one-dimensional motion, we have the following rule: Observers on different frames of reference that move at constant velocity relative to each other will measure the *same* acceleration for a moving particle.

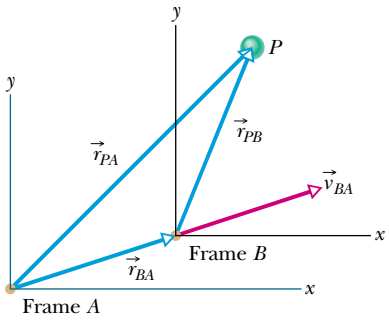

Figure 4-22 Frame B has the constant two-dimensional velocity $\vec{v}_{BA}$ relative to frame A. The position vector of B relative to A is $\vec{r}_{BA}$. The position vectors of particle P are $\vec{r}_{PA}$ relative to A and $\vec{r}_{PB}$ relative to B.

Figure 4-23 depicts a common situation that deals with relative velocity in two dimensions. Figure 4-23*a* shows a boat being carried downstream by a river; the engine of the boat is turned off. In Fig. 4-23*b*, the engine has been turned on, and now the boat moves across the river in a diagonal fashion because of the combined motion produced by the current and the engine. The list below gives the velocities for this type of motion and the objects relative to which they are measured:

$\vec{v}_{BW}$ = velocity of the boat relative to the water

$\vec{v}_{WS}$ = velocity of the water relative to the shore

$\vec{v}_{BS}$ = velocity of the boat relative to the shore

The velocity $\vec{v}_{BW}$ of the boat relative to the water is the velocity measured by an observer who, for instance, is floating on an inner tube and drifting downstream with the current. When the engine is turned off, the boat also drifts downstream with the current, and $\vec{v}_{BW}$ is zero. When the engine is turned on, however, the boat can move relative to the water, and $\vec{v}_{BW}$ is no longer zero. The velocity $\vec{v}_{WS}$ of the water relative to the shore is the velocity of the current measured by an observer on the shore. The velocity $\vec{v}_{BS}$ of the boat relative to the shore is due to the combined motion of the boat relative to the water and the motion of the water relative to the shore, which can be written as

$$\vec{v}_{BS} = \vec{v}_{BW} + \vec{v}_{WS}.$$

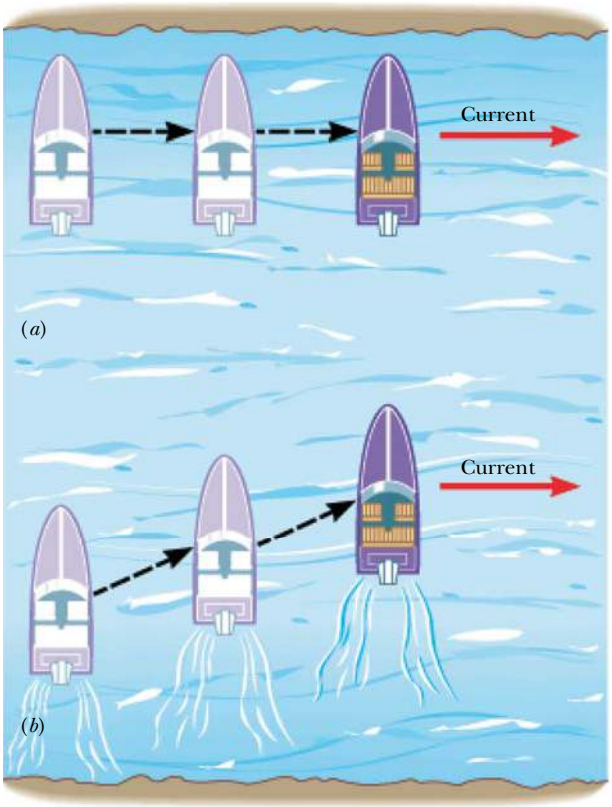

Figure 4-23 (*a*) A boat with its engine turned off is carried along by the current. (*b*) With the engine turned on, the boat moves across the river in a diagonal fashion

The concept of **combined** or **net motion** is of great importance in regulating the motion of aircrafts and ships. The pilot is concerned with the net motion of the aircraft starting from the point of origin to destination. The controls in the plane, however affect the motion of plane with respect to air. Similarly the motion of the ship or a boat is regulated by the net motion considering the velocities with respect to the shore, air and water.

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SAMPLE PROBLEM 4.12

Relative motion, two dimensional, airplanes

In Fig. 4-24*a*, a plane moves due east while the pilot points the plane somewhat south of east, toward a steady wind that blows to the northeast. The plane has velocity $\vec{v}_{PW}$ relative to the wind, with an airspeed (speed relative to the wind) of 215 km/h, directed at angle θ south of east. The wind has velocity $\vec{v}_{WG}$ relative to the ground with speed 65.0 km/h, directed 20.0° east of north. What is the magnitude of the velocity $\vec{v}_{PG}$ of the plane relative to the ground, and what is θ?

KEY IDEAS

The situation is like the one in Fig. 4-22. Here the moving particle P is the plane, frame A is attached to the ground (call it G), and frame B is "attached" to the wind (call it W). We need a vector diagram like Fig. 4-22 but with three velocity vectors.

Calculations: First we construct a sentence that relates the three vectors shown in Fig. 4-24*b*:

$$\underset{(PG)}{\text{velocity of plane relative to ground}} = \underset{(PW)}{\text{velocity of plane relative to ground}} + \underset{(WG)}{\text{velocity of wind relative to ground.}}$$

This relation is written in vector notation as

$$\vec{v}_{PG} = \vec{v}_{PW} + \vec{v}_{WG}. \qquad (4\text{-}43)$$

We need to resolve the vectors into components on the coordinate system of Fig. 4-24*b* and then solve Eq. 4-43 axis by axis. For the y components, we find

$$v_{PG,y} = v_{PW,y} + v_{WG,y}$$

or $$0 = -(215 \text{ km/h}) \sin\theta + (65.0 \text{ km/h})(\cos 20.0°).$$

Solving for θ gives us

$$\theta = \sin^{-1}\frac{(65.0 \text{ km/h})(\cos\ 20.0°)}{215 \text{ km/h}} = 16.5°. \qquad \text{(Answer)}$$

Similarly, for the x components we find

$$v_{PG,x} = v_{PW,x} + v_{WG,x}.$$

Here, because $\vec{v}_{PG}$ is parallel to the x axis, the component $v_{PG,x}$ is equal to the magnitude v_{PG}. Substituting this notation and the value $\theta = 16.5°$, we find

$$v_{PG} = (215 \text{ km/h})(\cos 16.5°) + (65.0 \text{ km/h})(\sin 20.0°)$$
$$= 228 \text{ km/h}. \qquad \text{(Answer)}$$

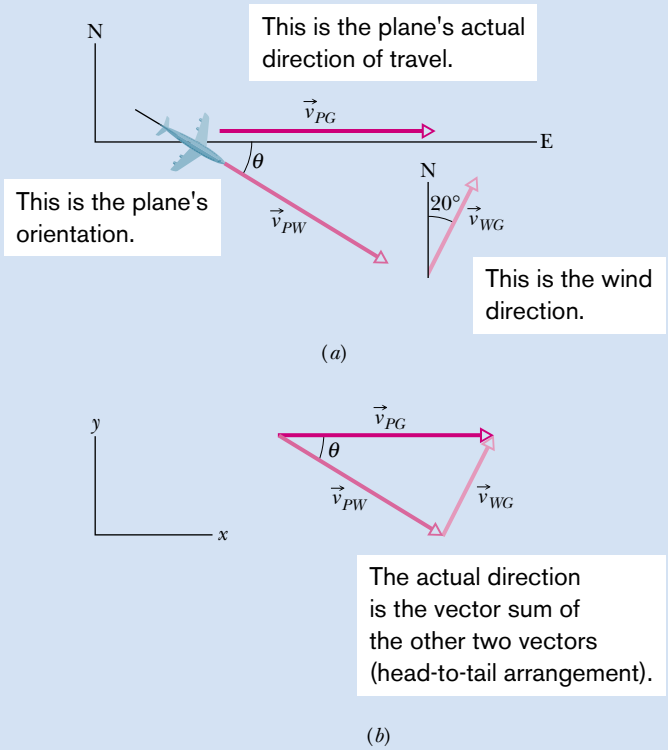

Figure 4-24 A plane flying in a wind.

SAMPLE PROBLEM 4.13

Relative motion, two dimensional, boats

A 200-m-wide river flows due east at a uniform speed of 2.0 m/s. A boat with a speed of 8.0 m/s relative to the water leaves the south bank pointed in a direction 30° west of north. What are the (a) magnitude and (b) direction of the boat's velocity relative to the ground? (c) How long does the boat take to cross the river?

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KEY IDEA

This is a classic problem involving two-dimensional relative motion. We align our coordinates so that *east* corresponds to $+x$ and *north* corresponds to $+y$. We write the vector addition equation as $\vec{v}_{BG} = \vec{v}_{BW} + \vec{v}_{WG}$. We have $\vec{v}_{WG} = (2.0\angle 0°)$ in the magnitude-angle notation (with the unit m/s understood), or $\vec{v}_{WG} = 2.0\hat{i}$ in unit-vector notation. We also have $\vec{v}_{BW} = (8.0\angle 120°)$ where the angle is measured counterclockwise from the $+x$ axis, in the standard way, or $\vec{v}_{BW} = (-4.0\hat{i} + 6.9\hat{j})$ m/s.

Calculate: (a) We can solve the vector addition equation for $\vec{v}_{BG}$:

$$\vec{v}_{BG} = v_{BW} + \vec{v}_{WG} = (2.0 \text{ m/s})\hat{i} + (-4.0\hat{i} + 6.9\hat{j}) \text{ m/s}$$
$$= (-2.0 \text{ m/s})\hat{i} + (6.9 \text{ m/s})\hat{j}.$$

Thus, we find $|\vec{v}_{BG}| = 7.2$ m/s.

(b) The direction of $\vec{v}_{BG}$ is $\theta = \tan^{-1}[(6.9 \text{ m/s})/[(6.9 \text{ m/s})/(-2.0 \text{ m/s})] = 106°$ (measured counterclockwise from the $+x$ axis), or 16° west of north.

(c) The velocity is constant, and we apply $y - y_0 = v_y t$ in a reference frame. Thus, in the *ground* reference frame, we have $(200 \text{ m}) = (7.2 \text{ m/s})\sin(106°)t \rightarrow t = 29$ s.

Note: If time obtained is "28 s," then you have probably neglected to take the y component properly (a common mistake).

SAMPLE PROBLEM 4.14

Relative motion, two dimensional, train and rain

A train travels due south at 30 m/s (relative to the ground) in a rain that is blown toward the south by the wind. The path of each raindrop makes an angle of 70° with the vertical, as measured by an observer stationary on the ground. An observer on the train, however, sees the drops fall perfectly vertically. Determine the speed of the raindrops relative to the ground.

KEY IDEA

This problem deals with relative motion in two dimensions. Raindrops appear to fall vertically by an observer on a moving train. Since the raindrops fall vertically relative to the train, the horizontal component of the velocity of a raindrop, $v_h = 30$ m/s, must be the same as the speed of the train, i.e., $v_h = v_{\text{train}}$ (see Fig. 4-25). On the other hand, if v_v is the vertical component of the velocity and θ is the angle between the direction of motion and the vertical, then $\tan\theta = v_h/v_v$. Knowing v_v and v_h allows us to determine the speed of the raindrops.

Calculation: With $\theta = 70°$, we find the vertical component of the velocity to be

$$v_v = v_h/\tan\theta = (30 \text{ m/s})/\tan 70° = 10.9 \text{ m/s}.$$

Therefore, the speed of a raindrop is

$$v = \sqrt{v_h^2 + v_v^2} = \sqrt{(30 \text{ m/s})^2 + (10.9 \text{ m/s})^2} = 32 \text{ m/s}.$$

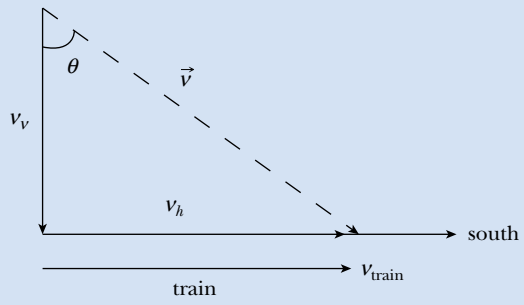

Figure 4-25 Relative motion of raindrops falling on a moving train.

SAMPLE PROBLEM 4.15

Relative motion, two dimensional, boxcar and bullet

A wooden boxcar is moving along a straight railroad track at speed v_1. A bullet (initial speed v_2) is fired at it from a high-powered rifle. The bullet passes through both lengthwise walls of the car, its entrance and exit holes being exactly opposite each other as viewed from within the car. From what direction, relative to the track,

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is the bullet fired? Assume that the bullet is not deflected upon entering the car, but that its speed decreases by 20%. Take v_1 85 km/h and v_2 650 m/s. (Why is the width of the boxcar not relevant in this calculation?)

KEY IDEA

The boxcar has velocity $\vec{v}_{cg} = v_1\hat{i}$ relative to the ground, and the bullet has velocity

$$\vec{v}_{0g} = v_2 \cos\theta\hat{i} + v_2 \sin\theta\hat{j}$$

relative to the ground before entering the car (we are neglecting the effects of gravity on the bullet).

Calculation: While in the car, its velocity relative to the outside ground is

$$\vec{v}_{bg} = 0.8\, v_2 \cos\theta\hat{i} + 0.8\, v_2 \sin\theta\hat{j}$$

(since there is a 20% reduction in the speed). The problem indicates that the velocity of the bullet in the car *relative to the car* is (with v_3 unspecified) $\vec{v}_{bc} = v_3\hat{j}$. Now, Eq. 4-41 provides the condition

$$\vec{v}_{bg} = \vec{v}_{bc} + \vec{v}_{cg}$$

$$0.8\, v_2 \cos\theta\hat{i} + 0.8\, v_2 \sin\theta\hat{j} = v_3\hat{j} + v_1\hat{i}$$

so that equating x components allows us to find θ. If one wished to find v_3 one could also equate the y components, and from this, if the car width were given, one could find the time spent by the bullet in the car, but this information is not asked for (which is why the width is irrelevant). Therefore, examining the x components in SI units leads to

$$\theta = \cos^{-1}\left(\frac{v_1}{0.8v_2}\right) = \cos^{-1}\left(\frac{85\text{ km/h}\left(\frac{1000\text{ m/km}}{3600\text{ s/h}}\right)}{0.8\ (650\text{ m/s})}\right)$$

which yields 87° for the direction of $\vec{v}_{bg}$ (measured from $\hat{i}$, which is the direction of motion of the car). The problem asks, "from what direction was it fired?"—which means the answer is not 87° but rather its supplement 93° (measured from the direction of motion). Stating this more carefully, in the coordinate system we have adopted in our solution, the bullet velocity vector is in the first quadrant, at 87° measured counterclockwise from the $+x$ direction (the direction of train motion), which means that the direction from which the bullet came (where the sniper is) is in the third quadrant, at

$-93°$ (that is, 93° measured clockwise from $+x$).

REVIEW AND SUMMARY

Position Vector The location of a particle relative to the origin of a coordinate system is given by a *position vector* $\vec{r}$, which in unit-vector notation is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}. \tag{4-1}$$

Here $x\hat{i}$, $y\hat{j}$, and $z\hat{k}$ are the vector components of position vector $\vec{r}$, and x, y, and z are its scalar components (as well as the coordinates of the particle). A position vector is described either by a magnitude and one or two angles for orientation, or by its vector or scalar components.

Displacement If a particle moves so that its position vector changes from $\vec{r}_1$ to $\vec{r}_2$, the particle's *displacement* $\Delta\vec{r}$ is

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1. \tag{4-2}$$

The displacement can also be written as

$$\Delta\vec{r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k} \tag{4-3}$$

$$= \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}. \tag{4-4}$$

Average Velocity and Instantaneous Velocity If a particle undergoes a displacement $\Delta\vec{r}$ in time interval Δt, its *average velocity* $\vec{v}_{avg}$ for that time interval is

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t}. \tag{4-8}$$

As Δt in Eq. 4-8 is shrunk to 0, $\vec{v}_{avg}$ reaches a limit called either the *velocity* or the *instantaneous velocity* $\vec{v}$:

$$\vec{v} = \frac{d\vec{r}}{dt}, \tag{4-10}$$

which can be rewritten in unit-vector notation as

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}, \tag{4-11}$$

where $v_x = dx/dt$, $v_y = dy/dt$, and $v_z = dz/dt$. The instantaneous velocity $\vec{v}$ of a particle is always directed along the tangent to the particle's path at the particle's position.

Average Acceleration and Instantaneous Acceleration If a particle's velocity changes from $\vec{v}_1$ to $\vec{v}_2$ in time interval Δt, its *average acceleration* during Δt is

$$\vec{a}_{avg} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\Delta\vec{v}}{\Delta t}. \tag{4-15}$$

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As Δt in Eq. 4-15 is shrunk to 0, $\vec{a}_{avg}$ reaches a limiting value called either the *acceleration* or the *instantaneous acceleration* $\vec{a}$:

$$\vec{a} = \frac{d\vec{v}}{dt}. \tag{4-16}$$

In unit-vector notation,

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}, \tag{4-17}$$

where $a_x = dv_x/dt$, $a_y = dv_y/dt$, and $a_z = dv_z/dt$.

Projectile Motion *Projectile motion* is the motion of a particle that is launched with an initial velocity $\vec{v}_0$. During its flight, the particle's horizontal acceleration is zero and its vertical acceleration is the free-fall acceleration $-g$. (Upward is taken to be a positive direction.) If $\vec{v}_0$ is expressed as a magnitude (the speed v_0) and an angle θ_0 (measured from the horizontal), the particle's equations of motion along the horizontal x axis and vertical y axis are

$$x - x_0 = (v_0 \cos\theta_0)t, \tag{4-21}$$

$$y - y_0 = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2, \tag{4-22}$$

$$v_y = v_0 \sin\theta_0 - gt, \tag{4-23}$$

$$v_y^2 = (v_0 \sin\theta_0)^2 - 2g(y - y_0). \tag{4-24}$$

The **trajectory** (path) of a particle in projectile motion is parabolic and is given by

$$y = (\tan\theta_0)x - \frac{gx^2}{2(v_0 \cos\theta_0)^2}, \tag{4-25}$$

if x_0 and y_0 of Eqs. 4-21 to 4-24 are zero. The particle's **horizontal range** R, which is the horizontal distance from the launch point to the point at which the particle returns to the launch height, is

$$R = \frac{v_0^2}{g}\sin 2\theta_0. \tag{4-26}$$

The maximum height to which a projectile can rise above the horizontal plane of projection is known as maximum height or vertical range.

$$H = \frac{v^2 \sin^2\theta}{2g}. \tag{4-28}$$

Time of flight is the total time taken by the projectile to return to the same level from where it was thrown. The time of flight, $T = 2t$ where t is the time of ascent or time taken by the projectile to reach the maximum height.

$$T = 2t = \frac{2v \sin\theta}{g}. \tag{4-29}$$

For projectile motion on an inclined plan: if the plane is inclined at an angle θ with the horizontal and the particle is projected up the slope with and angle a, the motion of the particle can be analyzed in terms of two mutually perpendicular horizontal and vertical directions, wherein coordinate along the incline is (x) and perpendicular to incline is (y).

Magnitude of the resultant velocity: $v = \sqrt{v^2 + g^2t^2 - 2v\,gt\sin\theta}$.

Direction of the velocity:

$$\tan\alpha = \frac{v_y}{v_x} = \frac{v\sin\theta - gt}{v\cos\theta} \Rightarrow \alpha = \tan^{-1}\left(\tan\theta - \frac{gt}{v\cos\theta}\right).$$

Time of flight is the time in which the projected particle strikes the inclined plane.

$$T = \frac{2v_0 \sin(\alpha - \theta)}{g\cos\theta}$$

The range along the inclined plane is

$$R = \frac{2v_0^2 \sin(\alpha - \theta)\cos\alpha}{g\cos^2\theta}$$

The maximum height relative to inclined plane is the maximum height attained by the projected particle relative to the inclined plane.

$$H = \frac{v_0^2 \sin 2(\alpha - \theta)}{2g\cos\theta}$$

Relative Motion When two frames of reference A and B are moving relative to each other at constant velocity, the velocity of a particle P as measured by an observer in frame A usually differs from that measured from frame B. The two measured velocities are related by

$$\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}, \tag{4-41}$$

where $\vec{v}_{BA}$ is the velocity of B with respect to A. Both observers measure the same acceleration for the particle:

$$\vec{a}_{PA} = \vec{a}_{PB}. \tag{4-42}$$

PROBLEMS

1. The position vector for an electron is $\vec{r} = (6.0\text{ m})\hat{i} - (4.0\text{ m})\hat{j} + (3.0\text{ m})\hat{k}$. (a) Find the magnitude of $\vec{r}$. (b) Sketch the vector on a right-handed coordinate system.
2. A watermelon seed has the following coordinates: $x = -5.0$ m, $y = 9.0$ m, and $z = 0$ m. Find its position vector (a) in unit-vector notation and as (b) a magnitude and (c) an angle relative to the positive direction of the x axis. (d) Sketch the vector on a right-handed coordinate system. If the seed is moved to the xyz coordinates (3.00 m, 0 m, 0 m), what is its displacement (e) in unit-vector notation

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and as (f) a magnitude and (g) an angle relative to the positive x direction?

3. An elementary particle is subjected to a displacement of $\Delta\vec{r} = 2.0\hat{i} - 4.0\hat{j} + 8.0\hat{k}$, ending with the position vector $\vec{r} = 4.0\hat{j} - 5.0\hat{k}$, in meters. What was the particle's initial position vector?

4. The minute hand of a wall clock measures 12 cm from its tip to the axis about which it rotates. The magnitude and angle of the displacement vector of the tip are to be determined for three time intervals. What are the (a) magnitude and (b) angle from a quarter after the hour to half past, the (c) magnitude and (d) angle for the next half hour, and the (e) magnitude and (f) angle for the hour after that?

5. A train at a constant 60.0 km/h moves east for 40.0 min, then in a direction 50.0° east of due north for 20.0 min, and then west for 50.0 min. What are the (a) magnitude and (b) angle of its average velocity during this trip?

6. An electron's position is given by $\vec{r} = 3.00t\hat{i} - 4.00t^2\hat{j} + 2.00\hat{k}$, with t in seconds and $\vec{r}$ in meters. (a) In unit-vector notation, what is the electron's velocity $\vec{r}(t)$? At $t = 3.00$ s, what is $\vec{v}$ (b) in unitvector notation and as (c) a magnitude and (d) an angle relative to the positive direction of the x axis?

7. In a particle accelerator, the position vector of a particle is initially estimated as $\vec{r} = 6.0\hat{i} - 7.0\hat{j} + 3.0\hat{k}$ and after 10 s, it is estimated to be $\vec{r} = -3.0\hat{i} + 9.0\hat{j} - 3.0\hat{k}$, all in meters. In unit vector notation, what is the average velocity of the particle?

8. A plane flies 483 km east from city A to city B in 48.0 min and then 966 km south from city B to city C in 1.50 h. For the total trip, what are the (a) magnitude and (b) direction of the plane's displacement, the (c) magnitude and (d) direction of its average velocity, and (e) its average speed?

9. Figure 4-26 gives the path of a squirrel moving about on level ground, from point A (at time $t = 0$), to points B (at $t = 5.00$ min), C (at $t = 10.0$ min), and finally D (at $t = 15.0$ min). Consider the average velocities of the squirrel from point A to each of the other three points. Of them, what are the (a) magnitude and (b) angle of the one with the least magnitude and the (c) magnitude and (d) angle of the one with the greatest magnitude?

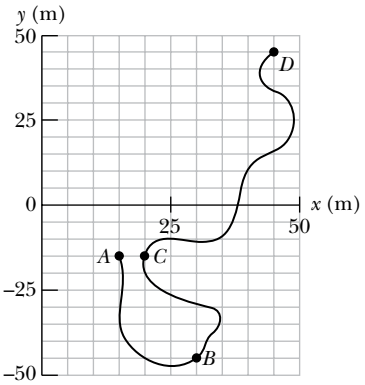

Figure 4-26 Problem 9.

10. The position vector $\vec{r} = 5.00t\hat{i} + (et + ft^2)\hat{j}$ locates a particle as a function of time t. Vector $\vec{r}$ is in meters, t is in seconds, and factors e and f are constants. Figure 4-27 gives the angle θ of the particle's direction of travel as a function of t (θ is measured from the positive x direction). What are (a) e and (b) f, including units?

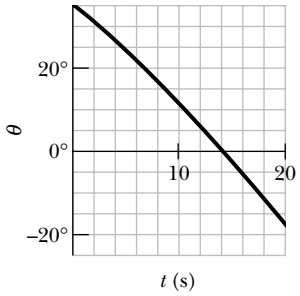

Figure 4-27 Problem 10.

11. A particle that is moving in an xy plane has a position vector given by $\vec{r} = (3.00t^3 - 6.00t)\hat{i} + (7.00 - 8.00t^4)\hat{j}$, where $\vec{r}$ is measured in meters and t is measured in seconds. For $t = 3.00$ s, in unitvector notation, find (a) $\vec{r}$, (b) $\vec{v}$, and (c) $\vec{a}$. (d) Find the angle between the positive direction of the x axis and a line that is tangent to the path of the particle at $t = 3.00$ s.

12. At one instant a bicyclist is 30.0 m due east of a park's flagpole, going due south with a speed of 10.0 m/s. Then 30.0 s later, the cyclist is 40.0 m due north of the flagpole, going due east with a speed of 10.0 m/s. For the cyclist in this 30.0 s interval, what are the (a) magnitude and (b) direction of the displacement, the (c) magnitude and (d) direction of the average velocity, and the (e) magnitude and (f) direction of the average acceleration?

13. An object moves in such a way that its position (in meters) as a function of time (in seconds) is $\vec{r} = \hat{i} + 3t^2\hat{j} + t\hat{k}$. Give expressions for (a) the velocity of the object and (b) the acceleration of the object as functions of time.

14. From the origin, a particle starts at $t = 0$ s with a velocity $\vec{v} = 7.0\hat{i}$ m/s and moves in the xy plane with a constant acceleration $\vec{a} = (-9.0\hat{i} + 3.0\hat{j})$ m/s^2. At the time the particle reaches the maximum x coordinate, what is its (a) velocity and (b) position vector?

15. The velocity $\vec{v}$ of a particle moving in the xy plane is given by $\vec{v} = (6.0t - 4.0t^2)\hat{i} + 8.0\hat{j}$, with $\vec{v}$ in meters per second and t (> 0) in seconds. (a) What is the acceleration when $t = 2.5$ s? (b) When (if ever) is the acceleration zero? (c) When (if ever) is the velocity zero? (d) When (if ever) does the speed equal 10 m/s?

16. A motorbike starts from the origin and moves over an xy plane with acceleration components $a_x = 6.0$ m/s^2 and $a_y = -3.0$ m/s^2. The initial velocity of the motorbike has components $v_{0x} = 12.0$ m/s and $v_{0y} = 18.0$ m/s. Find the velocity of the motorbike, in unit-vector notation, when it reaches its greatest y coordinate.

17. The acceleration of a particle moving only on a horizontal xy plane is given by $\vec{a} = 3t\hat{i} + 4t\hat{j}$, where $\vec{a}$ is in meters per secondsquared and t is in seconds. At $t = 0$, the position vector $\vec{r} = (20.0\text{ m})\hat{i} + (40.0\text{ m})\hat{j}$ locates the particle, which then has the velocity vector $\vec{v} = (5.00\text{ m/s})\hat{i} + (2.00\text{ m/s})\hat{j}$. At $t = 4.00$ s, what are (a) its position vector in unit-vector notation and (b) the angle between its direction of travel and the positive direction of the x axis?

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18. In Fig. 4-28, particle A moves along the line $y = 30$ m with a constant velocity $\vec{v}$ of magnitude 3.0 m/s and parallel to the x axis. At the instant particle A passes the y axis, particle B leaves the origin with a zero initial speed and a constant acceleration $\vec{a}$ of magnitude 0.40 m/s^2. What angle θ between $\vec{a}$ and the positive direction of the y axis would result in a collision?

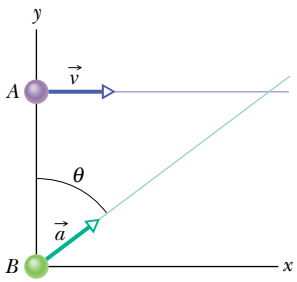

Figure 4-28 Problem 18.

19. A stone is thrown by aiming directly at the center P of a picture hanging on a wall. The stone leaves from the starting point horizontally with a speed of 6.75 m/s and strikes the target at point Q, which is 5.00 cm below P. Find the horizontal distance between the starting point of the stone and the target.

20. A small ball rolls horizontally off the edge of a tabletop that is 1.50 m high. It strikes the floor at a point 1.52 m horizontally from the table edge. (a) How long is the ball in the air? (b) What is its speed at the instant it leaves the table?

21. A shell, which is initially located at a distance of 40.4 m above a horizontal plane, is fired horizontally with a muzzle velocity of 285 m/s to strike a target on the horizontal plane. (a) How long does the projectile remain in the air? (b) At what horizontal distance from the firing point does the shell strike the plane? What are the magnitudes of the (c) horizontal and (d) vertical components of its velocity as it strikes the ground?

22. A stone is catapulted at time $t = 0$, with an initial velocity of magnitude 18.0 m/s and at an angle of 40.0° above the horizontal. What are the magnitudes of the (a) horizontal and (b) vertical components of its displacement from the catapult site at $t = 1.10$ s? Repeat for the (c) horizontal and (d) vertical components at $t = 1.80$ s, and for the (e) horizontal and (f) vertical components at $t = 5.00$ s.

23. A certain airplane has a speed of 290.0 km/h and is diving at an angle of $\theta = 30.0°$ below the horizontal when the pilot releases a radar decoy (Fig. 4-29). The horizontal distance between the release point and the point where the decoy strikes the ground is $d = 700$ m. (a) How long is the decoy in the air? (b) How high was the release point?

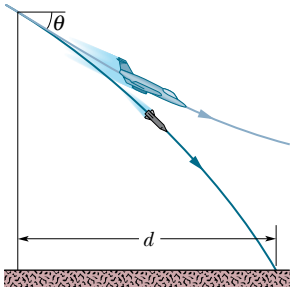

Figure 4-29 Problems 23.

24. In Fig. 4-30 a stone is projected at a cliff of height h with an initial speed of 42.0 m/s directed at angle $\theta_0 = 60.0°$ above the horizontal. The stone strikes at A, 5.50 s after launching. Find (a) the height h of the cliff, (b) the speed of the stone just before impact at A, and (c) the maximum height H reached above the ground.

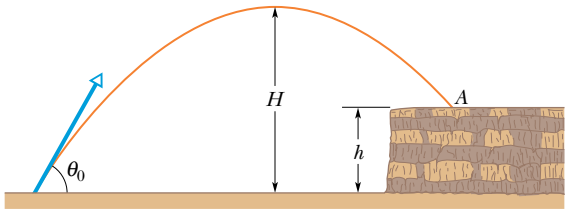

Figure 4-30 Problem 24.

25. A projectile's launch speed is 6.00 times that of its speed at its maximum height. Find the launch angle θ_0.

26. A soccer ball is kicked from the ground with an initial speed of 21.3 m/s at an upward angle of 45°. A player 55 m away in the direction of the kick starts running to meet the ball at that instant. What must be his average speed if he is to meet the ball just before it hits the ground?

27. A soccer player claims that he can kick the ball over a wall of height 3.5 m, which is 32 m away along a horizontal field. The player punts the ball from an elevation of 1.0 m and the ball is projected at an initial speed of 18 m/s in the direction 40° from the horizontal. Does the ball clear the wall?

28. You throw a ball toward a wall at speed 25.0 m/s and at angle $\theta_0 = 40.0°$ above the horizontal (Fig. 4-31). The wall is distance $d = 22.0$ m from the release point of the ball. (a) How far above the release point does the ball hit the wall? What are the (b) horizontal and (c) vertical components of its velocity as it hits the wall? (d) When it hits, has it passed the highest point on its trajectory?

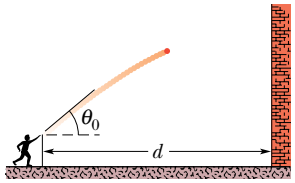

Figure 4-31 Problem 28.

29. A defense air force plane, diving with constant speed at an angle of 52.0° with the vertical, drops a shell at an altitude of 720 m. The shell reaches the ground 6.00 s after its release. (a) What is the speed of the plane? (b) How far does the shell travel horizontally during its flight? What are the (c) horizontal and (d) vertical components of its velocity just before reaching the ground? Assume an x axis in the direction of the horizontal motion and an upward y axis.

30. A rifle that shoots bullets at 460 m/s is to be aimed at a target 45.7 m away. If the center of the target is level with the rifle, how high above the target must the rifle barrel be pointed so that the bullet hits dead center?

31. A golf ball is struck at ground level. The speed of the golf ball as a function of the time is shown in Fig. 4-32, where $t = 0$ at the instant the ball is struck. The scaling on the vertical axis is set by $v_a = 19$ m/s and $v_b = 31$ m/s. (a) How far does the golf ball travel horizontally before returning to ground level? (b) What is the maximum height above ground level attained by the ball?

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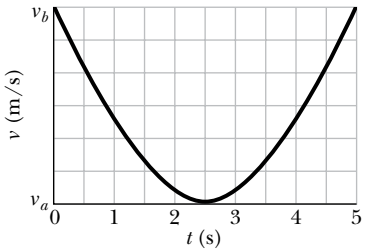

Figure 4-32 Problem 31.

32. In Fig. 4-33, a ball is thrown leftward from the left edge of the roof, at height h above the ground. The ball hits the ground 1.50 s later, at distance $d = 25.0$ m from the building and at angle $\theta = 60.0°$ with the horizontal. (a) Find h. (*Hint:* One way is to reverse the motion, as if on video.) What are the (b) magnitude and (c) angle relative to the horizontal of the velocity at which the ball is thrown? (d) Is the angle above or below the horizontal?

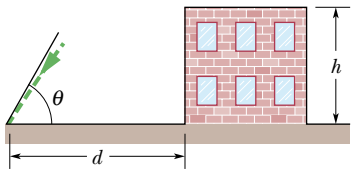

Figure 4-33 Problem 32.

33. Upon spotting an insect on a twig overhanging water, an archer fish squirts water drops at the insect to knock it into the water (Fig. 4-34). Although the insect is located along a straightline path at angle ϕ and distance d, a drop must be launched at a different angle θ_0 if its parabolic path is to intersect the insect. If $\phi = 36.0°$ and $d = 0.900$ m, what launch angle θ_0 is required for the drop to be at the top of the parabolic path when it reaches the insect?

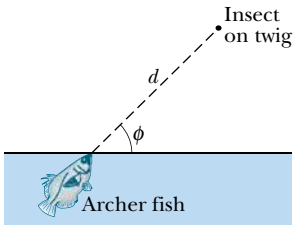

Figure 4-34 Problem 33.

34. A golfer hits a golf ball into the air over level ground. The velocity of the ball at a height of 10.3 m is $\vec{v} = (8.6\hat{i} + 7.2\hat{j})$ m/s, with $\hat{i}$ horizontal and $\hat{j}$ upward. Find (a) the maximum height of the ball and (b) the total horizontal distance traveled by the ball. What are the (c) magnitude and (d) angle (below the horizontal) of the ball's velocity just before it touches the ground?

35. A baseball leaves a pitcher's hand horizontally at a speed of 153 km/h. The distance to the batter is 18.3 m. (a) How long does the ball take to travel the first half of that distance? (b) The second half? (c) How far does the ball fall freely during the first half? (d) During the second half? (e) Why aren't the quantities in (c) and (d) equal?

36. A batter hits a pitched ball when the center of the ball is 1.22 m above the ground. The ball leaves the bat at an angle of 45° with the ground. With that launch, the ball should have a horizontal range (returning to the *launch* level) of 107 m. (a) Does the ball clear a 7.32-m-high fence that is 97.5 m horizontally from the launch point? (b) At the fence, what is the distance between the fence top and the ball center?

37. In Fig. 4-35, a ball is thrown up onto a roof, landing 4.50 s later at height $h = 20.0$ m above the release level. The ball's path just before landing is angled at $\theta = 60.0°$ with the roof. (a) Find the horizontal distance d it travels. (See the hint to Problem 32.) What are the (b) magnitude and (c) angle (relative to the horizontal) of the ball's initial velocity?

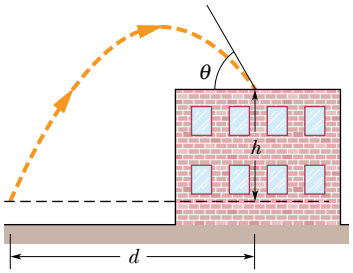

Figure 4-35 Problem 37.

38. Two seconds after being projected from ground level, a projectile is displaced 40 m horizontally and 58 m vertically above its launch point. What are the (a) horizontal and (b) vertical components of the initial velocity of the projectile? (c) At the instant the projectile achieves its maximum height above ground level, how far is it displaced horizontally from the launch point?

39. A ball is to be shot from level ground toward a wall at distance x (Fig. 4-36*a*). Figure 4-36*b* shows the y component v_y of the ball's velocity just as it would reach the wall, as a function of that distance x. The scaling is set by $v_{ys} = 5.0$ m/s and $x_s = 20$ m. What is the launch angle?

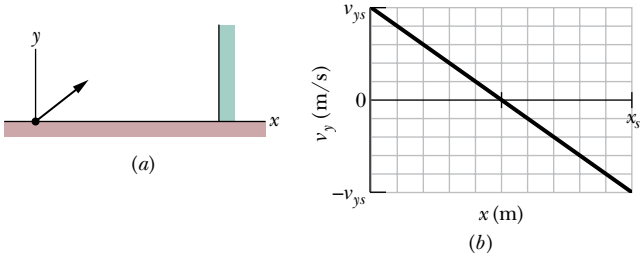

Figure 4-36 Problem 39.

40. In Fig. 4-37, a baseball is hit at a height $h = 1.00$ m and then caught at the same height. It travels alongside a wall, moving up past the top of the wall 1.00 s after it is hit and then down past the top of the wall 4.00 s later, at distance $D = 50.0$ m farther along the wall. (a) What horizontal distance is traveled by the ball from hit to catch? What are the (b) magnitude and (c) angle (relative to the horizontal) of the ball's velocity just after being hit? (d) How high is the wall?

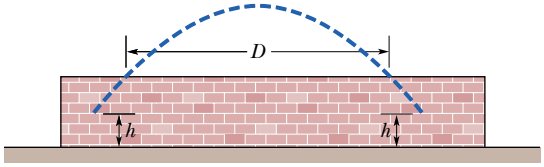

Figure 4-37 Problem 40.

41. A ball is to be shot from level ground with a certain speed. Figure 4-38 shows the range R it will have versus the launch angle θ_0. The value of θ_0 determines the flight time; let t_{max} represent the maximum flight time. What is

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the least speed the ball will have during its flight if θ_0 is chosen such that the flight time is $0.500t_{max}$?

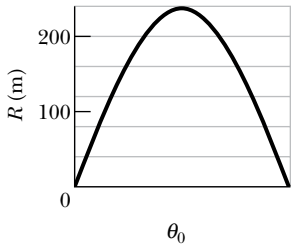

Figure 4-38 Problem 41.

42. A cameraman on a pickup truck is traveling westward at 20 km/h while he records a cheetah that is moving westward 30 km/h faster than the truck. Suddenly, the cheetah stops, turns, and then runs at 45 km/h eastward, as measured by a suddenly nervous crew member who stands alongside the cheetah's path. The change in the animal's velocity takes 2.0 s. What are the (a) magnitude and (b) direction of the animal's acceleration according to the cameraman and the (c) magnitude and (d) direction according to the nervous crew member?

43. A boat is traveling upstream in the positive direction of an x axis at 14 km/h with respect to the water of a river. The water is flowing at 8.2 km/h with respect to the ground. What are the (a) magnitude and (b) direction of the boat's velocity with respect to the ground? A child on the boat walks from front to rear at 6.0 km/h with respect to the boat. What are the (c) magnitude and (d) direction of the child's velocity with respect to the ground?

44. A suspicious-looking man runs as fast as he can along a moving sidewalk from one end to the other, taking 2.50 s. Then security agents appear, and the man runs as fast as he can back along the sidewalk to his starting point, taking 10.0 s. What is the ratio of the man's running speed to the sidewalk's speed?

45. A rugby player runs with the ball directly toward his opponent's goal, along the positive direction of an x axis. He can legally pass the ball to a teammate as long as the ball's velocity relative to the field does not have a positive x component. Suppose the player runs at speed 3.5 m/s relative to the field while he passes the ball with velocity $\vec{v}_{BP}$ relative to himself. If $\vec{v}_{BP}$ has magnitude 6.0 m/s, what is the smallest angle it can have for the pass to be legal?

46. Two highways intersect as shown in Fig. 4-39. At the instant shown, a police car P is distance $d_P = 800$ m from the intersection and moving at speed $v_P = 80$ km/h. Motorist M is distance $d_M = 600$ m from the intersection and moving at speed $v_M = 60$ km/h. (a) In unit-vector notation, what is the velocity of the motorist with respect to the police car? (b) For the instant shown in Fig. 4-39, what is the angle between the velocity found in (a) and the line of sight between the two cars? (c) If the cars maintain their velocities, do the answers to (a) and (b) change as the cars move nearer the intersection?

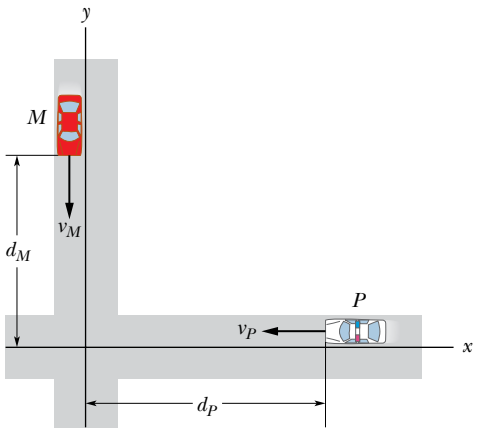

Figure 4-39 Problem 46.

47. After flying for 18 min in a wind blowing 42 km/h at an angle of 20° south of east, an airplane pilot is over a town that is 55 km due north of the starting point. What is the speed of the airplane relative to the air?

48. A light plane attains an airspeed of 500 km/h. The pilot sets out for a destination 900 km due north but discovers that the plane must be headed 20.0° east of due north to fly there directly. The plane arrives in 2.00 h. What were the (a) magnitude and (b) direction of the wind velocity?

49. Snow is falling vertically at a constant speed of 8.0 m/s. At what angle from the vertical do the snowflakes appear to be falling as viewed by the driver of a car traveling on a straight, level road with a speed of 50 km/h?

50. In the overhead view of Fig. 4-40, Jeeps P and B race along straight lines, across flat terrain, and past stationary border guard A. Relative to the guard, B travels at a constant speed of 25.0 m/s, at the angle $\theta_2 = 30.0°$. Relative to the guard, P has accelerated from rest at a constant rate of 0.400 m/s^2 at the angle $\theta_1 = 60.0°$. At a certain time during the acceleration, P has a speed of 40.0 m/s. At that time, what are the (a) magnitude and (b) direction of the velocity of P relative to B and the (c) magnitude and (d) direction of the acceleration of P relative to B?

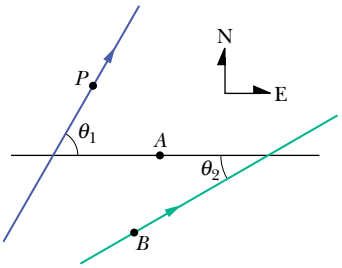

Figure 4-40 Problem 50.

51. Two ships, A and B, leave port at the same time. Ship A travels northwest at 24 knots, and ship B travels at 28 knots in a direction 40° west of south (1 knot = 1 nautical mile per hour; see Appendix D). What are the (a) magnitude and (b) direction of the velocity of ship A relative to B? (c) After what time will the ships be 160 nautical miles apart? (d) What will be the bearing of B (the direction of B's position) relative to A at that time?

52. Ship A is located 4.0 km north and 2.5 km east of ship B. Ship A has a velocity of 22 km/h toward the south, and ship B has a velocity of 40 km/h in a direction 37° north of east. (a) What is the velocity of A relative to B in unit-vector notation with toward the east? (b) Write an expression (in terms of $\hat{i}$ and $\hat{j}$) for the position of A relative to B as a function of t, where $t = 0$ when the ships are in the positions described above. (c) At what time is the separation between the ships least? (d) What is that least separation?

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PRACTICE QUESTIONS

Single Correct Choice Type

1. A diver springs upward from a board that is three meters above the water. At the instant she contacts the water her speed is 8.90 m/s and her body makes an angle of 75.0° with respect to the horizontal surface of the water. Determine her initial velocity, both magnitude and direction.

(a) 4.52 m/s, 59.4° **(b)** 2.49 m/s, 59.4°
(c) 4.21 m/s, 36.7° **(d)** 1.99 m/s, 36.7°

2. An arrow is shot horizontally from a height of 4.9 m above the ground. The initial speed of the arrow is 45 m/s. Neglecting friction, how long will it take the arrow to hit the ground?

(a) 9.2 s **(b)** 1.0 s
(c) 6.0 s **(d)** 1.4 s

3. A basketball player is running at a constant speed of 2.5 m/s when he tosses a basketball upward with a speed of 6.0 m/s. How far does the player run before he catches the ball? Ignore air resistance.

(a) 3.1 m
(b) 4.5 m
(c) The ball cannot be caught because it will fall behind the player.
(d) 6.0 m

4. The altitude of a hang glider is increasing at a rate of 6.80 m/s. At the same time, the shadow of the glider moves along the ground at a speed of 15.5 m/s when the Sun is directly overhead. Find the magnitude of the glider's velocity.

(a) 4.72 m/s **(b)** 14.1 m/s
(c) 9.44 m/s **(d)** 16.9 m/s

5. Rain is falling vertically with a speed of 20 m/s. A person is running in the rain with a velocity of 5 m/s and a wind also starts blowing with a speed of 15 m/s (both from the west). The angle with the vertical at which the person should hold his umbrella so that he may not get drenched is

(a) $\tan^{-1}\left(\frac{1}{2}\right)$ **(b)** $\tan^{-1}\left(\frac{1}{3}\right)$

(c) $\tan^{-1}\left(\frac{1}{4}\right)$ **(d)** $\tan^{-1}(2)$

6. A delivery truck leaves a warehouse and travels 2.60 km north. The truck makes a right turn and travels 1.33 km east before making another right turn and then travels 1.45 km south to arrive at its destination. What is the magnitude and direction of the truck's displacement from the warehouse?

(a) 1.76 km, 40.8° north of east
(b) 1.15 km, 59.8° north of east
(c) 1.33 km, 30.2° north of east
(d) 2.40 km, 45.0° north of east

7. In a football game a kicker attempts a field goal. The ball remains in contact with the kicker's foot for 0.050 s, during which time it experiences an acceleration of 340 m/s^2. The ball is launched at an angle of 51° above the ground. Determine the horizontal and vertical components of the launch velocity.

	v_x	v_y
(a)	13 m/s	11 m/s
(b)	21 m/s	11 m/s
(c)	11 m/s	13 m/s
(d)	17 m/s	21 m/s

8. At time $t = 0$ s, a disk is sliding on a horizontal table with a velocity 3.00 m/s, 65.0° above the $+x$ axis. As the disk slides, a constant acceleration acts on it that has the following components: $a_x = -0.460$ m/s^2 and $a_y = -0.980$ m/s^2. What is the velocity of the puck at time $t = 1.50$ s?

(a) 1.83 m/s, 62.0° above the $+x$ axis
(b) 2.04 m/s, 71.3° above the $+x$ axis
(c) 1.38 m/s, 65.2° above the $+x$ axis
(d) 1.06 m/s, 58.7° above the $+x$ axis

9. A volleyball is spiked so that it has an initial velocity of 15 m/s directed downward at an angle of 55° below the horizontal. What is the horizontal component of the ball's velocity when the opposing player fields the ball?

(a) 6.8 m/s **(b)** 8.6 m/s
(c) 12 m/s **(d)** 18 m/s

10. A pitcher can throw a baseball in excess of 41.0 m/s. If a ball is thrown horizontally at this speed, how much will it drop by the time it reaches a catcher who is 17.0 m away from the point of release?

(a) 0.422 m **(b)** 1.19 m
(c) 0.844 m **(d)** 1.68 m

11. A car drives straight off the edge of a cliff that is 54 m high. The police at the scene of the accident observe that the point of impact is 130 m from the base of the cliff. How fast was the car traveling when it went over the cliff?

(a) 39 m/s **(b)** 53 m/s
(c) 22 m/s **(d)** 45 m/s

12. A passenger at rest on a flatbed train car fires a bullet straight up. The event is viewed by observers at rest on the station platform as the train moves past the platform with constant velocity. What is the trajectory of the bullet as described by the observers on the platform?

(a) A straight horizontal path in the direction of the train's velocity.
(b) A straight vertical path up and down.
(c) A circular path centered on the gun.
(d) A parabolic path.

13. A particle moving with velocity v changes its direction of motion by an angle θ without change in speed. Which of the following statements is not correct?

(a) The magnitude of the change in its velocity is $2v\sin(\theta/2)$.
(b) The change in the magnitude of its velocity is zero.
(c) The change in its velocity makes an angle $(\pi/2 + \theta/2)$ with its initial direction of motion.
(d) The change in velocity is equal to the negative of the resultant of the initial and final velocities.

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14. A bicyclist is riding at a constant speed along a straight-line path. The rider throws a ball straight up to a height a few meters above her head. Ignoring air resistance, where will the ball land?
(a) Behind the rider.
(b) In front of the rider.
(c) In the same hand that threw the ball.
(d) In the opposite hand to the one that threw it.

15. Two trees have perfectly straight trunks and are both growing perpendicular to the flat horizontal ground beneath them. The sides of the trunks that face each other are separated by 1.3 m. A squirrel makes three jumps in rapid succession. First, he leaps from the foot of one tree to a spot that is 1.0 m above the ground on the other tree. Then, he jumps back to the first tree, landing on it at a spot that is 1.7 m above the ground. Finally, he leaps back to the other tree, now landing at a spot that is 2.5 m above the ground. What is the magnitude of the squirrel's displacement?
(a) 1.3 m **(b)** 2.8 m
(c) 2.5 m **(d)** 3.4 m

16. A spotlight S moves in a horizontal plane with a constant angular velocity of 0.1 rad/s. The spotlight P moves along the wall at a distance 3 m. When line SP makes an angle 45° with the wall, the velocity of spotlight P is
(a) 0.3 m/s **(b)** 0.6 m/s
(c) 0.8 m/s **(d)** 1.2 m/s

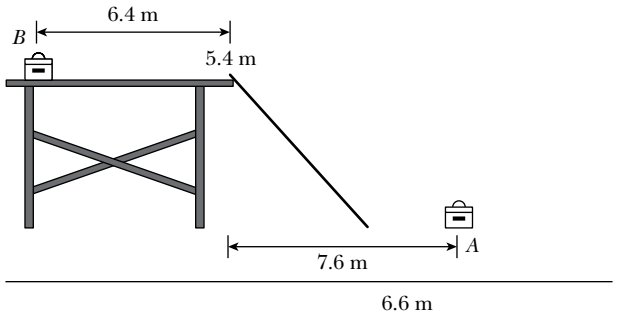
17. A particle moves in the plane x–y with constant acceleration a directed along the negative y-axis. The equation of motion of the particle has the form $y = px - qx^2$ where p and q are positive constants. Find the velocity of the particle at the origin of coordinates.

(a) $\sqrt{\left[\frac{a(p^2+1)}{2q}\right]}$ **(b)** $\sqrt{\left[\frac{(p^2+1)}{q}\right]}$

(c) $\sqrt{\left[\frac{2a(p^2-1)}{q}\right]}$ **(d)** $\sqrt{\left[\frac{a(1-p^2)}{2q}\right]}$

18. A bird watcher travels through the forest, walking 0.50 km due east, 0.75 km due south, and 2.15 km in a direction 35.0° north of west. The time required for this trip is 2.50 h. Determine the magnitude and direction (relative to due west) of the bird watcher's average velocity. Use kilometers and hours for distance and time, respectively.
(a) 0.540 km/h, 21° north of west
(b) 1.43 km/h, 17° south of west
(c) 1.22 km/h, 18° north of west
(d) 1.08 km/h, 25° north of west

19. A toolbox is carried from the base of a ladder at point A as shown in the figure. The toolbox comes to a rest on a platform 5.0 m above the ground. What is the magnitude of the displacement of the toolbox in its movement from point A to point B?
(a) 15 m **(b)** 19 m
(c) 11 m **(d)** 13 m



20. On a spacecraft, two engines are turned on for 684 s at a moment when the velocity of the craft has x and y components of $v_{ox} = 4370$ m/s and $v_{oy} = 6280$ m/s. While the engines are firing, the craft undergoes a displacement that has components of $x = 4.11 \times 10^6$ m and $y = 6.07 \times 10^6$ m. Find the x and y components of the spacecraft's acceleration.

	a_x	a_y
(a)	9.58 m/s²	5.06 m/s²
(b)	6.39 m/s²	10.1 m/s²
(c)	4.79 m/s²	7.59 m/s²
(d)	5.06 m/s²	9.58 m/s²

21. A man swims across the river which flows with a velocity of 3 km/h due east. If the velocity of man relative to water is 4 km/h due north, then what is his velocity and its direction relative to the shore of the river?
(a) 4 km/h, 63°50′ west of north
(b) 5 km/h, 36°52′ west of north
(c) 4 km/h, 63°50′ east of north
(d) 5 km/h, 36°52′ east of north

22. A golfer imparts a speed of 30.3 m/s to a ball, and it travels the maximum possible distance before landing on the green. The tee and the green are at the same elevation. How much time does the ball spend in the air?
(a) 4.37 s **(b)** 2.68 s
(c) 3.51 s **(d)** 1.84 s

23. A boat is traveling relative to the water at a speed of 5.0 m/s due south. Relative to the boat, a passenger walks toward the back of the boat at a speed of 1.5 m/s. What is the magnitude and direction of the passenger's velocity relative to the water?
(a) 5.2 m/s, south **(b)** 3.5 m/s, south
(c) 3.5 m/s, north **(d)** 6.5 m/s, south

24. A marble is thrown horizontally with a speed of 15 m/s from the top of a building. When it strikes the ground, the marble has a velocity that makes an angle of 65° with the horizontal. From what height above the ground was the marble thrown?
(a) 19 m **(b)** 38 m
(c) 53 m **(d)** 47 m

25. Baseball player A strikes the ball by hitting it in such a way that it acquires an initial velocity of 1.9 m/s parallel to the ground. Upon contact with the bat the ball is 1.2 m above the ground. Player B wishes to duplicate this strike, in so far as he also wants to give the ball a velocity parallel to the ground and have his ball travel the same horizontal

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distance as player A's ball does. However, player B hits the ball when it is 1.5 m above the ground. What is the magnitude of the initial velocity that player B's ball must be given?

(a) 1.3 m/s **(b)** 1.9 m/s
(c) 1.7 m/s **(d)** 2.1 m/s

26. A particle P is sliding down a frictionless hemispherical bowl. It passes the point A at $t = 0$. At this instant of time, the horizontal component of its velocity is v. A bead Q of same mass as P is ejected from A at $t = 0$ along the horizontal string AB, with a speed v. Friction between the bead and the string may be neglected. Let t_P and t_Q be the respective times taken by P and Q to reach the point B, then

(a) $t_P < t_Q$

(b) $t_P = t_Q$

(c) $t_P > t_Q$

(d) $\dfrac{t_P}{t_Q} = \dfrac{\text{length of arc } ACB}{\text{length of chord } AB}$

27. A baseball is hit into the air at an initial speed of 36.6 m/s and an angle of 50.0° above the horizontal. At the same time, the center fielder starts running away from the batter, and he catches the ball 0.914 m above the level at which it was hit. If the center fielder is initially 1.10×10^2 m from home plate, what must be his average speed?

(a) 3.5 m/s **(b)** 5.0 m/s
(c) 4.2 m/s **(d)** 6.9 m/s

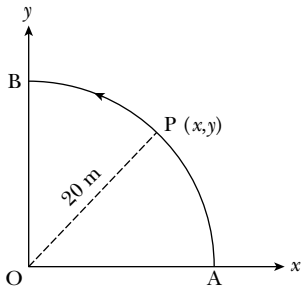
28. A player in the football team tries to kick a football so that it stays in the air for a long "hang time". If the ball is kicked with an initial velocity of 25.0 m/s at an angle of 60.0° above the ground, what is the "hang time"?

(a) 4.42 s **(b)** 3.36 s
(c) 2.21 s **(d)** 1.68 s

29. A rifle is used to shoot twice at a target, using identical cartridges. The first time, the rifle is aimed parallel to the ground and directly at the center of the bull's-eye. The bullet strikes the target at a distance of H_A below the center, however. The second time, the rifle is similarly aimed, but from twice the distance from the target. This time the bullet strikes the target at a distance of H_B below the center. Find the ratio H_B/H_A.

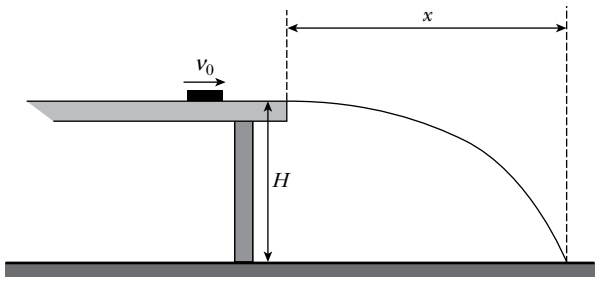
(a) 2 **(b)** 4
(c) 3 **(d)** 5

30. A point P moves in counter–clockwise direction on a circular path as shown in the figure. The movement of P is such that it sweeps out a length $s = t^3 + 5$, where s is in meters and t is in seconds. The radius of the path is 20 m. The acceleration of P when $t = 2$ s is nearly



(a) 14 m/s^2
(b) 13 m/s^2
(c) 12 m/s^2
(d) 7.2 m/s^2

31. A disk slides across a smooth, level tabletop at height H at a constant speed v_0. It slides off the edge of the table and hits the floor a distance x away as shown in the figure.

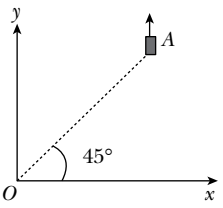


What is the relationship between the distances x and H?

(a) $x = v_0\sqrt{\dfrac{2H}{g}}$ **(b)** $x = \dfrac{v_0^2}{gH}$

(c) $x = \dfrac{v_0^2}{2gH}$ **(d)** $H = v_0\sqrt{\dfrac{2x}{g}}$

32. On a frictionless horizontal surface, assumed to be the x–y plane, a small trolley A is moving along a straight line parallel to the y-axis with a constant velocity of $(\sqrt{3} - 1)$ m/s. At a particular instant, when the line OA makes an angle 45° with the x-axis, a ball is thrown along the surface from the origin O. Its velocity makes an angle ϕ with the x-axis and it hits the trolley.



Find the speed of the ball with respect to the surface. If $\phi = 4\theta/3$

(a) 2 m/s **(b)** 5 m/s
(c) 7 m/s **(d)** 10 m/s

33. A bullet is fired from a rifle that is held 1.6 m above the ground in a horizontal position. The initial speed of the bullet is 1100 m/s. Find the time it takes for the bullet to strike the ground and the horizontal distance traveled by the bullet.

(a) 0.24 s, 693 m **(b)** 0.57 s, 630 m
(c) 0.32 s, 440 m **(d)** 0.63 s, 352 m

34. An airplane with a speed of 97.5 m/s is climbing upward at an angle of 50.0° with respect to the horizontal. When the plane's altitude is 732 m, the pilot releases a package. (a) Calculate the distance along the ground, measured from a point directly beneath the point of release, to where the package hits the earth.

(a) +425 m **(b)** +1380 m
(c) –678 m **(d)** –2880 m

35. A bullet is aimed at a target on the wall a distance L away from the firing position. Because of gravity, the bullet strikes the wall a distance Δy below the mark as suggested in the figure. The drawing is not to scale.

If the distance L was half as large and the bullet had the same initial velocity, how would Δy be affected?

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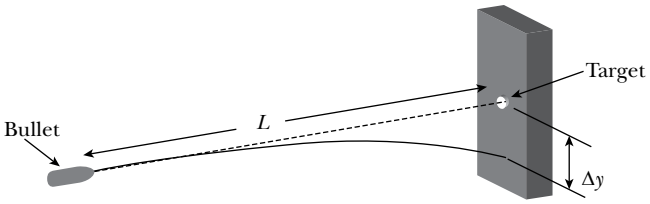
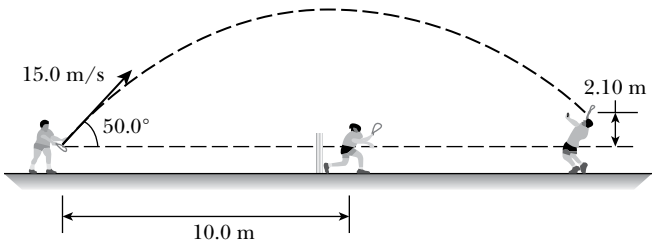


(a) Δy will double.
(b) Δy will be half as large.
(c) Δy will be four times larger.
(d) Δy will be one fourth as large.

36. In the javelin throw at a track-and-field event, the javelin is launched at a speed of 29 m/s at an angle of 36° above the horizontal. As the javelin travels upward, its velocity points above the horizontal at an angle that decreases as time passes. How much time is required for the angle to be reduced from 36° at launch to 18°?
(a) 0.96 s **(b)** 1.12 s
(c) 1.04 s **(d)** 1.16 s

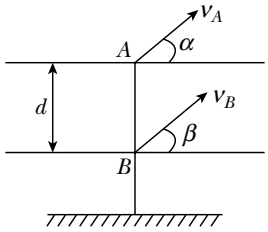
37. Take the z axis as vertical and the xy plane as horizontal. A particle A is projected at $4\sqrt{2}$ m/s at an angle of 45° to the horizontal, in the xz plane. Particle B is projected at 5 m/s at an angle $\theta = \tan^{-1}(4/3)$ to y axis, in the yz plane. Which of the following is not correct for the velocity of B with respect to A?
(a) Its initial magnitude is 5 m/s.
(b) Its magnitude will change with time.
(c) It lies in the xy plane.
(d) It will initially make an angle $(\theta + \pi/2)$ with the positive x axis.

38. An effective tactic in tennis, when your opponent is near the net, consists of lofting the ball over his head, forcing him to move quickly away from the net (see figure). Suppose that you loft the ball with an initial speed of 15.0 m/s, at an angle of 50.0° above the horizontal. At this instant your opponent is 10.0 m away from the ball. He begins moving away from you 0.30 s later, hoping to reach the ball and hit it back at the moment that it is 2.10 m above its launch point. With what minimum average speed must he move? (Ignore the fact that he can stretch, so that his racket can reach the ball before he does.)



(a) 4.98 m/s **(b)** 5.79 m/s
(c) 5.14 m/s **(d)** 9.64 m/s

39. Two particles A and B are thrown simultaneously from two different floors of tower having distance d between them. Velocities of particle A is v_A at angle α from horizontal while velocities of particle B is v_B at asn angle β from horizontal. If two particles collide in mid-air, then $\frac{v_A}{v_B}$.



(a) $\frac{\cos\beta}{\cos\alpha}$ **(b)** $\frac{\sin\alpha}{\sin\beta}$
(c) $\frac{\tan\alpha}{\tan\beta}$ **(d)** $\frac{\cot\alpha}{\cot\beta}$

40. On a spacecraft two engines fire for a time of 565 s. One gives the craft an acceleration in the x direction of $a_x = 5.10$ m/s^2, while the other produces an acceleration in the y direction of $a_y = 7.30$ m/s^2. At the end of the firing period, the craft has velocity components of $v_x = 3775$ m/s and $v_y = 4816$ m/s. Find the magnitude and direction of the initial velocity. Express the direction as an angle with respect to the $+x$ axis.
(a) 1130 m/s, 52.3° **(b)** 793 m/s, 37.7°
(c) 1130 m/s, 37.7° **(d)** 793 m/s, 52.3°

41. Two cars, A and B, are traveling in the same direction, although car A is 186 m behind car B. The speed of A is 24.4 m/s, and the speed of B is 18.6 m/s. How much time does it take for A to catch B?
(a) 32.1 s **(b)** 11.8 s
(c) 23.6 s **(d)** 8.33 s

42. Starting from one oasis, a camel walks 25 km in a direction 30° south of west and then walks 30 km towards the north to a second oasis. What distance separates the two cases?
(a) 15 km **(b)** 48 km
(c) 28 km **(d)** 53 km

43. A speed ramp at an airport is a moving conveyor belt on which you can either stand or walk. It is intended to get you from place to place more quickly. Suppose a speed ramp is 120 m long. When you walk at a comfortable speed on the ground, you cover this distance in 86 s. When you walk on the speed ramp at this same comfortable speed, you cover this distance in 35 s. Determine the speed at which the speed ramp is moving relative to the ground.
(a) 3.4 m/s **(b)** 1.4 m/s
(c) 2.0 m/s **(d)** 2.4 m/s

44. The captain of a plane wishes to proceed due west. The cruising speed of the plane is 245 m/s relative to the air. A weather report indicates that a 38.0 m/s wind is blowing from the south to the north. In what direction, measured with respect to due west, should the pilot head the plane relative to the air?
(a) 81.1° south of west **(b)** 17.8° south of west
(c) 8.92° south of west **(d)** 8.82° north of west

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45. A jetliner can fly 6.00 hours on a full load of fuel. Without any wind it flies at a speed of 2.40×10^2 m/s. The plane is to make a round-trip by heading due west for a certain distance, turning around, and then heading due east for the return trip. During the entire flight, however, the plane encounters a 57.8 m/s wind from the jet stream, which blows from west to east. What is the maximum distance that the plane can travel due west and just be able to return home?

(a) 1870 km **(b)** 2160 km
(c) 2440 km **(d)** 1950 km

46. You are traveling in a car with convertible roof with the top down. The car is moving at a constant velocity of 25 m/s, due east along flat ground. You throw a tomato straight upward at a speed of 11 m/s. How far has the car moved when you get a chance to catch the tomato?

(a) 66 m **(b)** 44 m
(c) 56 m **(d)** 28 m

47. The highest barrier that a projectile can clear is 13.5 m, when the projectile is launched at an angle of 15.0° above the horizontal. What is the projectile's launch speed?

(a) 15.7 m/s **(b)** 62.8 m/s
(c) 44.4 m/s **(d)** 22.2 m/s

48. Relative to the ground, a car has a velocity of 18.0 m/s, directed due north. Relative to this car, a truck has a velocity of 22.8 m/s, directed 52.1° south of east. Find the magnitude and direction of the truck's velocity relative to the ground.

(a) 4.8 m/s, 37.9° north of east
(b) 22.8 m/s, 37.9° south of east
(c) 14.0 m/s, due east
(d) 20.4 m/s, 68.2° north of east

49. Two boats are heading away from shore. Boat 1 heads due north at a speed of 3.00 m/s relative to the shore. Relative to Boat 1, Boat 2 is moving 30.0° north of east at a speed of 1.60 m/s. A passenger on Boat 2 walks due east across the deck at a speed of 1.20 m/s relative to Boat 2. What is the speed of the passenger relative to the shore?

(a) 4.60 m/s **(b)** 2.71 m/s
(c) 2.53 m/s **(d)** 3.14 m/s

More than One Correct Choice Type

50. The coordinates of a particle moving in a plane are given by $x(t) = a\cos(pt)$ and $y(t) = b\sin(pt)$ where a, $b(<a)$, and p are positive constants of appropriate dimensions. Then

(a) the path of the particle is an ellipse.
(b) the velocity and acceleration of the particle are normal to each other at $t = \pi/(2p)$.
(c) the acceleration of the particle is always directed toward a focus.
(d) the distance traveled by the particle in time interval $t = 0$ to $t = \pi/(2p)$ is a.

51. A man can swim with a velocity v relative to water. He has to cross a river of width d flowing with a velocity u $(u > v)$. The distance through which he is carried downstream by the river is x. Which of the following statements are correct?

(a) If he crosses the river in minimum time, $x = d \times \frac{v}{u}$
(b) x cannot be less than $d \times \frac{v}{u}$
(c) For x to be minimum, he has to swim in a direction making an angle of $\pi/2 + \sin^{-1}(v/u)$ with the direction of the flow of water.
(d) x will be maximum if he swims in a direction making an angle of $\pi/2 - \sin^{-1}(v/u)$ with the direction of the flow of water.

52. Two shells are fired from a canon successively with speed u each at angles of projection α and β, respectively. If the time interval between the firing of shells is dt and they collide in mid-air after a time t from the firing of the first shell. Then

(a) $t\cos\alpha = (t - dt)\cos\beta$
(b) $\alpha > \beta$
(c) $(t - dt)\cos\alpha = dt\cos\beta$
(d) $(u\sin\alpha)t - \frac{1}{2}gdt^2 = (u\sin\beta)t - dt - \frac{1}{2}g(t - dt)^2$

Linked Comprehension

***Paragraph for Questions 53 and 54*:** During a one-hour trip, a small boat travels 80.0 km north and then travels 60.0 km east.

53. What is the boat's displacement for the one-hour trip?

(a) 20 km **(b)** 140 km
(c) 100 km **(d)** 280 km

54. What is the direction of the boat's average velocity for the one-hour trip?

(a) due east **(b)** 41.7° north of east
(c) 53.1° north of east **(d)** 49.2° north of east

***Paragraph for Questions 55–59*:** A projectile fired from a gun has initial horizontal and vertical components of velocity equal to 30 m/s and 40 m/s, respectively.

55. At what angle is the projectile fired (measured with respect to the horizontal)?

(a) 37° **(b)** 45°
(c) 40° **(d)** 53°

56. Approximately how long does it take the projectile to reach the highest point in its trajectory?

(a) 1 s **(b)** 4 s
(c) 2 s **(d)** 8 s

57. What is the speed of the projectile when it is at the highest point in its trajectory?

(a) 0 m/s **(b)** 30 m/s
(c) 20 m/s **(d)** 40 m/s

58. What is the acceleration of the projectile when it reaches its maximum height?

(a) zero m/s^2
(b) 9.8 m/s^2, downward
(c) 4.9 m/s^2, downward
(d) less than 9.8 m/s^2 and non-zero.

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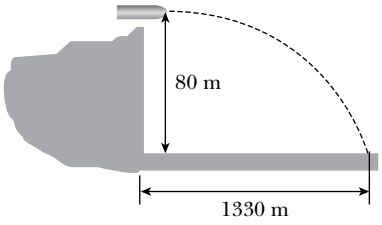
59. What is the magnitude of the projectile's velocity just before it strikes the ground?
(a) zero m/s **(b)** 30 m/s
(c) 50 m/s **(d)** 9.8 m/s

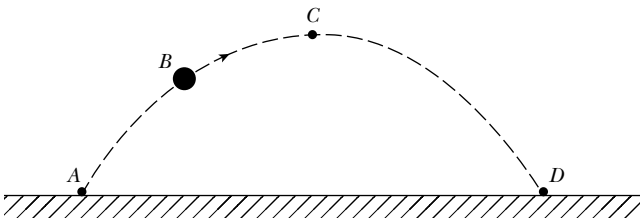
***Paragraph for Questions 60 and 61*:** A shell is fired with a horizontal velocity in the positive x direction from the top of an 80 m high cliff. The shell strikes the ground 1330 m from the base of the cliff. The drawing is not to scale.



60. What is the speed of the shell as it hits the ground?
(a) 4.0 m/s **(b)** 330 m/s
(c) 9.8 m/s **(d)** 170 m/s

61. What is the magnitude of the acceleration of the shell just before it strikes the ground?
(a) 4.0 m/s^2 **(b)** 82 m/s^2
(c) 9.8 m/s^2 **(d)** 170 m/s^2

***Paragraph for Questions 62–64*:** A tennis ball is thrown upward at an angle from point A. It follows a parabolic trajectory and hits the ground at point D. At the instant shown, the ball is at point B. Point C represents the highest position of the ball above the ground.

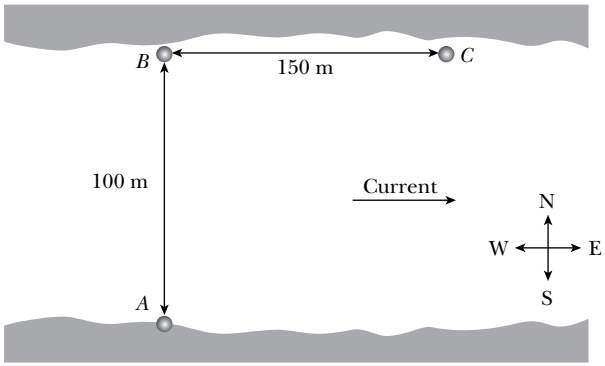


62. While in flight, how do the x and y components of the velocity vector of the ball compare at the points B and C?
(a) The velocity components are non-zero at B and zero at C.
(b) The x components are the same; the y component at C is zero m/s.
(c) The x components are the same; the y component has a larger magnitude at C than at B.
(d) The x component is larger at C than at B; the y component at B points up while at C, it points downward.

63. While in flight, how do the x and y components of the velocity vector of the ball compare at the points A and D?
(a) The velocity components are non-zero at A and are zero m/s at D.
(b) The velocity components are the same in magnitude and direction at both points.
(c) The velocity components have the same magnitudes at both points, but their directions are reversed.
(d) The velocity components have the same magnitudes at both points, but the directions of the y components are reversed.

64. Which statement is true concerning the ball when it is at C, the highest point in its trajectory?
(a) The ball's velocity and acceleration are both zero.
(b) The ball's velocity is perpendicular to its acceleration.
(c) The ball's velocity is not zero, but its acceleration is zero.
(d) The ball's velocity is zero, but its acceleration is not zero.

***Paragraph for Questions 65 and 66*:** A man at point A directs his rowboat due north toward point B, straight across a river of width 100 m. The river current is due east. The man starts across, rowing steadily at 0.75 m/s and reaches the other side of the river at point C, 150 m downstream from his starting point.



65. What is the speed of the river?
(a) 0.38 m/s **(b)** 1.1 m/s
(c) 0.67 m/s **(d)** 6.7 m/s

66. While the man is crossing the river, what is his velocity relative to the shore?
(a) 1.35 m/s, 34° north of east
(b) 2.11 m/s, 34° north of east
(c) 2.00 m/s, 56° north of east
(d) 1.74 m/s, 34° north of east

Matrix-Match

67. Trajectory of particle in a projectile motion is given as: $y = x - (x^2/80)$ (x and y are in meters). Take $g = 10$ m/s^2.

Column I	Column II
(a) Angle of projection	**(p)** 20 m
(b) Angle of velocity with horizontal after 4 s	**(q)** 80 m
(c) Maximum height	**(r)** 45°
(d) Horizontal range	**(s)** $\tan^{-1}(1/2)$

68. A body is projected with a velocity of 60 m/s at 30° to horizontal (assume positive x-axis and positive y-axis as horizontal and vertical directions, respectively, and take $g = 10\ \text{m/s}^2$).

Column I	Column II
(a) Initial velocity vector (in SI units)	**(p)** $60\sqrt{3}\hat{i} + 40\hat{j}$
(b) Velocity after 3 s (in SI units)	**(q)** $-10\hat{j}$
(c) Displacement after 2 s (in SI units)	**(r)** $30\sqrt{3}\hat{i} + 30\hat{j}$
(d) Centripetal acceleration after 2 s (in SI units)	**(s)** $30\sqrt{3}\hat{i}$

Directions for Questions 69 and 70: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

69. A ball is projected from the ground with velocity v such that its range is maximum. Answer the questions by appropriately matching the information given in the three columns of the following table.

Column I	Column II	Column III
(I) Velocity	**(i)** half of its maximum height	**(J)** $v_x\hat{i} + \frac{v_y}{\sqrt{2}}\hat{j}$
(II) Acceleration	**(ii)** at maximum height	**(K)** $\frac{v_x}{\sqrt{2}}\hat{i}$
(III) Change in velocity	**(iii)** At initial point	**(L)** $\frac{v_x}{\sqrt{2}}\hat{i} + \frac{v_y}{\sqrt{2}}\hat{j}$
(IV) Average velocity	**(iv)** At final point	**(M)** $\frac{v_x}{\sqrt{2}}\hat{i} + v_y\hat{j}$

(1) Which is the correct option to represent a point on trajectory at which the particle has only one component of which quantity?

(a) (III) (i) (L) **(b)** (IV) (iii) (J)
(c) (II) (ii) (K) **(d)** (I) (iv) (M)

(2) Which of the following is the correct columns match?

(a) (IV) (iii) (K) **(b)** (II) (iv) (L)
(c) (III) (ii) (M) **(d)** (I) (i) (J)

(3) In which case are the horizontal and vertical velocity components of the particle are the same?

(a) (III) (iii) (L) **(b)** (I) (ii) (K)
(c) (IV) (i) (M) **(d)** (II) (iii) (J)

70. The particle is projected in projectile motion with velocity u at an angle θ with horizontal is represented by $y = Px - Qx^2$. Answer the questions by appropriately matching the information given in the three columns of the following table.

Column I	Column II	Column III
(I) Range	**(i)** $\frac{P}{Q}$	**(J)** $\frac{2u}{g}\sin\theta$
(II) Maximum Height	**(ii)** P	**(K)** $\tan\theta$
(III) Time of flight	**(iii)** $\frac{P^2}{4Q}$	**(L)** $\frac{u^2}{g}\sin 2\theta$
(IV) Tangent of the angle of projection	**(iv)** $\left(\sqrt{\frac{2}{Qg}}\right)P$	**(M)** $\frac{u^2}{2g}\sin^2\theta$

(1) For which quantity of projectile motion P and Q are linearly dependent to each other and what is the relation between them?

(a) (II) (iv) (K) **(b)** (IV) (ii) (M)
(c) (III) (iii) (J) **(d)** (I) (i) (L)

(2) Which of the following combination shows the correct relation of quantities when total displacement of the particle is zero?

(a) (I) (i) (M) **(b)** (III) (iv) (J)
(c) (I) (iii) (L) **(d)** (IV) (ii) (K)

(3) At what point is the velocity of the particle is minimum?

(a) (II) (iii) (M) **(b)** (I) (iv) (L)
(c) (IV) (ii) (K) **(d)** (III) (i) (J)

Integer Type

71. Two swimmers leave a point A on one bank of a river to reach a point B lying right across on the other bank. One of them crosses the river along the straight line AB while the other swims at right angles to the stream and then walks the distance that he has been carried away by the stream to get to the point B. What was the velocity u (in km/h) of walking if both swimmers reached simultaneously? The stream velocity $v_0 = 2$ km/h and the velocity v' or each swimmer with respect to the water equals 2.5 km/h.

72. A cannon fires successively two shells with the same speed of 250 m/s, the first at an angle 60° and the second at 45° to the horizontal. The shells collide in midair. Find the time interval between the firings. The trajectories of the shells lie in the same plane.

ANSWER KEY

Checkpoints

1. (draw $\vec{v}$ tangent to path, tail on path) (a) first; (b) third
2. (take second derivative with respect to time) (1) and (3) a_x and a_y are both constant and thus $\vec{a}$ is constant; (2) and (4) a_y is constant but a_x is not, thus $\vec{a}$ is not
3. yes
4. (a) v_x constant; (b) v_y initially positive, decreases to zero, and then becomes progressively more negative; (c) $a_x = 0$ throughout; (d) $a_y = -g$ throughout

Problems

1. (a) 7.8 m
2. (a) $(-5.0\text{ m})\hat{i} + (9.0\text{ m})\hat{j}$; (b) 10 m; (c) –61° or 119° (d) 119°counterclockwise from the $+x$ direction; (e) $\Delta\vec{r} = (8.0\text{ m})\hat{i} - (9.0\text{ m})\hat{j}$; (f) 12 m; (g) (–42°, or 42° measured *clockwise* from $+x$)
3. $-(2.0\text{ m})\hat{i} + (8.0\text{ m})\hat{j} - (13\text{ m})\hat{k}$
4. (a) 17 cm; (b) –135°; (c) 24 cm; (d) 90°; (e) zero; (f) zero
5. (a) 7.59 km/h; (b) 67.5° (north of east), or 22.5° east of due north
6. (a) $(3.00\text{ m/s})\hat{i} - (8.00t\text{ m/s})\hat{j}$; (b) $(3.00\hat{i} - 24.0\hat{j})$m/s; (c) 24.2 m/s; (d) 82.9° measured *clockwise* from the $+x$ direction, or 277° counterclockwise from $+x$)
7. $(-0.90\hat{i} + 1.6\hat{j} - 0.60\hat{k})$ m/s
8. (a) 1.08×10^3 km; (b) 63.4° south of east, or 26.6° east of south; (c) 470 km/h 470 km/h$\angle -63.4°$; (d); (e) 630 km/h
9. (a) 0.0083 m/s; (b) 0° (measured counterclockwise from the $+x$ axis); (c) 0.11 m/s; (d) 297° (counterclockwise from $+x$) or –63° (which is equivalent to measuring 63° *clockwise* from the $+x$ axis).
10. (a) 3.50 m/s; (b) –0.125 m/s^2
11. (a) $(\vec{r})_{t=3s} = (63.0\hat{i} - 641\hat{j})$m; (b) $(\vec{v})_{t=3s} = (75.0\hat{i} - 864\hat{j})$m/s; (c) $(\vec{a})_{t=3s} = (54.0\hat{i} - 864\hat{j})$m/s^2; (d) (–85.0°, which is equivalent to 275° measured counterclockwise from the $+x$ axis
12. (a) 50.0 m; (b) 127° (53° north of due west); (c) 1.67 m/s; (d) 127° (53° north of due west); (e) 0.471 m/s^2; (f) 45° or 135°
13. (a) $(6t)\hat{j} + \hat{k}$; (b) $6\hat{j}$
14. (a) $(2.3\text{ m/s})\hat{j}$; (b) $(2.71\text{ m})\hat{i} + (0.91\text{ m})\hat{j}$
15. (a) $(-14\text{ m/s}^2)\hat{i}$; (b) 0.75 s; (c) it is never zero; (d) 2.2 s
16. $(48\text{ m/s})\hat{i}$
17. (a) $(72.0\text{ m})\hat{i} + (90.7\text{ m})\hat{j}$; (b) 49.5°
18. 60°
19. 0.682 m
20. (a) 0.553 s; (b) 2.75 m/s
21. (a) ≈2.87 s; (b) 818 m; (c) 285 m/s; (d) 28.1 m/s
22. (a) 15.17 m ≈ 15.2 m; (b) 6.80 m; (c) 24.8 m; (d) 4.95 m; (e) 32.6 m; (f) 0
23. (a) 10.0 s; (b) 897 m
24. (a) 5.50 s; (b) 27.4 m/s; (c) 67.5 m
25. 80.4°
26. 2.8 m/s
27. No
28. (a) 12.0 m; (b) 19.2 m/s; (c) 4.80 m/s; (d) does not reach the highest point
29. (a) 147 m/s; (b) 696 m; (c) 116 m/s; (d) 149 m/s
30. 0.0484 m = 4.84 cm
31. (a) 95 m; (b) 31 m
32. (a) 32.3 m; (b) 21.9 m/s; (c) 40.4°; (d) below
33. 55.5°
34. (a) ≈13 m; (b) ≈28 m; (c) 18 m/s; (d) –62° (which is equivalent to 298°)
35. (a) 0.215 s; (b) 0.215 s; (c) 0.227 m; (d) 0.681 m
36. (a) yes; (b) 2.56 m
37. (a) 45.7 m; (b) 28.4 m/s; (c) –68.9°
38. (a) 20 m/s; (b) ≈39 m/s; (c) 79 m
39. 14.3° ≈ 14°
40. (a) 75.0 m; (b) 31.9 m/s; (c) 67.0°; (d) 25.5 m
41. 42.0 m/s
42. (a) 13 m/s^2; (b) $+\hat{i}$, or eastward; (c) 13 m/s^2; (d) $+\hat{i}$, or eastward
43. (a) 6 km/h; (b) $+x$, or upstream; (c) $(0\text{ km/h})\hat{i}$
44. 1.67
45. 126°
46. (a) $(80\text{ km/h})\hat{i} - (60\text{ km/h})\hat{j}$; (b) 0; (c) No, they remain unchanged
47. ≈200 km/h
48. (a) 172 km/h; (b) 6.6 (south of west)
49. 60°
50. (a) 22.2 m/s; (b) –85.7°, which is 85.7 north of west; (c) 0.400 m/s^2; (d) 60.0° north of due east (or 30.0° east of north)
51. (a) 38.4 knots; (b) 1.5° east of north; (c) 4.2 h; (d) 1.5 west of due south
52. (a) $(-32\text{ km/h})\hat{i} - (46\text{ km/h})\hat{j}$; (b) $(2.5 - 3.2t)\hat{i} + (4.0 - 46t)\hat{j}$; (c) 0.084 h; (d) 2×10^2 m

Practice Questions

Single Correct Choice Type

1. (a)	2. (b)	3. (a)	4. (d)	5. (a)
6. (a)	7. (c)	8. (c)	9. (b)	10. (c)
11. (a)	12. (d)	13. (d)	14. (c)	15. (b)
16. (b)	17. (a)	18. (a)	19. (d)	20. (c)
21. (d)	22. (a)	23. (b)	24. (c)	25. (c)
26. (a)	27. (c)	28. (a)	29. (b)	30. (a)
31. (a)	32. (a)	33. (b)	34. (b)	35. (d)
36. (a)	37. (b)	38. (b)	39. (a)	40. (c)
41. (a)	42. (c)	43. (c)	44. (c)	45. (c)
46. (c)	47. (b)	48. (c)	49. (a)	

More than One Correct Choice Type

50. (a), (c) 51. (a), (c) 52. (a), (b), (d)

Linked Comprehension

53. (c)	54. (c)	55. (d)	56. (b)	57. (b)
58. (b)	59. (c)	60. (b)	61. (c)	62. (b)
63. (d)	64. (b)	65. (b)	66. (a)	

Matrix-Match

67. (a) $\rightarrow$ (r); (b) $\rightarrow$ (s); (c) $\rightarrow$ (p); (d) $\rightarrow$ (q)

68. (a) $\rightarrow$ (r); (b) $\rightarrow$ (s); (c) $\rightarrow$ (p); (d) $\rightarrow$ (q)

69. (1) $\rightarrow$ (c); (2) $\rightarrow$ (d); (3) $\rightarrow$ (a)

70. (1) $\rightarrow$ (d); (2) $\rightarrow$ (b); (3) $\rightarrow$ (a)

Integer Type

71. 3 72. 11

CHAPTER

5

Force and Motion – I

5.1 | WHAT IS PHYSICS?

We have seen that part of physics is a study of motion, including accelerations, which are changes in velocities. Physics is also a study of what can *cause* an object to accelerate. That cause is a **force**, which is, loosely speaking, a push or pull on the object. The force is said to *act* on the object to change its velocity. For example, when a dragster accelerates, a force from the track acts on the rear tires to cause the dragster's acceleration. When a defensive guard knocks down a quarterback, a force from the guard acts on the quarterback to cause the quarterback's backward acceleration. When a car slams into a telephone pole, a force on the car from the pole causes the car to stop. Science, engineering, legal, and medical journals are filled with articles about forces on objects, including people.

A Heads Up. Many students find this chapter to be more challenging than the preceding ones. One reason is that we need to use vectors in setting up equations—we cannot just sum some scalars. So, we need the vector rules from Chapter 3. Another reason is that we shall see a lot of different arrangements: objects will move along floors, ceilings, walls, and ramps. They will move upward on ropes looped around pulleys or by sitting in ascending or descending elevators. Sometimes, objects will even be tied together.

However, in spite of the variety of arrangements, we need only a single key idea (Newton's second law) to solve most of the homework problems. The purpose of this chapter is for us to explore how we can apply that single key idea to any given arrangement. The application will take experience—we need to solve lots of problems, not just read words. So, let's go through some of the words and then get to the sample problems.

5.2 | NEWTONIAN MECHANICS

The relation between a force and the acceleration it causes was first understood by Isaac Newton (1642–1727) and is the subject of this chapter. The study of that relation, as Newton presented it, is called *Newtonian mechanics*. We shall focus on its three primary laws of motion.

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Newtonian mechanics does not apply to all situations. If the speeds of the interacting bodies are very large—an appreciable fraction of the speed of light—we must replace Newtonian mechanics with Einstein's special theory of relativity, which holds at any speed, including those near the speed of light. If the interacting bodies are on the scale of atomic structure (for example, they might be electrons in an atom), we must replace Newtonian mechanics with quantum mechanics. Physicists now view Newtonian mechanics as a special case of these two more comprehensive theories. Still, it is a very important special case because it applies to the motion of objects ranging in size from the very small (almost on the scale of atomic structure) to astronomical (galaxies and clusters of galaxies).

5.3 | NEWTON'S FIRST LAW

Key Concept

- The velocity of an object can change (the object can accelerate) when the object is acted on by one or more forces (pushes or pulls) from other objects. Newtonian mechanics relates accelerations and forces.

Before Newton formulated his mechanics, it was thought that some influence, a "force," was needed to keep a body moving at constant velocity. Similarly, a body was thought to be in its "natural state" when it was at rest. For a body to move with constant velocity, it seemingly had to be propelled in some way, by a push or a pull. Otherwise, it would "naturally" stop moving.

These ideas were reasonable. If you send a puck sliding across a wooden floor, it does indeed slow and then stop. If you want to make it move across the floor with constant velocity, you have to continuously pull or push it.

Send a puck sliding over the ice of a skating rink, however, and it goes a lot farther. You can imagine longer and more slippery surfaces, over which the puck would slide farther and farther. In the limit you can think of a long, extremely slippery surface (said to be a **frictionless surface**), over which the puck would hardly slow. (We can in fact come close to this situation by sending a puck sliding over a horizontal air table, across which it moves on a film of air.)

From these observations, we can conclude that a body will keep moving with constant velocity if no force acts on it. That leads us to the first of Newton's three laws of motion:

Newton's First Law: If no force acts on a body, the body's velocity cannot change; that is, the body cannot accelerate.

In other words, if the body is at rest, it stays at rest. If it is moving, it continues to move with the same velocity (same magnitude *and* same direction).

5.4 | FORCE

Key Concepts

- Forces are vector quantities. Their magnitudes are defined in terms of the acceleration they would give the standard kilogram. A force that accelerates that standard body by exactly 1 m/s^2 is defined to have a magnitude of 1 N. The direction of a force is the direction of the acceleration it causes.
- Forces are combined according to the rules of vector algebra. The net force on a body is the vector sum of all the forces acting on the body.
- If there is no net force on a body, the body remains at rest if it is initially at rest or moves in a straight line at constant speed if it is in motion.
- Reference frames in which Newtonian mechanics holds are called inertial reference frames or inertial frames and those in which it does not hold are called noninertial reference frames or noninertial frames.

Before we begin working problems with forces, we need to discuss several features of forces, such as the force unit, the vector nature of forces, the combining of forces, and the circumstances in which we can measure forces (without being fooled by a fictitious force).

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Unit. We can define the unit of force in terms of the acceleration a force would give to the standard kilogram (Fig. 1-5), which has a mass defined to be exactly 1 kg. Suppose we put that body on a horizontal, frictionless surface and pull horizontally (Fig. 5-1) such that the body has an acceleration of 1 m/s^2. Then we can define our applied force as having a magnitude of 1 newton (abbreviated N). If we then pulled with a force magnitude of 2 N, we would find that the acceleration is 2 m/s^2. Thus, the acceleration is proportional to the force. If the standard body of 1 kg has an acceleration of magnitude a (in meters per second per second), then the force (in newtons) producing the acceleration has a magnitude equal to a. We now have a workable definition of the force unit.

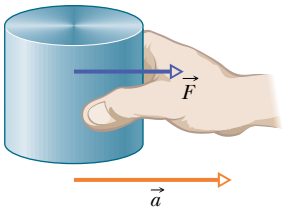

Figure 5-1 A force $\vec{F}$ on the standard kilogram gives that body an acceleration $\vec{a}$.

Vectors. Force is a vector quantity and thus has not only magnitude but also direction. So, if two or more forces act on a body, we find the **net force** (or **resultant force**) by adding them as vectors, following the rules of Chapter 3. A single force that has the same magnitude and direction as the calculated net force would then have the same effect as all the individual forces. This fact, called the **principle of superposition for forces**, makes everyday forces reasonable and predictable. The world would indeed be strange and unpredictable if, say, you and a friend each pulled on the standard body with a force of 1 N and somehow the net pull was 14 N and the resulting acceleration was 14 m/s^2.

In this book, forces are most often represented with a vector symbol such as $\vec{F}$, and a net force is represented with the vector symbol $\vec{F}_{net}$. As with other vectors, a force or a net force can have components along coordinate axes. When forces act only along a single axis, they are single-component forces. Then we can drop the overhead arrows on the force symbols and just use signs to indicate the directions of the forces along that axis.

The First Law. Instead of our previous wording, the more proper statement of Newton's first law is in terms of a *net* force:

Newton's First Law: If no *net* force acts on a body ($\vec{F}_{net} = 0$), the body's velocity cannot change; that is, the body cannot accelerate.

There may be multiple forces acting on a body, but if their net force is zero, the body cannot accelerate. So, if we happen to know that a body's velocity is constant, we can immediately say that the net force on it is zero.

Inertial Reference Frames

Newton's first law is not true in all reference frames, but we can always find reference frames in which it (as well as the rest of Newtonian mechanics) is true. Such special frames are referred to as **inertial reference frames**, or simply **inertial frames**.

An inertial reference frame is one in which Newton's laws hold.

For example, we can assume that the ground is an inertial frame provided we can neglect Earth's astronomical motions (such as its rotation).

That assumption works well if, say, a puck is sent sliding along a *short* strip of frictionless ice—we would find that the puck's motion obeys Newton's laws. However, suppose the puck is sent sliding along a *long* ice strip extending from the north pole (Fig. 5-2*a*). If we view the puck from a stationary frame in space, the puck moves south along a simple straight line because Earth's rotation around the north pole merely slides the ice beneath the puck. However, if we view the puck from a point on the ground so that we rotate with Earth, the puck's path is not a simple straight line. Because the eastward speed of the ground beneath the puck is greater the farther south the puck slides, from our ground-based view the puck appears to be deflected westward (Fig. 5-2*b*). However, this apparent deflection is caused not by a force as required by Newton's laws but by the fact that we see the puck from a rotating frame. In this situation, the ground is a **noninertial frame**, and trying to explain the deflection in terms of a force would lead us to a fictitious force. A more common example of inventing such a nonexistent force can

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occur in a car that is rapidly increasing in speed. You might claim that a force to the rear shoves you hard into the seat back.

In this book we usually assume that the ground is an inertial frame and that measured forces and accelerations are from this frame. If measurements are made in, say, a vehicle that is accelerating relative to the ground, then the measurements are being made in a noninertial frame and the results can be surprising.

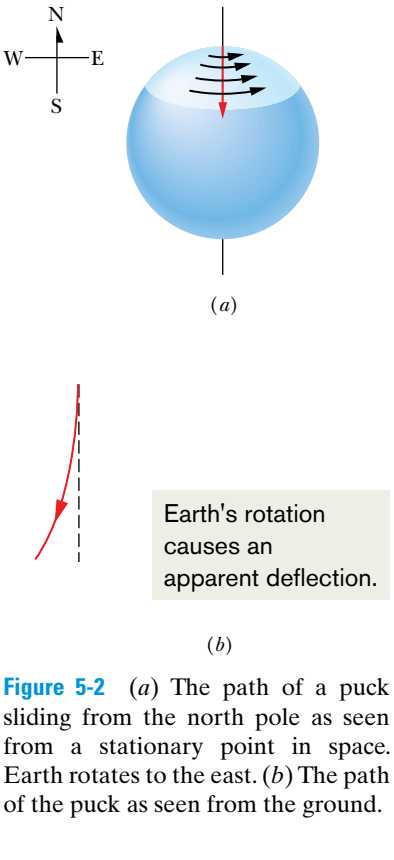
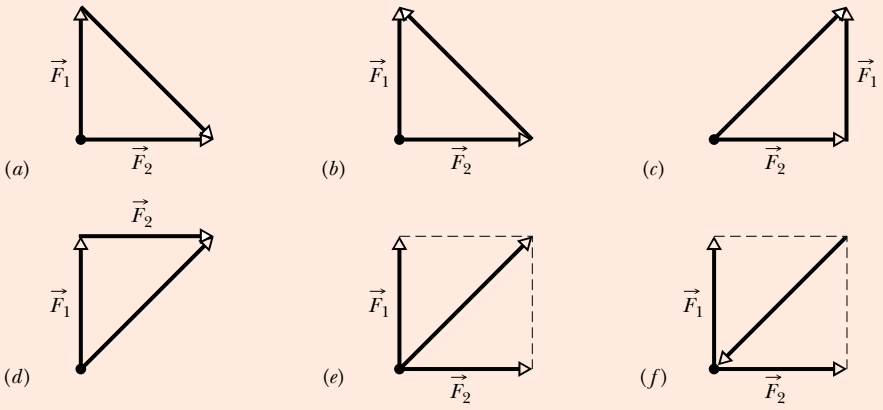


Figure 5-2 (*a*) The path of a puck sliding from the north pole as seen from a stationary point in space. Earth rotates to the east. (*b*) The path of the puck as seen from the ground.

CHECKPOINT 1

Which of the figure's six arrangements correctly show the vector addition of forces $\vec{F}_1$ and $\vec{F}_2$ to yield the third vector, which is meant to represent their net force $\vec{F}_{net}$?



5.5 | MASS

Key Concept

- The mass of a body is the characteristic of that body that relates the body's acceleration to the net force causing the acceleration. Masses are scalar quantities.

From everyday experience you already know that applying a given force to bodies (say, a baseball and a bowling ball) results in different accelerations. The common explanation is correct: The object with the larger mass is accelerated less. But we can be more precise. The acceleration is actually inversely related to the mass (rather than, say, the square of the mass).

Let's justify that inverse relationship. Suppose, as previously, we push on the standard body (defined to have a mass of exactly 1 kg) with a force of magnitude 1 N. The body accelerates with a magnitude of 1 m/s². Next we push on body X with the same force and find that it accelerates at 0.25 m/s². Let's make the (correct) assumption that with the same force,

$$\frac{m_X}{m_0} = \frac{a_0}{a_X},$$

and thus

$$m_X = m_0 \frac{a_0}{a_X} = (1.0\ \text{kg}) \frac{1.0\ \text{m/s}^2}{0.25\ \text{m/s}^2} = 4.0\ \text{kg}.$$

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Defining the mass of X in this way is useful only if the procedure is consistent. Suppose we apply an 8.0 N force first to the standard body (getting an acceleration of 8.0 m/s^2) and then to body X (getting an acceleration of 2.0 m/s^2). We would then calculate the mass of X as

$$m_X = m_0 \frac{a_0}{a_X} = (1.0\ \text{kg}) \frac{8.0\ \text{m/s}^2}{2.0\ \text{m/s}^2} = 4.0\ \text{kg},$$

which means that our procedure is consistent and thus usable.

The results also suggest that mass is an intrinsic characteristic of a body—it automatically comes with the existence of the body. Also, it is a scalar quantity. However, the nagging question remains: What, exactly, is mass?

Since the word *mass* is used in everyday English, we should have some intuitive understanding of it, maybe something that we can physically sense. Is it a body's size, weight, or density? The answer is no, although those characteristics are sometimes confused with mass. We can say only that *the mass of a body is the characteristic that relates a force on the body to the resulting acceleration*. Mass has no more familiar definition; you can have a physical sensation of mass only when you try to accelerate a body, as in the kicking of a baseball or a bowling ball.

5.6 | NEWTON'S SECOND LAW

Key Concepts

- The net force $\vec{F}_{\text{net}}$ on a body with mass m is related to the body's acceleration $\vec{a}$ by

  $$\vec{F}_{\text{net}} = m\vec{a},$$

  which may be written in the component versions

  $$F_{\text{net},x} = ma_x \quad F_{\text{net},y} = ma_y \quad \text{and} \quad F_{\text{net},z} = ma_z.$$

  The second law indicates that in SI units

  $$1\ \text{N} = 1\ \text{kg} \cdot \text{m/s}^2.$$

- A free-body diagram is a stripped-down diagram in which only *one* body is considered. That body is represented by either a sketch or a dot. The external forces on the body are drawn, and a coordinate system is superimposed, oriented so as to simplify the solution.

All the definitions, experiments, and observations we have discussed so far can be summarized in one neat statement:

Newton's Second Law: The net force on a body is equal to the product of the body's mass and its acceleration.

In equation form,

$$\vec{F}_{\text{net}} = m\vec{a} \quad \text{(Newton's second law).} \tag{5-1}$$

Identify the Body. This simple equation is the key idea for nearly all the homework problems in this chapter, but we must use it cautiously. First, we must be certain about which body we are applying it to. Then $\vec{F}_{\text{net}}$ must be the vector sum of *all* the forces that act on *that* body. Only forces that act on *that* body are to be included in the vector sum, not forces acting on other bodies that might be involved in the given situation. For example, if you are in a rugby scrum, the net force on *you* is the vector sum of all the pushes and pulls on *your* body. It does not include any push or pull on another player from you or from anyone else. Every time you work a force problem, your first step is to clearly state the body to which you are applying Newton's law.

Separate Axes. Like other vector equations, Eq. 5-1 is equivalent to three component equations, one for each axis of an xyz coordinate system:

$$F_{\text{net},x} = ma_x, \quad F_{\text{net},y} = ma_y, \quad \text{and} \quad F_{\text{net},z} = ma_z. \tag{5-2}$$

Each of these equations relates the net force component along an axis to the acceleration along that same axis. For example, the first equation tells us that the sum of all the force components along the x axis causes the

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x component a_x of the body's acceleration, but causes no acceleration in the y and z directions. Turned around, the acceleration component a_x is caused only by the sum of the force components along the x axis and is *completely* unrelated to force components along another axis. In general,

The acceleration component along a given axis is caused *only by* the sum of the force components along that *same* axis, and not by force components along any other axis.

Forces in Equilibrium. Equation 5-1 tells us that if the net force on a body is zero, the body's acceleration $\vec{a} = 0$. If the body is at rest, it stays at rest; if it is moving, it continues to move at constant velocity. In such cases, any forces on the body *balance* one another, and both the forces and the body are said to be in *equilibrium*. Commonly, the forces are also said to *cancel* one another, but the term "cancel" is tricky. It does *not* mean that the forces cease to exist (canceling forces is not like canceling dinner reservations). The forces still act on the body but cannot change the velocity.

Units. For SI units, Eq. 5-1 tells us that

$$1\ \text{N} = (1\ \text{kg})(1\ \text{m/s}^2) = 1\ \text{kg}\cdot\text{m/s}^2. \tag{5-3}$$

Some force units in other systems of units are given in Table 5-1 and Appendix D.

Table 5-1 Units in Newton's Second Law (Eqs. 5-1 and 5-2)

System	Force	Mass	Acceleration
SI	newton (N)	kilogram (kg)	m/s^2
CGS[a]	dyne	gram (g)	cm/s^2
British[b]	pound (lb)	slug	ft/s^2

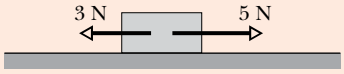
[a] $1\ \text{dyne} = 1\ \text{g}\cdot\text{cm/s}^2$.
[b] $1\ \text{lb} = 1\ \text{slug}\cdot\text{ft/s}^2$.

Diagrams. To solve problems with Newton's second law, we often draw a **free-body diagram** in which the only body shown is the one for which we are summing forces. A sketch of the body itself is preferred by some teachers but, to save space in these chapters, we shall usually represent the body with a dot. Also, each force on the body is drawn as a vector arrow with its tail anchored on the body. A coordinate system is usually included, and the acceleration of the body is sometimes shown with a vector arrow (labeled as an acceleration). This whole procedure is designed to focus our attention on the body of interest.

External Forces Only. A **system** consists of one or more bodies, and any force on the bodies inside the system from bodies outside the system is called an **external force**. If the bodies making up a system are rigidly connected to one another, we can treat the system as one composite body, and the net force $\vec{F}_{\text{net}}$ on it is the vector sum of all external forces. (We do not include **internal forces**—that is, forces between two bodies inside the system. Internal forces cannot accelerate the system.) For example, a connected railroad engine and car form a system. If, say, a tow line pulls on the front of the engine, the force due to the tow line acts on the whole engine–car system. Just as for a single body, we can relate the net external force on a system to its acceleration with Newton's second law, $\vec{F}_{\text{net}} = m\vec{a}$, where m is the total mass of the system.

CHECKPOINT 2

The figure here shows two horizontal forces acting on a block on a frictionless floor. If a third horizontal force $\vec{F}_3$ also acts on the block, what are the magnitude and direction of $\vec{F}_3$ when the block is (a) stationary and (b) moving to the left with a constant speed of 5 m/s?



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SAMPLE PROBLEM 5.01

One- and two-dimensional forces, puck

Here are examples of how to use Newton's second law for a puck when one or two forces act on it. Parts (*a*), (*b*), and (*c*) of Fig. 5-3 show three situations in which one or two forces act on a puck that moves over frictionless ice along an *x* axis, in one-dimensional motion. The puck's mass is $m = 0.20$ kg. Forces $\vec{F}_1$ and $\vec{F}_2$ are directed along the axis and have magnitudes $F_1 = 4.0$ N and $F_2 = 2.0$ N. Force $\vec{F}_3$ is directed at angle $\theta = 30°$ and has magnitude $F_3 = 1.0$ N. In each situation, what is the acceleration of the puck?

KEY IDEA

In each situation we can relate the acceleration $\vec{a}$ to the net force $\vec{F}_{net}$ acting on the puck with Newton's second law, $\vec{F}_{net} = m\vec{a}$. However, because the motion is along only the *x* axis, we can simplify each situation by writing the second law for *x* components only:

$$F_{net,x} = ma_x. \tag{5-4}$$

The free-body diagrams for the three situations are also given in Fig. 5-3, with the puck represented by a dot.

Calculations:

Situation A: For Fig. 5-3*b*, where only one horizontal force acts, Eq. 5-4 gives us

$$F_1 = ma_x,$$

which, with given data, yields

$$a_x = \frac{F_1}{m} = \frac{4.0\text{ N}}{0.20\text{ kg}} = 20\text{ m/s}^2. \qquad \text{(Answer)}$$

The positive answer indicates that the acceleration is in the positive direction of the *x* axis.

Situation B: In Fig. 5-3*d*, two horizontal forces act on the puck, $\vec{F}_1$ in the positive direction of *x* and $\vec{F}_2$ in the negative direction. Now Eq. 5-4 gives us

$$F_1 - F_2 = ma_x,$$

which, with given data, yields

$$a_x = \frac{F_1 - F_2}{m} = \frac{4.0\text{ N} - 2.0\text{ N}}{0.20\text{ kg}} = 10\text{ m/s}^2. \qquad \text{(Answer)}$$

Thus, the net force accelerates the puck in the positive direction of the *x* axis.

Situation C: In Fig. 5-3*f*, force $\vec{F}_3$ is not directed along the direction of the puck's acceleration; only *x* component $F_{3,x}$ is. (Force $\vec{F}_3$ is two-dimensional but the motion is only one-dimensional.) Thus, we write Eq. 5-4 as

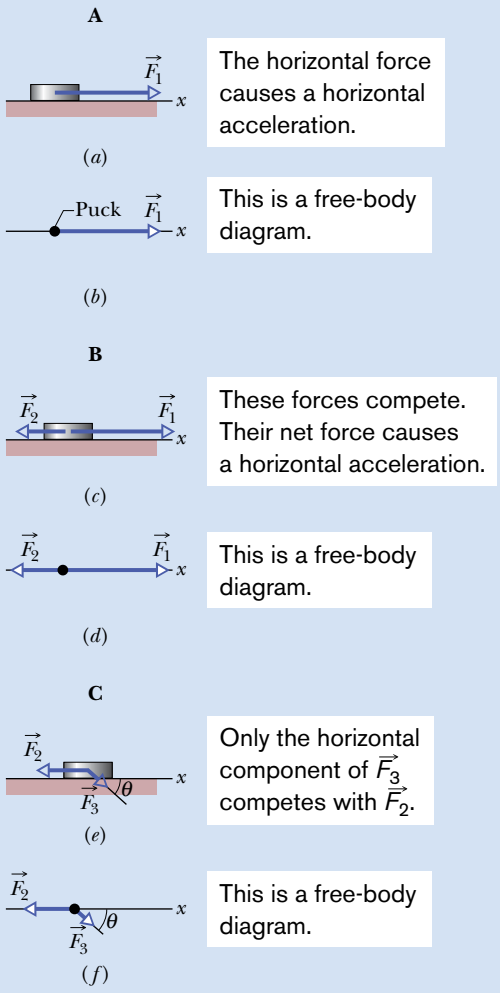

Figure 5-3 In three situations, forces act on a puck that moves along an *x* axis. Free-body diagrams are also shown.

$$F_{3,x} - F_2 = ma_x. \tag{5-5}$$

From the figure, we see that $F_{3,x} = F_3 \cos\theta$. Solving for the acceleration and substituting for $F_{3,x}$ yield

$$a_x = \frac{F_{3,x} - F_2}{m} = \frac{F_3\cos\theta - F_2}{m}$$
$$= \frac{(1.0\text{ N})(\cos 30°) - 2.0\text{ N}}{0.20\text{ kg}} = -5.7\text{ m/s}^2. \qquad \text{(Answer)}$$

Thus, the net force accelerates the puck in the negative direction of the *x* axis.

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SAMPLE PROBLEM 5.02

Two-dimensional forces, cookie tin

Here we find a missing force by using the acceleration. In the overhead view of Fig. 5-4*a*, a 2.0 kg cookie tin is accelerated at 3.0 m/s² in the direction shown by $\vec{a}$, over a frictionless horizontal surface. The acceleration is caused by three horizontal forces, only two of which are shown: $\vec{F}_1$ of magnitude 10 N and $\vec{F}_2$ of magnitude 20 N. What is the third force $\vec{F}_3$ in unit-vector notation and in magnitude-angle notation?

KEY IDEA

The net force $\vec{F}_{net}$ on the tin is the sum of the three forces and is related to the acceleration $\vec{a}$ via Newton's second law ($\vec{F}_{net} = m\vec{a}$). Thus,

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = m\vec{a}, \tag{5-6}$$

which gives us

$$\vec{F}_3 = m\vec{a} - \vec{F}_1 - \vec{F}_2. \tag{5-7}$$

Calculations: Because this is a two-dimensional problem, we *cannot* find $\vec{F}_3$ merely by substituting the magnitudes for the vector quantities on the right side of Eq. 5-7. Instead, we must vectorially add $m\vec{a}$, $-\vec{F}_1$ (the reverse of $\vec{F}_1$), and $-\vec{F}_2$ (the reverse of $\vec{F}_2$), as shown in Fig. 5-4*b*. This addition can be done directly on a vector-capable calculator because we know both magnitude and angle for all three vectors. However, here we shall evaluate the right side of Eq. 5-7 in terms of components, first along the *x* axis and then along the *y* axis. *Caution:* Use only one axis at a time.

x components: Along the *x* axis we have

$$\begin{aligned} F_{3,x} &= ma_x = F_{1,x} - F_{2,x} \\ &= m(a\cos 50°) - F_1\cos(-150°) - F_2\cos 90°. \end{aligned}$$

Then, substituting known data, we find

$$\begin{aligned} F_{3,x} &= (2.0\text{ kg})(3.0\text{ m/s}^2)\cos 50° - (10\text{ N})\cos(-150°) \\ &\quad - (20\text{ N})\cos 90° \\ &= 12.5\text{ N}. \end{aligned}$$

y components: Similarly, along the *y* axis we find

$$\begin{aligned} F_{3,y} &= ma_y - F_{1,y} - F_{2,y} \\ &= m(a\sin 50°) - F_1\sin(-150°) - F_2\sin 90° \\ &= (2.0\text{ kg})(3.0\text{ m/s}^2)\sin 50° - (10\text{ N})\sin(-150°) \\ &\quad - (20\text{ N})\sin 90° \\ &= -10.4\text{ N}. \end{aligned}$$

Vector: In unit-vector notation, we can write

$$\begin{aligned} \vec{F}_3 &= F_{3,x}\hat{i} + F_{3,y}\hat{j} = (12.5\text{ N})\hat{i} - (10.4\text{ N})\hat{j} \\ &\approx (13\text{ N})\hat{i} - (10\text{ N})\hat{j}. \end{aligned} \tag{Answer}$$

We can now use a vector-capable calculator to get the magnitude and the angle of $\vec{F}_3$. We can also use Eq. 3-6 to obtain the magnitude and the angle (from the positive direction of the *x* axis) as

$$F_3 = \sqrt{F_{3,x}^2 + F_{3,y}^2} = 16\text{ N} \tag{Answer}$$

and

$$\theta = \tan^{-1}\frac{F_{3,y}}{F_{3,x}} = -40°. \tag{Answer}$$

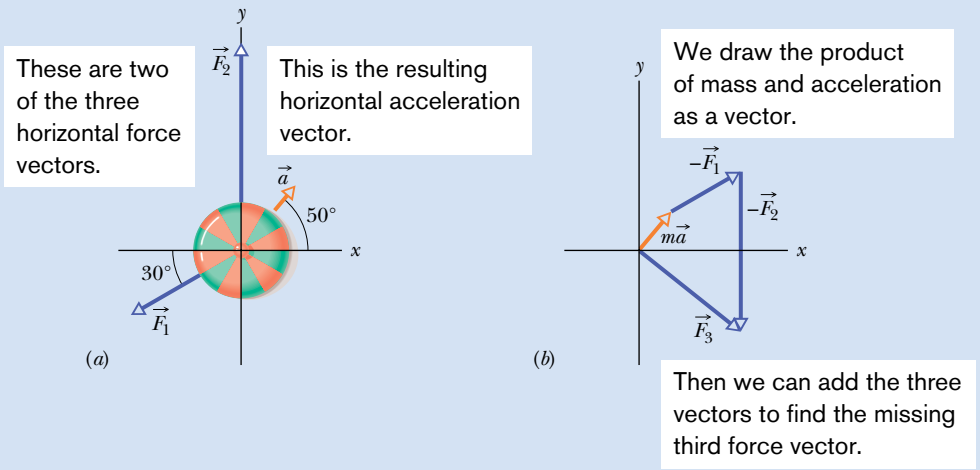
These are two of the three horizontal force vectors.
This is the resulting horizontal acceleration vector.
We draw the product of mass and acceleration as a vector.
Then we can add the three vectors to find the missing third force vector.
(a)
(b)

Figure 5-4 (*a*) An overhead view of two of three horizontal forces that act on a cookie tin, resulting in acceleration $\vec{a}$. $\vec{F}_3$ is not shown. (*b*) An arrangement of vectors $m\vec{a}$, $-\vec{F}_1$, and $-\vec{F}_2$ to find force $\vec{F}_3$.

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5.7 | NEWTON'S THIRD LAW

Key Concepts

- The net force $\vec{F}_{net}$ on a body with mass m is related to the body's acceleration $\vec{a}$ by

$$\vec{F}_{net} = m\vec{a},$$

which may be written in the component versions

$$F_{net,x} = ma_x, \quad F_{net,y} = ma_y, \quad \text{and} \quad F_{net,z} = ma_z.$$

- If a force $\vec{F}_{BC}$ acts on body B due to body C, then there is a force $\vec{F}_{CB}$ on body C due to body B:

$$\vec{F}_{BC} = -\vec{F}_{CB}.$$

The forces are equal in magnitude but opposite in directions.

Two bodies are said to *interact* when they push or pull on each other—that is, when a force acts on each body due to the other body. For example, suppose you position a book B so it leans against a crate C (Fig. 5-5*a*). Then the book and crate interact: There is a horizontal force $\vec{F}_{BC}$ on the book from the crate (or due to the crate) and a horizontal force $\vec{F}_{CB}$ on the crate from the book (or due to the book). This pair of forces is shown in Fig. 5-5*b*. Newton's third law states that

Newton's Third Law: When two bodies interact, the forces on the bodies from each other are always equal in magnitude and opposite in direction.

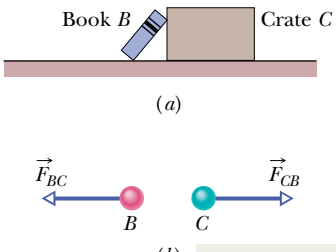

Figure 5-5 (*a*) Book B leans against crate C. (*b*) Forces $\vec{F}_{BC}$ (the force on the book from the crate) and $\vec{F}_{CB}$ (the force on the crate from the book) have the same magnitude and are opposite in direction.

For the book and crate, we can write this law as the scalar relation

$$F_{BC} = F_{CB} \quad \text{(equal magnitudes)}$$

or as the vector relation

$$\vec{F}_{BC} = -\vec{F}_{CB} \quad \text{(equal magnitudes and opposite directions)}, \qquad (5\text{-}8)$$

where the minus sign means that these two forces are in opposite directions. We can call the forces between two interacting bodies a **third-law force pair**. When any two bodies interact in any situation, a third-law force pair is present. The book and crate in Fig. 5-6*a* are stationary, but the third law would still hold if they were moving and even if they were accelerating.

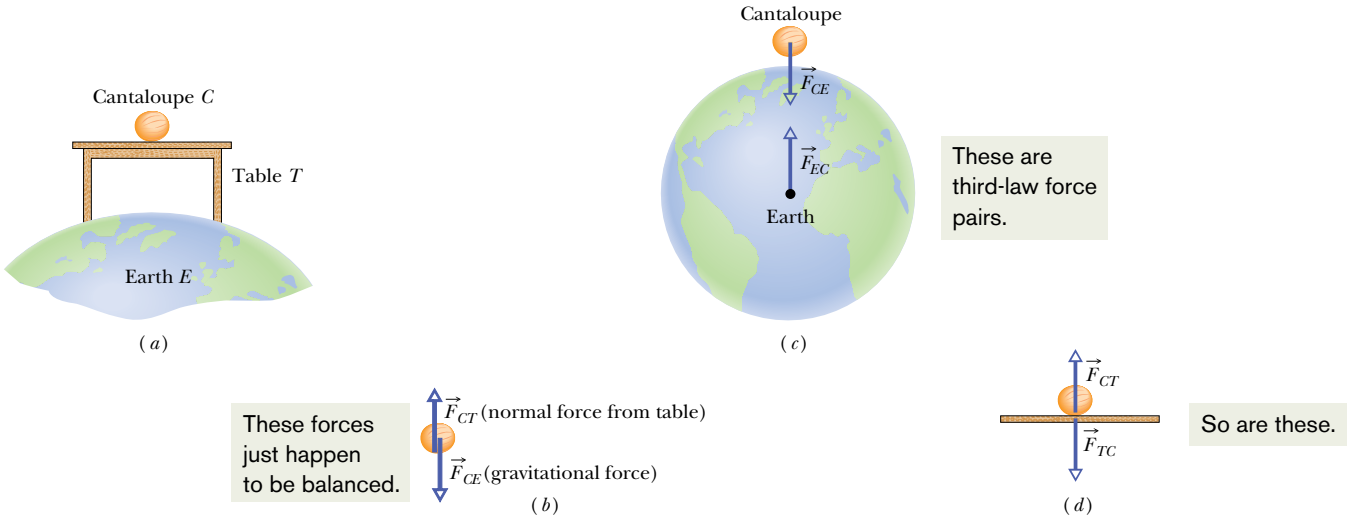

Figure 5-6 (*a*) A cantaloupe lies on a table that stands on Earth. (*b*) The forces *on the cantaloupe* are $\vec{F}_{CT}$ and $\vec{F}_{CE}$. (*c*) The third-law force pair for the cantaloupe–Earth interaction. (*d*) The third-law force pair for the cantaloupe–table interaction.

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As another example, let us find the third-law force pairs involving the cantaloupe in Fig. 5-6*a*, which lies on a table that stands on Earth. The cantaloupe interacts with the table and with Earth (this time, there are three bodies whose interactions we must sort out).

Let's first focus on the forces acting on the cantaloupe (Fig. 5-6*b*). Force $\vec{F}_{CT}$ is the normal force on the cantaloupe from the table, and force $\vec{F}_{CE}$ is the gravitational force on the cantaloupe due to Earth. Are they a third-law force pair? No, because they are forces on a single body, the cantaloupe, and not on two interacting bodies.

To find a third-law pair, we must focus not on the cantaloupe but on the interaction between the cantaloupe and one other body. In the cantaloupe–Earth interaction (Fig. 5-6*c*), Earth pulls on the cantaloupe with a gravitational force $\vec{F}_{CE}$ and the cantaloupe pulls on Earth with a gravitational force $\vec{F}_{EC}$. Are these forces a third-law force pair? Yes, because they are forces on two interacting bodies, the force on each due to the other. Thus, by Newton's third law,

$$\vec{F}_{CE} = -\vec{F}_{EC} \quad \text{(cantaloupe–Earth interaction)}.$$

Next, in the cantaloupe–table interaction, the force on the cantaloupe from the table is $\vec{F}_{CT}$ and, conversely, the force on the table from the cantaloupe is $\vec{F}_{TC}$ (Fig. 5-6*d*). These forces are also a third-law force pair, and so

$$\vec{F}_{CT} = -\vec{F}_{TC} \quad \text{(cantaloupe–table interaction)}.$$

CHECKPOINT 3

Suppose that the cantaloupe and table of Fig. 5-6 are in an elevator cab that begins to accelerate upward. (a) Do the magnitudes of $\vec{F}_{TC}$ and $\vec{F}_{CT}$ increase, decrease, or stay the same? (b) Are those two forces still equal in magnitude and opposite in direction? (c) Do the magnitudes of $\vec{F}_{CE}$ and $\vec{F}_{EC}$ increase, decrease, or stay the same? (d) Are those two forces still equal in magnitude and opposite in direction?

5.8 | SOME PARTICULAR FORCES

Key Concepts

- A gravitational force $\vec{F}_g$ on a body is a pull by another body. In most situations in this book, the other body is Earth or some other astronomical body. For Earth, the force is directed down toward the ground, which is assumed to be an inertial frame. With that assumption, the magnitude of $\vec{F}_g$ is

  $$F_g = mg,$$

  where m is the body's mass and g is the magnitude of the free-fall acceleration.
- The weight W of a body is the magnitude of the upward force needed to balance the gravitational force on the body. A body's weight is related to the body's mass by

  $$W = mg.$$
- A normal force $\vec{F}_N$ is the force on a body from a surface against which the body presses. The normal force is always perpendicular to the surface.
- A frictional force $\vec{f}$ is the force on a body when the body slides or attempts to slide along a surface. The force is always parallel to the surface and directed so as to oppose the sliding. On a frictionless surface, the frictional force is negligible.
- When a cord is under tension, each end of the cord pulls on a body. The pull is directed along the cord, away from the point of attachment to the body. For a massless cord (a cord with negligible mass), the pulls at both ends of the cord have the same magnitude T, even if the cord runs around a massless, frictionless pulley (a pulley with negligible mass and negligible friction on its axle to oppose its rotation).

Let us discuss some of the major forces in mechanics that we encounter very frequently in our day to day lives.

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The Gravitational Force

A **gravitational force** $\vec{F}_g$ on a body is a certain type of pull that is directed toward a second body. In these early chapters, we do not discuss the nature of this force and usually consider situations in which the second body is Earth. Thus, when we speak of *the* gravitational force $\vec{F}_g$ on a body, we usually mean a force that pulls on it directly toward the center of Earth—that is, directly down toward the ground. We shall assume that the ground is an inertial frame.

Free Fall. Suppose a body of mass m is in free fall with the free-fall acceleration of magnitude g. Then, if we neglect the effects of the air, the only force acting on the body is the gravitational force $\vec{F}_g$. We can relate this downward force and downward acceleration with Newton's second law ($\vec{F} = m\vec{a}$). We place a vertical y axis along the body's path, with the positive direction upward. For this axis, Newton's second law can be written in the form $F_{\text{net},y} = ma_y$, which, in our situation, becomes

$$-F_g = m(-g)$$

or

$$F_g = mg. \tag{5-9}$$

In words, the magnitude of the gravitational force is equal to the product mg.

At Rest. This same gravitational force, with the same magnitude, still acts on the body even when the body is not in free fall but is, say, at rest on a pool table or moving across the table. (For the gravitational force to disappear, Earth would have to disappear.)

We can write Newton's second law for the gravitational force in these vector forms:

$$\vec{F}_g = -F_g\hat{j} = -mg\hat{j} = m\vec{g}, \tag{5-10}$$

where $\hat{j}$ is the unit vector that points upward along a y axis, directly away from the ground, and $\vec{g}$ is the free-fall acceleration (written as a vector), directed downward.

Weight

The **weight** W of a body is the magnitude of the net force required to prevent the body from falling freely, as measured by someone on the ground. For example, to keep a ball at rest in your hand while you stand on the ground, you must provide an upward force to balance the gravitational force on the ball from Earth. Suppose the magnitude of the gravitational force is 2.0 N. Then the magnitude of your upward force must be 2.0 N, and thus the weight W of the ball is 2.0 N. We also say that the ball *weighs* 2.0 N and speak about the ball *weighing* 2.0 N.

A ball with a weight of 3.0 N would require a greater force from you—namely, a 3.0 N force—to keep it at rest. The reason is that the gravitational force you must balance has a greater magnitude—namely, 3.0 N. We say that this second ball is *heavier* than the first ball.

Now let us generalize the situation. Consider a body that has an acceleration $\vec{a}$ of zero relative to the ground, which we again assume to be an inertial frame. Two forces act on the body: a downward gravitational force $\vec{F}_g$ and a balancing upward force of magnitude W. We can write Newton's second law for a vertical y axis, with the positive direction upward, as

$$F_{\text{net},y} = ma_y.$$

In our situation, this becomes

$$W - F_g = m(0) \tag{5-11}$$

or

$$W = F_g \quad \text{(weight, with ground as inertial frame).} \tag{5-12}$$

This equation tells us (assuming the ground is an inertial frame) that

The weight W of a body is equal to the magnitude F_g of the gravitational force on the body.

Substituting mg for F_g from Eq. 5-9, we find

$$W = mg \quad \text{(weight)}, \tag{5-13}$$

which relates a body's weight to its mass.

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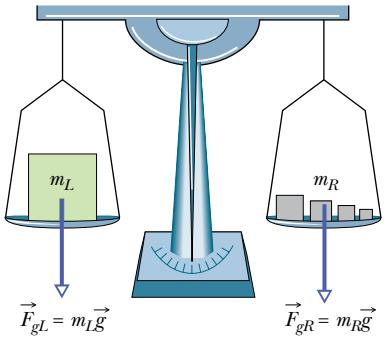

Figure 5-7 An equal-arm balance. When the device is in balance, the gravitational force $\vec{F}_{gL}$ on the body being weighed (on the left pan) and the total gravitational force $\vec{F}_{gR}$ on the reference bodies (on the right pan) are equal. Thus, the mass m_L of the body being weighed is equal to the total mass m_R of the reference bodies.

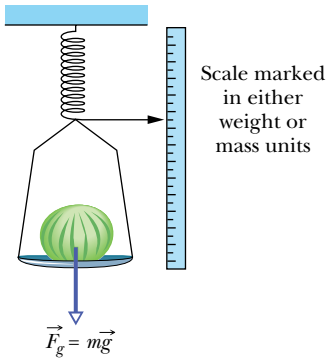

Figure 5-8 A spring scale. The reading is proportional to the *weight* of the object on the pan, and the scale gives that weight if marked in weight units. If, instead, it is marked in mass units, the reading is the object's weight only if the value of g at the location where the scale is being used is the same as the value of g at the location where the scale was calibrated.

Weighing. To *weigh* a body means to measure its weight. One way to do this is to place the body on one of the pans of an equal-arm balance (Fig. 5-7) and then place reference bodies (whose masses are known) on the other pan until we strike a balance (so that the gravitational forces on the two sides match). The masses on the pans then match, and we know the mass of the body. If we know the value of g for the location of the balance, we can also find the weight of the body with Eq. 5-13.

We can also weigh a body with a spring scale (Fig. 5-8). The body stretches a spring, moving a pointer along a scale that has been calibrated and marked in either mass or weight units. (Most bathroom scales in the United States work this way and are marked in the force unit pounds.) If the scale is marked in mass units, it is accurate only where the value of g is the same as where the scale was calibrated.

The weight of a body must be measured when the body is not accelerating vertically relative to the ground. For example, you can measure your weight on a scale in your bathroom or on a fast train. However, if you repeat the measurement with the scale in an accelerating elevator, the reading differs from your weight because of the acceleration. Such a measurement is called an *apparent weight*.

Caution: A body's weight is not its mass. Weight is the magnitude of a force and is related to mass by Eq. 5-13. If you move a body to a point where the value of g is different, the body's mass (an intrinsic property) is not different but the weight is. For example, the weight of a bowling ball having a mass of 7.2 kg is 71 N on Earth but only 12 N on the Moon. The mass is the same on Earth and Moon, but the free-fall acceleration on the Moon is only 1.6 m/s^2.

The Normal Force

If you stand on a mattress, Earth pulls you downward, but you remain stationary. The reason is that the mattress, because it deforms downward due to you, pushes up on you. Similarly, if you stand on a floor, it deforms (it is compressed, bent, or buckled ever so slightly) and pushes up on you. Even a seemingly rigid concrete floor does this (if it is not sitting directly on the ground, enough people on the floor could break it).

The push on you from the mattress or floor is a **normal force** $\vec{F}_N$ (also called normal contact force). The name comes from the mathematical term *normal*, meaning perpendicular: The force on you from, say, the floor is perpendicular to the floor.

When a body presses against a surface, the surface (even a seemingly rigid one) deforms and pushes on the body with a normal force $\vec{F}_N$ that is perpendicular to the surface.

Figure 5-9*a* shows an example. A block of mass m presses down on a table, deforming it somewhat because of the gravitational force $\vec{F}_g$ on the block. The table pushes up on the block with normal force $\vec{F}_N$. The free-body diagram

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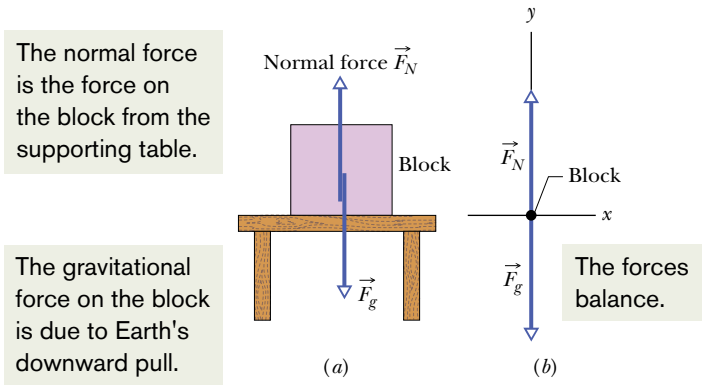

Figure 5-9 (*a*) A block resting on a table experiences a normal force $\vec{F}_N$ perpendicular to the tabletop. (*b*) The free-body diagram for the block.

for the block is given in Fig. 5-7*b*. Forces $\vec{F}_g$ and $\vec{F}_N$ are the only two forces on the block and they are both vertical. Thus, for the block we can write Newton's second law for a positive-upward y axis ($F_{\text{net},y} = ma_y$) as

$$F_N - F_g = ma_y.$$

From Eq. 5-9, we substitute mg for F_g, finding

$$F_N - mg = ma_y.$$

Then the magnitude of the normal force is

$$F_N = mg + ma_y = m(g + a_y) \tag{5-14}$$

for any vertical acceleration a_y of the table and block (they might be in an accelerating elevator). (*Caution:* We have already included the sign for g but a_y can be positive or negative here.) If the table and block are not accelerating relative to the ground, then $a_y = 0$ and Eq. 5-14 yields

$$F_N = mg. \tag{5-15}$$

CHECKPOINT 4

In Fig. 5-9, is the magnitude of the normal force $\vec{F}_N$ greater than, less than, or equal to mg if the block and table are in an elevator moving upward (a) at constant speed and (b) at increasing speed?

PROBLEM-SOLVING TACTICS

Tactic 1: Dimensions and Vectors When you are dealing with forces, you cannot just add or subtract their magnitudes to find their net force unless they happen to be directed *along the same axis*. If they are not, you must use vector addition, by finding components along axes, one axis at a time, as is done in Sample Problem 5.02.

Tactic 2: Reading Force Problems Read the problem statement several times until you have a good mental picture of what the situation is, what data are given, and what is requested. If you know what the problem is about but don't know what to do next, put the problem aside and reread the text. If you are hazy about Newton's second law, reread that section. Study the sample problems. And remember that solving physics problems (like repairing cars and designing computer chips) takes training.

Tactic 3: Draw Two Types of Figures You may need two figures. One is a rough sketch of the actual situation. When you draw the forces, place the tail of each force vector either on the boundary of or within the body on

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which that force acts. The other figure is a free-body diagram: the forces on a *single* body are drawn, with the body represented by a dot or a sketch. Place the tail of each force vector on the dot or sketch.

Tactic 4: What Is Your System? If you are using Newton's second law, you must know what body or system you are applying it to. In Sample Problem 5.01 it is the puck (not the ice). In Sample Problem 5.02, it is the cookie tin.

Tactic 5: Choose Your Axes Wisely Often, we can save a lot of work by choosing one of our coordinate axes to coincide with one of the forces.

Tactic 6: Normal Force Equation 5-15 for the normal force on a body holds only when $\vec{F}_N$ is directed upward and the body's vertical acceleration is zero; so we do *not* apply it for other orientations of or when the vertical acceleration is not zero. Instead, we must derive a new expression for $\vec{F}_N$ from Newton's second law. We are free to move $\vec{F}_N$ around in a figure as long as we maintain its orientation. For example, in Fig. 5-9*a* we can slide it downward so that its head is at the boundary between block and tabletop. However, $\vec{F}_N$ is least likely to be misinterpreted when its tail is either at that boundary or somewhere within the block (as shown). An even better technique is to draw a free-body diagram as in Fig. 5-9*b*, with the tail of $\vec{F}_N$ directly on the dot or sketch representing the block.

Tension

When a cord (or a rope, cable, or other such object) is attached to a body and pulled taut, the cord pulls on the body with a force $\vec{T}$ directed away from the body and along the cord (Fig. 5-10*a*). The force is often called a *tension force* because the cord is said to be in a state of *tension* (or to be *under tension*), which means that it is being pulled taut. The *tension in the cord* is the magnitude T of the force on the body. For example, if the force on the body from the cord has magnitude $T = 50$ N, the tension in the cord is 50 N.

A cord is often said to be *massless* (meaning its mass is negligible compared to the body's mass) and *unstretchable*. The cord then exists only as a connection between two bodies. It pulls on both bodies with the same force magnitude T, even if the bodies and the cord are accelerating and even if the cord runs around a *massless, frictionless pulley* (Figs. 5-10*b* and *c*). Such a pulley has negligible mass compared to the bodies and negligible friction on its axle opposing its rotation. If the cord wraps halfway around a pulley, as in Fig. 5-10*c*, the net force on the pulley from the cord has the magnitude $2T$.

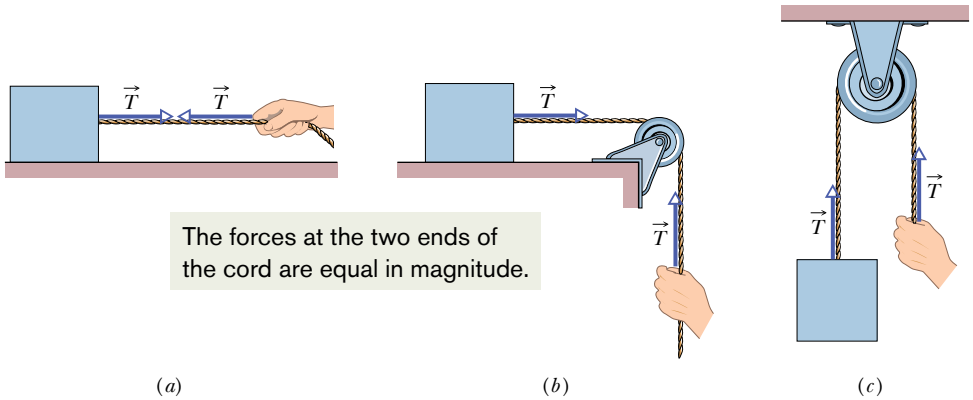

Figure 5-10 (*a*) The cord, pulled taut, is under tension. If its mass is negligible, the cord pulls on the body and the hand with force $\vec{T}$, even if the cord runs around a massless, frictionless pulley as in (*b*) and (*c*).

CHECKPOINT 5

The suspended body in Fig. 5-10*c* weighs 75 N. Is T equal to, greater than, or less than 75 N when the body is moving upward (a) at constant speed, (b) at increasing speed, and (c) at decreasing speed?

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Spring Force

The force exerted by a compressed or stretched spring on any object, which is attached to it is called **spring force**. An object, which compresses or stretches a spring is acted upon by a force that restores the object to its rest position (or translational equilibrium position). For most of the springs, specifically, which are said to obey **Hooke's Law**, the magnitude of the force is directly proportional to the amount of stretch or compression of the spring.

Hooke's law states that the restoring force of a spring is directly proportional to a small displacement. Mathematically, Hooke's law for an ideal spring is expressed as

$$F = -kx \tag{5-16}$$

where x is the measure of the displacement and k is the proportionality constant or **spring constant** for a particular spring.

Let us consider a massless (ideal) spring of length l and spring constant k lying on a smooth horizontal table. One end of the spring is tied to a rigid support as shown in the figure.

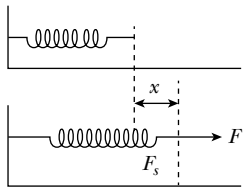

Figure 5-11 A massless spring attached to a rigid support exert a restoring force F_s.

Let us stretch the spring by x. To hold the spring in this position, we have to apply a force F. This is because a restoring force (F_s) comes into existence in the spring which tends to bring the spring back to its original shape and size. The magnitude of F_s is same as F but the direction is opposite to the applied force. Therefore, $F_s = -kx$.

Spring constant, *k* is defined as the force required to stretch the spring by unity. The value of k depends on the material of spring wire, diameter of spring wire, pitch and length of the spring.

If a spring of length l and spring constant x is cut into two parts of length l_1 and l_2 and if k_1 and k_2 are the spring constants of the new cut parts, respectively, then $kl = k_1 l_1 = k_2 l_2$.

SAMPLE PROBLEM 5.03

Mass suspended vertically by a spring

A mass 1 kg is attached to the hook of a spring and the spring is suspended vertically from a ceiling (see Fig. 5-12*a*). The spring is displaced from its equilibrium position by a distance x. The spring constant of the spring is 2.0×10^2 N/m. Calculate the displacement x.

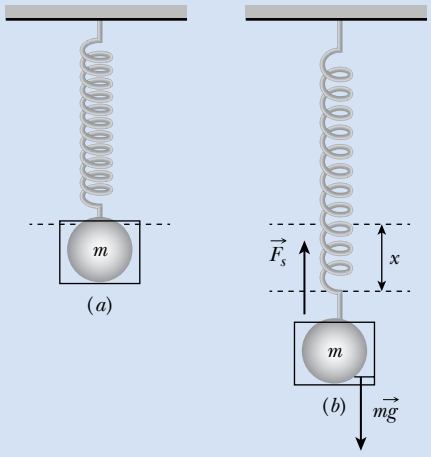

Figure 5-12 (*a*) Mass of 1 kg attached to a spring is suspended vertically. (*b*) Spring displace from equilibrium position by a distance x.

KEY IDEAS

(1) In Fig. 5-12, mass attached to the spring is shown. Figure 5-12*a* shows the relaxed position of spring. Figure 5-12*b* shows the distance x by which the spring is stretched due to displacement from equilibrium position.
(2) The force on the mass is balanced by its weight acting downwards and spring force acting upwards.

Calculation: At equilibrium, spring force = weight, that is,

$$\vec{F}_s = mg$$

That is, $$\left|\vec{F}_s\right| = kx = mg$$

Substituting $m = 1$ kg, $k = 2.0 \times 10^2$ N/m and $g = 10$ m/s^2, we obtain the displacement x as

$$x = \frac{mg}{x} = \frac{(1\text{ kg})(10\text{ m/s}^2)}{(2.0\times10^2\text{ N/m})}$$

$$= 5\times10^{-2}\text{ m} = 5\text{ cm} \qquad \text{(Answer)}$$

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Parallel and Series Combination of Springs

Parallel combination. When two springs are connected in parallel with each other, then the equivalent spring constant is the sum of the individual spring constants of the two springs. If the spring constants of the two springs are k_1 and k_2 then the equivalent spring constant is given by

$$k_{eq} = k_1 + k_2 \tag{5-17}$$

Consider the Fig. 5-13, if the mass attached to the springs being acted by a force, is stretched by the same amount x. Then, $F_1 = k_1 x$ and $F_2 = k_2 x$. Now if both the springs are replaced by a single equivalent spring, then the force is given by $F = k_{eq} x$; and $F_1 \neq F_2$. Now since spring constants are not equal, for equilibrium, we have

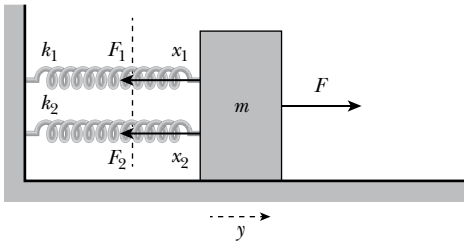

Figure 5-13 Two springs are connected in parallel and pulled by a force F.

$$F = F_1 + F_2 = k_1 x + k_2 x = k_{eq} x$$
$$\Rightarrow k_{eq} = k_1 + k_2$$

Therefore, for more than two springs in parallel, the equivalent spring constant is

$$k_{eq} = k_1 + k_2 + k_3 + \cdots$$

Series combination. When two springs having spring constants k_1 and k_2 are connected in series with each other, then the equivalent spring constant is given by

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} \tag{5-18}$$

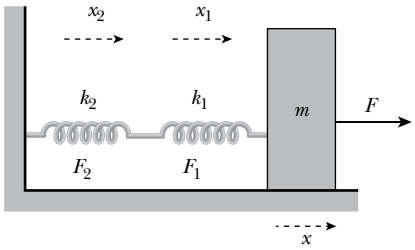

Figure 5-14 Two springs are connected in series and pulled by a force F.

Consider the Fig. 5-14, if the mass attached to the springs being acted by a force, the same amount of force is exerted on both the springs because we considered massless springs. Therefore, $F_1 = F_2 = F$. Now if both the springs are replaced by a single equivalent spring, then the force is given by $F = k_{eq} x$. Since the spring constants are different for both the springs, therefore, the extensions produced by the two springs are also different. Thus, total extension can be written as the sum of individual extensions of the springs as $x = x_1 + x_2$. Therefore,

$$\frac{F}{k_{eq}} = \frac{F}{k_1} + \frac{F}{k_2}$$
$$\Rightarrow \frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

Therefore, for more than two springs in series, the equivalent spring constant is

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \cdots$$

5.9 | CONSTRAINT MOTION: BODIES WITH LINKED MOTION

Key Concepts

- When different parts of a system are free to move differently the accelerations of different parts are not same. Motion of such systems are constraint motion.
- In Atwood's machine, acceleration of each block is

$$a = \left[\frac{m_2 - m_1}{m_1 + m_2}\right] g$$

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There are situations in which the accelerations of different parts of the system may not be the same. We get such situations in case of moveable pulleys or bodies in contact where each body is free to move. In such cases, a relationship between accelerations can be found by considering physical properties of system. We call such relations as **constrained relation**.

Sometimes the motions of two bodies are linked. This is true for blocks A and B in each of the situations in Fig. 5-15. In Fig. 5-15a, A and B clearly move together; their velocities and accelerations must therefore be the same at every instant. In Figs. 5.14b–c, if the string remains taut and does not stretch, the magnitudes of A's velocity and acceleration must be the same as B's, even though the directions are different.

In all the three situations, there is a force exerted on B equal and opposite to one that is exerted on A. In Fig. 5-15a, this is because Newton's third law tells us that the normal forces that A and B exert on each other must be equal and opposite (see Fig. 5-16). In Figs. 5-15b and c, blocks A and B do not make direct contact with each other. However, as long as the string connecting them and the pulley are light enough to be treated as massless and there is no slippage between the string and the pulley, the forces that the string exerts at its two ends will both be equal in magnitude to the tension in the string.

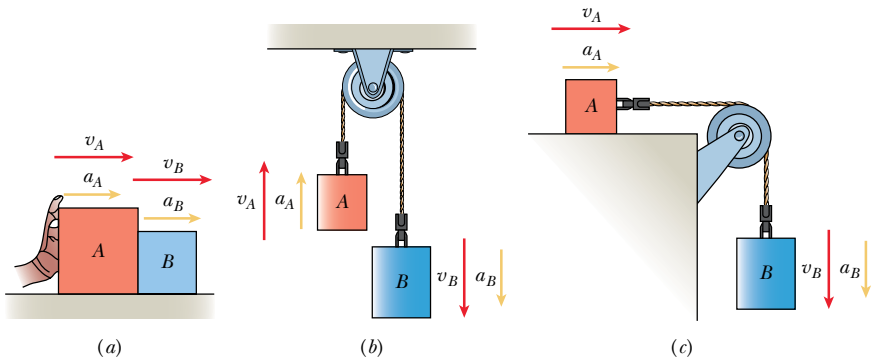

Figure 5-15 Examples of situations involving two bodies with linked motions. The setup in (b) is sometimes called an Atwood's machine and the setup in (c) a modified Atwood's machine.

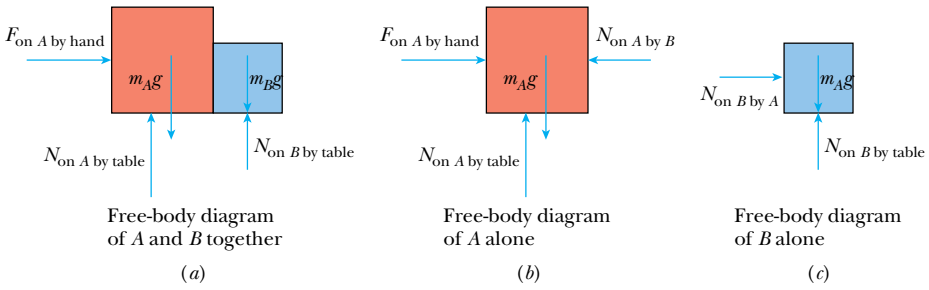

Figure 5-16 Free-body diagrams for Fig. 5-15a. For legibility, we have placed the heads rather than the tails of the normal force vectors at their points of application.

Motion of Connected Block Over a Pulley (Atwood's Machine)

Figure 5-17 shows a free-body diagram of Atwood's machine. The given diagram is of a pulley which is used to reduce human effort to lift heavier bodies. It is given that the two masses m_1 and m_2 are attached with a string having tension T over pulley. Now, to find the acceleration of each block, tension in string and force on pulley, we will make the following assumptions that

1. the pulley is massless and smooth.
2. string is inextensible and massless.

Since the string is inextensible, the acceleration of both blocks are same in magnitude, say, a.

Now, the free-body diagram of mass m_1 is as shown in Fig. 5-18a:

$$m_1a = T_1 - m_1g \tag{5-19}$$

The free-body diagram of mass m_2 is as shown in Fig. 5-18b:

$$M_2a = m_2g - T_1 \tag{5-20}$$

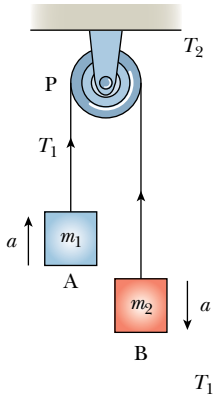

Figure 5-17 Atwood's machine.

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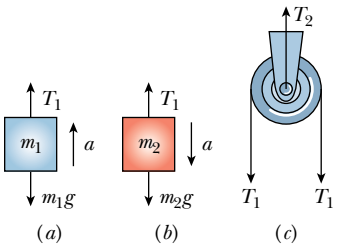

Figure 5-18 Free-body diagram of (*a*) mass m_1, (*b*) mass m_2, and (*c*) pulley.

Now, solving Eqs. (5-19) and (5-20), we get

$$a = \left[\frac{m_2 - m_1}{m_1 + m_2}\right] g$$

and

$$T_1 = \frac{2 m_1 m_2}{m_1 + m_2} g$$

Now, force on pulley is $T_2 = 2T_1$.

As shown in the free body diagram in Fig. (5-18*c*), force on pulley is

$$T_2 = \frac{4 m_1 m_2}{m_1 + m_2} g$$

SAMPLE PROBLEM 5.04

Secret agent pushes two mysterious crates across river

A secret agent pushes two mysterious crates across a frozen river in the dark of night. The crates slide frictionlessly. Their masses are 150 kg (*A*) and 50 kg (*B*). The agent's hands exert a 100 N force on *A* (see Fig. 5-19). How much force does *A* exert on *B*? If there is no friction, can either block's acceleration be zero? Therefore, can $\sum \vec{F}_{\text{on } A} = 0$? Therefore, can the force exerted on *A* by *B* (which is equal in magnitude to that exerted on *B* by *A*) "balance" (be equal and opposite to) the force exerted by the agent?

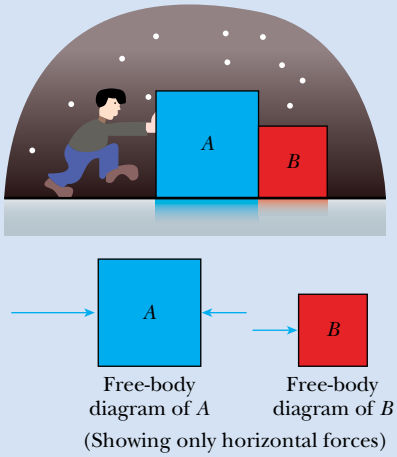

Figure 5-19

KEY IDEA

With no friction, the unopposed force exerted by the agent makes **both** crates accelerate. If crate *A* is accelerating, the forces on *A* cannot be equal and opposite.

Calculations: (a) To find a force on block *B*, we apply Newton's second law to *B*. Using a free-body diagram of *B* (left), we get

$$\sum F_{x \text{ on } B} = F_{\text{on } B \text{ by } A} = m_B a_{Bx} = m_B a_B. \quad (5\text{-}21)$$

This leaves us with two unknowns, $F_{\text{on } B \text{ by } A}$ and a_B.

(b) However, if we apply the second law to block *A* as well (see free-body diagram of *A* at left), we get

$$\sum F_{x \text{ on } A} = F_{\text{on } A \text{ by agent}} - F_{\text{on } A \text{ by } B} = m_A a_{Ax} = m_A a_A. \quad (5\text{-}22)$$

However, we know that $F_{\text{on } A \text{ by } B} = F_{\text{on } B \text{ by } A}$. Also, $a_A = a_B$ (label this common acceleration *a*). Thus, from Eqs. (5-21) and (5-22), we get

$$F_{\text{on } B \text{ by } A} = m_B a \quad (5\text{-}23)$$

and

$$F_{\text{on } A \text{ by agent}} - F_{\text{on } B \text{ by } A} = m_A a, \quad (5\text{-}24)$$

which we can solve for the two unknowns $F_{\text{on } B \text{ by } A}$ and *a*. We use Eq. (5-23) to substitute for $F_{\text{on } B \text{ by } A}$ in Eq. (5-24):

$$F_{\text{on } A \text{ by agent}} - m_B a = m_A a.$$

Then

$$F_{\text{on } A \text{ by agent}} = m_B a + m_A a = (m_B + m_A) a$$

and

$$a = \frac{F_{\text{on } A \text{ by agent}}}{m_B + m_A} = \frac{100 \text{ N}}{50 \text{ kg} + 150 \text{ kg}} = 0.50 \text{ m/s}^2 \quad (5\text{-}25)$$

Substituting this value and $m_A = 50$ kg into Eq. (5-23) gives us

$$F_{\text{on } B \text{ by } A} = (50 \text{ kg})(0.50 \text{ m/s}^2) = 25 \text{ N}. \quad \text{(Answer)}$$

We have found that $F_{\text{on } B \text{ by } A}$ is one quarter of the force exerted by the hand on *A*. Likewise, m_B is one quarter of the combined mass $m_A + m_B$ pushed by the hand. As you would expect from Newton's second law of motion, to impart the same acceleration to a mass one-fourth as great requires one fourth the force.

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5.10 | APPLYING NEWTON'S LAWS

The rest of this chapter consists of sample problems. You should pore over them, learning the procedures for tackling these. Especially important is knowing how to translate a sketch of a situation into a free-body diagram with appropriate axes, so that Newton's laws can be applied.

SAMPLE PROBLEM 5.05

Block on table, block hanging

Figure 5-20 shows a block S (the *sliding block*) with mass $M = 3.3$ kg. The block is free to move along a horizontal frictionless surface and connected, by a cord that wraps over a frictionless pulley, to a second block H (the *hanging block*), with mass $m = 2.1$ kg. The cord and pulley have negligible masses compared to the blocks (they are "massless"). The hanging block H falls as the sliding block S accelerates to the right. Find (a) the acceleration of block S, (b) the acceleration of block H, and (c) the tension in the cord.

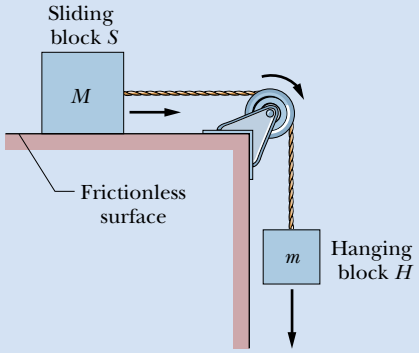

Figure 5-20 A block S of mass M is connected to a block H of mass m by a cord that wraps over a pulley.

Reasoning:

Q *What is this problem all about?*

You are given two bodies—sliding block and hanging block—but must also consider *Earth*, which pulls on both bodies. (Without Earth, nothing would happen here.) A total of five forces act on the blocks, as shown in Fig. 5-21:

1. The cord pulls to the right on sliding block S with a force of magnitude T.
2. The cord pulls upward on hanging block H with a force of the same magnitude T. This upward force keeps block H from falling freely.
3. Earth pulls down on block S with the gravitational force $\vec{F}_{gS}$, which has a magnitude equal to Mg.
4. Earth pulls down on block H with the gravitational force $\vec{F}_{gH}$, which has a magnitude equal to mg.
5. The table pushes up on block S with a normal force $\vec{F}_N$.

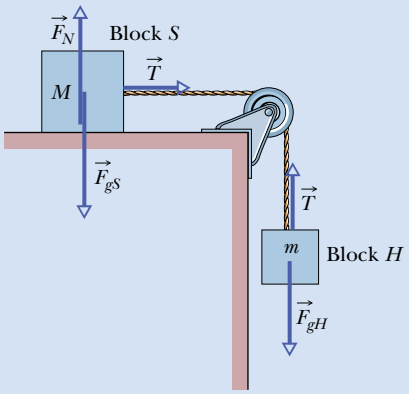

Figure 5-21 The forces acting on the two blocks of Fig. 5-20.

There is another thing you should note. We assume that the cord does not stretch, so that if block H falls 1 mm in a certain time, block S moves 1 mm to the right in that same time. This means that the blocks move together and their accelerations have the same magnitude a.

Q *How do I classify this problem? Should it suggest a particular law of physics to me?*

Yes. Forces, masses, and accelerations are involved, and they should suggest Newton's second law of motion, $\vec{F}_{net} = m\vec{a}$. That is our starting key idea.

Q *If I apply Newton's second law to this problem, to which body should I apply it?*

We focus on two bodies, the sliding block and the hanging block. Although they are *extended objects* (they are not points), we can still treat each block as a particle because every part of it moves in exactly the same way. A second key idea is to apply Newton's second law separately to each block.

Q *What about the pulley?*

We cannot represent the pulley as a particle because different parts of it move in different ways. When we discuss rotation, we shall deal with pulleys in detail. Meanwhile, we eliminate the pulley from consideration by assuming its mass to be

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negligible compared with the masses of the two blocks. Its only function is to change the cord's orientation.

Q *OK. Now how do I apply* $\vec{F}_{net} = m\vec{a}$ *to the sliding block?*

Represent block S as a particle of mass M and draw *all* the forces that act *on* it, as in Fig. 5-22*a*. This is the block's free-body diagram. Next, draw a set of axes. It makes sense to draw the x axis parallel to the table, in the direction in which the block moves.

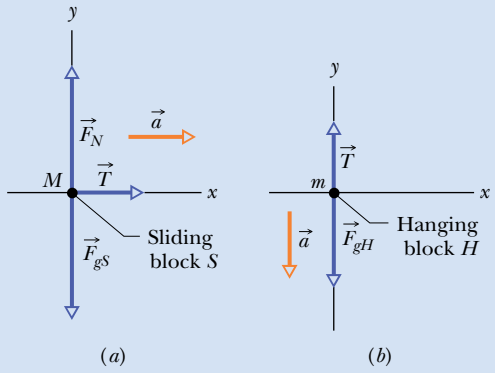

Figure 5-22 (*a*) A free-body diagram for block S of Fig. 5-22. (*b*) A free-body diagram for block H of Fig. 5-22.

Q *Thanks, but you still haven't told me how to apply* $\vec{F}_{net} = m\vec{a}$ *to the sliding block. All you've done is explain how to draw a free-body diagram.*

You are right, and here's the third key idea: The expression $\vec{F}_{net} = M\vec{a}$ is a vector equation, so we can write it as three component equations:

$$F_{net,x} = Ma_x, \quad F_{net,y} = Ma_y, \quad F_{net,z} = Ma_z, \qquad (5\text{-}26)$$

in which $F_{net,x}$, $F_{net,y}$, and $F_{net,z}$ are the components of the net force along the three axes. Now we apply each component equation to its corresponding direction. Because block S does not accelerate vertically, $F_{net,y} = Ma_y$ becomes

$$F_N - F_{gS} = 0 \quad \text{or} \quad F_N = F_{gS}. \qquad (5\text{-}27)$$

Thus in the y direction, the magnitude of the normal force is equal to the magnitude of the gravitational force.

No force acts in the z direction, which is perpendicular to the page.

In the x direction, there is only one force component, which is T. Thus, $F_{net,x} = Ma_x$ becomes

$$T = Ma. \qquad (5\text{-}28)$$

This equation contains two unknowns, T and a; so we cannot yet solve it. Recall, however, that we have not said anything about the hanging block.

Q *I agree. How do I apply* $\vec{F}_{net} = m\vec{a}$ *to the hanging block?*

We apply it just as we did for block S: Draw a free-body diagram for block H, as in Fig. 5-22*b*. Then apply $\vec{F}_{net} = m\vec{a}$ in component form. This time, because the acceleration is along the y axis, we use the y part of Eq. 5-26 ($F_{net,y} = ma_y$) to write

$$T - F_{gH} = ma_y. \qquad (5\text{-}29)$$

We can now substitute mg for F_{gH} and $-a$ for a_y (negative because block H accelerates in the negative direction of the y axis). We find

$$T - mg = -ma. \qquad (5\text{-}30)$$

Now note that Eqs. 5-28 and 5-30 are simultaneous equations with the same two unknowns, T and a. Subtracting these equations eliminates T. Then solving for a yields

$$a = \frac{m}{M+m}g. \qquad (5\text{-}31)$$

Substituting this result into Eq. 5-28 yields

$$T = \frac{Mm}{M+m}g. \qquad (5\text{-}32)$$

Putting in the numbers gives, for these two quantities,

$$a = \frac{m}{M+m}g = \frac{2.1\ \text{kg}}{3.3\ \text{kg} + 2.1\ \text{kg}}(9.8\ \text{m/s}^2)$$
$$= 3.8\ \text{m/s}^2 \qquad \text{(Answer)}$$

and $$T = \frac{Mm}{M+m}g = \frac{(3.3\ \text{kg})(2.1\ \text{kg})}{3.3\ \text{kg} + 2.1\ \text{kg}}(9.8\ \text{m/s}^2)$$
$$= 13\ \text{N}. \qquad \text{(Answer)}$$

Q *The problem is now solved, right?*

That's a fair question, but the problem is not really finished until we have examined the results to see whether they make sense. (If you made these calculations on the job, wouldn't you want to see whether they made sense before you turned them in?)

Look first at Eq. 5-31. Note that it is dimensionally correct and that the acceleration a will always be less than g (because of the cord, the hanging block is not in free fall).

Look now at Eq. 5-32, which we can rewrite in the form

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$$T = \frac{M}{M+m}mg. \qquad (5\text{-}33)$$

In this form, it is easier to see that this equation is also dimensionally correct, because both T and mg have dimensions of forces. Equation 5-33 also lets us see that the tension in the cord is always less than mg, and thus is always less than the gravitational force on the hanging block. That is a comforting thought because, if T were *greater* than mg, the hanging block would accelerate upward.

Learn: We can also check the results by studying special cases, in which we can guess what the answers must be. A simple example is to put $g = 0$, as if the experiment were carried out in interstellar space. We know that in that case, the blocks would not move from rest, there would be no forces on the ends of the cord, and so there would be no tension in the cord. Do the formulas predict this? Yes, they do. If you put $g = 0$ in Eqs. 5-31 and 5-32, you find $a = 0$ and $T = 0$. Two more special cases you might try are $M = 0$ and $m \to \infty$.

SAMPLE PROBLEM 5.06

Cord accelerates box up a ramp

In Fig. 5-23*a*, a cord pulls a box of sea biscuits up along a frictionless plane inclined at angle $\theta = 30.0°$. The box has mass $m = 5.00$ kg, and the force from the cord has magnitude $T = 25.0$ N. What is the box's acceleration a along the inclined plane?

KEY IDEA

The acceleration along the plane is set by the force components along the plane (not by force components perpendicular to the plane), as expressed by Newton's second law (Eq. 5-1).

Calculations: We need to write Newton's second law for motion along an axis. Because the box moves along the inclined plane, placing an x axis along the plane seems reasonable (Fig. 5-23*b*). (There is nothing wrong with using our usual coordinate system, but the expressions for components would be a lot messier because of the misalignment of the x axis with the motion.)

After choosing a coordinate system, we draw a freebody diagram with a dot representing the box (Fig. 5-23*b*). Then we draw all the vectors for the forces acting on the box, with the tails of the vectors anchored on the dot. (Drawing the vectors willy-nilly on the diagram can easily lead to errors, especially on exams, so always anchor the tails.)

Force $\vec{T}$ from the cord is up the plane and has magnitude $T = 25.0$ N. The gravitational force $\vec{F}_g$ is downward (of course) and has magnitude $mg = (5.00\text{ kg})(9.80\text{ m/s}^2) = 49.0$ N. That direction means that only a component of the force is along the plane, and only that component (not the full force) affects the box's acceleration along the plane. Thus, before we can write Newton's second law for motion along the x axis,we need to find an expression for that important component.

Figures 5-23*c* to *h* indicate the steps that lead to the expression. We start with the given angle of the plane and work our way to a triangle of the force components (they are the legs of the triangle and the full force is the hypotenuse). Figure 5-23*c* shows that the angle between the ramp and $\vec{F}_g$ is $90° - \theta$. (Do you see a right triangle there?) Next, Figs. 5-23*d* to *f* show $\vec{F}_g$ and its components: One component is parallel to the plane (that is the one we want) and the other is perpendicular to the plane.

Because the perpendicular component *is* perpendicular, the angle between it and $\vec{F}_g$ must be θ (Fig. 5-23*d*). The component we want is the far leg of the component right triangle. The magnitude of the hypotenuse is mg (the magnitude of the gravitational force). Thus, the component we want has magnitude $mg \sin\theta$ (Fig. 5-23*g*).

We have one more force to consider, the normal force $\vec{F}_N$ shown in Fig. 5-23*b*. However, it is perpendicular to the plane and thus cannot affect the motion along the plane. (It has no component along the plane to accelerate the box.)

We are now ready to write Newton's second law for motion along the tilted x axis:

$$F_{\text{net},x} = ma_x.$$

The component a_x is the only component of the acceleration (the box is not leaping up from the plane, which would be strange, or descending into the plane, which would be even stranger). So, let's simply write a for the acceleration along the plane. Because $\vec{T}$ is in the positive

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x direction and the component $mg \sin \theta$ is in the negative x direction, we next write

$$T - mg \sin \theta = ma. \qquad (5\text{-}34)$$

Substituting data and solving for a, we find

$$a = 0.100 \text{ m/s}^2. \qquad \text{(Answer)}$$

The result is positive, indicating that the box accelerates up the inclined plane, in the positive direction of the tilted x axis. If we decreased the magnitude of $\vec{T}$ enough to make $a = 0$, the box would move up the plane at constant speed. And if we decreased the magnitude of $\vec{T}$ even more, the acceleration would be negative in spite of the cord's pull.

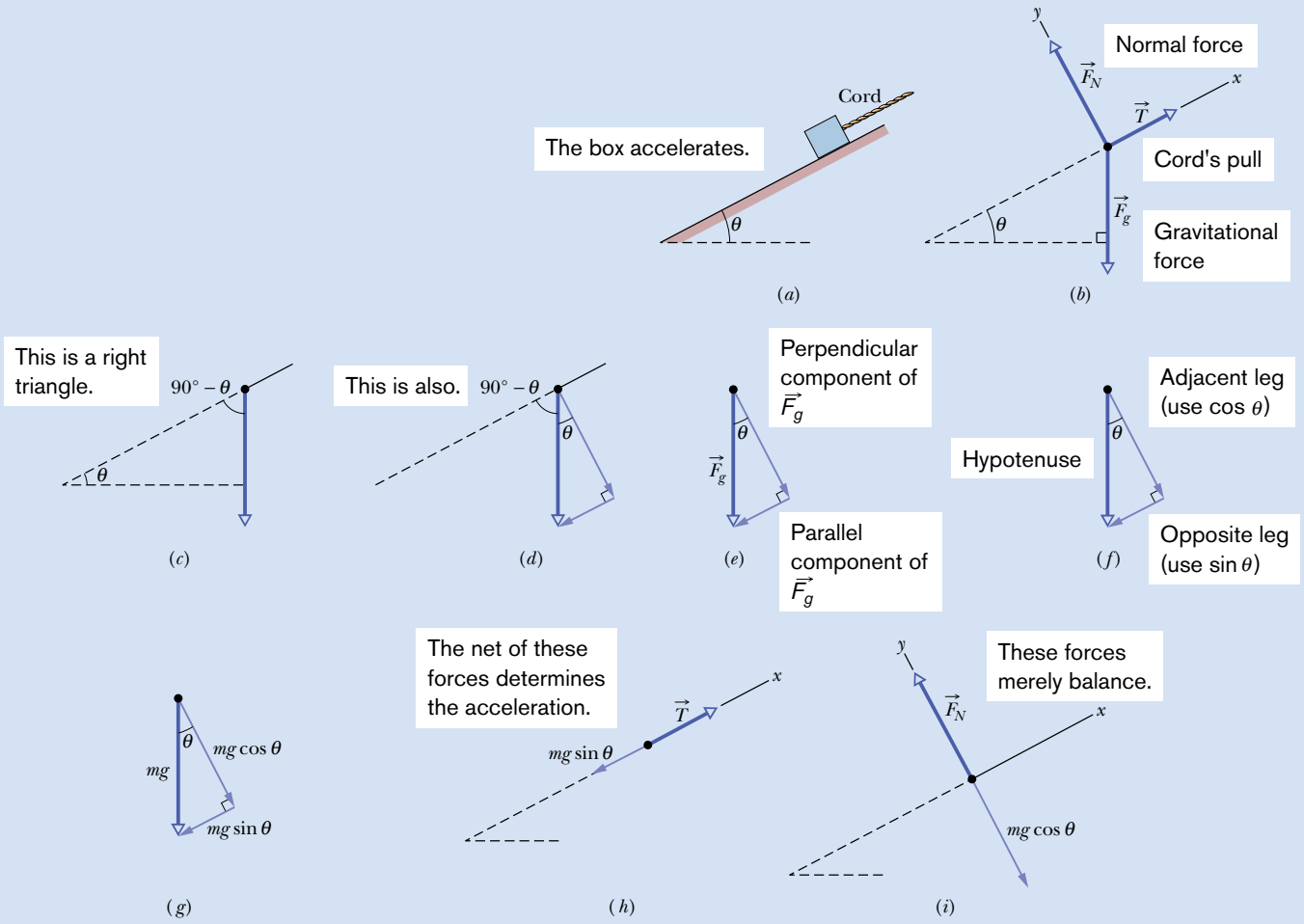

Figure 5-23 (*a*) A box is pulled up a plane by a cord. (*b*) The three forces acting on the box: the cord's force $\vec{T}$, the gravitational force $\vec{F}_g$, and the normal force $\vec{F}_N$. (*c*)–(*i*) Finding the force components along the plane and perpendicular to it.

SAMPLE PROBLEM 5.07

Reading a force graph

Here is an example of where you must dig information out of a graph, not just read off a number. In Fig. 5-24*a*, two forces are applied to a 4.00 kg block on a frictionless floor, but only force $\vec{F}_1$ is indicated. That force has a fixed magnitude but can be applied at an adjustable angle θ to the positive direction of the x axis. Force $\vec{F}_2$ is horizontal and fixed in both magnitude and angle. Figure 5-24*b* gives the horizontal acceleration a_x of the block for any given value of θ from 0° to 90°. What is the value of a_x for $\theta = 180°$?

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KEY IDEAS

(1) The horizontal acceleration a_x depends on the net horizontal force $F_{net,x}$, as given by Newton's second law. (2) The net horizontal force is the sum of the horizontal components of forces $\vec{F}_1$ and $\vec{F}_2$.

Calculations: The x component of $\vec{F}_2$ is F_2 because the vector is horizontal. The x component of $\vec{F}_1$ is $F_1 \cos\theta$. Using these expressions and a mass m of 4.00 kg, we can write Newton's second law ($\vec{F}_{net} = m\vec{a}$) for motion along the x axis as

$$F_1 \cos\theta + F_2 = 4.00a_x. \qquad (5\text{-}35)$$

From this equation we see that when angle $\theta = 90°$, $F_1 \cos\theta$ is zero and $F_2 = 4.00a_x$. From the graph we see that the corresponding acceleration is 0.50 m/s². Thus, $F_2 = 2.00$ N and $\vec{F}_2$ must be in the positive direction of the x axis.

From Eq. 5-35, we find that when $\theta = 0°$,

$$F_1 \cos 0° + 2.00 = 4.00a_x. \qquad (5\text{-}36)$$

From the graph we see that the corresponding acceleration is 3.0 m/s². From Eq. 5-36, we then find that $F_1 = 10$ N.

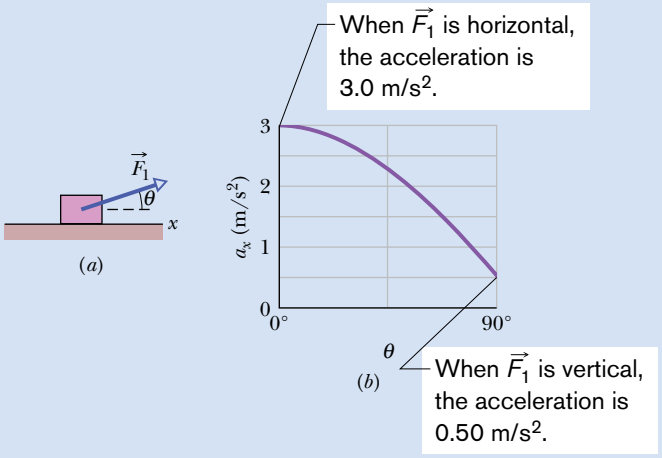

Figure 5-24 (*a*) One of the two forces applied to a block is shown. Its angle θ can be varied. (*b*) The block's acceleration component a_x versus θ.

Substituting $F_1 = 10$ N, $F_2 = 2.00$ N, and $\theta = 180°$ into Eq. 5-35 leads to

$$a_x = -2.00 \text{ m/s}^2. \qquad \text{(Answer)}$$

SAMPLE PROBLEM 5.08

Forces within an elevator cab

In Fig. 5-25*a*, a passenger of mass $m = 72.2$ kg stands on a platform scale in an elevator cab. We are concerned with the scale readings when the cab is stationary and when it is moving up or down.

(a) Find a general solution for the scale reading, whatever the vertical motion of the cab.

KEY IDEAS

(1) The reading is equal to the magnitude of the normal force $\vec{F}_N$ on the passenger from the scale. The only other force acting on the passenger is the gravitational force $\vec{F}_g$, as shown in the free-body diagram of Fig. 5-25*b*. (2) We can relate the forces on the passenger to his acceleration $\vec{a}$ by using Newton's second law ($\vec{F}_{net} = m\vec{a}$). However, recall that we can use this law only in an inertial frame. If the cab accelerates, then it is *not* an inertial frame. So we choose the ground to be our inertial frame and make any measure of the passenger's acceleration relative to it.

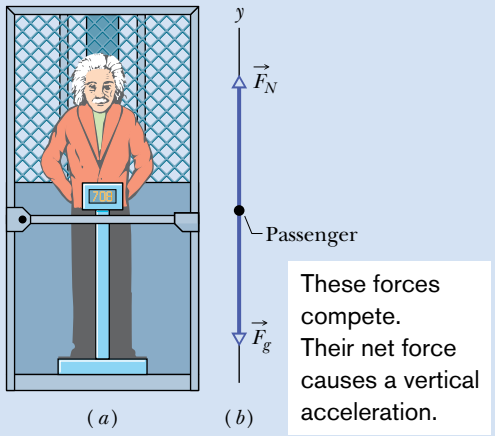

Figure 5-25 (*a*) A passenger stands on a platform scale that indicates either his weight or his apparent weight. (*b*) The free-body diagram for the passenger, showing the normal force $\vec{F}_N$ on him from the scale and the gravitational force $\vec{F}_g$.

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Calculations: Because the two forces on the passenger and his acceleration are all directed vertically, along the y axis in Fig. 5-25*b*, we can use Newton's second law written for y components ($F_{net,y} = ma_y$) to get

$$F_N - F_g = ma$$

or
$$F_N = F_g + ma. \quad (5\text{-}37)$$

This tells us that the scale reading, which is equal to normal force magnitude F_N, depends on the vertical acceleration. Substituting mg for F_g gives us

$$F_N = m(g + a) \quad \text{(Answer)} \quad (5\text{-}38)$$

for any choice of acceleration a. If the acceleration is upward, a is positive; if it is downward, a is negative.

(b) What does the scale read if the cab is stationary or moving upward at a constant 0.50 m/s?

KEY IDEA

For any constant velocity (zero or otherwise), the acceleration a of the passenger is zero.

Calculation: Substituting this and other known values into Eq. 5-38, we find

$$F_N = (72.2 \text{ kg})(9.8 \text{ m/s}^2 + 0) = 708 \text{ N}. \quad \text{(Answer)}$$

This is the weight of the passenger and is equal to the magnitude F_g of the gravitational force on him.

(c) What does the scale read if the cab accelerates upward at 3.20 m/s^2 and downward at 3.20 m/s^2?

Calculations: For $a = 3.20$ m/s^2, Eq. 5-38 gives

$$F_N = (72.2 \text{ kg})(9.8 \text{ m/s}^2 + 3.20 \text{ m/s}^2)$$
$$= 939 \text{ N}. \quad \text{(Answer)}$$

and for $a = -3.20$ m/s^2, it gives

$$F_N = (72.2 \text{ kg})(9.8 \text{ m/s}^2 - 3.20 \text{ m/s}^2)$$
$$= 477 \text{ N}. \quad \text{(Answer)}$$

For an upward acceleration (either the cab's upward speed is increasing or its downward speed is decreasing), the scale reading is greater than the passenger's weight. That reading is a measurement of an apparent weight, because it is made in a noninertial frame. For a downward acceleration (either decreasing upward speed or increasing downward speed), the scale reading is less than the passenger's weight.

(d) During the upward acceleration in part (c), what is the magnitude F_{net} of the net force on the passenger, and what is the magnitude $a_{p,cab}$ of his acceleration as measured in the frame of the cab? Does $\vec{F}_{net} = m\vec{a}_{p,cab}$?

Calculation: The magnitude F_g of the gravitational force on the passenger does not depend on the motion of the passenger or the cab; so, from part (b), F_g is 708 N. From part (c), the magnitude F_N of the normal force on the passenger during the upward acceleration is the 939 N reading on the scale. Thus, the net force on the passenger is

$$F_{net} = F_N - F_g = 939 \text{ N} - 708 \text{ N} = 231 \text{ N} \quad \text{(Answer)}$$

during the upward acceleration. However, his acceleration $a_{p,cab}$ relative to the frame of the cab is zero. Thus, in the noninertial frame of the accelerating cab, F_{net} is not equal to $ma_{p,cab}$, and Newton's second law does not hold.

SAMPLE PROBLEM 5.09

Acceleration of block pushing on block

In Fig. 5-26*a*, a constant horizontal force $\vec{F}_{app}$ of magnitude 20 N is applied to block A of mass $m_A = 4.0$ kg, which pushes against block B of mass $m_B = 6.0$ kg. The blocks slide over a frictionless surface, along an x axis.

(a) What is the acceleration of the blocks?

Serious Error: Because force $\vec{F}_{app}$ is applied directly to block A, we use Newton's second law to relate that force to the acceleration $\vec{a}$ of block A. Because the motion is along the x axis, we use that law for x components ($F_{net,x} = ma_x$), writing it as

$$F_{app} = m_A a.$$

However, this is seriously wrong because $\vec{F}_{app}$ is not the only horizontal force acting on block A. There is also the force $\vec{F}_{AB}$ from block B (Fig. 5-26*b*).

Dead-End Solution: Let us now include force $\vec{F}_{AB}$ by writing, again for the x axis,

$$F_{app} - F_{AB} = m_A a.$$

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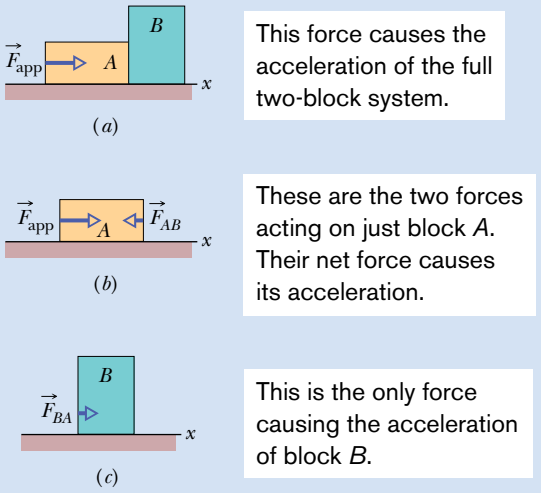

Figure 5-26 (*a*) A constant horizontal force $\vec{F}_{app}$ is applied to block *A*, which pushes against block *B*. (*b*) Two horizontal forces act on block *A*. (*c*) Only one horizontal force acts on block *B*.

(We use the minus sign to include the direction of $\vec{F}_{AB}$.) Because F_{AB} is a second unknown, we cannot solve this equation for a.

Successful Solution: Because of the direction in which force $\vec{F}_{app}$ is applied, the two blocks form a rigidly connected system. We can relate the net force *on the system* to the acceleration *of the system* with Newton's second law. Here, once again for the x axis, we can write that law as

$$F_{app} = (m_A + m_B)a,$$

where now we properly apply $\vec{F}_{app}$ to the system with total mass $m_A + m_B$. Solving for a and substituting known values, we find

$$a = \frac{F_{app}}{m_A + m_B} = \frac{20\text{ N}}{4.0\text{ kg} + 6.0\text{ kg}} = 2.0\text{ m/s}^2. \quad \text{(Answer)}$$

Thus, the acceleration of the system and of each block is in the positive direction of the x axis and has the magnitude 2.0 m/s^2.

(b) What is the (horizontal) force $\vec{F}_{BA}$ on block *B* from block *A* (Fig. 5-26*c*)?

KEY IDEA

We can relate the net force on block *B* to the block's acceleration with Newton's second law.

Calculation: Here we can write that law, still for components along the x axis, as

$$F_{BA} = M_B a,$$

which, with known values, gives

$$F_{BA} = (6.0\text{ kg})(2.0\text{ m/s}^2) = 12\text{ N}. \quad \text{(Answer)}$$

Thus, force $\vec{F}_{BA}$ is in the positive direction of the x axis and has a magnitude of 12 N.

SAMPLE PROBLEM 5.10

Tension and normal force on block

In Fig. 5-27, a block attached to a chord is resting on an inclined plane. Let the mass of the block be 8.5 kg and the angle θ be 30°.

(a) Find the tension in the cord and normal force acting on the block.

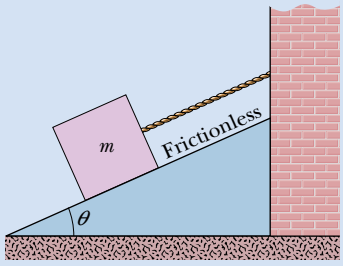

Figure 5-27 A block attached to a chord resting on an incline.

KEY IDEA

(1) We apply Newton's second law to solve for the tension in the cord and the normal force on the block.

(2) The free-body diagram of the problem is shown in the Fig. 5-28. Since the acceleration of the block is zero, the components of Newton's second law equation yield

$$T - mg\sin\theta = 0 \quad (5\text{-}39)$$

$$F_N - mg\cos\theta = 0, \quad (5\text{-}40)$$

where T is the tension in the cord, and F_N is the normal force on the block.

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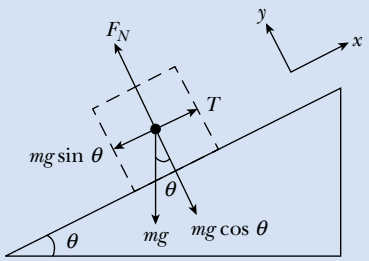

Figure 5.28 Free body diagram of the given problem.

Calculation: Solving Eq. (5-39) for the tension in the string, we find

$$T = mg \sin\theta = (8.5\text{ kg})(9.8\text{ m/s}^2)\sin 30°$$
$$= 42\text{ N}. \qquad \text{(Answer)}$$

We solve Eq. (5-40) above for the normal force F_N:

$$F_N = \text{mg}\cos\theta = (8.5\text{ kg})(9.8\text{ m/s}^2)\cos 30°$$
$$= 72\text{ N}. \qquad \text{(Answer)}$$

(b) If the cord is cut, find the magnitude of the resulting acceleration of the block.

Calculation: When the cord is cut, it no longer exerts a force on the block and the block accelerates. The x component of the second law becomes $-mg\sin\theta = ma$, so the acceleration becomes

$$a = -g\sin\theta = -(9.8\text{ m/s}^2)\sin 30°$$
$$= -4.9\text{ m/s}^2. \qquad \text{(Answer)}$$

The negative sign indicates the acceleration is down the plane. The magnitude of the acceleration is 4.9 m/s^2.

Note: The normal force F_N on the block must be equal to $mg\cos\theta$ so that the block is in contact with the surface of the incline at all time. When the cord is cut, the block has an acceleration $a = -g\sin\theta$, which in the limit $\theta \to 90°$ becomes $-g$, as in the case of a free fall.

SAMPLE PROBLEM 5.11

Block pulled along horizontal floor by a cord

In Fig. 5-29, a block of mass $m = 5.00$ kg is pulled along a horizontal frictionless floor by a cord that exerts a force of magnitude $F = 12.0$ N at an angle $\theta = 25.0°$.

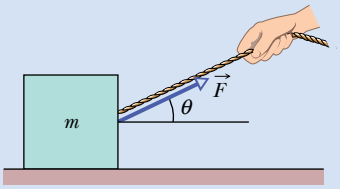

Figure 5-29 A block is pulled along a horizontal floor.

(a) What is the magnitude of the block's acceleration?

KEY IDEA

The free-body diagram (not to scale) for the block is shown in Fig. 5-30. $\vec{F}_N$ is the normal force exerted by the floor and $m\vec{g}$ is the force of gravity.

Calculation: The x component of Newton's second law is $F\cos\theta = ma$, where m is the mass of the block and a is the x component of its acceleration. We obtain

$$a = \frac{F\cos\theta}{m} = \frac{(12.0\text{ N})\cos 25.0°}{5.00\text{ kg}} = 2.18\text{ m/s}^2. \qquad \text{(Answer)}$$

This is its acceleration provided it remains in contact with the floor. Assuming it does, we find the value of F_N (and if F_N is positive, then the assumption is true but if F_N is negative then the block leaves the floor). The y component of Newton's second law becomes

$$F_N + F\sin\theta - mg = 0,$$

so

$$F_N = mg - F\sin\theta$$
$$= (5.00\text{ kg})(9.80\text{ m/s}^2) - (12.0\text{ N})\sin 25.0°$$
$$= 43.9\text{ N}.$$

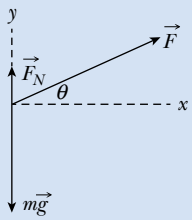

Figure 5-30 Free body diagram for the block.

Hence, the block remains on the floor and its acceleration is $a = 2.18$ m/s^2.

(b) The force magnitude F is slowly increased. What is its value just before the block is lifted (completely) off the floor?

KEY IDEA

If F is the minimum force for which the block leaves the floor, then $F_N = 0$ and the y component of the acceleration vanishes.

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Calculation: The y component of the second law becomes

$$F\sin\theta - mg = 0$$

$$\Rightarrow \quad F = \frac{mg}{\sin\theta} = \frac{(5.00\text{ kg})(9.80\text{ m/s}^2)}{\sin 25.0^\circ} = 116\text{ N}. \quad \text{(Answer)}$$

(c) What is the magnitude of the block's acceleration just before it is lifted (completely) off the floor?

Calculation: The acceleration is still in the x direction and is still given by the equation developed in part (a):

$$a = \frac{F\cos\theta}{m} = \frac{(116\text{ N})\cos 25.0^\circ}{5.00\text{ kg}} = 21.0\text{ m/s}^2. \quad \text{(Answer)}$$

SAMPLE PROBLEM 5.12

Three boxes connected by a cord

In Fig. 5-31, three ballot boxes are connected by cords, one of which wraps over a pulley having negligible friction on its axle and negligible mass. The three masses are $m_A = 30.0$ kg, $m_B = 40.0$ kg, and $m_C = 10.0$ kg.

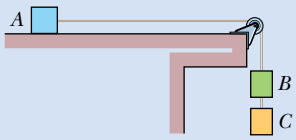

Figure 5-31 Three ballot boxes are connected by cords.

(a) When the assembly is released from rest, what is the tension in the cord connecting B and C?

KEY IDEA

We use Newton's second law of motion to find the tension.

Calculation: The net force on the *system* (of total mass $M = 80.0$ kg) is the force of gravity acting on the total overhanging mass ($m_{BC} = 50.0$ kg). The magnitude of the acceleration is therefore $a = (m_{BC}\ g)/M = 6.125$ m/s^2. Next we apply Newton's second law to block C itself (choosing *down* as the $+y$ direction) and obtain

$$m_C g - T_{BC} = m_C a.$$

This leads to $T_{BC} = 36.8$ N.

(b) When the assembly is released from rest, how far does A move in the first 0.250 s (assuming it does not reach the pulley)?

Calculation: We use equation of motion (choosing *rightward* as the $+x$ direction):

$$\Delta x = 0 + \frac{1}{2}at^2 = 0.191\text{ m}. \quad \text{(Answer)}$$

SAMPLE PROBLEM 5.13

Two blocks connected by a pulley

A block of mass $m_1 = 3.70$ kg on a frictionless plane inclined at angle 30.0° is connected by a cord over a massless, frictionless pulley to a second block of mass $m_2 = 2.30$ kg (see Fig. 5-32).

(a) What is the magnitude of the acceleration of each block?

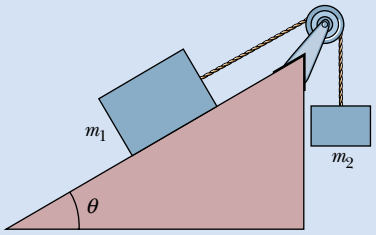

Figure 5-32 Block on frictionless inclined plane connected by a cord over a pulley to another block.

KEY IDEA

The free-body diagram for each block is shown in Fig. 5-33. T is the tension in the cord and $\theta = 30^\circ$ is the angle of the incline. For block 1, we take the $+x$ direction to be up the incline and the $+y$ direction to be in the direction of the normal force $\vec{F}_N$ that the plane exerts on the block. For block 2, we take the $+y$ direction to be down.

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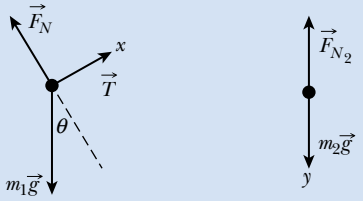

Figure 5-33 Free-body diagram for each of the block.

In this way, the accelerations of the two blocks can be represented by the same symbol a, without ambiguity. Applying Newton's second law to the x and y axes for block 1 and to the y axis of block 2, we obtain

$$T - m_1 g \sin\theta = m_1 a \qquad (5\text{-}41)$$

$$F_N - m_1 g \cos\theta = 0 \qquad (5\text{-}42)$$

$$m_2 g - T = m_2 a \qquad (5\text{-}43)$$

respectively. The first and third of these equations provide a simultaneous set for obtaining values of a and T. The second equation is not needed in this problem, since the normal force is neither asked for nor is it needed as part of some further computation (such as can occur in formulas for friction).

Calculation: We add Eqs. (5-41) and (5-43) above:

$$m_2 g - m_1 g \sin\theta = m_1 a + m_2 a.$$

Consequently, we find

$$a = \frac{(m_2 - m_1 \sin\theta)g}{m_1 + m_2}$$

$$= \frac{[2.30\text{ kg} - (3.70\text{ kg})\sin 30.0°](9.80\text{ m/s}^2)}{3.70\text{ kg} + 2.30\text{ kg}}$$

$$= 0.735\text{ m/s}^2. \qquad \text{(Answer)}$$

(b) What is the direction of the acceleration of the hanging block?

Reasoning: The result for a is positive, indicating that the acceleration of block 1 is indeed up the incline and that the acceleration of block 2 is vertically down.

(c) What is the tension in the cord?

Calculation: The tension in the cord is

$$T = m_1 a + m_1 g \sin\theta$$

$$= (3.70\text{ kg})(0.735\text{ m/s}^2) + (3.70\text{ kg})(9.80\text{ m/s}^2)\sin 30.0°$$

$$= 20.8\text{ N}. \qquad \text{(Answer)}$$

SAMPLE PROBLEM 5.14

Three blocks attached by a cord over pulleys

Figure 5-34 shows three blocks attached by cords that loop over frictionless pulleys. Block B lies on a frictionless table; the masses are $m_A = 6.00$ kg, $m_B = 8.00$ kg, and $m_C = 10.0$ kg. When the blocks are released, what is the tension in the cord at the right?

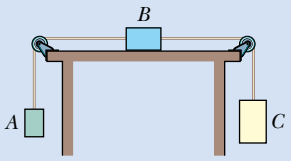

Figure 5-34 Three blocks attached over two pulleys by a cord.

KEY IDEA

First we analyze the entire *system* with "clockwise" motion considered positive (that is, downward is positive for block C, rightward is positive for block B, and upward is positive for block A) and find acceleration. Then we analyze forces on block C only.

Calculation: The equation for the system is

$$m_C g - m_A g = Ma \qquad \text{(where } M = \text{mass of the system} = 24.0\text{ kg).}$$

This yields an acceleration of

$$a = g(m_C - m_A)/M = 1.63\text{ m/s}^2.$$

Next we analyze the forces just on block C: $m_C g - T = m_C a$. Thus the tension is

$$T = m_C g(2m_A + m_B)/M = 81.7\text{ N}. \qquad \text{(Answer)}$$

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SAMPLE PROBLEM 5.15

Block is pulled along a horizontal surface

A block of mass M is pulled along a horizontal frictionless surface by a rope of mass m, as shown in Fig. 5-35. A horizontal force acts on one end of the rope.

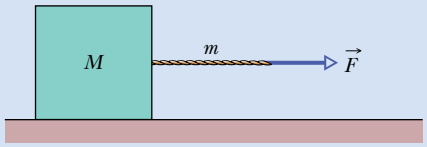

Figure 5-35 Block of mass M is pulled by rope of mass m by a force F.

(a) Show that the rope *must* sag, even if only by an imperceptible amount.

Reasoning: A small segment of the rope has mass and is pulled down by the gravitational force of the Earth. Equilibrium is reached because neighboring portions of the rope pull up sufficiently on it. Since tension is a force *along* the rope, at least one of the neighboring portions must slope up away from the segment we are considering. Then, the tension has an upward component, which means the rope sags.

(b) Then, assuming that the sag is negligible, find the acceleration of rope and block.

KEY IDEA

The only force acting with a horizontal component is the applied force $\vec{F}$. Treat the block and rope as a single object.

Calculation: We write Newton's second law for it: $F = (M + m)a$, where a is the acceleration and the positive direction is taken to be to the right. The acceleration is given by $a = F/(M + m)$.

(c) Find the force on the block from the rope.

Calculation: The force of the rope F_r is the only force with a horizontal component acting on the block. Then Newton's second law for the block gives

$$F_r = Ma = \frac{MF}{M+m} \qquad \text{(Answer)}$$

where the expression found above for a has been used.

(d) Find the tension in the rope at its midpoint.

Calculation: Treating the block and half the rope as a single object, with mass $M + 1/2\,m$, where the horizontal force on it is the tension T_m at the midpoint of the rope, we use Newton's second law:

$$T_m = \left(M + \frac{1}{2}m\right)a$$
$$= \frac{(M + m/2)F}{(M+m)}$$
$$= \frac{(2M+m)F}{2(M+m)}. \qquad \text{(Answer)}$$

5.11 | MOTION IN ACCELERATED FRAMES: FICTITIOUS/PSEUDO FORCE

Key Concepts

- A reference frame, in which the net force or the net acceleration is non-zero, is known as noninertial frame.
- Pseudo or fictitious force acts on all the occupants of noninertial frame (an accelerated frame).

So far in this chapter we have learnt about the Newton's laws of motion that hold good in inertial frame of reference. Now we will discuss about noninertial frame of reference. Objects in noninertial reference frames do not obey Newton's first law.

Fictitious/Pseudo Force

It is an imaginary or fictitious force, which acts on all the occupants of non-inertial frame (an accelerated frame). The direction of the force is opposite to the direction of the acceleration. It is fictitious in the sense that it has no physical origin that is not caused by one of the basic interactions in nature. Its action does not have the reaction required by Newton's third law.

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Examples

1. Suppose a ball is on the frictionless floor of a car and the brakes are applied. Since there is no net force on the ball, an observer on the ground will see the ball continue to move at the velocity of the car just before the brakes were applied. However, relative to an observer in the car, the ball accelerates in the forward direction, even though there is no net force on it. It is due to pseudo force.
2. In all rotating frame due to centripetal force it is noninertial frame, so a pseudo force always acts in opposite direction known as centrifugal force.
3. Apparent weights in an accelerating elevator (see Fig. 5-36):

   (a) When the elevator accelerates upward with acceleration a_0: The forces on the man, exerted by the scale platform on which it rests, are reaction force N and the force of gravity Mg.

   In the reference frame attached to the elevator, an additional pseudo force, $F_P = Ma_0$, acts opposite to the acceleration. The free-body diagrams of the man with respect to ground and the elevator are shown in the figure. Applying Newton's second law, we have

   $$N - Mg = Ma_0$$

   or
   $$N = M(g + a_0) \qquad (5\text{-}44)$$

   So, the person feels heavier inside elevator.

   (b) When the elevator accelerates downward with acceleration a_0: Applying Newton's second law,

   $$Mg - N = Ma_0$$

   or
   $$N = M(g - a_0) \qquad (5\text{-}45)$$

   Thus, when the elevator accelerating down, the apparent weight of the man is lesser than his true weight by Ma_0, thus, the person feels lighter inside elevator.

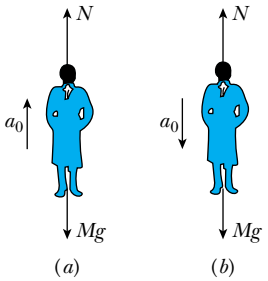

Figure 5-36 Free-body diagram of man with respect to ground (*a*) when elevator is accelerating upward (*b*) when elevator is accelerating downward.

SAMPLE PROBLEM 5.16

Pseudo force acting on man and coin in an accelerated frame

A customer sits in an amusement park ride in which the compartment is to be pulled downward in the negative direction of a y axis with an acceleration magnitude of $1.24\,g$, with $g = 9.80\ \text{m/s}^2$. A 0.567 g coin rests on the customer's knee. Once the motion begins and in unit-vector notation,

(a) What is the coin's acceleration relative to the ground?

KEY IDEA

The customer sitting and the coin on his knee both are in the noninertial frame of reference as the compartment is accelerating downward. Thus the concept of pseudo force comes into picture and we use this concept here.

Calculation: The coin undergoes free fall. Therefore, with respect to ground, its acceleration is

$$\vec{a}_{\text{coin}} = \vec{g} = (-9.8\ \text{m/s}^2)\hat{j}. \qquad \text{(Answer)}$$

(b) What is the coin's acceleration relative to the customer?

Calculation: Since the customer is being pulled down with an acceleration of $\vec{a}'_{\text{customer}} = 1.24\vec{g} = (-12.15\ \text{m/s}^2)\hat{j}$, the acceleration of the coin with respect to the customer is

$$\begin{aligned}\vec{a}_{\text{rel}} &= \vec{a}_{\text{coin}} - \vec{a}'_{\text{customer}} \\ &= (-9.8\ \text{m/s}^2)\hat{j} - (-12.15\ \text{m/s}^2)\hat{j} \\ &= (+2.35\ \text{m/s}^2)\hat{j}. \qquad \text{(Answer)}\end{aligned}$$

(c) How long does the coin take to reach the compartment ceiling, 2.20 m above the knee?

Calculation: The time it takes for the coin to reach the ceiling is

$$t = \sqrt{\frac{2h}{a_{\text{rel}}}} = \sqrt{\frac{2(2.20\ \text{m})}{2.35\ \text{m/s}^2}} = 1.37\ \text{s}. \qquad \text{(Answer)}$$

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(d) In unit-vector notation, what is the actual force on the coin?

Calculation: Since gravity is the only force acting on the coin, the actual force on the coin is

$$\begin{aligned}\vec{F}_{\text{coin}} &= m\vec{a}_{\text{coin}} = m\vec{g}\\ &= (0.567\times10^{-3}\text{ kg})(-9.8\text{ m/s}^2)\hat{j}\\ &= (-5.56\times10^{-3}\text{ N})\hat{j}. \qquad \text{(Answer)}\end{aligned}$$

(e) In unit-vector notation, what is the apparent force according to the customer's measure of the coin's acceleration?

Calculation: In the customer's frame, the coin travels upward at a constant acceleration. Therefore, the apparent force on the coin is

$$\begin{aligned}\vec{F}_{\text{app}} &= m\vec{a}_{\text{rel}}\\ &= (0.567\times10^{-3}\text{ kg})(+2.35\text{ m/s}^2)\hat{j}\\ &= (+1.33\times10^{-3}\text{ N})\hat{j}. \qquad \text{(Answer)}\end{aligned}$$

REVIEW AND SUMMARY

Newtonian Mechanics The velocity of an object can change (the object can accelerate) when the object is acted on by one or more **forces** (pushes or pulls) from other objects. *Newtonian mechanics* relates accelerations and forces.

Force Forces are vector quantities. Their magnitudes are defined in terms of the acceleration they would give the standard kilogram. A force that accelerates that standard body by exactly 1 m/s^2 is defined to have a magnitude of 1 N. The direction of a force is the direction of the acceleration it causes. Forces are combined according to the rules of vector algebra. The **net force** on a body is the vector sum of all the forces acting on the body.

Newton's First Law If there is no net force on a body, the body remains at rest if it is initially at rest or moves in a straight line at constant speed if it is in motion.

Inertial Reference Frames Reference frames in which Newtonian mechanics holds are called *inertial reference frames* or *inertial frames*. Reference frames in which Newtonian mechanics does not hold are called *noninertial reference frames* or *noninertial frames*.

Mass The **mass** of a body is the characteristic of that body that relates the body's acceleration to the net force causing the acceleration. Masses are scalar quantities.

Newton's Second Law The net force $\vec{F}_{\text{net}}$ on a body with mass m is related to the body's acceleration $\vec{a}$ by

$$\vec{F}_{\text{net}} = m\vec{a}, \qquad (5\text{-}1)$$

which may be written in the component versions

$$F_{\text{net},x} = ma_x, \quad F_{\text{net},y} = ma_y, \quad \text{and} \quad F_{\text{net},z} = ma_z. \qquad (5\text{-}2)$$

The second law indicates that in SI units,

$$1\text{ N} = 1\text{ kg}\cdot\text{m/s}^2. \qquad (5\text{-}3)$$

A **free-body diagram** is a stripped-down diagram in which only *one* body is considered. That body is represented by either a sketch or a dot. The external forces on the body are drawn, and a coordinate system is superimposed, oriented so as to simplify the solution.

Newton's Third Law If a force $\vec{F}_{BC}$ acts on body B due to body C, then there is a force $\vec{F}_{CB}$ on body C due to body B:

$$\vec{F}_{BC} = -\vec{F}_{CB}.$$

Some Particular Forces A **gravitational force** $\vec{F}_g$ on a body is a pull by another body. In most situations in this book, the other body is Earth or some other astronomical body. For Earth, the force is directed down toward the ground, which is assumed to be an inertial frame. With that assumption, the magnitude of $\vec{F}_g$ is

$$F_g = mg, \qquad (5\text{-}8)$$

where m is the body's mass and g is the magnitude of the free-fall acceleration.

The **weight** W of a body is the magnitude of the upward force needed to balance the gravitational force on the body. A body's weight is related to the body's mass by

$$W = mg. \qquad (5\text{-}12)$$

A **normal force** $\vec{F}_N$ is the force on a body from a surface against which the body presses. The normal force is always perpendicular to the surface.

When a cord is under **tension**, each end of the cord pulls on a body. The pull is directed along the cord, away from the point of attachment to the body. For a *massless* cord (a cord with negligible mass), the pulls at both ends of the cord have the same magnitude T, even if the cord runs around a *massless, frictionless pulley* (a pulley with negligible mass and negligible friction on its axle to oppose its rotation).

A **spring force** is the force exerted on any object by a compressed or stretched spring.

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When two springs are connected in **parallel** with each other, then the equivalent spring constant is the sum of the individual spring constants of the two springs, given by

$$k_{eq} = k_1 + k_2 \tag{5-17}$$

When two springs having spring constants k_1 and k_2 are connected in **series** with each other, then the equivalent spring constant is given by

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} \tag{5-18}$$

Constraint Motion When motion of a body is dependent on the motion of another body, then the motion is called constraint motion.

Pseudo Force It is an imaginary or fictitious force, which acts on all the occupants of non-inertial frame (an accelerated frame). The direction of the force is opposite to the direction of the acceleration.

When the elevator is **accelerating upward with acceleration** a_0, the person feels heavier inside elevator, the apparent weight of the man is more than his true weight by Ma_0, thus, the person feels heavier inside elevator.

When the elevator is **accelerating downward with acceleration** a_0, the apparent weight of the man is lesser than his true weight by Ma_0, thus, the person feels lighter inside elevator.

PROBLEMS

1. When two perpendicular forces 9.0 N (toward positive x) and 7.0 N (toward positive y) act on a body of mass 6.0 kg, what are the (a) magnitude and (b) direction of the acceleration of the body?

2. Two horizontal forces act on a 2.5 kg chopping block that can slide over a frictionless kitchen counter, which lies in an xy plane. One force is $\vec{F}_1 = (3.0\text{ N})\hat{i} + (4.0\text{ N})\hat{j}$. Find the acceleration of the chopping block in unit-vector notation when the other force is (a) $\vec{F}_2 = (-3.0\text{ N})\hat{i} + (-4.0\text{ N})\hat{j}$, (b) $\vec{F}_2 = (-3.0\text{ N})\hat{i} + (4.0\text{ N})\hat{j}$, and (c) $\vec{F}_2 = (3.0\text{ N})\hat{i} + (-4.0\text{ N})\hat{j}$.

3. A body has an acceleration of 3.00 m/s^2 at 30.0° to the positive direction of an x axis. The mass of the body is 2.00 kg. Find (a) the x component and (b) the y component of the net force acting on the body. (c) What is the net force in unit-vector notation?

4. A particle is to move along a line at the constant velocity $\vec{v} = (2\text{ m/s})\hat{i} - (3\text{ m/s})\hat{j}$. During the motion of the particle, we assume that two forces are acting on it. If one of the forces is $\vec{F} = (2\text{ N})\hat{i} + (-5\text{ N})\hat{j}$, find the other force.

5. Three astronauts, propelled by jet backpacks, push and guide a 120 kg asteroid toward a processing dock, exerting the forces shown in Fig. 5-37, with $F_1 = 32$ N, $F_2 = 55$ N, $F_3 = 41$ N, $\theta_1 = 30°$, and $\theta_3 = 60°$. What is the asteroid's acceleration (a) in unit-vector notation and as (b) a magnitude and (c) a direction relative to the positive direction of the x axis?

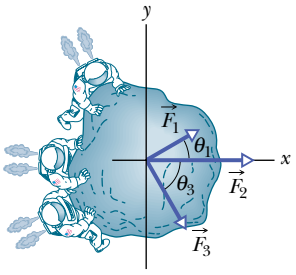

Figure 5-37 Problem 5.

6. There are two forces on the 2.00 kg box in the overhead view of Fig. 5-38, but only one is shown. For $F_1 = 20.0$ N, $a = 12.0$ m/s^2, and $\theta = 30.0°$, find the second force (a) in unit-vector notation and as (b) a magnitude and (c) an angle relative to the positive direction of the x axis.

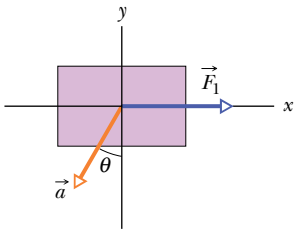

Figure 5-38 Problem 6.

7. A 1.50 kg object is subjected to three forces that give it an acceleration $\vec{a} = -(8.00\text{ m/s}^2)\hat{i} + (6.00\text{ m/s}^2)\hat{j}$. If two of the three forces are $\vec{F}_1 = (30.0\text{ N})\hat{i} + (16.0\text{ N})\hat{j}$ and $\vec{F}_2 = -(12.0\text{ N})\hat{i} + (8.00\text{ N})\hat{j}$, find the third force.

8. In an xy plane, a 0.450 kg object moves in such a way that $x(t) = -16.0 + 3.00t - 5.00t^3$ and $y(t) = 26.0 + 8.00t - 10.0t^2$, where x and y are measured in meters and t in seconds. At $t = 0.800$ s, find (a) the magnitude and (b) the angle, relative to the positive direction of the x axis, of the net force on the object, and (c) the angle of the object's travel direction.

9. A 0.150 kg particle moves along an x axis according to $x(t) = -13.00 + 2.00t + 4.00t^2 - 3.00t^3$, with x in meters and t in seconds. In unit-vector notation, what is the net force acting on the particle at $t = 2.60$ s?

10. A 3.0 kg object is driven along an x axis by a variable force that is directed along that axis. Its position is given by $x = 4.0\text{ m} + (5.0\text{ m/s})t + kt^2 - (3.0\text{ m/s}^3)t^3$, where x is measured in meters and t in seconds. The factor k is a constant. At $t = 4.0$ s, the force on the particle has a magnitude of 37 N and is in the negative direction of the axis. Find the value of k.

11. Two horizontal forces $\vec{F}_1$ and $\vec{F}_2$ act on a 4.0 kg disk that slides over frictionless ice, on which an xy coordinate system is laid out. Force $\vec{F}_1$ is in the positive direction of the x axis and has a magnitude of 7.0 N. Force $\vec{F}_2$ has a magnitude of 9.0 N. Figure 5-39 gives the x component v_x of the velocity of the disk as a function of time t during

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the sliding. What is the angle between the constant directions of forces $\vec{F}_1$ and $\vec{F}_2$?

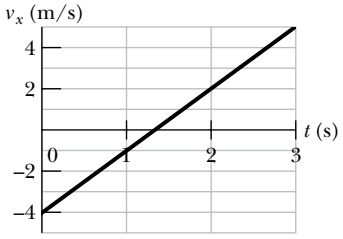

Figure 5-39 Problem 11.

12. Two particles of masses m and $2m$ are placed on a smooth horizontal table. A string, which joins these two masses, hangs over the edge supporting a pulley, which suspends a particle of mass $3m$, as shown in Fig. 5-40. The pulley has negligible mass. The two parts of the string on the table are parallel and perpendicular to the edge of the table. The hanging parts of the string are vertical. Find the acceleration of the particle of mass $3m$.

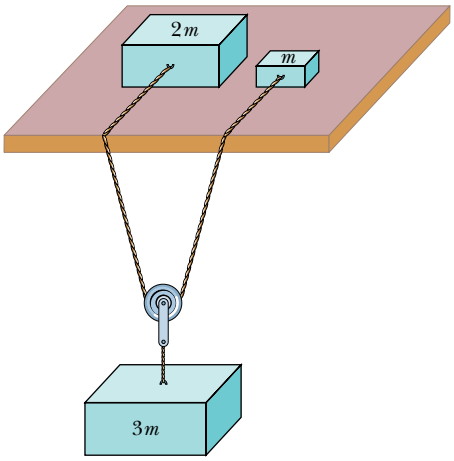

Figure 5-40 Problem 12.

13. A block with a weight of 4.0 N is at rest on a horizontal surface. A 1.0 N upward force is applied to the block by means of an attached vertical string. What are the (a) magnitude and (b) direction of the force of the block on the horizontal surface?

14. (a) An 11.0 kg salami is supported by a cord that runs to a spring scale, which is supported by a cord hung from the ceiling (Fig. 5-41*a*). What is the reading on the scale, which is marked in SI weight units? (This is a way to measure weight by a deli owner.) (b) In Fig. 5-41*b* the salami is supported by a cord that runs around a pulley and to a scale. The opposite end of the scale is attached by a cord to a wall. What is the reading on the scale? (This is the way by a physics major.) (c) In Fig. 5-41*c* the wall has been replaced with a second 11.0 kg salami, and the assembly is stationary. What is the reading on the scale? (This is the way by a deli owner who was once a physics major.)

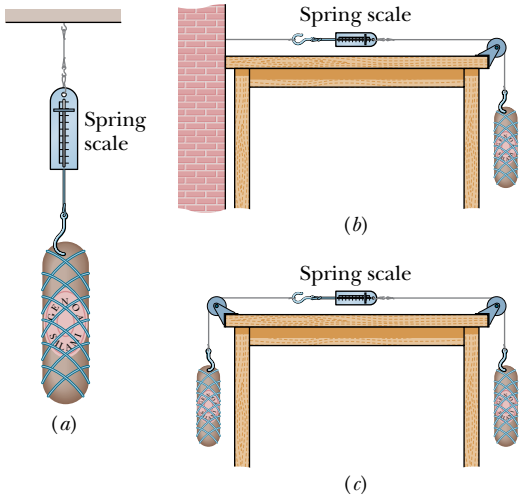

Figure 5-41 Problem 14.

15. A 550 kg rocket sled can be accelerated at a constant rate from rest to 1650 km/h in 2.0 s. What is the magnitude of the required net force?

16. A car traveling at 63 km/h hits a bridge abutment. A passenger in the car moves forward a distance of 65 cm (with respect to the road) while being brought to rest by an inflated air bag. What magnitude of force (assumed constant) acts on the passenger's upper torso, which has a mass of 41 kg?

17. A constant horizontal force $\vec{F}_a$ pushes a 2.00 kg package across a frictionless floor on which an xy coordinate system has been drawn. Figure 5-42 gives the package's x and y velocity components versus time t. What are the (a) magnitude and (b) direction of $\vec{F}_a$?

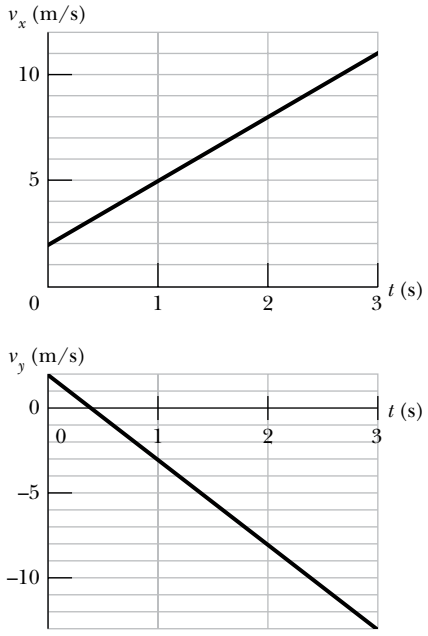

Figure 5-42 Problem 17.

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18. Tarzan, who weighs 860 N, swings from a cliff at the end of a 20.0 m vine that hangs from a high tree limb and initially makes an angle of 22.0° with the vertical. Assume that an x axis extends horizontally away from the cliff edge and a y axis extends upward. Immediately after Tarzan steps off the cliff, the tension in the vine is 760 N. Just then, what are (a) the force on him from the vine in unit-vector notation and the net force on him (b) in unit-vector notation and as (c) a magnitude and (d) an angle relative to the positive direction of the x axis? What are the (e) magnitude and (f) angle of Tarzan's acceleration just then?

19. There are two horizontal forces on the 2.0 kg box in the overhead view of Fig. 5-43 but only one (of magnitude $F_1 = 30$ N) is shown. The box moves along the x axis. For each of the following values for the acceleration a_x of the box, find the second force in unit-vector notation: (a) 10 m/s², (b) 20 m/s², (c) 0, (d) –10 m/s², and (e) –20 m/s².

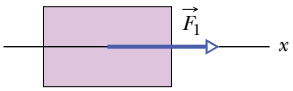

Figure 5-43 Problem 19.

20. The tension at which a fishing line snaps is commonly called the line's "strength." What minimum strength is needed for a line that is to stop a salmon of weight 90 N in 11 cm if the fish is initially drifting at 2.8 m/s? Assume a constant deceleration.

21. A subatomic particle moves horizontally, with a speed of 1.5×10^7 m/s, into a region where a uniform vertical electric force of 5.5×10^{-16} N acts on it. Assuming the subatomic particle is an electron (the mass of the electron is 9.11×10^{-31} kg), find the vertical distance the particle is deflected during the time it has moved 35 mm horizontally.

22. A car that weighs 1.30×10^4 N is initially moving at 35 km/h when the brakes are applied and the car is brought to a stop in 15 m. Assuming the force that stops the car is constant, find (a) the magnitude of that force and (b) the time required for the change in speed. If the initial speed is doubled, and the car experiences the same force during the braking, by what factors are (c) the stopping distance and (d) the stopping time multiplied? (There could be a lesson here about the danger of driving at high speeds.)

23. A firefighter who weighs 689 N slides down a vertical pole with an acceleration of 2.00 m/s², directed downward. What are the (a) magnitude and (b) direction (up or down) of the vertical force on the firefighter from the pole and the (c) magnitude and (d) direction of the vertical force on the pole from the firefighter?

24. Figure 5-44 shows an overhead view of a 0.0250 kg lemon half and two of the three horizontal forces that act on it as it is on a frictionless table. Force $\vec{F}_1$ has a magnitude of 6.00 N and is at $\theta_1 = 30.0°$. Force $\vec{F}_2$ has a magnitude of 7.00 N and is at $\theta_2 = 30.0°$. In unit-vector notation, what is the third force if the lemon half (a) is stationary, (b) has the constant velocity $\vec{v} = (13.0\hat{i} - 14.0\hat{j})$ m/s, and (c) has the varying velocity $\vec{v} = (13.0t\hat{i} - 14.0t\hat{j})$ m/s², where t is time?

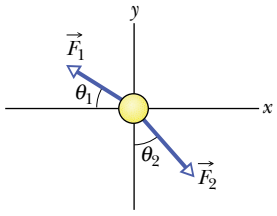

Figure 5-44 Problem 24.

25. A 1500 kg cable car moves vertically by means of a cable that connects the ground and the top of a hill. What is the tension in the supporting cable when the cab, originally moving downward at a speed of 9.0 m/s, is brought to rest with constant acceleration in a distance of 38 m.

26. In Fig. 5-45, a crate of mass $m = 115$ kg is pushed at constant speed up a frictionless ramp ($\theta = 30.0°$) by a horizontal force $\vec{F}$. What are the magnitudes of (a) $\vec{F}$ and (b) the force on the crate from the ramp?

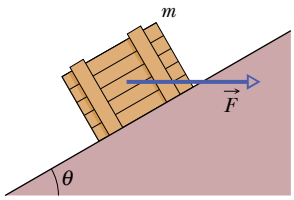

Figure 5-45 Problem 26.

27. An object weighs 2.50 kg. In time t, measured in seconds, the velocity of the object is given by $\vec{v} = (7.00t\hat{i} + 2.00t^2\hat{j})$ m/s. At the instant the net force on the object has a magnitude of 35.0 N, what are (a) the direction of the net force and (b) the object's direction of travel?

28. Holding on to a tow rope moving parallel to a frictionless ski slope, a 45 kg skier is pulled up the slope, which is at an angle of 8.0° with the horizontal. What is the magnitude F_{rope} of the force on the skier from the rope when (a) the magnitude v of the skier's velocity is constant at 2.0 m/s and (b) $v = 2.0$ m/s as v increases at a rate of 0.10 m/s²?

29. A boy with a mass of 35 kg and a sled with a mass of 6.5 kg are on the frictionless ice of a frozen lake, 12 m apart but connected by a rope of negligible mass. The boy exerts a horizontal 4.2 N force on the rope. What are the acceleration magnitudes of (a) the sled and (b) the boy? (c) How far from the boy's initial position do they meet?

30. A 50 kg skier skis directly down a frictionless slope angled at 10° to the horizontal. Assume the skier moves in the negative direction of an x axis along the slope. A wind force with component F_x acts on the skier. What is F_x if the magnitude of the skier's velocity is (a) constant, (b) increasing at a rate of 1.0 m/s², and (c) increasing at a rate of 2.0 m/s²?

31. A bead of mass 2.5×10^{-4} kg is suspended from a cord. A steady horizontal breeze pushes the bead so that the cord makes a constant angle of 40° with the vertical. Find (a) the tension in the cord and (b) the push magnitude.

32. A dated box of dates, of mass 4.50 kg, is sent sliding up a frictionless ramp at an angle of θ to the horizontal. Figure 5-46 gives, as a function of time t, the component v_x of the box's velocity along an x axis that extends directly up the ramp. What is the magnitude of the normal force on the box from the ramp?

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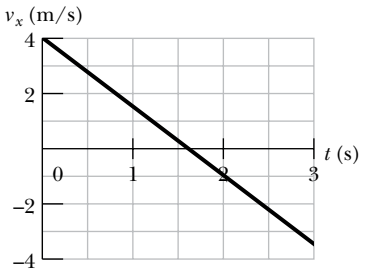

Figure 5-46 Problem 32.

33. In earlier days, horses pulled barges down canals in the manner shown in Fig. 5-47. Suppose the horse pulls on the rope with a force of 8600 N at an angle of $\theta = 18°$ to the direction of motion of the barge, which is headed straight along the positive direction of an x axis. The mass of the barge is 9500 kg, and the magnitude of its acceleration is 0.12 m/s^2. What are the (a) magnitude and (b) direction (relative to positive x) of the force on the barge from the water?

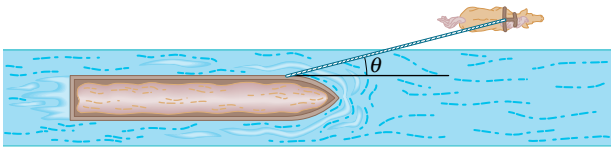

Figure 5-47 Problem 33.

34. In Fig. 5-48, a chain consisting of five links, each of mass 0.100 kg, is lifted vertically with constant acceleration of magnitude $a = 2.50$ m/s^2. Find the magnitudes of (a) the force on link 1 from link 2, (b) the force on link 2 from link 3, (c) the force on link 3 from link 4, and (d) the force on link 4 from link 5. Then find the magnitudes of (e) the force $\vec{F}$ on the top link from the person lifting the chain and (f) the *net* force accelerating each link.

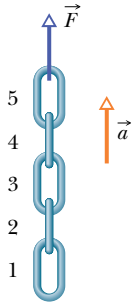

Figure 5-48 Problem 34.

35. A lamp hangs vertically from a cord in a descending elevator that decelerates at 2.4 m/s^2. (a) If the tension in the cord is 93 N, what is the lamp's mass? (b) What is the cord's tension when the elevator ascends with an upward acceleration of 2.4 m/s^2?

36. An elevator cab that weighs 29.0 kN moves upward. What is the tension in the cable if the cab's speed is (a) increasing at a rate of 1.50 m/s^2 and (b) decreasing at a rate of 1.50 m/s^2?

37. An elevator cab is pulled upward by a cable. The cab and its single occupant have a combined mass of 2000 kg. When that occupant drops a coin, its acceleration relative to the cab is 8.30 m/s^2 downward. What is the tension in the cable?

38. The Zacchini family was renowned for their human-cannonball act in which a family member was shot from a cannon using either elastic bands or compressed air. In one version of the act, Emanuel Zacchini was shot over three Ferris wheels to land in a net at the same height as the open end of the cannon and at a range of 69 m. He was propelled inside the barrel for 5.2 m and launched at an angle of 53°. If his mass was 85 kg and he underwent constant acceleration inside the barrel, what was the magnitude of the force propelling him? (*Hint:* Treat the launch as though it were along a ramp at 53°. Neglect air drag.)

39. In Fig. 5-49, elevator cabs A and B are connected by a short cable and can be pulled upward or lowered by the cable above cab A. Cab A has mass 1700 kg; cab B has mass 1200 kg. A 12.0 kg box of catnip lies on the floor of cab A. The tension in the cable connecting the cabs is 1.91×10^4 N. What is the magnitude of the normal force on the box from the floor?

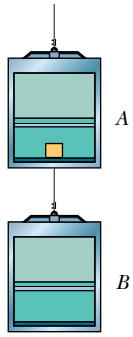

Figure 5-49 Problem 39.

40. Figure 5-50 shows two blocks connected by a cord (of negligible mass) that passes over a frictionless pulley (also of negligible mass). The arrangement is known as *Atwood's machine*. One block has mass $m_1 = 1.30$ kg; the other has mass $m_2 = 2.80$ kg. What are (a) the magnitude of the blocks' acceleration and (b) the tension in the cord?

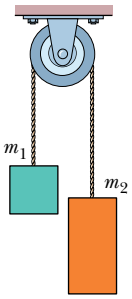

Figure 5-50 Problems 40 and 53.

41. A 93 kg man lowers himself to the ground from a height of 10.0 m by holding onto a rope that runs over a frictionless pulley to a 65 kg sandbag. With what speed does the man hit the ground if he started from rest?

42. As shown in Fig. 5-51, body C (2.9 kg) and body D (1.9 kg) are suspended from a rigid support by inextensible wires B and A, each of length 1.0 m. Wire B has negligible mass; wire A has a uniform density of 0.20 kg/m. The whole system undergoes an upward acceleration of magnitude 0.50 m/s^2. Find the tension at the midpoint in (a) wire A and (b) wire B.

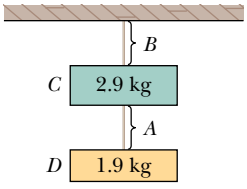

Figure 5-51 Problem 42.

43. Figure 5-52 shows four penguins that are being playfully pulled along very slippery (frictionless) ice by a curator. The masses of three penguins and the tension in two of the cords are $m_1 = 12$ kg, $m_3 = 15$ kg, $m_4 = 20$ kg, $T_2 = 111$ N, and $T_4 = 222$ N. Find the penguin mass m_2 that is not given.

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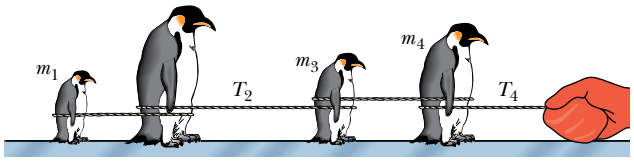

Figure 5-52 Problem 43.

44. Two blocks are in contact on a frictionless table. A horizontal force is applied to the larger block, as shown in Fig. 5-53. (a) If $m_1 = 2.3$ kg, $m_2 = 1.2$ kg, and $F = 3.2$ N, find the magnitude of the force between the two blocks. (b) Show that if a force of the same magnitude F is applied to the smaller block but in the opposite direction, the magnitude of the force between the blocks is 2.1 N, which is not the same value calculated in (a). (c) Explain the difference.

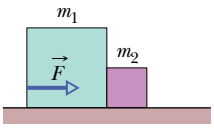

Figure 5-53 Problem 44.

45. In Fig. 5-54*a*, a constant horizontal force $\vec{F}_a$ is applied to block A, which pushes against block B with a 15.0 N force directed horizontally to the right. In Fig. 5-54*b*, the same force $\vec{F}_a$ is applied to block B; now block A pushes on block B with a 10.0 N force directed horizontally to the left. The blocks have a combined mass of 12.0 kg. What are the magnitudes of (a) their acceleration in Fig. 5-54*a* and (b) force $\vec{F}_a$?

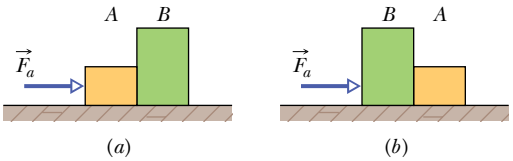

Figure 5-54 Problem 45.

46. Figure 5-55 shows a man sitting in a bosun's chair that dangles from a massless rope, which runs over a massless, frictionless pulley and back down to the man's hand. The combined mass of man and chair is 103.0 kg. With what force magnitude must the man pull on the rope if he is to rise (a) with a constant velocity and (b) with an upward acceleration of 1.30 m/s²? (*Hint:* A free-body diagram can really help.) If the rope on the right extends to the ground and is pulled by a co-worker, with what force magnitude must the co-worker pull for the man to rise (c) with a constant velocity and (d) with an upward acceleration of 1.30 m/s²? What is the magnitude of the force on the ceiling from the pulley system in (e) part a, (f) part b, (g) part c, and (h) part d?

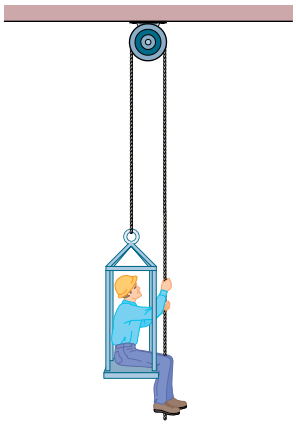

Figure 5-55 Problem 46.

47. A 10 kg monkey climbs up a massless rope that runs over a frictionless tree limb and back down to a 15 kg package on the ground (Fig. 5-56). (a) What is the magnitude of the least acceleration the monkey must have if it is to lift the package off the ground? If, after the package has been lifted, the monkey stops its climb and holds on to the rope, what are the (b) magnitude and (c) direction of the monkey's acceleration and (d) the tension in the rope?

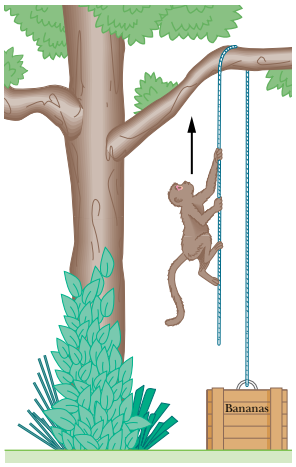

Figure 5-56 Problem 47.

48. Figure 5-57 shows a 5.00 kg block being pulled along a frictionless floor by a cord that applies a force of constant magnitude 15.0 N but with an angle $\theta(t)$ that varies with time. When angle $\theta = 25.0°$, at what rate is the acceleration of the block changing if (a) $\theta(t) = (2.00 \times 10^{-2}\ \text{deg/s})t$ and (b) $\theta(t) = -(2.00 \times 10^{-2}\ \text{deg/s})t$? (*Hint:* The angle should be in radians.)

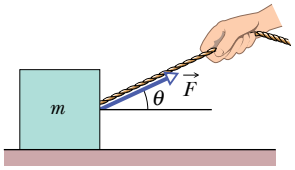

Figure 5-57 Problems 48.

49. A hot-air balloon of mass M is descending vertically with downward acceleration of magnitude a. How much mass (ballast) must be thrown out to give the balloon an upward acceleration of magnitude a? Assume that the upward force from the air (the lift) does not change because of the decrease in mass.

50. In shot putting, many athletes elect to launch the shot at an angle that is smaller than the theoretical one (about 42°) at which the distance of a projected ball at the same speed and height is greatest. One reason has to do with the speed the athlete can give the shot during the acceleration phase of the throw. Assume that a 7.260 kg shot is accelerated along a straight path of length 1.650 m by a constant applied force of magnitude 380.0 N, starting with an initial speed of 2.500 m/s (due to the athlete's preliminary motion). What is the shot's speed at the end of the acceleration phase if the angle between the path and the horizontal is (a) 30.00° and (b) 42.00°? (*Hint:* Treat the motion as though it were along a ramp at the given angle.) (c) By what percentage is the launch speed decreased if the athlete increases the angle from 30.00° to 42.00°?

51. Figure 5-58 gives, as a function of time t, the force component F_x that acts on a 3.00 kg ice block that can move only along the x axis. At $t = 0$, the block is moving in the positive direction of the axis, with a speed of 3.0 m/s. What are its (a) speed and (b) direction of travel at $t = 11$ s?

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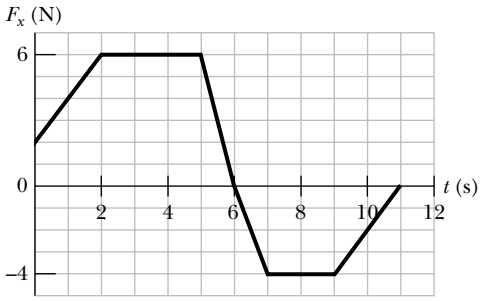

Figure 5-58 Problem 51.

52. Figure 5-59 shows a box of mass $m_2 = 1.0$ kg on a frictionless plane inclined at angle $\theta = 30°$. It is connected by a cord of negligible mass to a box of mass $m_1 = 2.5$ kg on a horizontal frictionless surface. The pulley is frictionless and massless. (a) If the magnitude of horizontal force $\vec{F}$ is 2.3 N, what is the tension in the connecting cord? (b) What is the largest value the magnitude of $\vec{F}$ may have without the cord becoming slack?

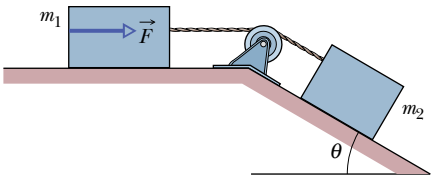

Figure 5-59 Problem 52.

53. Figure 5-50 shows *Atwood's machine,* in which two containers are connected by a cord (of negligible mass) passing over a frictionless pulley (also of negligible mass). At time $t = 0$, container 1 has mass 1.30 kg and container 2 has mass 2.80 kg, but container 1 is losing mass (through a leak) at the constant rate of 0.200 kg/s. At what rate is the acceleration magnitude of the containers changing at (a) $t = 0$ and (b) $t = 3.00$ s? (c) When does the acceleration reach its maximum value?

54. Figure 5-60 shows a section of a cable-car system. The maximum permissible mass of each car with occupants is 2750 kg. The cars, riding on a support cable, are pulled by a second cable attached to the support tower on each car. Assume that the cables are taut and inclined at angle $\theta = 35°$. What is the difference in tension between adjacent sections of pull cable if the cars are at the maximum permissible mass and are being accelerated up the incline at 0.81 m/s^2?

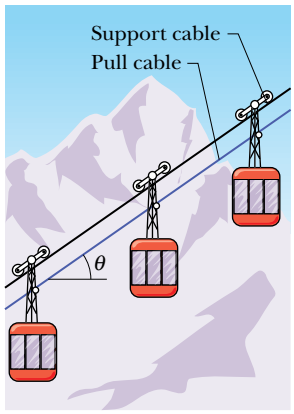

Figure 5-60 Problem 54.

55. A shot putter launches a 7.260 kg shot by pushing it along a straight line of length 1.650 m and at an angle of 34.10° from the horizontal, accelerating the shot to the launch speed from its initial speed of 2.500 m/s (which is due to the athlete's preliminary motion). The shot leaves the hand at a height of 2.110 m and at an angle of 34.10°, and it lands at a horizontal distance of 15.98 m. What is the magnitude of the athlete's average force on the shot during the acceleration phase? (*Hint:* Treat the motion during the acceleration phase as though it were along a ramp at the given angle.)

PRACTICE QUESTIONS

Single Correct Choice Type

1. A boat has a mass of 6800 kg. Its engines generate a drive force of 4100 N, due west, while the wind exerts a force of 800 N, due east, and the water exerts a resistive force of 1200 N due east. What is the magnitude and direction of the boat's acceleration?
(a) 0.54 m/s^2, due west **(b)** 0.54 m/s^2, due east
(c) 0.66 m/s^2, due west **(d)** 0.31 m/s^2, due west

2. Complete the following statement: An inertial reference frame is one in which
(a) Newton's first law of motion is valid.
(b) the inertias of objects within the frame are zero.
(c) the frame is accelerating.
(d) the acceleration due to gravity is greater than zero m/s^2.

3. When a horse pulls a cart, the force that helps the horse to move forward is the force exerted by
(a) the cart on the horse.
(b) the ground on the horse.
(c) the ground on the cart.
(d) the horse on the ground.

4. Sixteen balls of equal masses are connected like beads on a string. Some balls are placed on a smooth inclined plane of inclination $\sin^{-1}(1/3)$ and the remaining balls hang over the top of the plane. The number of balls hanging so as to produce an acceleration of $g/2$ is
(a) 8 **(b)** 9
(c) 10 **(d)** 11

5. The system shown in the following figure is in equilibrium and at rest. The spring and string are massless. Now, the

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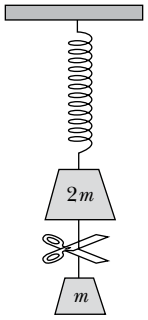
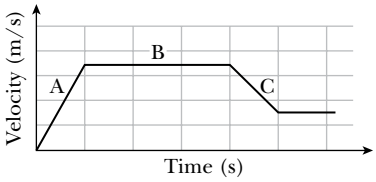
string is cut. The acceleration of mass $2m$ and m, just after the string is cut, will be



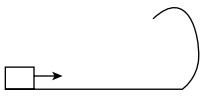
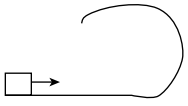
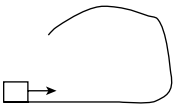
(a) $g/2$ upwards, g downwards.
(b) g upwards, $g/2$ downwards.
(c) g upwards, $2g$ downwards.
(d) $2g$ upwards, g downwards.

6. The given figure shows the velocity versus time curve for a car traveling along a straight line. Which of the following statements is false?



(a) Net forces act on the car during intervals A and C.
(b) Opposing forces may be acting on the car during interval B.
(c) Opposing forces may be acting on the car during interval C.
(d) The magnitude of the net force acting during interval A is less than that during C.

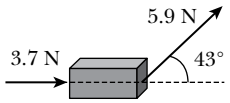
7. A small block is projected into each of the four tracks as shown below. Each of the tracks rises to the same height. The speed with which the block enters the track is the same in all cases. At the highest point of the track, the normal reaction is maximum in

(a) 
(b)
(c) 
(d) 

8. A rock is suspended from a string; and it accelerates upward. Which statement is true concerning the tension in the string?
(a) The tension points downward.
(b) The tension is less than the weight of the rock.
(c) The tension is equal to the weight of the rock.
(d) The tension is greater than the weight of the rock.

9. A paraglider is flying horizontally at a constant speed. Assume that only two forces act on it in the vertical direction, its weight and a vertical lift force exerted on its wings by the air. The lift force has a magnitude of 1800 N. If the lift force should suddenly decrease to 1200 N, what would be the vertical acceleration of the glider? For question, take the upward direction to be the $+y$ direction.
(a) $-0.67\ m/s^2$ **(b)** $-3.3\ m/s^2$
(c) $-4.9\ m/s^2$ **(d)** $-6.6\ m/s^2$

10. Two forces act on a 4.5 kg block resting on a frictionless surface as shown in the given figure. What is the magnitude of the horizontal acceleration of the block?

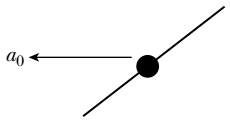


(a) $1.8\ m/s^2$ **(b)** $1.2\ m/s^2$
(c) $0.82\ m/s^2$ **(d)** $3.2\ m/s^2$

11. A force F_1 acts on a particle so as to accelerate it from rest to a velocity v. The force F_1 is then replaced by F_2 which decelerates it to rest.
(a) F_1 must be equal to F_2
(b) F_1 may be equal to F_2
(c) F_1 must be unequal to F_2
(d) None of these.

12. A 15 N net force is applied for 6.0 s to a 12 kg box initially at rest. What is the speed of the box at the end of the 6.0 s interval?
(a) 1.8 m/s **(b)** 15 m/s
(c) 3.0 m/s **(d)** 7.5 m/s

13. A bead of mass m moves on a rod without friction. Initially the bead is at the middle of the rod and the rod moves translationally in a vertical plane with an acceleration a_0 in a direction forming angle θ with the horizontal as shown in the given figure. The acceleration of bead with respect to rod is



(a) $g \sin \theta$
(b) $(g + a_0) \sin \theta$
(c) $g \sin \theta + a_0 \cos \theta$
(d) $g \sin \theta - a_0 \cos \theta$

14. Two satellites of different masses are in the same circular orbit about the Earth. Which one of the following statements is true concerning the magnitude of the gravitational force that acts on each of them?
(a) The magnitude of the gravitational force is zero newtons for both satellites.
(b) The magnitude of the gravitational force is the same for both satellites, but not zero newtons.
(c) The magnitude of the gravitational force is zero newtons for one, but not for the other.
(d) The magnitude of the gravitational force depends on their masses.

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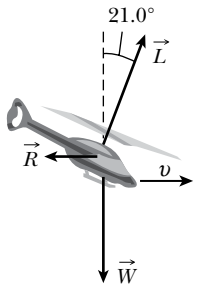
15. Neglect the effect of rotation of the Earth. Suppose the Earth suddenly stops attracting objects placed near its surface. A person standing on the surface of the earth will
(a) fly up.
(b) slip along the surface.
(c) fly along a tangent to the Earth's surface.
(d) remain standing.

16. A 810 kg car accelerates from rest to 27 m/s in a distance of 120 m. What is the magnitude of the average net force acting on the car?
(a) 740 N **(b)** 2500 N
(c) 91 N **(d)** 1300 N

17. A 1580 kg car is traveling with a speed of 15.0 m/s. What is the magnitude of the horizontal net force that is required to bring the car to a halt in a distance of 50.0 m?
(a) 1030 N **(b)** 2490 N
(c) 3560 N **(d)** 4010 N

18. An elevator and its load weigh a total of 1600 kg. Find the tension T in the supporting cable when the elevator, originally moving downward at 20 m/s is brought to rest with constant acceleration in a distance of 50 m.
(a) 2024×10^2 N **(b)** 1024×10^4 N
(c) 2024×10^4 N **(d)** 2024×10^{-4} N

19. The helicopter in the drawing is moving horizontally to the right at a constant velocity. The weight of the helicopter is $W = 53\,800$ N. The lift force $\vec{L}$ generated by the rotating blade makes an angle of 21.0° with respect to the vertical as shown in the following figure. What is the magnitude of the lift force?



(a) 18 800 N **(b)** 57 600 N
(c) 20 600 N **(d)** 37 600 N

20. A rock is suspended from a string; and it moves downward at constant speed. Which one of the following statements is true concerning the tension in the string *if air resistance is not ignored?*
(a) The tension is zero newtons.
(b) The tension points downward.
(c) The tension is equal to the weight of the rock.
(d) The tension is less than the weight of the rock.

21. A 71 kg man stands on a bathroom scale in an elevator. What does the scale read if the elevator is ascending with an acceleration of 3.0 m/s^2?
(a) 140 N **(b)** 480 N
(c) 690 N **(d)** 910 N

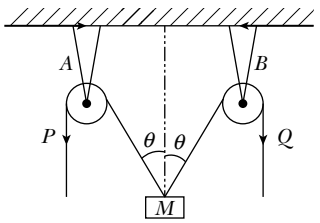
22. Two bodies of masses m and M are attached to the two ends of a light string passing over a fixed ideal pulley ($M \gg m$). When the bodies are in motion, the tension in the string is approximately
(a) $(M - m)g$ **(b)** mg
(c) $2\,mg$ **(d)** $(m/M)mg$

23. Two forces act on a 16 kg object. The first force has a magnitude of 68 N and is directed 24° north of east. The second force is 32 N, 48° north of west. What is the acceleration of the object resulting from the action of these two forces?
(a) 1.6 m/s^2, 5.5° north of east
(b) 1.9 m/s^2, 18° north of west
(c) 2.4 m/s^2, 34° north of east
(d) 4.1 m/s^2, 52° north of east

24. On earth, two parts of a space probe weigh 11 000 N and 3400 N. These parts are separated by a center-to-center distance of 12 m and may be treated as uniform spherical objects. Find the magnitude of the gravitational force that each part exerts on the other out in space, far from any other objects.
(a) 1.7×10^{-6} N **(b)** 3.4×10^{-7} N
(c) 1.8×10^{-7} N **(d)** 3.6×10^{-5} N

25. A certain crane can provide a maximum lifting force of 25 000 N. It hoists a 2000 kg load starting at ground level by applying the maximum force for a 2 second interval; then, it applies just sufficient force to keep the load moving upward at constant speed. Approximately how long does it take to raise the load from ground level to a height of 30 m?
(a) 2 s **(c)** 7 s
(b) 5 s **(d)** 9 s

26. In the arrangement, shown in the given figure, the ends P and Q of an unstretchable string move downwards with uniform speed u. Pulleys A and B are fixed. Mass M moves upwards with a speed



(a) $2u \cos\theta$ **(b)** $u/\cos\theta$
(c) $2u/\cos\theta$ **(d)** $u \cos\theta$

27. Two point masses m and M are separated by a distance d. If the separation d remains fixed and the masses are increased to the values $3m$ and $3M$, respectively, how does the gravitational force between them change?
(a) The force will be one-third as great.
(b) The force will be one-ninth as great.
(c) The force will be three times as great.
(d) The force will be nine times as great.

28. A car is towing a boat on a trailer. The driver starts from rest and accelerates to a velocity of +11 m/s in a time of 28 s. The combined mass of the boat and trailer is 410 kg. The frictional force acting on the trailer can be ignored.

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What is the tension in the hitch that connects the trailer to the car?

(a) 3.2×10^2 N **(b)** 1.6×10^2 N
(c) 6.5×10^2 N **(d)** 4.0×10^3 N

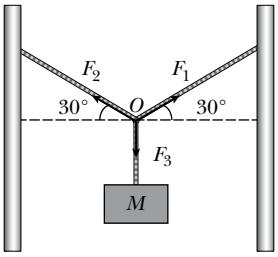
29. A marble is dropped straight down from a distance h above the floor.

Let F_m = the magnitude of the gravitational force *on the marble* due to the Earth;
F_e = the magnitude of the gravitational force *on the Earth* due to the marble;
a_m = the magnitude of the acceleration *of the marble* toward the Earth;
a_e = the magnitude of the acceleration *of the Earth* toward the marble.

Which set of conditions is true as the marble falls toward the earth? Neglect any effects of air resistance.

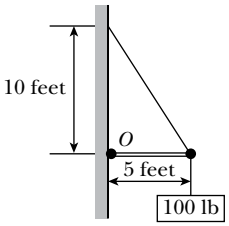
(a) $F_m = F_e$ and $a_m < a_e$
(b) $F_m < F_e$ and $a_m > a_e$
(c) $F_m < F_e$ and $a_m = a_e$
(d) $F_m = F_e$ and $a_m > a_e$

30. A block of mass M is hung by ropes as shown in the following figure. The system is in equilibrium. The point O represents the knot, the junction of the three ropes. Which of the following statements is true concerning the magnitudes of the three forces in equilibrium?



(a) $F_1 = F_2 = F_3$ **(b)** $F_2 = 2F_3$
(c) $F_2 < F_3$ **(d)** $F_1 = F_2 = F_3/2$

31. A massless horizontal strut is attached to the wall at the hinge O as shown in the following figure. Which one of the following phrases best describes the force that the hinge pin applies *to the strut* if the weight of the cables is also neglected?

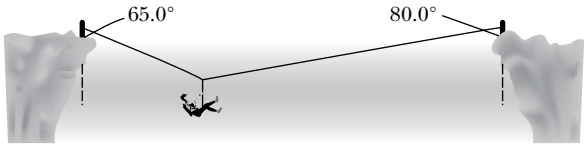


(a) 50 lb, to the right
(b) 100 lb, straight up
(c) 200 lb, to the right
(d) 244 lb, 27° above the strut

32. A woman stands on a scale in a moving elevator. Her mass is 60.0 kg, and the combined mass of the elevator and scale is an additional 815 kg. Starting from rest, the elevator accelerates upward. During the acceleration, the hoisting cable applies a force of 9410 N. What does the scale read during the acceleration?

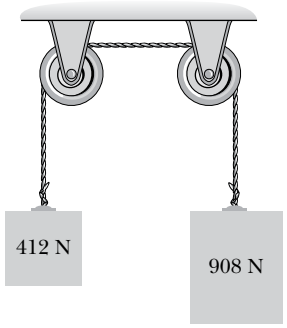
(a) 588 N **(b)** 606 N
(c) 645 N **(d)** 720 N

33. A mountain climber, in the process of crossing between two cliffs by a rope, pauses to rest. She weighs 535 N. As the drawing shows, she is closer to the left cliff than to the right cliff, with the result that the tensions in the left and right sides of the rope are not the same. Find the tensions in the rope to the left and to the right of the mountain climber (see given figure).



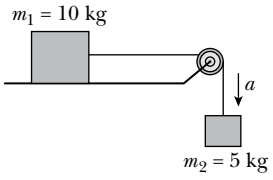
	T_L	T_R
(a)	919 N	845 N
(b)	909 N	998 N
(c)	919 N	919 N
(d)	998 N	845 N

34. As shown in the given figure, two blocks are connected by a rope that passes over a set of pulleys. One block has a weight of 412 N, and the other has a weight of 908 N. The rope and the pulleys are massless and there is no friction. What is the acceleration of the lighter block?



(a) 4.44 m/s^2 **(b)** 6.34 m/s^2
(c) 2.66 m/s^2 **(d)** 3.68 m/s^2

35. A body of mass 10 kg is placed on the horizontal smooth table (see given figure). A string is tied with it which passes over a frictionless pulley. The other end of the string is tied with a body of mass 5 kg. When the bodies move, the acceleration produced in them, is



(a) 9.8 m/s^2 **(b)** 4.8 m/s^2
(c) 4.25 m/s^2 **(d)** 3.27 m/s^2

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36. When a parachute opens, the air exerts a large drag force on it. This upward force is initially greater than the weight of the sky diver and, thus, slows him down. Suppose the weight of the sky diver is 915 N and the drag force has a magnitude of 1027 N. The mass of the sky diver is 93.4 kg. What are the magnitude and direction of his acceleration?
(a) 1.20 m/s^2, downward
(b) 11.0 m/s^2, downward
(c) 1.20 m/s^2, upward
(d) 4.90 m/s^2, upward

More than One Correct Choice Type

37. A block A kept on an inclined surface just begins to slide if the inclination is 30°. The block is replaced by another block B and it is found that it just begins to slide if the inclination is 40°. Then,
(a) mass of A > mass of B **(b)** mass of A < mass of B
(c) mass of A = mass of B **(d)** insufficient information

38. A ball of mass m is attached to lower end of light vertical spring of force constant K. The upper end of spring is fixed. The ball is released from rest with the spring at its normal (unstretched) length and comes to rest again after descending through a distance x
(a) $x = \frac{mg}{K}$
(b) $x = \frac{2mg}{K}$
(c) The ball will have no acceleration at the position where it has descent through $x/2$.
(d) The ball will have an upward acceleration equal to g at its lower most position.

39. A particle is moving horizontally with constant acceleration.
(a) The sum of the horizontal components of the forces acting on the particle is not zero.
(b) The sum of the vertical component of the forces acting on the particle is not zero.
(c) The forces acting on the particle are not in equilibrium.
(d) All of the above.

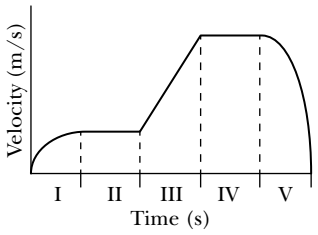
40. A balloon of mass m is rising up with an acceleration a.
(a) The upthrust on the balloon is $m(g + a)$.
(b) The upthrust on the balloon is ma.
(c) $ma/(2a + g)$ mass must be detached in order to double its acceleration.
(d) $m/2$ mass must be detached in order to double its acceleration.

41. The two ends of a spring are displaced along the length of the spring. All displacements have equal magnitudes. In which case or cases the extension or compression in the spring will have a maximum magnitude?
(a) the right end is displaced towards right and the left end towards left.
(b) both ends are displaced towards right.
(c) both ends are displaced towards left.
(d) the right end is displaced towards left and the left end towards right.

Linked Comprehension

Paragraph for Questions 42–44: A 2.0 kg object moves in a straight line on a horizontal frictionless surface. The graph in the given figure shows the velocity of the object as a function of time. The various equal time intervals are labeled using Roman numerals: I, II, III, IV, and V.

The net force on the object always acts along the line of motion of the object.



42. Which section of the graph corresponds to the application of the *largest constant* net force?
(a) I **(b)** II
(c) III **(d)** IV

43. In which section of the graph is the magnitude of the net force decreasing?
(a) I **(b)** II
(c) III **(d)** IV

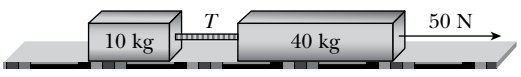
44. In which section(s) of the graph is the net force changing?
(a) I and III **(b)** II and IV
(c) III **(d)** I and V

Paragraph for Questions 45 and 46: A 70.0 kg astronaut pushes to the left on a spacecraft with a force $\vec{F}$ in "gravity-free" space. The spacecraft has a total mass of 1.0×10^4 kg. During the push, the astronaut accelerates to the right with an acceleration of 0.36 m/s^2.

45. Which one of the following statements concerning this situation is true?
(a) The spacecraft does not move, but the astronaut moves to the right with a constant speed.
(b) The astronaut stops moving after he stops pushing on the spacecraft.
(c) The force exerted on the astronaut is larger than the force exerted on the spacecraft.
(d) The velocity of the astronaut increases while he is pushing on the spacecraft.

46. Determine the magnitude of the acceleration of the spacecraft.
(a) 51.4 m/s^2
(b) 0.36 m/s^2
(c) 2.5×10^{-3} m/s^2
(d) 7.0×10^{-3} m/s^2

Paragraph for Questions 47 and 48: A 10 kg block is connected to a 40 kg block as shown in the figure. The surface on that the blocks slide is frictionless. A force of 50 N pulls the blocks to the right.



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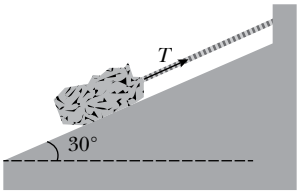
47. What is the magnitude of the acceleration of the 40 kg block?

(a) 0.5 m/s^2 **(b)** 1 m/s^2
(c) 2 m/s^2 **(d)** 4 m/s^2

48. What is the magnitude of the tension T in the rope that connects the two blocks?

(a) 0 N **(b)** 10 N
(c) 20 N **(d)** 40 N

Paragraph for Questions 49–51: A rope holds a 10 kg rock at rest on a *frictionless* inclined plane as shown in the figure.



49. Determine the tension in the rope.

(a) 9.8 N **(b)** 20 N
(c) 49 N **(d)** 85 N

50. Which one of the following statements concerning the force exerted *on* the plane *by* the rock is true?

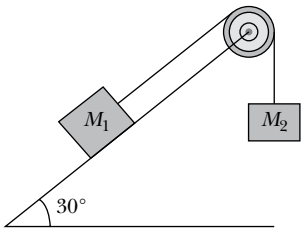
(a) It is 0 N.
(b) It is 98 N.
(c) It is greater than 98 N.
(d) It is less than 98 N, but greater than zero newtons.

51. Determine the magnitude of the acceleration of the rock down the inclined plane if the rope breaks?

(a) zero m/s^2 **(b)** 4.9 m/s^2
(c) 5.7 m/s^2 **(d)** 8.5 m/s^2

Matrix-Match

52. In the system shown in the given figure, the inline is frictionless, the string is massless, and inextensible pulley is light and frictionless. The system is released from rest.



Column I	Column II
(a) $M_1 > M_2$	**(p)** M_2 accelerates down
(b) $M_2 > M_1$	**(q)** M_2 accelerates up
(c) $M_1 = M_2$	**(r)** M_1 and M_2 are in equilibrium
(d) $M_1 \gg M_2$	**(s)** Tension in string equals the weight of either block

Directions for Questions 53 and 54: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

53. In the given table, Column I shows masses of the objects, Column II shows velocities of the objects before collision and Column III shows velocity/velocities of the object/objects after collision.

Column I	Column II	Column III
(I) Mass of one of the objects is 4 kg, mass of the other object is 0.05 kg.	**(i)** Velocity of one object before collision is $u_1 = 10$ m/s and velocity of another object before collision is $u_2 = 0$ m/s.	**(J)** Velocity of one object after collision $v_1 = -0.4375$ m/s.
(II) Mass of one of the objects is 0.1 kg, mass of the other object is 0.2 kg.	**(ii)** Velocity of one object before collision is $u_1 = 2$ m/s and velocity of another object before collision is $u_2 = 1$ m/s.	**(K)** Velocity of one object after collision $v_1 = 0$ m/s
(III) Mass of one of the objects is 1.5 kg, mass of the other object is 1.5 kg.	**(iii)** Velocity of one object before collision is $u_1 = 2.5$ m/s and velocity of another object before collision is $u_2 = -2.5$ m/s.	**(L)** Velocity of one object after collision $v_1 = 1.67$ m/s
(IV) Mass of one of the objects is 1 kg, mass of the other object is 5 kg.	**(iv)** Velocity of one object before collision is $u_1 = 0$ m/s and velocity of another object before collision is $u_2 = 0$ m/s.	**(M)** Velocity of the combined object after collision is 5/3 m/s

(1) In which condition is the velocity of one of the objects after collision 35 m/s?

(a) (I) (iv) (J)
(b) (IV) (ii) (L)
(c) (II) (i) (K)
(d) (I) (ii) (M)

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(2) In which condition is the total momentum after collision 10 kg m/s?

(a) (I) (ii) (J) **(b)** (IV) (i) (M)
(c) (II) (iii) (K) **(d)** (I) (i) (L)

(3) In which condition is the velocity of one of the objects after collision 1.165 m/s?

(a) (II) (ii) (J) **(b)** (III) (iv) (M)
(c) (IV) (i) (L) **(d)** (II) (ii) (L)

54. In the given table, Column I shows masses of the objects, Column II shows the initial and final velocity of the object after force is applied on it and Column III shows the time span for which force is applied on the object.

Column I	Column II	Column III
(I) Mass = 0.5 kg	**(i)** $u = 50$ m/s, $v = 0$ m/s	**(J)** $t = 3$ s
(II) Mass = 300 kg	**(ii)** $u = 25$ m/s, $v = 5$ m/s	**(K)** $t = 0.23$ s
(III) Mass = 1200 kg	**(iii)** $u = 20$ m/s, $v = 4$ m/s	**(L)** $t = 0.01$ s
(IV) Mass = 3.5 kg	**(iv)** $u = 20$ m/s, $v = 30$ m/s	**(M)** $t = 4$ s

(1) Conditions for a motorcar with the force of 6000 N applied in opposite direction of motion are:

(a) (I) (iii) (L) **(b)** (II) (ii) (K)
(c) (III) (ii) (M) **(d)** (I) (iii) (J)

(2) Conditions for a hammer with the force of 2500 N applied in opposite direction of motion are:

(a) (I) (i) (L) **(b)** (III) (ii) (J)
(c) (II) (ii) (K) **(d)** (III) (i) (M)

(3) Conditions for a dumbbell with the force of 1000 N applied in same direction of motion are:

(a) (III) (i) (L) **(b)** (IV) (i) (K)
(c) (I) (iv) (M) **(d)** (II) (iv) (J)

Integer Type

55. An elevator starts from rest with a constant upward acceleration. It moves 5 m in the first 2 s. A passenger in the elevator is holding a 4 kg package by a vertical string. Find the tension in the string during the acceleration process in M.

56. Workers are loading equipment into a freight elevator at the top floor of a building. However, they overload the elevator and the worn cable snaps. The mass of the loaded elevator at the time of the accident is 1600 kg. As the elevator falls, the guide rails exert a constant retarding force of 3700 N on the elevator. At what speed does the elevator hit the bottom of the shaft 72 m below?

57. Three blocks A, B and C of masses 5 kg, 10 kg and 15 kg respectively connected by two ideal strings are present on a smooth horizontal surface. An external horizontal force of 30 N acts on the block A to pull the system. Find the difference in the tensions in strings connecting A and B and, B and C.

ANSWER KEY

Checkpoints

1. c, d, and e ($\vec{F}_1$ and $\vec{F}_2$ must be head to tail, $\vec{F}_{net}$ must be from tail of one of them to head of the other)
2. (a) and (b) 2 N, leftward (acceleration is zero in each situation)
3. (a) equal; (b) greater (acceleration is upward, thus net force on body must be upward)
4. (a) equal; (b) greater; (c) less
5. (a) increase; (b) yes; (c) same; (d) yes

Problems

1. (a) 1.9 m/s^2; (b) 38° **2.** (a) 0; (b) $(3.2\text{ m/s})\hat{j}$; (c) $(2.4\text{ m/s})\hat{i}$ **3.** (a) 5.20 N; (b) 3.00 N; (c) $(5.20\text{ N})\hat{i}+(3.00\text{ N})\hat{j}$
4. $(-2\text{ N})\hat{i}+(5\text{ N})\hat{j}$ **5.** (a) $(0.86\text{ m/s}^2)\hat{i}-(0.16\text{ m/s}^2)\hat{j}$; (b) 0.88 m/s^2; (c) –11°
6. (a) $(-32.0\text{ N})\hat{i}-(20.8\text{ N})\hat{j}$; (b) 38.2 N; (c) –147° **7.** $(-30\text{ N})\hat{i}-(15\text{ N})\hat{j}$ **8.** (a) 14.1 N; (b) –140°; (c) –130°
9. $(-5.82\text{ N})\hat{i}$ **10.** 30 m/s^2 **11.** 56° **12.** 5.2 m/s^2 **13.** (a) 3.0 N; (b) Down
14. (a) 108 N; (b) 108 N; (c) 108 N **15.** 1.3×10^5 N **16.** 9.7×10^3 N
17. (a) 11.7 N; (b) –59.0° **18.** (a) $(285\text{ N})\hat{i}+(705\text{ N})\hat{j}$; (b) $(285\text{ N})\hat{i}-(155\text{ N})\hat{j}$; (c) 324 N; (d) –29.0°; (e) 3.70 m/s^2; (f) –29.0°
19. (a) $(-10\text{ N})\hat{i}$; (b) $(10\text{ N})\hat{i}$; (c) $(-30\text{ N})\hat{i}$; (d) $(-50\text{ N})\hat{i}$; (e) $(-70\text{ N})\hat{i}$;
20. 3.3×10^2 N **21.** 1.6 mm **22.** (a) 4.2×10^3 N; (b) 3.1 s; (c) 4.0; (d) 2.0
23. (a) 548 N; (b) up; (c) 548 N; (d) down **24.** (a) $(1.70\text{ N})\hat{i}+(3.06\text{ N})\hat{j}$; (b) same; (c) $(2.02\text{ N})\hat{i}+(2.71\text{ N})\hat{j}$;
25. 1.6×10^4 N **26.** (a) 651 N; (b) 1.30×10^3 N **27.** (a) 60.0× ; (b) 40.9° **28.** (a) 61 N; (b) 66 N
29. (a) 0.65 m/s^2; (b) 0.12 m/s^2; (c) 1.9 m **30.** (a) 85 N; (b) 35 N; (c) 12 N

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31. (a) 3.2 mN; (b) 2.1 mN **32.** 42.6 N **33.** (a) 7.5×10^3 N; (b) 200.7° counterclockwise
34. (a) 1.23 N; (b) 2.46 N; (c) 3.69 N; (d) 4.92 N; (e) 6.15 N; (f) 0.250 N **35.** (a) 7.6 kg; (b) 93 N
36. (a) 33.4 kN; (b) 24.6 kN **37.** 16.6 kN **38.** 6.4×10^3 N **39.** 191 N
40. (a) 3.6 m/s^2; (b) 17 N **41.** 5.9 m/s **42.** (a) 21 N; (b) 52 N **43.** 23 kg **44.** (a) 1.1 N
45. (a) 2.08 m/s^2; (b) 25.0 N
46. (a) 505 N; (b) 572 N; (c) 1.01 kN; (d) 1.14 kN; (e) 1.01 kN; (f) 1.14 kN; (g) 2.02 kN; (h) 2.29 kN
47. (a) 4.9 m/s^2; (b) 2.0 m/s^2; (c) up; (d) 120 N **48.** (a) -4.43×10^{-4} m/s^3; (b) $+4.43 \times 10^{-4}$ m/s^3 **49.** $2Ma/(a+g)$
50. (a) 47.44 m/s; (b) 12.54 m/s; (c) 1.69% **51.** (a) 8.0 m/s; (b) $+x$ **52.** (a) 2.8 N; (b) 12 N
53. (a) 0.653 m/s^3; (b) 0.896 m/s^3; (c) 6.50 s **54.** 1.8×10^4 N **55.** 336.6 N

Practice Problems

Single Correct Choice Type

1. (d) **2.** (a) **3.** (b) **4.** (c) **5.** (a)
6. (d) **7.** (a) **8.** (d) **9.** (b) **10.** (a)
11. (b) **12.** (d) **13.** (d) **14.** (d) **15.** (d)
16. (b) **17.** (c) **18.** (c) **19.** (b) **20.** (d)
21. (d) **22.** (c) **23.** (d) **24.** (c) **25.** (c)
26. (b) **27.** (d) **28.** (b) **29.** (d) **30.** (a)
31. (a) **32.** (c) **33.** (a) **34.** (d) **35.** (d)
36. (c)

More than One Correct Choice Type

37. (a), (b), (c) **38.** (b), (c), (d) **39.** (a), (c) **40.** (a), (c) **41.** (a), (d)

Linked Comprehension

42. (c) **43.** (a) **44.** (d) **45.** (d) **46.** (c)
47. (b) **48.** (b) **48.** (c) **50.** (d) **51.** (b)

Matrix-Match

52. (a) → (p), (q), (r), (s); (b) → (p), (s); (c) → (p); (d) → (q). **53.** (1) → (a), (2) → (b), (3) → (d)
54. (1) → (c), (2) → (a), (3) → (d)

Integer Type

55. 50 **56.** 33 **57.** 10

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CHAPTER 6

Force and Motion – II

6.1 | WHAT IS PHYSICS?

In this chapter, we focus on the physics of three common types of forces: frictional force, drag force, and centripetal force. An engineer preparing a car for the Indianapolis 500 must consider all three types. Frictional forces acting on the tires are crucial to the car's acceleration out of the pit and out of a curve (if the car hits an oil slick, the friction is lost and so is the car). Drag forces acting on the car from the passing air must be minimized or else the car will consume too much fuel and have to pit too early (even one 14 s pit stop can cost a driver the race). Centripetal forces are crucial in the turns (if there is insufficient centripetal force, the car slides into the wall). We start our discussion with frictional forces.

6.2 | FRICTION

Key Concept

- When a force $\vec{F}$ tends to slide a body along a surface, a frictional force from the surface acts on the body. The frictional force is parallel to the surface and directed so as to oppose the sliding. It is due to bonding between the body and the surface.

  If the body does not slide, the frictional force is a static frictional force $\vec{f}_s$. If there is sliding, the frictional force is a kinetic frictional force $\vec{f}_k$.

Frictional forces are unavoidable in our daily lives. If we were not able to counteract them, they would stop every moving object and bring to a halt every rotating shaft. About 20% of the gasoline used in an automobile is needed to counteract friction in the engine and in the drive train. On the other hand, if friction were totally absent, we could not get an automobile to go anywhere, and we could not walk or ride a bicycle. We could not hold a pencil, and, if we could, it would not write. Nails and screws would be useless, woven cloth would fall apart, and knots would untie.

Contents

6.1 What is Physics?
6.2 Friction
6.3 Properties of Friction
6.4 Some More Applications of Properties of Friction
6.5 The Drag Force and Terminal Speed

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Three Experiments. Here we deal with the frictional forces that exist between dry solid surfaces, either stationary relative to each other or moving across each other at slow speeds. Consider three simple thought experiments:

1. Send a book sliding across a long horizontal counter. As expected, the book slows and then stops. This means the book must have an acceleration parallel to the counter surface, in the direction opposite the book's velocity. From Newton's second law, then, a force must act on the book parallel to the counter surface, in the direction opposite its velocity. That force is a frictional force.
2. Push horizontally on the book to make it travel at constant velocity along the counter. Can the force from you be the only horizontal force on the book? No, because then the book would accelerate. From Newton's second law, there must be a second force, directed opposite your force but with the same magnitude, so that the two forces balance. That second force is a frictional force, directed parallel to the counter.
3. Push horizontally on a heavy crate. The crate does not move. From Newton's second law, a second force must also be acting on the crate to counteract your force. Moreover, this second force must be directed opposite your force and have the same magnitude as your force, so that the two forces balance. That second force is a frictional force. Push even harder. The crate still does not move. Apparently the frictional force can change in magnitude so that the two forces still balance. Now push with all your strength. The crate begins to slide. Evidently, there is a maximum magnitude of the frictional force. When you exceed that maximum magnitude, the crate slides.

Two Types of Friction. Figure 6-1 shows a similar situation. In Fig. 6-1*a*, a block rests on a tabletop, with the gravitational force $\vec{F}_g$ balanced by a normal force $\vec{F}_N$. In Fig. 6-1*b*, you exert a force $\vec{F}$ on the block, attempting to pull it to the left. In response, a frictional force $\vec{f}_s$ is directed to the right, exactly balancing your force. The force $\vec{f}_s$ is called the **static frictional force**. The block does not move.

Figures 6-1*c* and 6-1*d* show that as you increase the magnitude of your applied force, the magnitude of the static frictional force $\vec{f}_s$ also increases and the block remains at rest. When the applied force reaches a certain magnitude, however, the block "breaks away" from its intimate contact with the tabletop and accelerates leftward (Fig. 6-1*e*). The frictional force that then opposes the motion is called the **kinetic frictional force** $\vec{f}_k$.

Usually, the magnitude of the kinetic frictional force, which acts when there is motion, is less than the maximum magnitude of the static frictional force, which acts when there is no motion. Thus, if you wish the block to move across the surface with a constant speed, you must usually decrease the magnitude of the applied force once the block begins to move, as in Fig. 6-1*f*. As an example, Fig. 6-1*g* shows the results of an experiment in which the force on a block was slowly increased until breakaway occurred. Note the reduced force needed to keep the block moving at constant speed after breakaway.

Microscopic View. A frictional force is, in essence, the vector sum of many forces acting between the surface atoms of one body and those of another body. If two highly polished and carefully cleaned metal surfaces are brought together in a very good vacuum (to keep them clean), they cannot be made to slide over each other. Because the surfaces are so smooth, many atoms of one surface contact many atoms of the other surface, and the surfaces *cold-weld* together instantly, forming a single piece of metal. If a machinist's specially polished gage blocks are brought together in air, there is less atom-to-atom contact, but the blocks stick firmly to each other and can be separated only by means of a wrenching motion. Usually, however, this much atom-to-atom contact is not possible. Even a highly polished metal surface is far from being flat on the atomic scale. Moreover, the surfaces of everyday objects have layers of oxides and other contaminants that reduce cold-welding.

When two ordinary surfaces are placed together, only the high points touch each other. (It is like having the Alps of Switzerland turned over and placed down on the Alps of Austria.) The actual *microscopic* area of contact is much less than the apparent *macroscopic* contact area, perhaps by a factor of 10^4. Nonetheless, many contact points do cold-weld together. These welds produce static friction when an applied force attempts to slide the surfaces relative to each other.

If the applied force is great enough to pull one surface across the other, there is first a tearing of welds (at breakaway) and then a continuous re-forming and tearing of welds as movement occurs and chance contacts are made (Fig. 6-2). The kinetic frictional force $\vec{f}_k$ that opposes the motion is the vector sum of the forces at those many chance contacts.

If the two surfaces are pressed together harder, many more points cold-weld. Now getting the surfaces to slide relative to each other requires a greater applied force: The static frictional force $\vec{f}_s$ has a greater maximum value. Once the surfaces are sliding, there are many more points of momentary cold-welding, so the kinetic frictional force $\vec{f}_k$ also has a greater magnitude.

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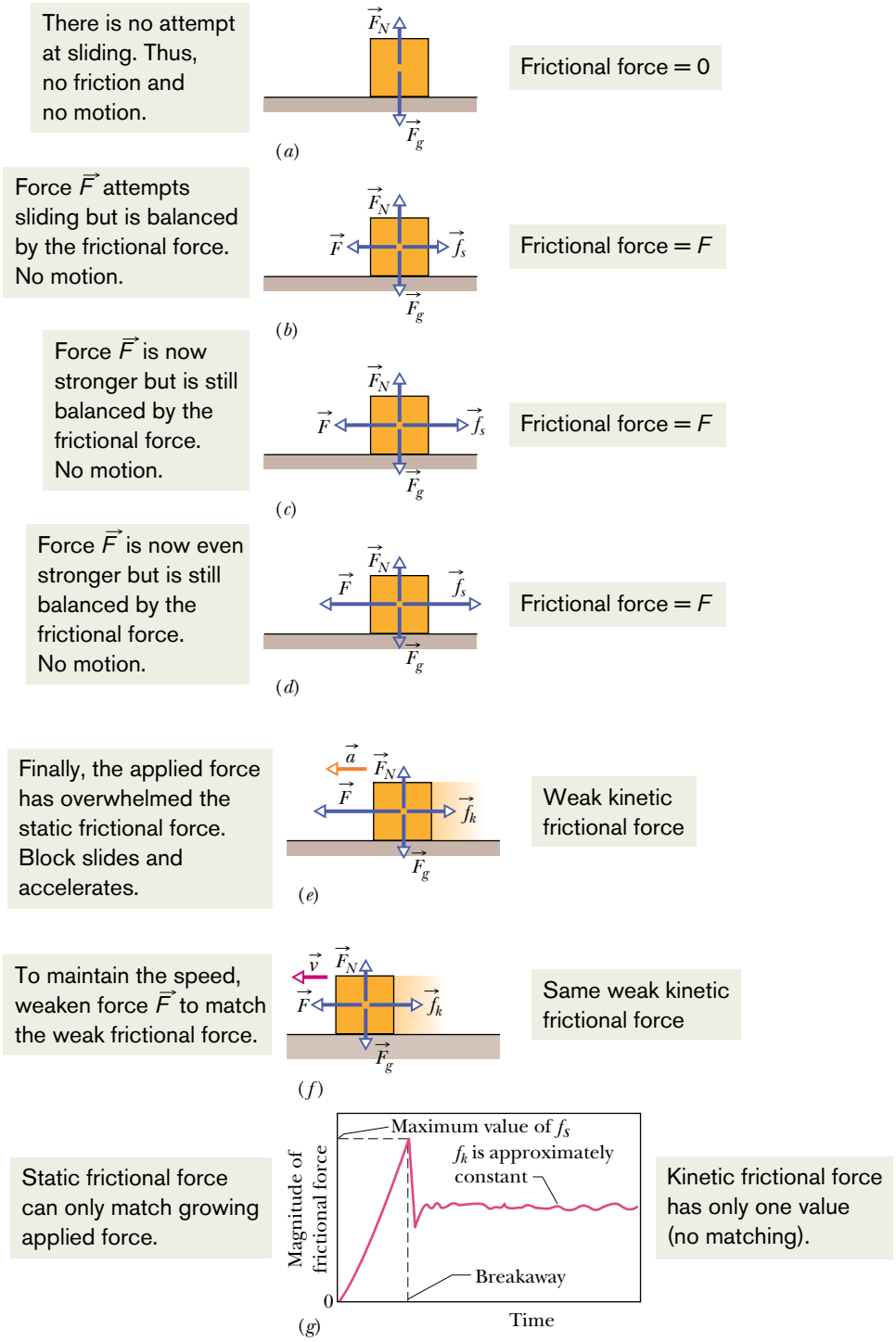
There is no attempt at sliding. Thus, no friction and no motion.
Frictional force = 0
(a)
Force $\vec{F}$ attempts sliding but is balanced by the frictional force. No motion.
Frictional force = F
(b)
Force $\vec{F}$ is now stronger but is still balanced by the frictional force. No motion.
Frictional force = F
(c)
Force $\vec{F}$ is now even stronger but is still balanced by the frictional force. No motion.
Frictional force = F
(d)
Finally, the applied force has overwhelmed the static frictional force. Block slides and accelerates.
Weak kinetic frictional force
(e)
To maintain the speed, weaken force $\vec{F}$ to match the weak frictional force.
Same weak kinetic frictional force
(f)
Static frictional force can only match growing applied force.
Maximum value of f_s
f_k is approximately constant
Breakaway
Magnitude of frictional force
0
Time
Kinetic frictional force has only one value (no matching).
(g)

Figure 6-1 (*a*) The forces on a stationary block. (*b–d*) An external force $\vec{F}$, applied to the block, is balanced by a static frictional force $\vec{f}_s$. As F is increased, f_s also increases, until f_s reaches a certain maximum value. (*e*) Once f_s reaches its maximum value, the block "breaks away," accelerating suddenly in the direction of $\vec{F}$. (*f*) If the block is now to move with constant velocity, F must be reduced from the maximum value it had just before the block broke away. (*g*) Some experimental results for the sequence (*a*) through (*f*).

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Often, the sliding motion of one surface over another is "jerky" because the two surfaces alternately stick together and then slip. Such repetitive *stick-and-slip* can produce squeaking or squealing, as when tires skid on dry pavement, fingernails scratch along a chalkboard, or a rusty hinge is opened. It can also produce beautiful and captivating sounds, as in music when a bow is drawn properly across a violin string.

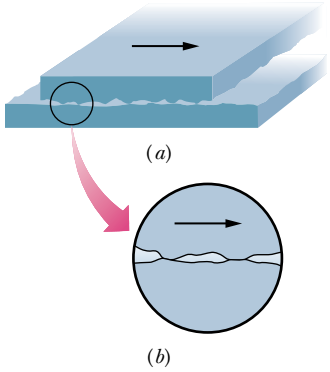

Figure 6-2 The mechanism of sliding friction. (*a*) The upper surface is sliding to the right over the lower surface in this enlarged view. (*b*) A detail, showing two spots where cold-welding has occurred. Force is required to break the welds and maintain the motion.

6.3 | PROPERTIES OF FRICTION

Key Concepts

- If a body does not move, the static frictional force $\vec{f}_s$ and the component of $\vec{F}$ parallel to the surface are equal in magnitude, and $\vec{f}_s$ is directed opposite that component. If the component increases, f_s also increases.
- The magnitude of $\vec{f}_s$ has a maximum value $\vec{f}_{s,\max}$ given by
$$f_{s,\max} = \mu_s F_N,$$
where μ_s is the coefficient of static friction and F_N is the magnitude of the normal force. If the component of $\vec{F}$ parallel to the surface exceeds $f_{s,\max}$, the body slides on the surface.
- If the body begins to slide on the surface, the magnitude of the frictional force rapidly decreases to a constant value f_k given by
$$f_k = \mu_k F_N,$$
where μ_k is the coefficient of kinetic friction.

Experiment shows that when a dry and unlubricated body presses against a surface in the same condition and a force $\vec{F}$ attempts to slide the body along the surface, the resulting frictional force has three properties:

Property 1. If the body does not move, then the static frictional force $\vec{f}_s$ and the component of $\vec{F}$ that is parallel to the surface balance each other. They are equal in magnitude, and $\vec{f}_s$ is directed opposite that component of $\vec{F}$.

Property 2. The magnitude of $\vec{f}_s$ has a maximum value $f_{s,\max}$ that is given by

$$f_{s,\max} = \mu_s F_N, \tag{6-1}$$

where μ_s is the **coefficient of static friction** and F_N is the magnitude of the normal force on the body from the surface. If the magnitude of the component of $\vec{F}$ that is parallel to the surface exceeds $f_{s,\max}$, then the body begins to slide along the surface.

Property 3. If the body begins to slide along the surface, the magnitude of the frictional force rapidly decreases to a value f_k given by

$$f_k = \mu_k F_N, \tag{6-2}$$

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where μ_k is the **coefficient of kinetic friction**. Thereafter, during the sliding, a kinetic frictional force $\vec{f}_k$ with magnitude given by Eq. 6-2 opposes the motion.

The magnitude F_N of the normal force appears in properties 2 and 3 as a measure of how firmly the body presses against the surface. If the body presses harder, then, by Newton's third law, F_N is greater. Properties 1 and 2 are worded in terms of a single applied force $\vec{F}$, but they also hold for the net force of several applied forces acting on the body. Equations 6-1 and 6-2 are *not* vector equations; the direction of $\vec{f}_s$ or $\vec{f}_k$ is always parallel to the surface and opposed to the attempted sliding, and the normal force $\vec{F}_N$ is perpendicular to the surface.

The coefficients μ_s and μ_k are dimensionless and must be determined experimentally. Their values depend on certain properties of both the body and the surface; hence, they are usually referred to with the preposition "between," as in "the value of μ_s *between* an egg and a Teflon-coated skillet is 0.04, but that *between* rock-climbing shoes and rock is as much as 1.2." We assume that the value of μ_k does not depend on the speed at which the body slides along the surface.

CHECKPOINT 1

A block lies on a floor. (a) What is the magnitude of the frictional force on it from the floor? (b) If a horizontal force of 5 N is now applied to the block, but the block does not move, what is the magnitude of the frictional force on it? (c) If the maximum value $f_{s,\max}$ of the static frictional force on the block is 10 N, will the block move if the magnitude of the horizontally applied force is 8 N? (d) If it is 12 N? (e) What is the magnitude of the frictional force in part (c)?

SAMPLE PROBLEM 6.01

Angled force applied to an initially stationary block

This sample problem involves a tilted applied force, which requires that we work with components to find a frictional force. The main challenge is to sort out all the components. Figure 6-3*a* shows a force of magnitude F = 12.0 N applied to an 8.00 kg block at a downward angle of θ = 30.0°. The coefficient of static friction between block and floor is μ_s = 0.700; the coefficient of kinetic friction is μ_k = 0.400. Does the block begin to slide or does it remain stationary? What is the magnitude of the frictional force on the block?

KEY IDEAS

(1) When the object is stationary on a surface, the static frictional force balances the force component that is attempting to slide the object along the surface. (2) The maximum possible magnitude of that force is given by Eq. 6-1 ($f_{s,\max} = \mu_s F_N$). (3) If the component of the applied force along the surface exceeds this limit on the static friction, the block begins to slide. (4) If the object slides, the kinetic frictional force is given by Eq. 6-2 ($f_k = \mu_k F_N$).

Calculations: To see if the block slides (and thus to calculate the magnitude of the frictional force), we must compare the applied force component F_x with the maximum magnitude $f_{s,\max}$ that the static friction can have. From the triangle of components and full force shown in Fig. 6-3*b*, we see that

$$F_x = F\cos\theta = (12.0\text{ N})\cos 30° = 10.39\text{ N}. \quad (6\text{-}3)$$

From Eq. 6-1, we know that $f_{s,\max} = \mu_s F_N$, but we need the magnitude F_N of the normal force to evaluate $f_{s,\max}$.

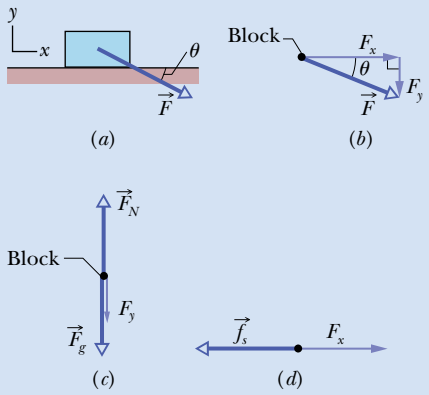

Figure 6-3 (*a*) A force is applied to an initially stationary block. (*b*) The components of the applied force. (*c*) The vertical force components. (*d*) The horizontal force components.

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Because the normal force is vertical, we need to write Newton's second law ($F_{net,y} = ma_y$) for the vertical force components acting on the block, as displayed in Fig. 6-3*c*. The gravitational force with magnitude mg acts downward. The applied force has a downward component $F_y = F \sin\theta$. And the vertical acceleration a_y is just zero. Thus, we can write Newton's second law as

$$F_N - mg - F\sin\theta = m(0), \tag{6-4}$$

which gives us

$$F_N = mg + F\sin\theta. \tag{6-5}$$

Now we can evaluate $f_{s,max} = \mu_s F_N$:

$$\begin{aligned} f_{s,max} &= \mu_s(mg + F\sin\theta) \\ &= (0.700)((8.00\text{ kg})(9.8\text{ m/s}^2) + (12.0\text{ N})(\sin 30°)) \\ &= 59.08\text{ N}. \end{aligned} \tag{6-6}$$

Because the magnitude F_x (= 10.39 N) of the force-component attempting to slide the block is less than $f_{s,max}$ (= 59.08 N), the block remains stationary. That means that the magnitude f_s of the frictional force *matches* F_x. From Fig. 6-3*d*, we can write Newton's second law for x components as

$$F_x - f_s = m(0), \tag{6-7}$$

and thus $f_s = F_x = 10.39\text{ N} \approx 10.4\text{ N}.$ (Answer)

SAMPLE PROBLEM 6.02

Sliding to a stop on icy roads, horizontal and inclined

Some of the funniest videos on the web involve motorists sliding uncontrollably on icy roads. Here let's compare the typical stopping distances for a car sliding to a stop from an initial speed of 10.0 m/s on a dry horizontal road, an icy horizontal road, and (everyone's favorite) an icy hill.

(a) How far does the car take to slide to a stop on a horizontal road (Fig. 6-4*a*) if the coefficient of kinetic friction is $\mu_k = 0.60$, which is typical of regular tires on dry pavement? Let's neglect any effect of the air on the car, assume that the wheels lock up and the tires slide, and extend an x axis in the car's direction of motion.

KEY IDEAS

(1) The car accelerates (its speed decreases) because a horizontal frictional force acts against the motion, in the negative direction of the x axis. (2) The frictional force is a kinetic frictional force with a magnitude given by Eq. 6-2 ($f_k = \mu_k F_N$), in which F_N is the magnitude of the normal force on the car from the road. (3) We can relate the frictional force to the resulting acceleration by writing Newton's second law ($F_{net,x} = ma_x$) for motion along the road.

Calculations: Figure 6-4*b* shows the free-body diagram for the car. The normal force is upward, the gravitational force is downward, and the frictional force is horizontal. Because the frictional force is the only force with an x component, Newton's second law written for motion along the x axis becomes

$$-f_k = ma_x. \tag{6-8}$$

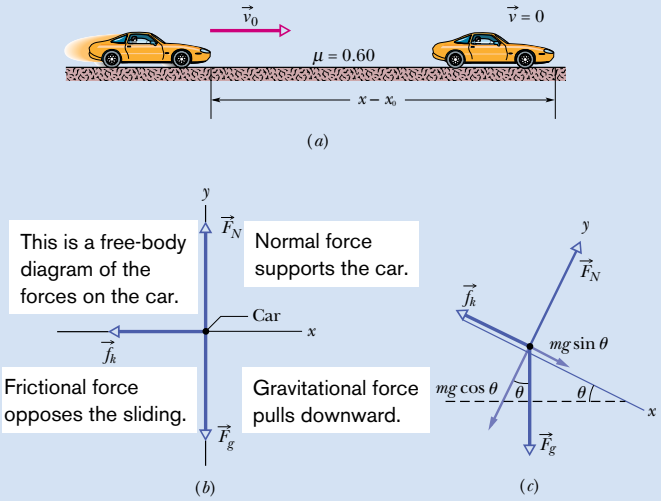

Figure 6-4 (*a*) A car sliding to the right and finally stopping. A free-body diagram for the car on (*b*) the same horizontal road and (*c*) a hill.

Substituting $f_k = \mu_k F_N$ gives us

$$-\mu_k F_N = ma_x. \tag{6-9}$$

From Fig. 6-4*b* we see that the upward normal force balances the downward gravitational force, so in Eq. 6-9 let's replace magnitude F_N with magnitude mg. Then we can cancel m (the stopping distance is thus independent of the car's mass—the car can be heavy or light, it does not matter). Solving for a_x we find

$$a_x = -\mu_k g. \tag{6-10}$$

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Because this acceleration is constant, we can use the constant-acceleration equations of Table 2-1. The easiest choice for finding the sliding distance $x - x_0$ is Eq. 2-16 ($v^2 = v_0^2 + 2a(x - x_0)$), which gives us

$$x - x_0 = \frac{v^2 - v_0^2}{2a_x}. \tag{6-11}$$

Substituting from Eq. 6-10, we then have

$$x - x_0 = \frac{v^2 - v_0^2}{-2\mu_k g}. \tag{6-12}$$

Inserting the initial speed $v_0 = 10.0$ m/s, the final speed $v = 0$, and the coefficient of kinetic friction $\mu_k = 0.60$, we find that the car's stopping distance is

$$x - x_0 = 8.50 \text{ m} \approx 8.5 \text{ m}. \qquad \text{(Answer)}$$

(b) What is the stopping distance if the road is covered with ice with $\mu_k = 0.10$?

Calculation: Our solution is perfectly fine through Eq. 6-12 but now we substitute this new μ_k, finding

$$x - x_0 = 51 \text{ m}. \qquad \text{(Answer)}$$

Thus, a much longer clear path would be needed to avoid the car hitting something along the way.

(c) Now let's have the car sliding down an icy hill with an inclination of $\theta = 5.00°$ (a mild incline, nothing like the hills of San Francisco). The free-body diagram shown in Fig. 6-4*c* is like the ramp in Sample Problem 5.06 except, to be consistent with Fig. 6-4*b*, the positive direction of the *x* axis is *down* the ramp. What now is the stopping distance?

Calculations: Switching from Fig. 6-4*b* to *c* involves two major changes. (1) Now a component of the gravitational force is along the tilted *x* axis, pulling the car down the hill. From Sample Problem 5.06 and Fig. 5-23, that down-the-hill component is $mg \sin \theta$, which is in the positive direction of the *x* axis in Fig. 6-4*c*. (2) The normal force (still perpendicular to the road) now balances only a component of the gravitational force, not the full force. From Sample Problem 5.04 (see Fig. 5-23*i*), we write that balance as

$$F_N = mg \cos \theta.$$

In spite of these changes, we still want to write Newton's second law ($F_{\text{net},x} = ma_x$) for the motion along the (now tilted) *x* axis. We have

$$-f_k + mg \sin \theta = ma_x,$$

$$-\mu_k F_N + mg \sin \theta = ma_x,$$

and

$$-\mu_k mg \cos \theta + mg \sin \theta = ma_x.$$

Solving for the acceleration and substituting the given data now give us

$$\begin{aligned} a_x &= -\mu_k g \cos \theta + g \sin \theta \\ &= -(0.10)(9.8 \text{ m/s}^2) \cos 5.00° + (9.8 \text{ m/s}^2) \sin 5.00° \\ &= -0.122 \text{ m/s}^2. \end{aligned} \tag{6-13}$$

Substituting this result into Eq. 6-11 gives us the stopping distance down the hill:

$$x - x_0 = 409 \text{ m} \approx 400 \text{ m}, \qquad \text{(Answer)}$$

which is about 1/4 mile. Such icy hills separate people who can do this calculation (and thus know to stay home) from people who cannot (and thus end up in web videos).

SAMPLE PROBLEM 6.03

Sliding of a block along a floor while a force is applied to it

In Fig. 6-5*a*, a block of mass $m = 3.0$ kg slides along a floor while a force $\vec{F}$ of magnitude 12.0 N is applied to it at an upward angle θ. The coefficient of kinetic friction between the block and the floor is $\mu_k = 0.40$. We can vary θ from 0 to 90° (the block remains on the floor). What θ gives the maximum value of the block's acceleration magnitude *a*?

KEY IDEAS

Because the block is moving, a *kinetic* frictional force acts on it. The magnitude is given by Eq. 6-2 ($f_k = \mu_k F_N$, where F_N is the normal force). The direction is opposite the motion (the friction opposes the sliding).

Calculating F_N: Because we need the magnitude f_k of the frictional force, we first must calculate the magnitude F_N of the normal force. Figure 6-5*b* is a free-body diagram showing the forces along the vertical *y* axis. The normal force is upward, the gravitational force with magnitude *mg* is downward, and (note) the vertical component F_y of the applied force is upward. That component is shown in Fig. 6-5*c*, where we can see that $F_y = F \sin \theta$. We can write Newton's second law ($\vec{F}_{\text{net}} = m\vec{a}$) for those forces along the *y* axis as

$$F_N + F \sin \theta - mg = m(0), \tag{6-14}$$

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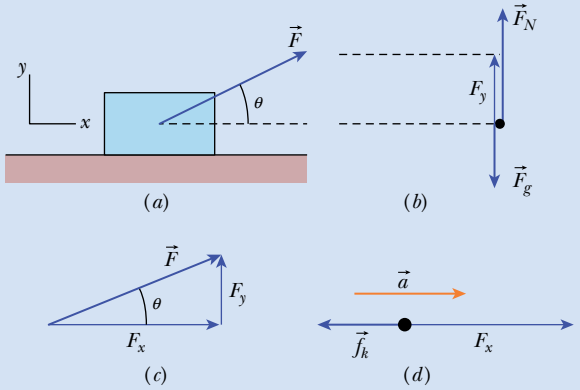

Figure 6-5 (*a*) A force is applied to a moving block. (*b*) The vertical forces. (*c*) The components of the applied force. (*d*) The horizontal forces and acceleration.

where we substituted zero for the acceleration along the y axis (the block does not even move along that axis). Thus,

$$F_N = mg - F\sin\theta. \tag{6-15}$$

Calculating acceleration a: Figure 6-5*d* is a free-body diagram for motion along the x axis. The horizontal component F_x of the applied force is rightward; from Fig. 6-5*c*, we see that $F_x = F\cos\theta$. The frictional force has magnitude f_k $(= kF_N)$ and is leftward. Writing Newton's second law for motion along the x axis gives us

$$F\cos\theta - \mu_k F_N = ma. \tag{6-16}$$

Substituting for F_N from Eq. 6-15 and solving for a lead to

$$a = \frac{F}{m}\cos\theta - \mu_k\left(g - \frac{F}{m}\sin\theta\right). \tag{6-17}$$

Finding a maximum: To find the value of θ that maximizes a, we take the derivative of a with respect to θ and set the result equal to zero:

$$\frac{da}{d\theta} = -\frac{F}{m}\sin\theta + \mu_k\frac{F}{m}\cos\theta = 0.$$

Rearranging and using the identity $(\sin\theta)/(\cos\theta) = \tan\theta$ give us

$$\tan\theta = \mu_k.$$

Solving for θ and substituting the given $\mu_k = 0.40$, we find that the acceleration will be maximum if

$$\theta = \tan^{-1}\mu_k = 21.8^\circ \approx 22^\circ. \quad \text{(Answer)}$$

Comment: As we increase θ from 0, more of the applied force $\vec{F}$ is upward, relieving the normal force. The decrease in the normal force causes a decrease in the frictional force, which opposes the block's motion. Thus, the block's acceleration tends to increase. However, the increase in θ also decreases the horizontal component of $\vec{F}$, and so the block's acceleration tends to decrease. These opposing tendencies produce a maximum acceleration at $\theta = 22^\circ$.

SAMPLE PROBLEM 6.04

Sliding of a block along an incline while a force is applied to it

A 68 kg crate is dragged across a floor by pulling on a rope attached to the crate and inclined 15° above the horizontal. (a) If the coefficient of static friction is 0.50, what minimum force magnitude is required from the rope to start the crate moving?

KEY IDEA

Since the crate is being pulled by a rope at an angle with the horizontal, we need to analyze the force components in both the x and y-directions.

Reasoning: The free-body diagram for the crate is shown in Fig. 6-6. Here, $\vec{T}$ is the tension force of the rope on the crate, $\vec{F}_N$ is the normal force of the floor on the crate, $m\vec{g}$ is the force of gravity, and $\vec{f}$ is the force of friction. We take the $+x$ direction to be horizontal to the right and the $+y$ direction to be up. We assume the crate is motionless.

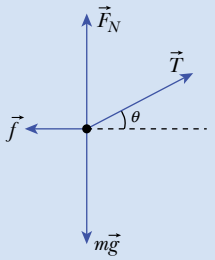

Figure 6-6 The free-body diagram for the crate discussed in Sample Problem 6.04.

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The equations for the x- and the y-components of the force according to Newton's second law are given as follows:

$$T\cos\theta - f = 0,$$
$$T\sin\theta - F_N - mg = 0.$$

where $\theta = 15°$ is the angle between the rope and the horizontal. The first equation gives $f = T\cos\theta$ and the second gives $F_N = mg - T\sin\theta$. If the crate is to remain at rest, f must be less than $\mu_s F_N$, or $T\cos\theta < \mu_s(mg - T\sin\theta)$. When the tension force is sufficient to just start the crate moving, we must have $T\cos\theta = \mu_s(mg - T\sin\theta)$.

Calculations: We solve for the tension:

$$T = \frac{\mu_s mg}{\cos\theta + \mu_s \sin\theta} = \frac{(0.50)(68\text{ kg})(9.8\text{ m/s}^2)}{\cos 15° + 0.50\sin 15°}$$
$$= 304\text{ N} \approx 3.0\times 10^2\text{ N}$$

(b) If $\mu_k = 0.35$, what is the magnitude of the initial acceleration of the crate?

Calculations: The second law equations for the moving crate are as given as follows:

$$T\cos\theta - f = ma,$$
$$T\sin\theta + F_N - mg = 0$$

Now, $f = \mu_k F_N$, and the second equation above gives $F_N = mg - T\sin\theta$, which then yields $f = \mu_k(mg - T\sin\theta)$. This expression is substituted for f in the first equation to obtain $T\cos\theta - \mu_k(mg - T\sin\theta) = ma$, so the acceleration is

$$a = \frac{T(\cos\theta + \mu_k\sin\theta)}{m} - \mu_k g$$
$$= \frac{(304\text{ N})(\cos 15° + 0.35\sin 15°)}{68\text{ kg}} - (0.35)(9.8\text{ m/s}^2)$$
$$= 1.3\text{ m/s}^2.$$

Static Friction

The force of friction between two surfaces as long as there is no relative motion between them is called **static friction** and it is equal to the applied force. Let us consider a man pushes a box on which a book is placed (Fig. 6-7*a*). When the man pushes the box, we can see that the book and box move together. Since the box and book together are moving with respect to ground, but they are not moving with respect to each other, this force of friction between the book and box is of static nature (Fig. 6-7*b*).

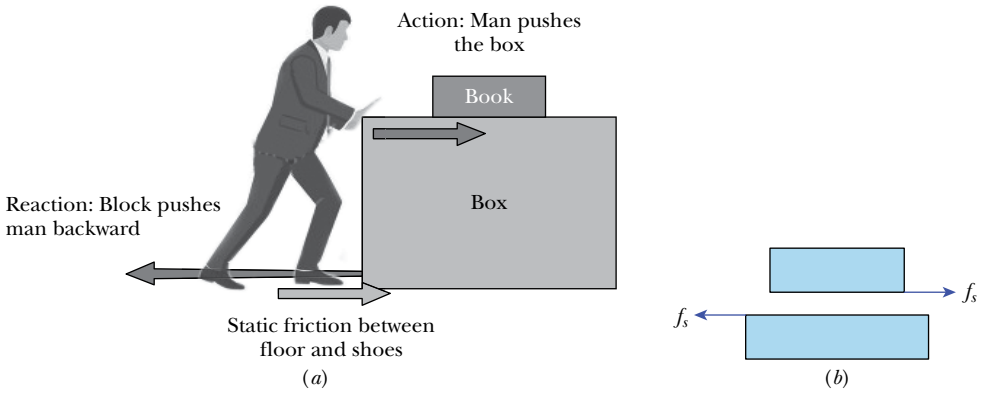

Figure 6-7 (*a*) A box on which a book is placed and the box is pushed. (*b*) Free-body diagram of the box and the book on the box.

A person walking on surface of Earth is another simple example for static friction (Fig. 6-8). You may be surprised but the foot in contact with ground is not moving while the other foot is moving. During this time friction acting between foot and ground is static friction. If we increase our speed then static friction is acting in forward direction. This happens because we try to pull our leg backward while walking in forward direction. This means foot is trying to move in backward direction with respect to ground and ground applies friction in forward direction on foot to oppose the direction of relative motion.

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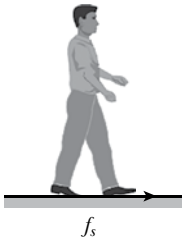


Figure 6-8 A person walking on surface of Earth due to static friction.

Both the direction and magnitude of static friction are self-adjusting such that relative motion is opposed. In fact, in the previous case of book placed on the box also, we can see that static friction by acting in forward direction on upper book has given it some motion so it can move with the lower book. Another point to be noted here is that even the third law pair of the static friction is opposing relative motion by trying to slow down the lower book (which is being accelerated externally). In Fig. 6-9*a* and *b* if we assume that the block is stationary, and we can see that static friction is acting in such a direction so as to oppose the relative motion. F represents external force and f_s represents friction.

The magnitude of the static friction between any two surfaces in contact can have the values

$$f_s \leq \mu F_N \tag{6-18}$$

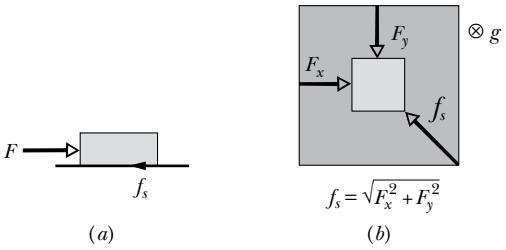


Figure 6-9 Stationary block and static friction.

where the dimensionless constant μ_s is known as the coefficient of static friction and F_N is the magnitude of the normal contact force exerted by one surface on the other. The equality in Eq. 6-18 holds when the surfaces are on the verge of slipping, that is, when $f_s = f_{s,\max} = \mu_s F_N$. This situation is called impending motion. The inequality holds when the surfaces are not on the verge of slipping. Maximum strength of the joints formed is directly proportional to the normal contact force because higher the normal contact force, higher is the joint strength that is, $f_{s,\max} \propto F_N$.

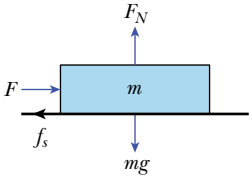


Figure 6-10 Forces acting on block of mass *m*.

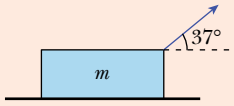
Maximum strength also depends on the roughness of contact surface $f_{s,\max}$ (also called f_{limiting}) $= \mu_s N$. Magnitude of static friction is self-adjusting such that relative motion does not start (but still it has maximum value). Let us say we are applying force F on a block kept on horizontal rough surface (Fig. 6-10) with coefficient of static friction $\mu_s = 0.1$ and mass of block is 5 kg. When applied force F is less than 5 N the value of static friction is not 5 N. It is the maximum value of friction. But when applied force F is more than 5 N, the value of static friction is 5 N as given in Table 6-1.

Table 6-1 Values Static Friction and of Limiting friction on Application of Force

Applied Force (N)	Static Friction, f_s (N)	Limiting Friction, $f_{s,\max}$ (N)
1	1	5
3	3	5
6	5	5
10	5	5

CHECKPOINT 2

If a block kept on rough horizontal surface is pulled to an angle by applying 100 N force at 37° with horizontal as depicted in the figure, find the force of friction acting on the block. Given that $m = 20$ kg and $\mu_s = 0.5$.



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SAMPLE PROBLEM 6.05

Experimental determination of μ_s up the side

Although many ingenious schemes have been attributed to the building of the Great Pyramid, the stone blocks were probably hauled up the side of the pyramid by men pulling on ropes. Figure 6-5*a* represents a 2000 kg stone block in the process of being pulled up the finished (smooth) side of the Great Pyramid, which forms a plane inclined at angle $\theta = 52°$. The block is secured to a wood sled and is pulled by multiple ropes (only one is shown). The sled's track is lubricated with water to decrease the coefficient of static friction to 0.40. Assume negligible friction at the (lubricated) point where the ropes pass over the edge at the top of the side. If each man on top of the pyramid pulls with a (reasonable) force of 686 N, how many men are needed to put the block on the verge of moving?

KEY IDEAS

(1) Because the block is on the *verge* of moving, the static frictional force must be at its maximum possible value; that is, $f_s = f_{s,\max}$. (2) Because the block is on the verge of moving *up* the plane, the frictional force must be *down* the plane (to oppose the pending motion). (3) From Sample Problem 5.05, we know that the component of the gravitational force down the plane is $mg \sin\theta$ and the component perpendicular to (and inward from) the plane is $mg \cos\theta$ (Fig. 6-11*b*).

Calculations: Figure 6-11*c* is a free-body diagram for the block, showing the force $\vec{F}$ applied by the ropes, the static frictional force $\vec{f}_s$, and the two components of the gravitational force. We can write Newton's second law ($\vec{F}_{\text{net}} = m\vec{a}$) for forces along the x axis as

$$F - mg\sin\theta - f_s = m(0). \tag{6-19}$$

Because the block is on the verge of sliding and the frictional force is at the maximum possible value $f_{s,\max}$, we use Eq. 6-1 to replace f_s with $\mu_s F_N$:

$$f_s = f_{s,\max} = \mu_s F_N. \tag{6-20}$$

From Fig. 6-11*c*, we see that along the y axis Newton's second law becomes

$$F_N - mg\cos\theta = m(0). \tag{6-21}$$

Solving Eq. 6-21 for F_N and substituting the result into Eq. 6-20, we have

$$f_s = \mu_s mg\cos\theta. \tag{6-22}$$

Substituting this expression into Eq. 6-19 and solving for F lead to

$$F = \mu_s mg\cos\theta + mg\sin\theta. \tag{6-23}$$

Substituting m = 2000 kg, $\theta = 52°$, and μ_s = 0.40, we find that the force required to put the stone block on the verge of moving is 2.027×10^4 N. Dividing this by the assumed pulling force of 686 N from each man, we find that the required number of men is

$$N = \frac{2.027\times10^4 \text{ N}}{686 \text{ N}} = 29.5 \approx 30 \text{ men.} \qquad \text{(Answer)}$$

Comment: Once the stone block began to move, the friction was kinetic friction and the coefficient was about 0.20. You can show that the required number of men was then 26 or 27. Thus, the huge stone blocks of the Great Pyramid could be pulled up into position by reasonably small teams of men.

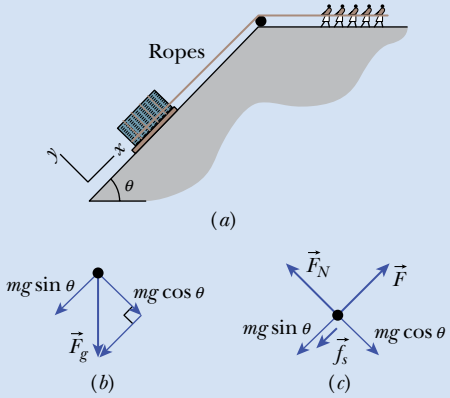

Figure 6-11 (*a*) A stone block on the verge of being pulled up the side of the Great Pyramid. (*b*) The components of the gravitational force. (*c*) A free-body diagram for the block.

SAMPLE PROBLEM 6.06

Maximum weight of a block hanging from a knotted position to maintain equilibrium

Block B in Fig. 6-12 weighs 711 N. The coefficient of static friction between block and table is 0.25; angle θ is 30°; assume that the cord between B and the knot is horizontal. Find the maximum weight of block A for which the system will be stationary.

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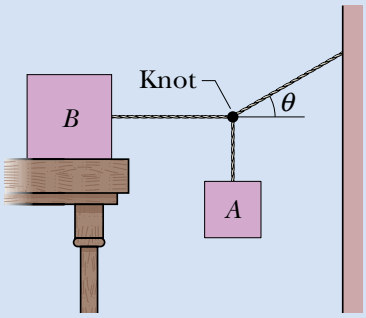

Figure 6-12 A block hanging from a knotted position.

KEY IDEA

In order that the two blocks remain in equilibrium, friction must be present between block B and the surface.

Reasoning: The free-body diagrams for block B and for the knot just above block A is shown in Fig. 6-13. In this case, $\vec{T}_1$ is the tension force of the rope pulling on block B or pulling on the knot (as the case may be), $\vec{T}_2$ is the tension force exerted by the second rope (at angle $\theta = 30°$) on the knot, $\vec{f}$ is the force of static friction exerted by the horizontal surface on block B, $\vec{F}_N$ is normal force exerted by the surface on block B, W_A is the weight of block A (W_A is the magnitude of $m_A\vec{g}$), and W_B is the weight of block B ($W_B = 711$ N is the magnitude of $m_B\vec{g}$).

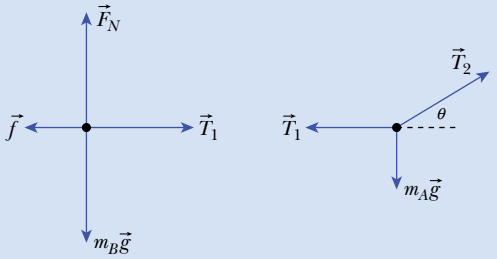

Figure 6-13 The free-body diagrams for block B and for the knot just above block A.

Calculations: For each object, we take $+x$ horizontally rightward and $+y$ upward. Applying Newton's second law in the x- and y-directions for block B and then doing the same for the knot results in four equations:

$$T_1 - f_{s,\max} = 0,$$
$$F_N - W_B = 0,$$
$$T_2 \cos\theta - T_1 = 0,$$
$$T_2 \cos\theta - T_1 = 0,$$
$$T_2 \sin\theta - W_A = 0,$$

where we assume the static friction to be at its maximum value (permitting us to use Eq. 6-1). The above equations yield $T_1 = \mu_s F_N$, $F_N = W_B$ and $T_1 = T_2 \cos\theta$.

Solving these equations with $\mu_s = 0.25$, we obtain

$$W_A = T_2 \sin\theta = T_1 \tan\theta = \mu_s F_N \tan\theta = \mu_s W_B \tan\theta$$
$$= (0.25)(711\text{ N})\tan 30° = 1.0\times10^2\text{ N}$$

As expected, the maximum weight of A is proportional to the weight of B, as well as the coefficient of static friction. In addition, we see that W_A is proportional to $\tan\theta$ (the larger the angle, the greater the vertical component of T_2 that supports its weight).

SAMPLE PROBLEM 6.07

Impending motion

A block of mass m_1 is kept on a fixed inclined plane and attached to a block of mass m_2 by a rope as shown in Fig. 6-14*a*. The incline makes angle θ with horizontal and coefficient of friction is μ. Find range of m_2 for which m_1 remains at rest (given $\theta > \tan^{-1}\mu_s$).

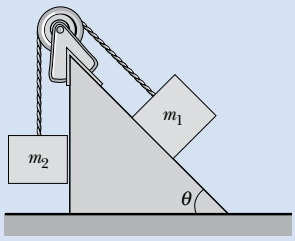

Figure 6-14 (*a*) A block of mass m_1 is kept on a fixed rough inclined plane and attached to a block of mass m_2 by a rope.

KEY IDEAS

(1) This problem is about finding range. If we make m_2 very light such that m_2 becomes zero, m_1 will tend to go down the inclined plane hence friction will act in upward direction. On the other hand, if we make m_2 very heavy then m_1 will definitely move up with the friction acting downward.

(2) It involves static friction as it is given in the sample problem that m_1 remains at rest and also m_1 at rest implies that m_2 must be at rest. Also, from above discussion it is clear that static friction is going to change direction. So, we need to solve this problem in two parts.

(a) To find minimum value, assuming block m_1 is in impending state to go down the inclined plane.

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Calculation: If m_2 is small then m_1 will tend to go down the incline for the smallest possible value of m_2, for equilibrium m_1 will experience maximum friction upward. Make free-body diagram as shown in Fig. 6-14*b*. For equilibrium of m_2

$$T = m_2 g$$

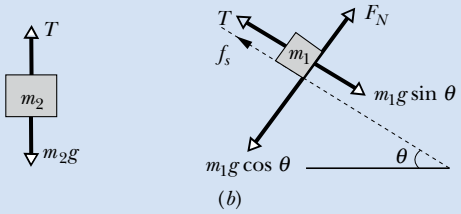

Figure 6-14 (*b*) Free-body diagrams.

For equilibrium of m_1 along incline

$$T + \mu_s F_N - m_1 g \sin\theta = 0$$

and perpendicular to incline

$$m_1 g \cos\theta - F_N = 0$$

Solving, we get

$$m_2 = m_1 (\sin\theta - \mu_s \cos\theta)$$

(b) To find maximum value, assuming block m_1 is in impending state to go up the inclined plane.

Calculation: Now if m_2 is increased, friction on m_1 will decrease as tension is increasing, but if we keep on increasing m_2 then friction will become directed down the incline and for maximum value of m_1 it will be directed downward and equal to $f_{s,\max}$ because m_1 is about to move up. Make free-body diagram as shown in Fig 6-14*c*.

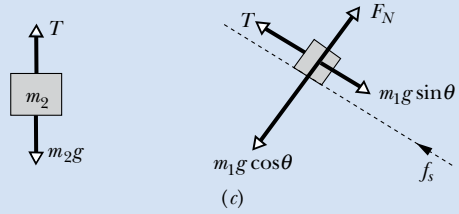

Figure 6-14 (*c*) Free-body diagrams.

For equilibrium of m_2

$$T = m_2 g$$

For equilibrium of m_1 along incline

$$T - \mu_s F_N - m_1 g \sin\theta = 0$$

and perpendicular to incline

$$m_1 g \cos\theta - F_N = 0$$

Solving above equation

$$m_2 = m_1 (\sin\theta + \mu_s \cos\theta)$$

Learn: As expected we are getting two different values of m_2 for two different impending states of motion of m_1.

In Case of: $m_2 = m_1 \sin\theta$, find value of friction force acting on m_1.

Friction force is not required as $T = m_2 g$ balances m_2 and $T = m_1 g \sin\theta$ balances m_1. At this value of m_2 friction shifts the direction, hence it is zero at this value. If we have mass m_2 lower than this, m_1 tends to go down with friction acting up the incline and vice versa.

Kinetic Friction

Kinetic friction acts when there is relative motion between two surfaces in contact. It acts always opposite to the relative velocity as we can see in Fig. 6-15. The magnitude is not self-adjusting as in static friction, it is always equal to $\mu_k F_N$.

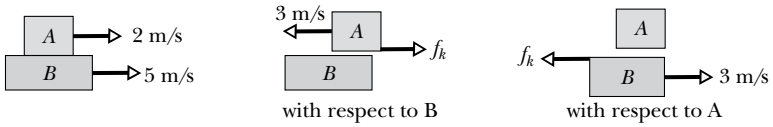

Figure 6-15 Two blocks kept one over another with the given instantaneous velocities.

CHECKPOINT 3

A student wants to determine the coefficients of static friction and kinetic friction between a box and a plank. She places the box on the plank and gradually raises one end of the plank. When the angle of inclination with the horizontal reaches 30°, the box starts to slip, and it then slides 2.5 m down the plank in 4.0 s at constant acceleration. What are (a) the coefficient of static friction and (b) the coefficient of kinetic friction between the box and the plank?

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SAMPLE PROBLEM 6.08

Force applied to accelerate a crate horizontally a level floor

A person pushes horizontally with a force of 220 N on a 55 kg crate to move it across a level floor. The coefficient of kinetic friction between the crate and the floor is 0.35. What is the magnitude of (a) the frictional force?

KEY IDEA

A force is being applied to accelerate a crate in the presence of friction. We apply Newton's second law to solve for the acceleration.

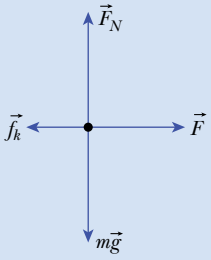

Figure 6-16 The free-body diagram for the crate discussed in Sample Problem 6.07.

Express: The free-body diagram for the crate is shown I Fig. 6-16. We denote $\vec{F}$ as the horizontal force of the person exerted on the crate (in the $+x$ direction), $\vec{f}_k$ is the force of kinetic friction (in the $-x$ direction), F_N is the vertical normal force exerted by the floor (in the $+y$ direction), and $m\vec{g}$ is the force of gravity. The magnitude of the force of friction is given by Eq. 6-2: $f_k = \mu_k F_N$. Applying Newton's second law to the x and y axes, we obtain

$$F - f_k = ma$$
$$F_N - mg = 0$$

respectively.

Calculation: The second equation here yields the normal force $F_N = mg$, so that the friction is

$$f_k = \mu_k F_N = \mu_k mg = (0.35)(55\text{ kg})(9.8\text{ m/s}^2)$$
$$= 1.9 \times 10^2\text{ N}$$

(b) What is the magnitude of (a) the acceleration of the crate?

Calculation: The first equation here becomes

$$F - \mu_k mg = ma$$

which (with $F = 220$ N) we solve to find

$$a = \frac{F}{m} - \mu_k g = \frac{220\text{ N}}{55\text{ kg}} - (0,35)(9.8\text{ m/s}^2)$$
$$= 0.56\text{ m/s}^2$$

Learn: For the crate to accelerate, the condition $F > f_k = \mu_k g$ must be met. As can be seen from the equation above, the greater the value of μ_k, the smaller the acceleration under the same applied force.

SAMPLE PROBLEM 6.09

Unit vector representation of kinetic friction

Find unit vector in direction of friction force acting on block B kept on moving platform P (Fig. 6-17). Given,

$$\vec{v}_P = 7\hat{i} - 2\hat{j}, \quad \vec{v}_B = 3\hat{i} + \hat{j}$$

Calculation: $\vec{v}_{B/P} = 3\hat{i} + \hat{j} - (7\hat{i} - 2\hat{j}) = -4\hat{i} + 3\hat{j}$

Friction on B due to P must be opposite of the velocity of B with respect to P.

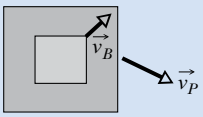

Figure 6-17 Top view of a block kept on a moving platform.

Friction on B due to P,

$$\hat{f}_k = -v_{B/P} = +\frac{4}{5}\hat{i} - \frac{3}{5}\hat{j} \qquad \text{(Answer)}$$

SAMPLE PROBLEM 6.10

Force applied at an angle to accelerate two objects connected by a pulley

A block m_1 of mass 10 kg kept on a rough, horizontal surface is connected to a sphere m_2 of mass 1 kg by a string over an ideal pulley as shown in Fig. 6-18*a*. A force F of magnitude 50 N at an angle 37° with the horizontal is applied to the block as shown and the block slides to the right. The coefficient of kinetic friction between the

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block and surface is 0.1. Determine the magnitude of the acceleration of the two objects.

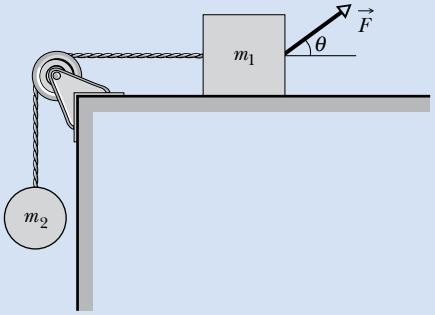

Figure 6-18 (*a*) A block of mass m_1 connected to sphere of mass m_2 by a string over an ideal pulley.

KEY IDEA

We cannot write the correct direction of kinetic friction till we know the correct direction of motion of the block. The way out is imagine if there was no friction then in which direction the bock would have moved. It is easy to calculate that in absence of friction the block would have moved toward right as the sphere will pull by its weight that is 10 N which is smaller than the horizontal component of force 40 N.

Calculation: Now we can draw free-body diagrams for the two objects, as shown in Fig. 6-18*b* and Fig. 6-18*c*. The applied force has *x* and *y* components 40 N and 30 N, respectively. The magnitudes of the acceleration of the block and the acceleration of the sphere is $\vec{a}$.

Applying Newton's second law to the block in the horizontal direction:

$$F\cos\theta - f_k - T = m_1 a \qquad (6\text{-}24)$$

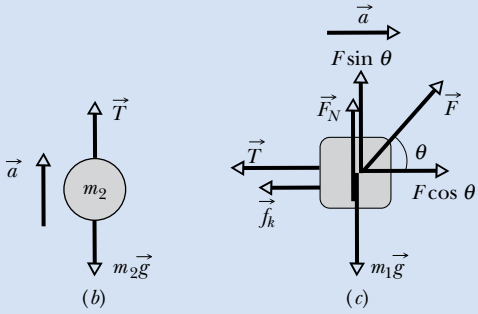

Figure 6-18 (*b*) and (*c*) Free-body diagrams.

Applying Newton's second law to the block in the vertical direction:

$$F_N + F\sin\theta - m_1 g = 0 \qquad (6\text{-}25)$$

Applying Newton's second law to the ball in the vertical direction:

$$T - m_2 g = m_2 a \qquad (6\text{-}26)$$

Solve Eq. 6-25 for F_N:

$$F_N = m_1 g - F\sin\theta$$

Substituting F_N into $f_k = \mu_k F_N$,

$$f_k = \mu_k\left(m_1 g - F\sin\theta\right) = 0.1(100 - 30) = 7N \qquad (6\text{-}27)$$

Substituting Eq. 6-27 and the value of T from Eq. 6-26 into Eq. 6-24,

$$40 - 7 - 1(a + g) = 10a$$

and solving for $\vec{a}$:

$$\vec{a} = 23/11\ \text{m/s}^2 \qquad \text{(Answer)}$$

Note: If friction force would have been larger, the block would have been at rest.

6.4 | SOME MORE APPLICATIONS OF PROPERTIES OF FRICTION

In addition to the Sample Problems discussed in Section 6.3, the properties of friction can be applied to solve problems that involve the following variations in conditions:

1. Nature of friction may change during the course of motion
2. Magnitude of static friction varies when blocks are in contact (block over block)
3. Magnitude of applied force is varied.

The above scenarios can be explored with the help of the following Sample Problems

Condition 1: Nature of friction changes during the course of motion.

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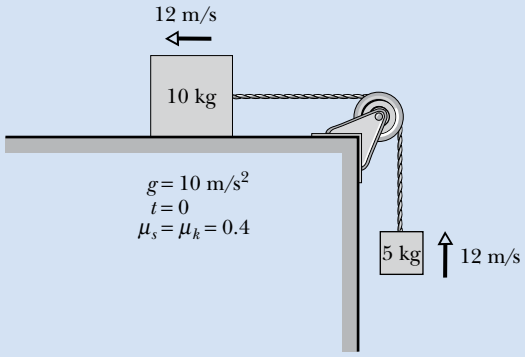
SAMPLE PROBLEM 6.11

Change in friction for two connected blocks in motion

Blocks are given velocities as shown in Fig. 6-19*a* at $t = 0$ s. Find velocity and position of 10 kg block at given values of t: (a) $t = 1$ s and (b) $t = 4$ s, (c) $t = 4$ s and $\mu = 0.6$.



(*a*)

Figure 6-19 (*a*) A block kept on rough horizontal rough table and attached to another block of mass 5 kg moving with initial speeds.

(a) $t = 1$ s.

KEY IDEA

Initially frictional force acting on 10 kg block is kinetic friction acting opposite to the velocity (Fig. 6-19*a*). We can also imagine that as time passes velocities of the blocks will keep on decreasing as both the forces (friction and tension) on 10 kg block are opposing the motion.

This problem looks similar to the previous example except that a horizontal force on 10 kg block is missing. So, to find acceleration initially we will proceed similar to the previous problem.

Calculation: Kinetic friction acts opposite to relative velocity so on the 10 kg block it will act toward right. Free-body diagrams of the above are shown in Fig. 6-19*b*.

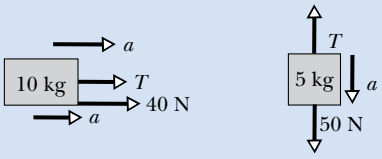


Figure 6-19 (*b*) Free-body diagrams.

Both the blocks will have same acceleration. Writing Newton's second law for both blocks:

$$40 + T = 10a, \quad 50 - T = 5a$$

Solving, we get $a = 6$ m/s^2

Writing parameters of motion:

$$v_0 = 12 \text{ m/s}, \quad a = -6 \text{ m/s}^2$$

Applying equations for uniform acceleration at $t = 1$ s, we get

$$v = 12 - 6 \times 1 = 6 \text{ m/s}; \quad s = 12 \times 1 - 3 \times 1 = 9 \text{ m}$$

(b) $t = 4$ s.

KEY IDEA

This part seems to be very simple as we have to put value of time in equations for uniform acceleration at $t = 4$ s and should get the answer

$$v = -12 \text{ m/s}$$

which means the direction of *motion changed during* these 3 seconds ($t = 1$ to $t = 4$ s). Since kinetic friction acts opposite to the relative velocity, it must have changed the direction. If kinetic friction changes direction then value of acceleration will be different, so we have to first of all find when and where the blocks stop. To do that we have to make new free-body diagram and proceed further.

Calculation: Writing parameters of motion

$$v_0 = 12 \text{ m/s}; a = -6 \text{ m/s}^2 \text{ (till velocity becomes zero)}$$

Using equations of uniform motion, we get $v = 0$.

$$\Rightarrow t = 2 \text{ s}; s = 12 \times 2 - 3 \times 4 = 12 \text{ m (towards left)}$$

The new free-body diagrams after $t = 2$ s are as shown in Fig. 6-19*c*.

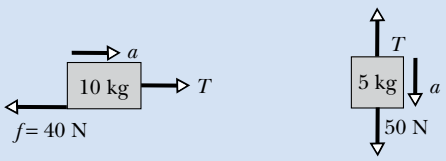


Figure 6-19 (*c*) Free-body diagrams.

The 10 kg block has tendency to move toward right hence friction will act toward left and applying Newton's second law, we get

$$50 - T = 5a \quad \text{or} \quad T - 40 = 10a$$

Writing parameter of motion:

$$a = 2/3 \text{ m/s}^2 \quad \text{(after } t = 2 \text{ s)}$$

$$v = 0 \text{ m/s}, a = 2/3 \text{ m/s}^2, \Delta t = 2 \text{ s}$$

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Applying equation for uniform acceleration, we get

$$s = \frac{1}{2} \times \frac{2}{3} \times 4 = 1.33 \text{ m}$$

Total displacement will be = 12 – 1.33 = 10.67 m (Answer)

Learn: We have learnt that kinetic friction opposes relative velocity not relative acceleration. We have also seen that kinetic friction changes direction when motion is reversed, so in problems involving kinetic friction we should not take acceleration as constant.

(c) In the same question, if $\mu_s = 0.6$ and all other parameters are same. The velocity of blocks at $t = 4$ s would be?

KEY IDEA

The effect of different coefficient of static friction will come into play only when block is at rest. Kinetic friction acts on the block while it is moving and has same value as in previous question, thus at $t = 2$ s blocks will stop. At the instant when the blocks are at rest, the force of friction on 10 kg block is static. Now we must check whether it will allow the motion or not.

Calculation: At t = 2s making free-body diagrams (Fig. 6-19*d*).

Direction of static friction on the 10 kg block will be toward left as it tends to go toward right due to tension. Now applying Newton's second law.

$$5g - T = 5a$$
$$T - f_s = 10a$$

Let's assume friction has limiting value that is,

$$f_s = \mu_s F_N = 60 \text{ N}$$

Solving, we get $a = -2/3 \text{ m/s}^2$

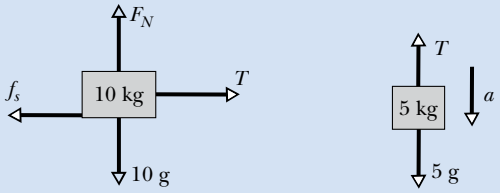

Figure 6-19 (*d*) Free-body diagrams.

The negative acceleration means that the block is accelerating toward left which is not possible. Our assumption that friction has attained limiting value must be wrong.

Here, for value of static friction less than limiting value, blocks can be in equilibrium. When friction is 50 N, we can have blocks in equilibrium.

Thus answer is $v = 0$ as after $t = 2$ s, the blocks will not move.

Learn: When $\mu_s \neq \mu_k$ then to find whether relative motion can start or not we must check using μ_s. Once relative motion starts, we use μ_k to find friction.

CHECKPOINT 4

A block of mass m slips on a rough horizontal table under the action of a horizontal force applied to it. The coefficient of friction between the block and the table is μ. The table does not move on the floor. Find the total frictional force applied by the floor on the legs of the table. Do you need the friction coefficient between the table and the floor or the mass of the table?

Condition 2: Magnitude of static friction varies when block is placed over another block.

Consider two blocks kept one over the other as in Fig. 6-20*a* where m_1 is kept on top of m_2. Let's say coefficient of friction between m_1 and m_2 is m_1 and between m_2 and ground is m_2. To find acceleration of m_1 and m_2 we make their free-body diagrams (Fig. 6-20*b*) and write Newton's second law as

Along y axis

$$F_{N_1} - m_1 g = 0 \quad \text{and} \quad F_{N_1} - m_2 g - F_{N_2} = 0$$

Solving the two gives,

$$F_{N_1} - m_1 g \quad \text{and} \quad F_{N_2} = (m_1 + m_2) g$$

Along x axis

$$P - f_1 = m_1 a_1 \quad \text{and} \quad f_1 = m_2 a_2$$

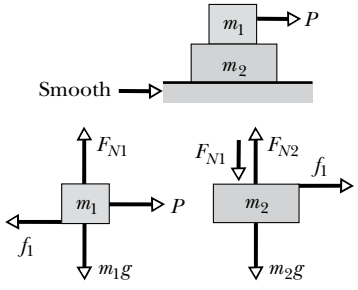

Figure 6-20 (*a*) Two blocks kept one over the other with friction between them. (*b*) Free-body diagram.

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Also, we know

$$f_1 < \mu_1 F_{N_1}$$

We now have two equations and three unknowns in equations along x direction.

This problem cannot be solved unless we make some assumptions.

Case I: If we assume both blocks are moving together then friction between them need not be equal to maximum value. Thus, taking $a_1 = a_2$, we are down to two unknowns and we can solve the equations, but we must verify our answer by checking $f_1 \leq \mu_1 F_{N_1}$. If this check fails we go to next possibility.

Case II: If we assume blocks are moving relative to each other, then friction between them must have reached maximum value

$$a_1 \neq a_2 \text{ but } f_1 = \mu_1 F_{N_1}$$

Again, we get two unknowns and two equations. We can solve it, but we must verify our answer. We will check the answer by verifying that friction is opposite the direction of relative motion.

SAMPLE PROBLEM 6.12

Change in static friction in motion of block over a block

(a) Friction coefficient between blocks is 0.5 and between ground and 10 kg block is 0.2. Find acceleration of blocks if force F = 40 N is applied on 5 kg block as shown in Fig. 6-21*a*.

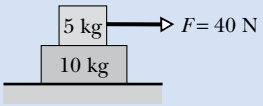

Figure 6-21 (*a*) Two blocks kept one over the other and force of 40 N applied on the upper block.

KEY IDEA

First, we find the values of limiting friction at all contact surfaces ($f_{s,\max}$) (Fig. 6-21*b*).

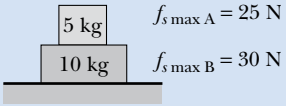

Figure 6-21 (*b*) Free-body diagram.

Calculation: The only driving force that the block B can experience is the one applied by the lower surface of block A on block B.

The maximum value of this force is 25 N. This is lower than the minimum force required to move with respect to ground. Hence only the block A will move as shown in Fig. 6-21*c*.

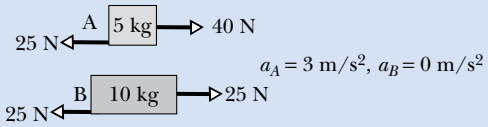

Figure 6-21 (*c*) Free-body diagram.

(b) Friction coefficient between the blocks is 0.5 and ground is smooth, if force of 30 N is applied on the upper block as shown. Find accelerations of the blocks (Fig. 6-22*a*).

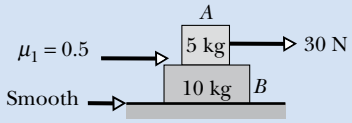

Figure 6-22 (*a*) Force of 30 N applied on upper block.

KEY IDEA

Maximum friction force between blocks is 25 N and ground is smooth. Block B must move because some force on upper surface will act on it. Block B can either move with same velocity and acceleration as block A or it can move relative to A. Let's assume block B moves with block A and solve.

Calculations: Making free-body diagram as shown in Fig. 6-22*b* and applying Newton's second law, we get

$$30 - f = 5a \quad \text{(for } A\text{)}$$

$$f = 10a \quad \text{(for } B\text{)}$$

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Solving, we get

$$a = 2 \text{ m/s}^2 \quad \text{and} \quad f = 20 \text{ N}$$

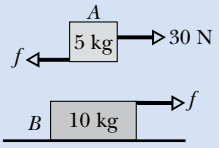

Figure 6-22 (*b*) Free-body diagrams.

Now check if this acceleration is possible by verifying $f \le f_{\max A}$ (Fig. 6-22*c*).

Figure 6-22 (*c*) Free-body diagrams.

Writing Newton's second law, we get

$$30 - f = 5 \times 2 \quad \text{(for } A\text{)}$$

$$f = 10 \times 2 = 20 \text{ N} \quad \text{(for } B\text{)}$$

Therefore, $f = 20 < 25$

Hence our assumption is true. We can try by assuming they are not moving together and solve the equations to get an absurd result.

(c) Friction coefficient between the blocks is 0.5 and ground is smooth. If force of 60 N is applied on the upper block as shown in Fig. 6-23*a* find accelerations of the blocks.

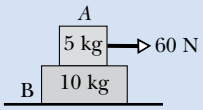

Figure 6-23 (*a*) Force of 60 N applied on upper block.

KEY IDEA

Let's assume they have same acceleration. The value of acceleration is $60/(10 + 5) = 4 \text{ m/s}^2$.

Calculation: Making free-body diagram as shown in Fig. 6-23*b*.

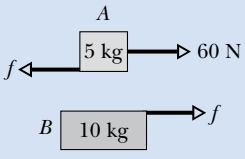

Figure 6-23 (*b*) Free-body diagrams.

Writing Newton's second law,

$$60 - f = 5 \times 4 \quad \text{(for } A\text{)}$$

$$f = 10 \times 4 = 40 \text{ N} \quad \text{(for } B\text{)}$$

Solving we get, $a = 4 \text{ m/s}^2$ and $f = 40$ N while limiting value of friction is 25 N, hence our assumption is wrong.

Let's say they are not moving together. Then the free-body diagram is as shown in Fig. 6-23*c*.

$$a_A = 7 \text{ m/s}^2, a_B = 2.5 \text{ m/s}^2$$

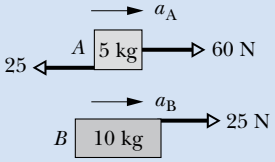

Figure 6-23 (*c*) Free-body diagrams.

Learn: If we look at block *A* from frame attached to block *B* we will find it is moving toward right and friction is acting towards left. This is satisfying friction's tendency to oppose relative motion.

In Case of: In part (b) if we assume that blocks are moving separately and solve, how do we verify that answer is incorrect?

Calculations: If we assume they are moving separately, friction attains maximum value.

Again, making free-body diagram as shown in Fig. 6-23*d*.

$$a_A = \frac{30 - 25}{5} = 1 \text{ m/s}^2$$

$$a_B = \frac{25}{10} = 2.5 \text{ m/s}^2$$

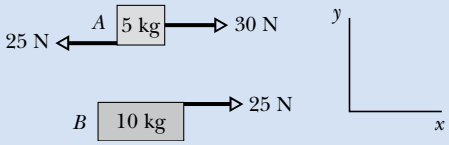

Figure 6-23 (*d*) Free-body diagrams.

If we look at block A from reference frame attached to block *B*, we get

$$a_{A/B} = a_A - a_B = 1.5 \text{ m/s}^2$$

that is, block *A* will be moving towards left and friction on it also acting in the same direction. This is not possible as friction must oppose relative motion.

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CHECKPOINT 5

Find the maximum force for which the blocks of the above problem can move together.

Condition 3: Magnitude of applied force is varied.

SAMPLE PROBLEM 6.13

Motion of a bar kept over a plank as a function of applied horizontal force

A bar of mass m_1 is placed on a plank of mass m_2, which rests on a smooth horizontal plane (Fig. 6-24*a*). The coefficient of friction between the surfaces of the bar and the plank is equal to μ. The plank is subjected to the horizontal force F depending on time t as $F = at$ (a is a constant).

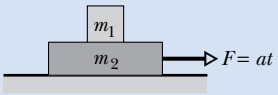

Figure 6-24 (*a*) A bar kept on a plank which is subjected to a horizontal force.

Find (a) the moment of time t_0 at which the plank starts sliding from under the bar and (b) the acceleration of the bar a_1 and that of plank a_2 during motion.

KEY IDEA

This problem is similar to problems that we have already solved. Here as the force F grows, so does the static friction force f_s. However, the friction force f has the limiting value $f_{\text{limit}} = \mu m_1 g$. Unless this value is reached, both bodies move together with equal accelerations. But as soon as the force f reaches this limit, mass m_2 starts sliding under mass m_1.

Calculations: Let us make free-body diagram as shown in Fig. 6-24*b*.

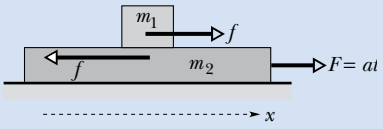

Figure 6-24 (*b*) Free-body diagram.

Writing Newton's second law for the plank and the bar, having taken the positive direction of the x axis as shown in Fig. 6-24*b*.

$$f = m_1 a_1,\ F - f = m_2 a_2 \qquad (6\text{-}28)$$

Acceleration of m_2 must be always greater than or equal to the acceleration of m_1 that is $a_2 \geq a_1$ where the sign "=" corresponds to the moment $t = t_0$. Hence, when $f = \mu m_1 g$, then sliding begins. Putting $f = \mu m_1 g$ in Eq. 6-28, we get

$$t_0 = (m_1 + m_2)\frac{\mu g}{a}$$

When $t \leq t_0$, then

$$a_1 = a_2 = \frac{at}{(m_1 + m_2)}$$

and when $t > t_0$, then they separate. Only force acting on m_1 is friction, whose value is constant.

Thus, $a_1 = \mu g = \text{constant}$

Now m_2 is experiencing force F and constant friction thus, $at - \mu m_1 g = m_2 a_2$

Solving we get

$$a_2 = \frac{(at - \mu m_1 g)}{m_2}$$

Note: You may have been tempted to think that when external force $F(= at)$ is equal to $\mu m_1 g$, slipping will begin. You can check that at this instant t_2 they are moving with same acceleration.

$$at_2 = \mu m_1 g$$

$$t_2 = \frac{\mu m_1 g}{a}$$

Let's find force of friction between the blocks at this instant.

$$\mu m_1 g - f = m_2 a$$

$$f = m_1 a$$

Solving for friction

$$f = \frac{\mu m_1 g}{\left(1 + \frac{m_2}{m_1}\right)}$$

Since friction is less than the limiting value, slipping has not yet begun.

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6.5 | THE DRAG FORCE AND TERMINAL SPEED

Key Concepts

- When there is relative motion between air (or some other fluid) and a body, the body experiences a drag force $\vec{D}$ that opposes the relative motion and points in the direction in which the fluid flows relative to the body. The magnitude of $\vec{D}$ is related to the relative speed v by an experimentally determined drag coefficient C according to

$$D = \frac{1}{2} C \rho A v^2,$$

where ρ is the fluid density (mass per unit volume) and A is the effective cross-sectional area of the body (the area of a cross section taken perpendicular to the relative velocity $\vec{v}$).

- When a blunt object has fallen far enough through air, the magnitudes of the drag force $\vec{D}$ and the gravitational force $\vec{F}_g$ on the body become equal. The body then falls at a constant terminal speed v_t given by

$$v_t = \sqrt{\frac{2F_g}{C\rho A}}.$$

A **fluid** is anything that can flow—generally either a gas or a liquid. When there is a relative velocity between a fluid and a body (either because the body moves through the fluid or because the fluid moves past the body), the body experiences a **drag force** $\vec{D}$ that opposes the relative motion and points in the direction in which the fluid flows relative to the body.

Here we examine only cases in which air is the fluid, the body is blunt (like a baseball) rather than slender (like a javelin), and the relative motion is fast enough so that the air becomes turbulent (breaks up into swirls) behind the body. In such cases, the magnitude of the drag force $\vec{D}$ is related to the relative speed v by an experimentally determined **drag coefficient** C according to

$$D = \frac{1}{2} C \rho A v^2, \tag{6-29}$$

where ρ is the air density (mass per volume) and A is the **effective cross-sectional area** of the body (the area of a cross section taken perpendicular to the velocity $\vec{v}$). The drag coefficient C (typical values range from 0.4 to 1.0) is not truly a constant for a given body because if v varies significantly, the value of C can vary as well. Here, we ignore such complications.

Downhill speed skiers know well that drag depends on A and v^2. To reach high speeds a skier must reduce D as much as possible by, for example, riding the skis in the "egg position" (Fig. 6-25) to minimize A.

Karl-Josef Hildenbrand/dpa/Landov LLC

Figure 6-25 This skier crouches in an "egg position" so as to minimize her effective cross-sectional area and thus minimize the air drag acting on her.

Falling. When a blunt body falls from rest through air, the drag force $\vec{D}$ is directed upward; its magnitude gradually increases from zero as the speed of the body increases. This upward force $\vec{D}$ opposes the downward gravitational force $\vec{F}_g$ on the body. We can relate these forces to the body's acceleration by writing Newton's second law for a vertical y axis ($F_{net,y} = ma_y$) as

$$D - F_g = ma, \tag{6-30}$$

where m is the mass of the body. As suggested in Fig. 6-26, if the body falls long enough, D eventually equals F_g. From Eq. 6-30, this means that $a = 0$, and so the body's speed no longer increases. The body then falls at a constant speed, called the **terminal speed** v_t.

To find v_t, we set $a = 0$ in Eq. 6-30 and substitute for D from Eq. 6-29, obtaining

$$\frac{1}{2} C \rho A v_t^2 - F_g = 0,$$

which gives

$$v_t = \sqrt{\frac{2F_g}{C\rho A}}. \tag{6-31}$$

Table 6-2 gives values of v_t for some common objects.

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Table 6-2 Some Terminal Speeds in Air

Object	Terminal Speed (m/s)	95% Distance[a] (m)
Shot (from shot put)	145	2500
Sky diver (typical)	60	430
Baseball	42	210
Tennis ball	31	115
Basketball	20	47
Ping-Pong ball	9	10
Raindrop (radius = 1.5 mm)	7	6
Parachutist (typical)	5	3

[a] This is the distance through which the body must fall from rest to reach 95% of its terminal speed. Based on Peter J. Brancazio, *Sport Science*, 1984, Simon & Schuster, New York.

According to calculations* based on Eq. 6-29, a cat must fall about six floors to reach terminal speed. Until it does so, $F_g > D$ and the cat accelerates downward because of the net downward force. Recall from Chapter 3 that your body is an accelerometer, not a speedometer. Because the cat also senses the acceleration, it is frightened and keeps its feet underneath its body, its head tucked in, and its spine bent upward, making A small, v_t large, and injury likely.

However, if the cat does reach v_t during a longer fall, the acceleration vanishes and the cat relaxes somewhat, stretching its legs and neck horizontally outward and straightening its spine (it then resembles a flying squirrel). These actions increase area A and thus also, by Eq. 6-29, the drag D. The cat begins to slow because now $D > F_g$ (the net force is upward), until a new, smaller v_t is reached. The decrease in v_t reduces the possibility of serious injury on landing. Just before the end of the fall, when it sees it is nearing the ground, the cat pulls its legs back beneath its body to prepare for the landing.

Humans often fall from great heights for the fun of skydiving. However, in April 1987, during a jump, sky diver Gregory Robertson noticed that fellow sky diver Debbie Williams had been knocked unconscious in a collision with a third sky diver and was unable to open her parachute. Robertson, who was well above Williams at the time and who had not yet opened his parachute for the 4 km plunge, reoriented his body head-down so as to minimize A and maximize his downward speed. Reaching an estimated v_t of 320 km/h, he caught up with Williams and then went into a horizontal "spread eagle" (as in Fig. 6-27) to increase D so that he could grab her. He opened her parachute and then, after releasing her, his own, a scant 10 s before impact. Williams received extensive internal injuries due to her lack of control on landing but survived.

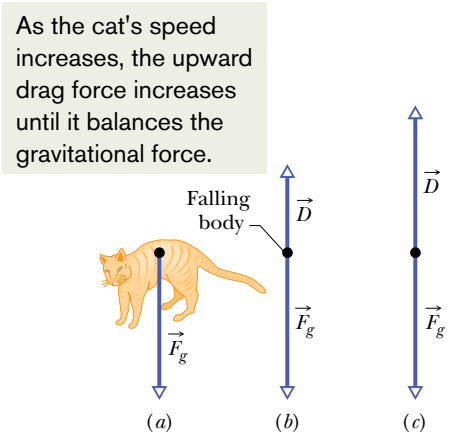

Figure 6-26 The forces that act on a body falling through air: (*a*) the body when it has just begun to fall and (*b*) the freebody diagram a little later, after a drag force has developed. (*c*) The drag force has increased until it balances the gravitational force on the body. The body now falls at its constant terminal speed.

Steve Fitchett/Taxi/Getty Images

Figure 6-27 Sky divers in a horizontal "spread eagle" maximize air drag.

*W. O. Whitney and C. J. Mehlhaff, "High-Rise Syndrome in Cats." *The Journal of the American Veterinary Medical Association*, 1987.

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SAMPLE PROBLEM 6.14

Terminal speed of falling raindrop

A raindrop with radius $R = 1.5$ mm falls from a cloud that is at height $h = 1200$ m above the ground. The drag coefficient C for the drop is 0.60. Assume that the drop is spherical throughout its fall. The density of water ρ_w is 1000 kg/m^3, and the density of air ρ_a is 1.2 kg/m^3.

(a) As Table 6-2 indicates, the raindrop reaches terminal speed after falling just a few meters. What is the terminal speed?

KEY IDEA

The drop reaches a terminal speed v_t when the gravitational force on it is balanced by the air drag force on it, so its acceleration is zero. We could then apply Newton's second law and the drag force equation to find v_t, but Eq. 6-30 does all that for us.

Calculations: To use Eq. 6-30, we need the drop's effective cross-sectional area A and the magnitude F_g of the gravitational force. Because the drop is spherical, A is the area of a circle (πR^2) that has the same radius as the sphere. To find F_g, we use three facts: (1) $F_g = mg$, where m is the drop's mass; (2) the (spherical) drop's volume is $V = 4/3\,\pi R^3$ and (3) the density of the water in the drop is the mass per volume, or $\rho_w = m/V$. Thus, we find

$$F_g = V\rho_w g = \frac{4}{3}\pi R^3 \rho_w g.$$

We next substitute this, the expression for A, and the given data into Eq. 6-30. Being careful to distinguish between the air density ρ_a and the water density ρ_w, we obtain

$$v_t = \sqrt{\frac{2F_g}{C\rho_a A}} = \sqrt{\frac{8\pi R^3 \rho_w g}{3C\rho_a \pi R^2}} = \sqrt{\frac{8R\rho_w g}{3C\rho_a}}$$

$$= \sqrt{\frac{(8)(1.5\times 10^{-3}\text{ m})(1000\text{ kg/m}^3)(9.8\text{ m/s}^2)}{(3)(0.60)(1.2\text{ kg/m}^3)}}$$

$$= 7.4\text{ m/s} \approx 27\text{ km/h}. \qquad \text{(Answer)}$$

Note that the height of the cloud does not enter into the calculation.

(b) What would be the drop's speed just before impact if there were no drag force?

KEY IDEA

With no drag force to reduce the drop's speed during the fall, the drop would fall with the constant free-fall acceleration g, so the constant-acceleration equations of Table 2-2 apply.

Calculation: Because we know the acceleration is g, the initial velocity v_0 is 0, and the displacement $x - x_0$ is $-h$, we use Eq. 2-30 to find v:

$$v = \sqrt{2gh} = \sqrt{(2)(9.8\text{ m/s}^2)(1200\text{ m})}$$

$$= 153\text{ m/s} \approx 550\text{ km/h}. \qquad \text{(Answer)}$$

Had he known this, Shakespeare would scarcely have written, "it droppeth as the gentle rain from heaven, upon the place beneath." In fact, the speed is close to that of a bullet from a large-caliber handgun!

REVIEW AND SUMMARY

Friction When a force $\vec{F}$ tends to slide a body along a surface, a **frictional force** from the surface acts on the body. The frictional force is parallel to the surface and directed so as to oppose the sliding. It is due to bonding between the atoms on the body and the atoms on the surface, an effect called cold-welding.

If the body does not slide, the frictional force is a **static frictional force** $\vec{f}_s$. If there is sliding, the frictional force is a **kinetic frictional force** $\vec{f}_k$.

1. If a body does not move, the static frictional force $\vec{f}_s$ and the component of $\vec{F}$ parallel to the surface are equal in magnitude, and $\vec{f}_s$ is directed opposite that component. If the component increases, f_s also increases.
2. The magnitude of $\vec{f}_s$ has a maximum value $f_{s,\max}$ given by

$$f_{s,\max} = \mu_s F_N \qquad (6\text{-}1)$$

where μ_s is the coefficient of static friction and F_N is the magnitude of the normal force. If the component of $\vec{F}$ parallel to the surface exceeds $f_{s,\max}$, the static friction is overwhelmed and the body slides on the surface.

3. If the body begins to slide on the surface, the magnitude of the frictional force rapidly decreases to a constant value f_k given by

$$f_k = \mu_k F_N, \qquad (6\text{-}2)$$

where μ_k is the **coefficient of kinetic friction**.

Static Friction The force of friction between two surfaces as long as there is no relative motion between them is called **static friction** and it is equal to the applied force.

The magnitude of static friction f_s (static frictional force) has a maximum value $f_{s,\max}$, which is given by

$$f_s = f_{s,\max} = \mu_s F_N,$$

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where μ_s is the coefficient of kinetic friction and F_N is the normal force.

Limiting Friction This maximum value of force of friction is called **limiting friction** which is otherwise defined as follows: *Limiting friction is the maximum value of force of friction between two surfaces as long as there is no relative motion between them.*

The frictional force between two surfaces after sliding begins is the product of the coefficient of kinetic friction and the normal force:

$$f_k = \mu_k F_N$$

Drag Force When there is relative motion between air (or some other fluid) and a body, the body experiences a **drag force** $\vec{D}$ that opposes the relative motion and points in the direction in which the fluid flows relative to the body. The magnitude of $\vec{D}$ is related to the relative speed v by an experimentally determined **drag coefficient** C according to

$$D = \frac{1}{2} C \rho A v^2, \tag{6-29}$$

where ρ is the fluid density (mass per unit volume) and A is the **effective cross-sectional area** of the body (the area of a cross section taken perpendicular to the relative velocity $\vec{v}$).

Terminal Speed When a blunt object has fallen far enough through air, the magnitudes of the drag force $\vec{D}$ and the gravitational force $\vec{F}_g$ on the body become equal. The body then falls at a constant **terminal speed** v_t given by

$$v_t = \sqrt{\frac{2F_g}{C\rho A}}. \tag{6-31}$$

PROBLEMS

1. Figure 6-28 shows a 6.0 kg block on a 60° ramp with a coefficient of static friction of 0.60. A force $\vec{F}$ is applied up the ramp. What magnitude of that force puts the block on the verge of sliding down the ramp?

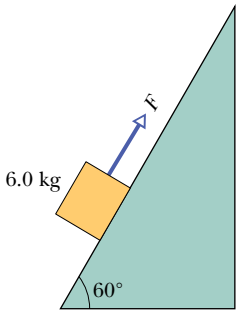

Figure 6-28 Problem 1.

2. In a pickup game of dorm shuffleboard, students crazed by final exams use a broom to propel a calculus book along the dorm hallway. If the 3.5 kg book is pushed from rest through a distance of 1.20 m by the horizontal 25 N force from the broom and then has a speed of 1.75 m/s, what is the coefficient of kinetic friction between the book and floor?

3. In Fig. 6-29, a 2.0 kg block is placed on top of a 3.0 kg block, which lies on a frictionless surface. The coefficient of kinetic friction between the two blocks is 0.30; they are connected via a pulley and a string. A hanging block of mass 10 kg is connected to the 3.0 kg block via another pulley and string. Both strings have negligible mass and both pulleys are frictionless and have negligible mass. When the assembly is released, what are (a) the acceleration magnitude of the blocks, (b) the tension in string 1, and (c) the tension in string 2?

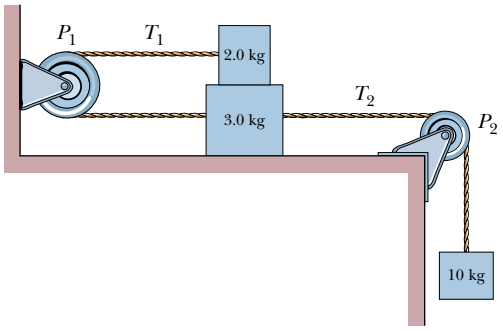

Figure 6-29 Problem 3.

4. Figure 6-30 shows a block of mass m connected to a block of mass M = 2.00 kg, both on 45° inclined planes where the coefficient of static friction is 0.28. Find the (a) minimum and (b) maximum values of m for which the system is at rest.

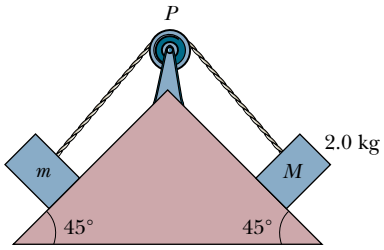

Figure 6-30 Problem 4.

5. A 2.5 kg block is initially at rest on a horizontal surface. A horizontal force $\vec{F}$ of magnitude 6.0 N and a vertical force $\vec{P}$ are then applied to the block (Fig. 6-31). The coefficients of friction for the block and surface are μ_s = 0.40 and μ_k = 0.25. Determine the magnitude of the frictional force acting on the block if the magnitude of $\vec{P}$ is (a) 8.0 N, (b) 10 N, and (c) 12 N.

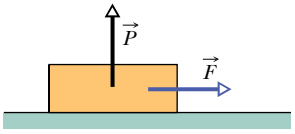

Figure 6-31 Problem 5.

6. A baseball player with mass m = 83 kg, sliding into second base, is retarded by a frictional force of magnitude 485 N. What is the coefficient of kinetic friction μ_k between the player and the ground?

7. *The mysterious sliding stones.* Along the remote Racetrack Playa in Death Valley, California, stones sometimes gouge out prominent trails in the desert floor, as if the stones had been migrating (Fig. 6-32). For years curiosity mounted about why the stones moved. One explanation was that strong winds during occasional rainstorms would drag the rough stones over ground softened by rain. When the desert dried out, the trails behind the stones were hard-baked in place. According to measurements, the coefficient

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Figure 6-32 Problem 7. What moved the stone?

of kinetic friction between the stones and the wet playa ground is about 0.80. What horizontal force must act on a 20 kg stone (a typical mass) to maintain the stone's motion once a gust has started it moving? (Story continues with Problem 34.)

8. A 3.5 kg block is pushed along a horizontal floor by a force $\vec{F}$ of magnitude 15 N at an angle $\theta = 40°$ with the horizontal (Fig. 6-33). The coefficient of kinetic friction between the block and the floor is 0.25. Calculate the magnitudes of (a) the frictional force on the block from the floor and (b) the block's acceleration.

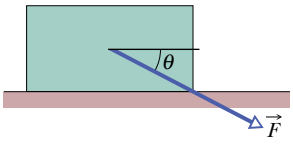

Figure 6-33 Problem 8.

9. In Fig. 6-34 a block of weight W experiences two applied forces, each of magnitude $W/2$. What coefficient of static friction between the block and the floor puts the block on the verge of sliding?

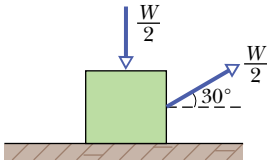

Figure 6-34 Problem 9.

10. In about 1915, Henry Sincosky of Philadelphia suspended himself from a rafter by gripping the rafter with the thumb of each hand on one side and the fingers on the opposite side (Fig. 6-35). Sincosky's mass was 79 kg. If the coefficient of static friction between hand and rafter was 0.70, what was the least magnitude of the normal force on the rafter from each thumb or opposite fingers? (After suspending himself, Sincosky chinned himself on the rafter and then moved hand-over-hand along the rafter. If you do not think Sincosky's grip was remarkable, try to repeat his stunt.)

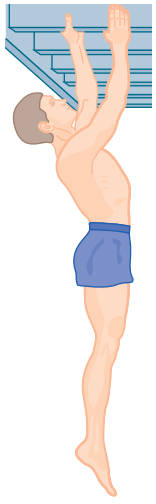

Figure 6-35 Problem 10.

11. A worker pushes horizontally on a 35 kg crate with a force of magnitude 110 N. The coefficient of static friction between the crate and the floor is 0.37. (a) What is the value of $f_{s,max}$ under the circumstances? (b) Does the crate move? (c) What is the frictional force on the crate from the floor? (d) Suppose, next, that a second worker pulls directly upward on the crate to help out. What is the least vertical pull that will allow the first worker's 110 N push to move the crate? (e) If, instead, the second worker pulls horizontally to help out, what is the least pull that will get the crate moving?

12. Figure 6-36 shows the cross section of a road cut into the side of a mountain. The solid line AA' represents a weak bedding plane along which sliding is possible. Block B directly above the highway is separated from uphill rock by a large crack (called a *joint*), so that only friction between the block and the bedding plane prevents sliding. The mass of the block is 1.5×10^7 kg, the *dip angle* θ of the bedding plane is 24°, and the coefficient of static friction between block and plane is 0.63. (a) Show that the block will not slide under these circumstances. (b) Next, water seeps into the joint and expands upon freezing, exerting on the block a force $\vec{F}$ parallel to AA'. What minimum value of force magnitude F will trigger a slide down the plane?

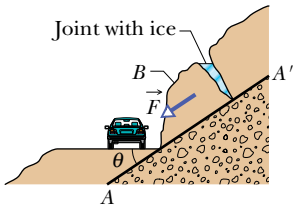

Figure 6-36 Problem 12.

13. In Fig. 6-37, a block of mass m = 5.0 kg is at rest on a ramp. The coefficient of static friction between the block and ramp is not known. Find the magnitude of the net force exerted by the ramp on the block.

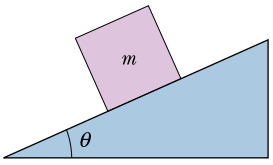

Figure 6-37 Problem 13.

14. In Fig. 6-38, a small block of mass m is sent sliding with velocity v along a slab of mass 10 m, starting at a distance of l from the far end of the slab. The coefficient of kinetic friction between the slab and the floor is μ_1; that between the block and the slab is μ_2, with $\mu_2 > 11\mu_1$. (a) Find the minimum value of v such that the block reaches the far end of the slab. (b) For that value of v, how long does the block take to reach the far end?

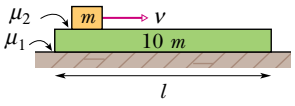

Figure 6-38 Problem 14.

15. In Fig. 6-29, a force $\vec{P}$ acts on a block weighing 45 N. The block is initially at rest on a plane inclined at angle θ = 15° to the horizontal. The positive direction of the x axis is up the plane. Between block and plane, the coefficient of static friction is μ_s = 0.50 and the coefficient of kinetic friction is μ_k = 0.34. In unit-vector notation, what is the frictional force on the block from the plane when $\vec{P}$ is (a) (–5.0 N)$\hat{i}$, (b) (–8.0 N)$\hat{i}$, and (c) (–15 N)$\hat{i}$?

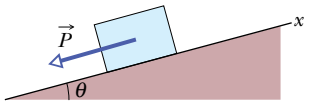

Figure 6-39 Problem 15.

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16. You testify as an *expert witness* in a case involving an accident in which car A slid into the rear of car B, which was stopped at a red light along a road headed down a hill (Fig. 6-40). You find that the slope of the hill is $\theta = 12.0°$, that the cars were separated by distance $d = 30.0$ m when the driver of car A put the car into a slide (it lacked any automatic anti-brake-lock system), and that the speed of car A at the onset of braking was $v_0 = 18.0$ m/s. With what speed did car A hit car B if the coefficient of kinetic friction was (a) 0.60 (dry road surface) and (b) 0.10 (road surface covered with wet leaves)?

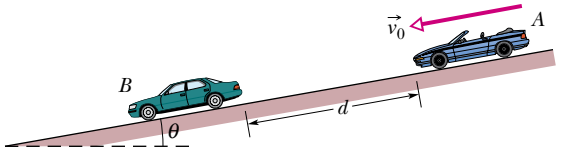

Figure 6-40 Problem 16.

17. A 12 N horizontal force $\vec{F}$ pushes a block weighing 5.0 N against a vertical wall (Fig. 6-41). The coefficient of static friction between the wall and the block is 0.60, and the coefficient of kinetic friction is 0.40. Assume that the block is not moving initially. (a) Will the block move? (b) In unit-vector notation, what is the force on the block from the wall?

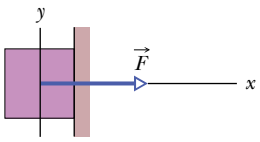

Figure 6-41 Problem 17.

18. In Fig. 6-42, a box of Cheerios (mass $m_C = 1.0$ kg) and a box of Wheaties (mass $m_W = 3.0$ kg) are accelerated across a horizontal surface by a horizontal force $\vec{F}$ applied to the Cheerios box. The magnitude of the frictional force on the Cheerios box is 2.0 N, and the magnitude of the frictional force on the Wheaties box is 3.5 N. If the magnitude of $\vec{F}$ is 12 N, what is the magnitude of the force on the Wheaties box from the Cheerios box?

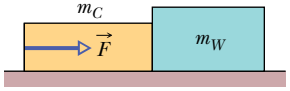

Figure 6-42 Problem 18.

19. In Fig. 6-43, a 15 kg sled is attached to a 2.0 kg sand box by a string of negligible mass, wrapped over a pulley of negligible mass and friction. The coefficient of kinetic friction between the sled and table top is 0.040. Find (a) the acceleration of the sled and (b) the tension of the string.

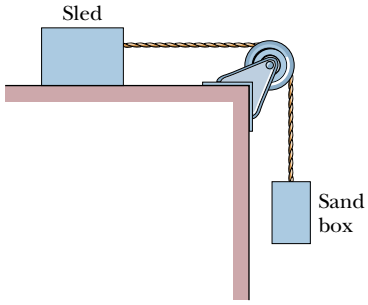

Figure 6-43 Problem 19.

20. In Fig. 6-44*a*, a sled is held on an inclined plane by a cord pulling directly up the plane. The sled is to be on the verge of moving up the plane. In Fig. 6-44*b*, the magnitude F required of the cord's force on the sled is plotted versus a range of values for the coefficient of static friction μ_s between sled and plane: $F_1 = 2.0$ N, $F_2 = 5.0$ N, and $\mu_2 = 0.25$. At what angle θ is the plane inclined?

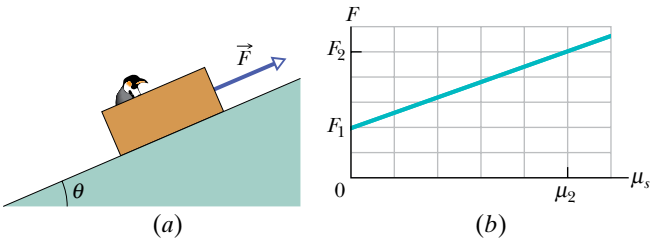

Figure 6-44 Problem 20.

21. When the three blocks in Fig. 6-45 are released from rest, they accelerate with a magnitude of 0.500 m/s². Block 1 has mass M, block 2 has $2M$, and block 3 has $2M$. What is the coefficient of kinetic friction between block 2 and the table?

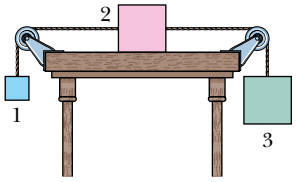

Figure 6-45 Problem 21.

22. A 4.10 kg block is pushed along a floor by a constant applied force that is horizontal and has a magnitude of 50.0 N. Figure 6-46 gives the block's speed v versus time t as the block moves along an x axis on the floor. The scale of the figure's vertical axis is set by $v_s = 5.0$ m/s. What is the coefficient of kinetic friction between the block and the floor?

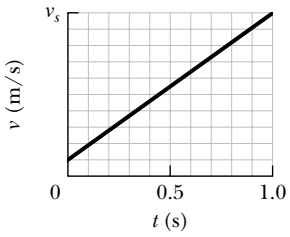

Figure 6-46 Problem 22.

23. Figure 6-47 shows three crates being pushed over a concrete floor by a horizontal force $\vec{F}$ of magnitude 425 N. The masses of the crates are $m_1 = 30.0$ kg, $m_2 = 10.0$ kg, and $m_3 = 20.0$ kg. The coefficient of kinetic friction between the floor and each of the crates is 0.700. (a) What is the magnitude F_{32} of the force on crate 3 from crate 2? (b) If the crates then slide onto a polished floor, where the coefficient of kinetic friction is less than 0.700, is magnitude F_{32} more than, less than, or the same as it was when the coefficient was 0.700?

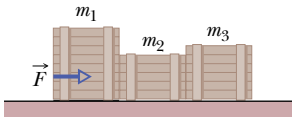

Figure 6-47 Problem 23.

24. In Fig. 6-48, a 2.0 kg block lies on a 20 kg trolley that can roll across a floor on frictionless bearings. Between the block and the trolley, the coefficient of kinetic friction is 0.20 and the coefficient of static friction is 0.25. When a horizontal 2.0 N force is applied to the block, what are the magnitudes of (a) the frictional force between the block and the trolley and (b) the acceleration of the trolley?

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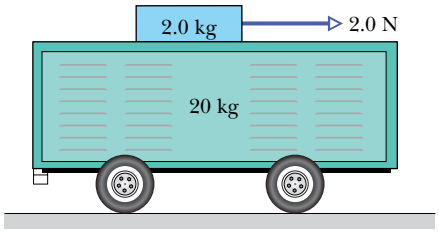

Figure 6-48 Problem 24.

25. In Fig. 6-49, two blocks are connected over a pulley. The mass of block A is 15 kg, and the coefficient of kinetic friction between A and the incline is 0.20. Angle θ of the incline is 30°. Block A slides down the incline at constant speed. What is the mass of block B?

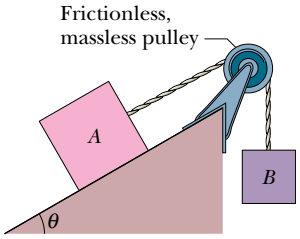

Figure 6-49 Problem 25.

26. In Fig. 6-50, block A of mass 2.0 kg, block B of 3.0 kg, and block C of 6.0 kg are connected by strings of negligible mass that run over pulleys of negligible mass and friction. The coefficient of kinetic friction between block B and the table top is 0.40. When the system is released, the blocks move. What is the magnitude of their acceleration?

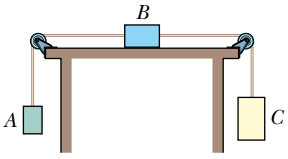

Figure 6-50 Problem 26.

27. A toy chest and its contents have a combined weight of 200 N. The coefficient of static friction between toy chest and floor is 0.47. The child in Fig. 6-51 attempts to move the chest across the floor by pulling on an attached rope. (a) If θ is 42°, what is the magnitude of the force $\vec{F}$ that the child must exert on the rope to put the chest on the verge of moving? (b) Write an expression for the magnitude F required to put the chest on the verge of moving as a function of the angle θ. Determine (c) the value of θ for which F is a minimum and (d) that minimum magnitude.

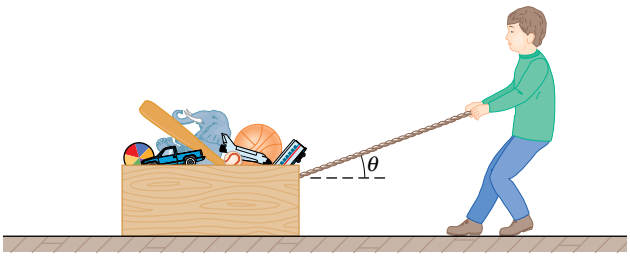

Figure 6-51 Problem 27.

28. In Fig. 6-52, two blocks, in contact, slide down an inclined plane AC of inclination 30°. The coefficient of kinetic friction between the 2.0 kg block and the incline is $\mu_1 = 0.20$ and that between the 4.0 kg block and the incline is $\mu_2 = 0.30$. Find the magnitude of the acceleration.

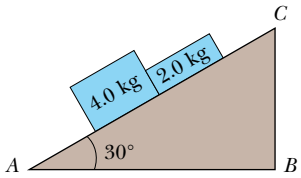

Figure 6-52 Problem 28.

29. A block is pushed across a floor by a constant force that is applied at downward angle θ (Fig. 6-33). Figure 6-53 gives the acceleration magnitude a versus a range of values for the coefficient of kinetic friction μ_k between block and floor: $a_1 = 3.0$ m/s^2, $\mu_{k2} = 0.20$, and $\mu_{k3} = 0.40$. What is the value of θ?

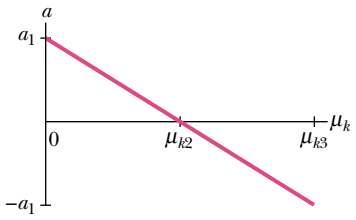

Figure 6-53 Problem 29.

30. A 1000 kg boat is traveling at 100 km/h when its engine is shut off. The magnitude of the frictional force $\vec{f}_k$ between boat and water is proportional to the speed v of the boat: $f_k = 70v$, where v is in meters per second and f_k is in newtons. Find the time required for the boat to slow to 45 km/h.

31. In Fig. 6-54, a slab of mass $m_1 = 40$ kg rests on a frictionless floor, and a block of mass $m_2 = 12$ kg rests on top of the slab. Between block and slab, the coefficient of static friction is 0.60, and the coefficient of kinetic friction is 0.40. A horizontal force $\vec{F}$ of magnitude 120 N begins to pull directly on the block, as shown. In unit-vector notation, what are the resulting accelerations of (a) the block and (b) the slab?

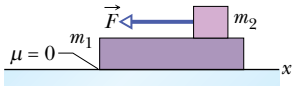

Figure 6-54 Problem 31.

32. The two blocks ($m = 16$ kg and $M = 88$ kg) in Fig. 6-55 are not attached to each other. The coefficient of static friction between the blocks is $\mu_s = 0.33$, but the surface beneath the larger block is frictionless. What is the minimum magnitude of the horizontal force $\vec{F}$ required to keep the smaller block from slipping down the larger block?

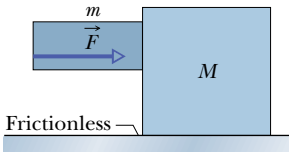

Figure 6-55 Problem 32.

33. A water droplet 4.0 mm in diameter is falling with a speed of 10 km/h at an altitude of 20 km. Another droplet 6.0 mm in diameter is falling at 25% of that speed and at 25% of that altitude. The density of air at 20 km is

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0.20 kg/m³ and that at 5.0 km is 0.70 kg/m³. Assume that the drag coefficient C is the same for the two drops. Find the ratio of the drag force on the higher drop to that on the lower drop.

34. *Continuation of Problem 7.* Now assume that Eq. 6-29 gives the magnitude of the air drag force on the typical 20 kg stone, which presents to the wind a vertical cross-sectional area of 0.040 m² and has a drag coefficient C of 0.80. Take the air density to be 1.21 kg/m³, and the coefficient of kinetic friction to be 0.80. (a) In kilometers per hour, what wind speed V along the ground is needed to maintain the stone's motion once it has started moving? Because winds along the ground are retarded by the ground, the wind speeds reported for storms are often measured at a height of 10 m. Assume wind speeds are 2.00 times those along the ground. (b) For your answer to (a), what wind speed would be reported for the storm? (c) Is that value reasonable for a high speed wind in a storm?

35. Assume Eq. 6-29 gives the drag force on a pilot plus ejection seat just after they are ejected from a plane traveling horizontally at 1300 km/h. Assume also that the mass of the seat is equal to the mass of the pilot and that the drag coefficient is that of a sky diver. Making a reasonable guess of the pilot's mass and using the appropriate v_t value from Table 6-2, estimate the magnitudes of (a) the drag force on the *pilot + seat* and (b) their horizontal deceleration (in terms of g), both just after ejection. (The result of (a) should indicate an engineering requirement: The seat must include a protective barrier to deflect the initial wind blast away from the pilot's head.)

36. Calculate the ratio of the drag force on a jet flying at 1200 km/h at an altitude of 15 km to the drag force on a prop-driven transport flying at half that speed and altitude. The density of air is 0.38 kg/m³ at 10 km and 0.67 kg/m³ at 5.0 km. Assume that the airplanes have the same effective cross-sectional area and drag coefficient C.

PRACTICE QUESTIONS

Single Correct Choice Type

1. A person, sunbathing on a warm day, is lying horizontally on the deck of a boat. Her mass is 59 kg, and the coefficient of static friction between the deck and her is 0.70. Assume that the person is moving horizontally, and that the static frictional force is the only force acting on her in this direction. What is the magnitude of the static frictional force when the boat moves with a constant velocity of +8.0 m/s?

(a) 94 N **(b)** 370 N
(c) zero N **(d)** 130 N

2. The wheels of an automobile are locked as it slides to a stop from an initial speed of 30.0 m/s. If the coefficient of kinetic friction is 0.600 and the road is horizontal, approximately how long does it take the car to stop?

(a) 4.22 s **(b)** 5.10 s
(c) 8.75 s **(d)** 10.4 s

3. A body of mass M is kept on a rough horizontal surface (friction coefficient = μ). A person is trying to pull the body by applying a horizontal force but the body is not moving. The force by the surface on A is F, where

(a) $F = Mg$
(b) $F = \mu Mg$
(c) $Mg \le F \le Mg\sqrt{1+\mu^2}$
(d) $Mg \ge F \ge Mg\sqrt{1-\mu^2}$

4. A smooth block is released at rest on a 45° incline and then slides a distance d. The time taken to slide is n times as much to slide on rough incline than on a smooth incline. The coefficient of friction is

(a) $\mu_k = 1 - \frac{1}{n^2}$ **(b)** $\mu_k = \sqrt{1 - \frac{1}{n^2}}$
(c) $\mu_s = 1 - \frac{1}{n^2}$ **(d)** $\mu_s = \sqrt{1 - \frac{1}{n^2}}$

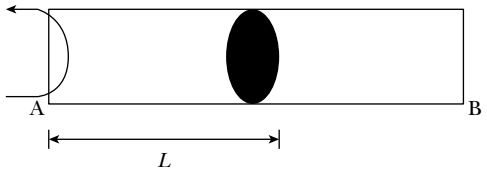
5. A boy pulls a sled of mass 5.0 kg with a rope that makes a 60.0° angle with respect to the horizontal surface of a frozen pond. The boy pulls on the rope with a force of 10.0 N, and the sled moves with constant velocity. What is the coefficient of friction between the sled and the ice?

(a) 0.09 **(b)** 0.18
(c) 0.24 **(d)** 0.12

6. A 6.00 kg box is sliding across the horizontal floor of an elevator. The coefficient of kinetic friction between the box and the floor is 0.360. Determine the kinetic frictional force that acts on the box when the elevator is stationary.

(a) 18.6 N **(b)** 22.4 N
(c) 21.2 N **(d)** 23.8 N

7. A long horizontal rod has a bead which can slide along its length and initially placed at a distance L from one end A of the rod. The rod is set in angular motion about A with constant angular acceleration α. If the coefficient of friction between the rod and the bead is μ, and gravity is neglected, then the time after which the bead starts slipping is



(a) $\sqrt{\frac{\mu}{a}}$ **(b)** $\frac{\mu}{\sqrt{\alpha}}$
(c) $\frac{L}{\sqrt{\mu\alpha}}$ **(d)** infinitesimal

8. A scooter, which is starting from rest moves with a constant acceleration for a time Δt_1, then with a constant velocity for the next Δt_2 and finally with a constant deceleration for the next Δt_3 to come to rest. A 500 N man

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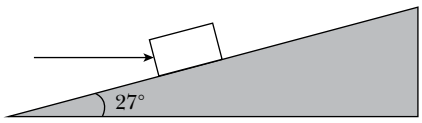
sitting on the scooter behind the driver manages to stay at rest with respect to the scooter without touching any other part. The force exerted by the seat on the man is
(a) 500 N throughout the journey.
(b) less than 500 N throughout the journey.
(c) more than 500 N throughout the journey.
(d) >500 N for time Δt_1 and Δt_3 and 500 N for Δt_2.

9. A 10 kg block is set moving with an initial speed of 6 m/s on a rough horizontal surface. If the force of friction is 20 N, approximately how far does the block travel before it stops?
(a) 1.5 m **(b)** 6 m
(c) 3 m **(d)** 9 m

10. Traveling at a speed of 16.1 m/s, the driver of an automobile suddenly locks the wheels by slamming on the brakes. The coefficient of kinetic friction between the tires and the road is 0.720. What is the speed of the automobile after 1.30 s have elapsed? Ignore the effects of air resistance.
(a) 5.2 m/s **(b)** 6.9 m/s
(c) 9.2 m/s **(d)** 5.7 m/s

11. A block of mass 2 kg rests on a rough inclined plane making an angle of 30° (with the horizontal. The coefficient of static friction between the block and the plane is 0.7. The frictional force on the block is
(a) 9.8 N
(b) $0.7 \times 9.8 \times \sqrt{3}$ N
(c) $9.8 \times \sqrt{3}$ N
(d) 0.7×9.8 N

12. A 250 N force is directed horizontally as shown in the figure to push a 29 kg box up an inclined plane at a constant speed. Determine the magnitude of the normal force, F_N, and the coefficient of kinetic friction, μ_k.



	F_N	μ_k
(a)	330 N	0.31
(b)	310 N	0.33
(c)	290 N	0.30
(d)	370 N	0.26

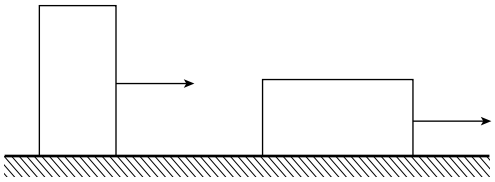
13. A body of mass m kept on the floor of a lift moving downwards is pulled horizontally. If μ is the coefficient of friction between the surface in contact, then
(a) frictional resistance offered by the floor is 2 μmg, when lift moves up with a uniform velocity of 5 m/s.
(b) frictional resistance offered by the floor is μmg, when lift moves lift moves up with uniform velocity of 3 m/s.
(c) frictional resistance offered by the floor is 4.8 μmg, when lift accelerates down with an acceleration of 4.8 m/s^2.
(d) frictional resistance offered by the floor must lie in the range $0 \le f < \infty$.

14. A boy of mass M is applying a horizontal force to slide a box of mass M' on a rough horizontal surface. The coefficient of friction between the shoes of the boy and the floor is μ and that between the box and the floor is μ'. In which of the following cases, it is certainly not possible to slide the box?
(a) $\mu < \mu', M < M'$
(b) $\mu > \mu', M < M'$
(c) $\mu < \mu', M > M'$
(d) $\mu > \mu', M > M'$

15. An ice skater is gliding horizontally across the ice with an initial velocity of +6.3 m/s. The coefficient of kinetic friction between the ice and the skate blades is 0.081, and air resistance is negligible. How much time elapses before her velocity is reduced to +2.8 m/s?
(a) 7.6 s **(b)** 6.0 s
(c) 4.4 s **(d)** 5.2 s

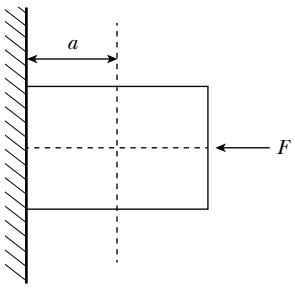
16. Large brake on a bicycle wheel and a small one will have same effect because
(a) the force of friction is independent of the area of contact.
(b) the force of friction is directly proportional to the area of contact.
(c) the force of friction is dependent on the frame of reference.
(d) the force of friction is independent on the frame of reference.

17. Two identical blocks are pulled along a rough surface as suggested in the figure. Which one of the following statements is *false*?



(a) The coefficient of kinetic friction is the same in each case.
(b) A force of the same magnitude is needed to keep each block moving.
(c) A force of the same magnitude was required to start each block moving.
(d) The magnitude of the force of kinetic friction is greater for the block on the right.

18. A block of mass m is at rest under the action of force F against a wall as shown in the figure. Which of the following statements is incorrect?



(a) $f = mg$ (where f is frictional force)
(b) $F = N$ (where N is normal force)
(c) No net torque acts on the block
(d) N will not produce torque.

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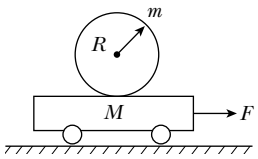
19. A block of mass 0.1 kg is held against a wall applying a horizontal force of 5 N on the block. If the co-efficient of friction between the block and the wall is 0.5, the magnitude of the frictional force acting on the block is

(a) 2.5 N **(b)** 0.98 N
(c) 4.9 N **(d)** 0.49 N

20. A block of mass m is placed on a surface with a vertical cross-section given by $y = x^3/6$. If the coefficient of friction is 0.5, the maximum height above the ground at which the block can be placed without slipping is

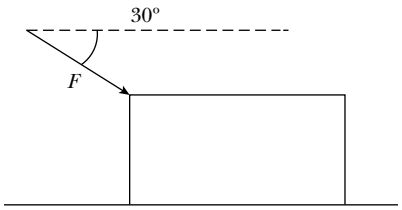
(a) $\frac{1}{6}$ m **(b)** $\frac{2}{3}$ m
(c) $\frac{1}{3}$ m **(d)** $\frac{1}{2}$ m

21. A sphere of mass m and radius R is kept on a trolley of mass M as shown in the figure. The coefficient of static and kinetic friction between the sphere and the trolley are μ_s and μ_k, respectively. The maximum horizontal force F that can be applied to the trolley for which the solid sphere does not slip is



(a) $\mu_s g\left(m+\frac{7}{2}M\right)$ **(b)** $\mu_k g\left(m+\frac{7}{2}M\right)$
(c) $\mu_s g\left(m+\frac{5}{2}M\right)$ **(d)** $\frac{9}{2}\mu_s mg$

22. A force of F = 12.0 N is applied to a 8.00 kg block at a downward angle of 30°, as shown in the figure.



The coefficient of static friction between the block and the floor is 0.700 and the coefficient of kinetic friction is 0.400. What is the magnitude of the frictional force on the block?

(a) 19 N **(b)** 59 N
(c) $12\sqrt{3}$ N **(d)** $6\sqrt{3}$ N

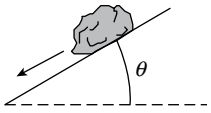
23. A crate rests on the flatbed of a truck that is initially traveling at 15 m/s on a level road. The driver applies the brakes and the truck is brought to a halt in a distance of 38 m. If the deceleration of the truck is constant, what is the minimum coefficient of friction between the crate and the truck that is required to keep the crate from sliding?

(a) 0.20 **(b)** 0.39
(c) 0.30 **(d)** 0.59

24. A 225 kg crate rests on a surface that is inclined above the horizontal at an angle of 20.0°. A horizontal force (magnitude = 535 N and parallel to the ground, not the incline) is required to start the crate moving down the incline. What is the coefficient of static friction between the crate and the incline?

(a) 0.425 **(b)** 0.592
(c) 0.665 **(d)** 0.740

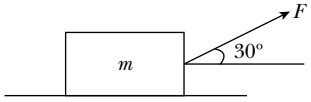
25. A 2.0 N rock slides on a frictionless inclined plane as shown in the figure. Which one of the following statements is true concerning the normal force that the plane exerts on the rock?



(a) The normal force is 0 N.
(b) The normal force is 2.0 N.
(c) The normal force is less than 2.0 N, but greater than 0 N.
(d) The normal force is greater than 2.0 N.

More than One Correct Choice Type

26. A mass m is at rest under the action of a force F, as shown in the figure, on a horizontal surface. The coefficient of friction between mass and surface is μ. Then, find the correct choice(s).



(a) The force of friction between the mass and surface is $F\sqrt{3}/2$.
(b) The force of friction between the mass and surface is μmg.
(c) Normal force is $(mg + F)$.
(d) Normal force is $(mg - F/2)$.

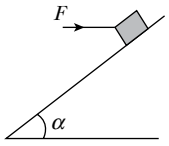
27. Consider a vehicle going on a horizontal road towards east. Neglect any force by the air. The frictional forces on the vehicle by the road

(a) is towards east if the vehicle is accelerating.
(b) is zero if the vehicle is moving with a uniform velocity.
(c) must be towards east.
(d) must be towards west.

28. Mark the correct statement(s) regarding friction.

(a) Friction force can be zero, even though the contact surface is rough.
(b) Even though there is no relative motion between surfaces, frictional force may exist between them.
(c) The expressions $f_L = \mu_s F_N$ or $f_K = \mu_k F_N$ are empirical relations.
(d) The expression $f_L = \mu_s F_N$ says that direction of f_L and F_N are the same.

29. A block of mass m in the equilibrium on a rough inclined plane with inclination α and coefficient of friction as shown in the figure ($\mu < \tan\alpha$). A force F is applied on the block which makes an angle θ with the horizontal as shown in diagram.



(a) Normal force on the block = $mg\cos\alpha + F\sin\theta$.
(b) Minimum force f to keep the block in equilibrium $F_{\min} = \frac{mg(\sin\alpha - \mu\cos\alpha)}{2\mu}\sqrt{1+\mu^2}$.
(c) $\mu = \tan\theta$.
(d) None of these.

30. In the given figure, the pulley can rotate about its fixed horizontal axis (axle) without friction. There is friction between light inextensible string and the pulley. The mass of the blocks are m_1 and m_2, where $m_2 > m_1$. The system is initially released from rest as shown. Before the block of mass m_1 touches the pulley, pick up the correct statements.

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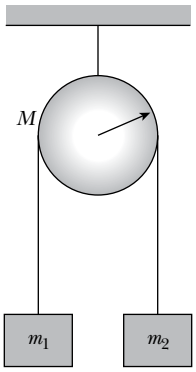
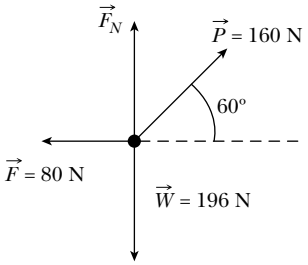


(a) The magnitude of acceleration of any small length dl of the string is constant throughout the motion.
(b) Magnitude of force exerted by the string on mass m_2 is larger as compared to that exerted by the string on mass m_1.
(c) Magnitude of acceleration of both the blocks is same.
(d) The acceleration of a small length dl of the string in contact with the block of mass m_2 remains constant.

Linked Comprehension

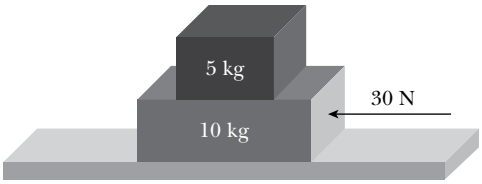
Paragraph for Questions 31 and 32: A force $\vec{P}$ pulls on a crate of mass m that is in contact with a rough surface. The figure shows the magnitudes and directions of the forces that act on the crate in this situation. $\vec{W}$ represents the weight of the crate. $\vec{F}_N$ represents the normal force on the crate, and $\vec{F}$ represents the frictional force.



31. Which statement best describes the motion of the crate?
(a) The crate must be at rest.
(b) The crate must be moving with constant velocity.
(c) The crate must be moving with constant acceleration.
(d) The crate may be either at rest or moving with constant velocity.

32. What is the magnitude of F_N the normal force on the crate?
(a) 57 N **(b)** 160 N
(c) 230 N **(d)** 80 N

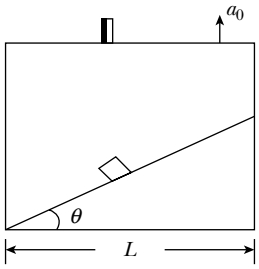
Paragraph for Questions 33 and 34: Two blocks rest on a horizontal *frictionless* surface as shown in the figure. The surface between the top and bottom blocks is roughened so that there is no slipping between the two blocks. A 30-N force is applied to the bottom block as suggested in the figure.



33. What is the magnitude of the force of static friction between the top and bottom blocks?
(a) 0 N **(b)** 20 N
(c) 30 N **(d)** 10 N

34. What is the minimum coefficient of static friction necessary to keep the top block from slipping on the bottom block?
(a) 0.05 **(b)** 0.20
(c) 0.10 **(d)** 0.30

Paragraph for Questions 35 and 36: A particle slides down a smooth inclined plane of elevation fixed in an elevator going with an acceleration a as shown in the figure. The base of the incline has a length L.



35. The time taken by the particle to reach the bottom:
(a) $t = \left(\dfrac{2L}{g\sin\theta}\right)^{1/2}$
(b) $\left(\dfrac{2L}{(g+a_0)\sin\theta\cos\theta}\right)^{1/2}$
(c) $\left(\dfrac{2L}{a_0\sin\theta}\right)^{1/2}$
(d) $\left(\dfrac{2L}{(g+a_0)\sin\theta}\right)^{1/2}$

36. If the elevator going up with constant velocity, the time taken by the particle to reach the bottom is
(a) $\left(\dfrac{2L}{g\sin\theta\cos\theta}\right)^{1/2}$
(b) $\left(\dfrac{2L}{g\sin\theta}\right)^{1/2}$
(c) $\left(\dfrac{2L}{g\cos\theta}\right)^{1/2}$
(d) none of these

Paragraph for Questions 37 and 38: A block is pulled along a rough level surface at constant speed by the force $\vec{P}$. The figure shows the free-body diagram for the block.

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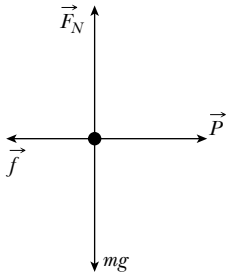
$\vec{F}_N$ represents the normal force on the block; and $\vec{f}$ represents the force of kinetic friction.



37. What is the magnitude of $\vec{F}_N$?

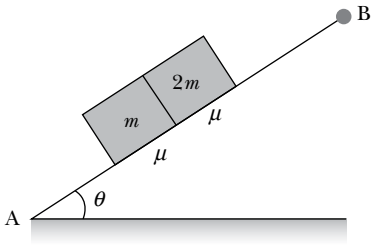
(a) $2mg$ **(b)** f
(c) P **(d)** mg

38. If the coefficient of kinetic friction, μ_k, between the block and the surface is 0.30 and the magnitude of the frictional force is 80.0 N, what is the weight of the block?

(a) 1.6 N **(b)** 160 N
(c) 4.0 N **(d)** 270 N

Matrix-Match

39. Two blocks of mass m and $2m$ are slowly just placed in contact with each other on a rough fixed inclined plane as shown in the figure initially both the blocks are rest on inclined plane. The coefficient of friction between the block together and inclined surface is m. There is no friction between both blocks. Neglect the tendency of rotation of blocks on the inclined surface.



Column I	Column II
(a) The magnitude of acceleration of both blocks are same if	**(p)** $\mu = 0$
(b) The normal reaction between both blocks is zero if	**(q)** $\mu > 0$
(c) The net reaction exerted by inclined surface on each block make same angle with inclined surface (AB) if	**(r)** $\mu = \tan\theta$
(d) The net reaction exerted by inclined surface on block of mass $2m$ is doubled that of net reaction exerted by inclined surface on block of mass m if	**(s)** $\mu < \tan\theta$

Directions for Questions 40 and 41: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

40. Types of friction and their formulas.

Column I	Column II	Column III
(I) Force of friction between two surfaces	**(i)** when there is some relative motion between them	**(J)** $f_r = \mu_r F_N$
(II) Force of friction which comes into play between two surfaces	**(ii)** when there is no relative motion between them	**(K)** $f_{s,\max} = \mu_s F_N$
(III) Force of friction which comes into play, between two surfaces	**(iii)** when a body moves through a fluid	**(L)** $f_{\min} = \mu F_N$
(IV) Opposing force which comes into play	**(iv)** while one is rolling over the other	**(M)** $f_k = \mu_k F_N$

(1) What are the characteristics of rolling friction?

(a) (IV) (iv) (J) **(b)** (IV) (ii) (L)
(c) (II) (i) (K) **(d)** (I) (ii) (L)

(2) What are the characteristics of kinetic friction?

(a) (I) (ii) (J) **(b)** (IV) (iii) (J)
(c) (II) (i) (M) **(d)** (I) (iii) (K)

(3) What are the characteristics of static friction?

(a) (II) (iii) (J) **(b)** (I) (ii) (K)
(c) (IV) (i) (L) **(d)** (II) (iii) (M)

41. Coefficients of limiting and kinetic friction of different surfaces

Column I	Column II	Column III
(I) Rough	**(i)** Coefficient of limiting friction = 0.74	**(J)** Coefficient of kinetic friction = 0.40
(II) Rough or Smooth	**(ii)** Coefficient of limiting friction = 0.78	**(K)** Coefficient of kinetic friction = 0.57
(III) Polished	**(iii)** Coefficient of limiting friction = 0.50	**(L)** Coefficient of kinetic friction = 0.42
(IV) Not polished	**(iv)** Coefficient of limiting friction = 0.70	**(M)** Coefficient of kinetic friction = 0.05

(1) What are the coefficients of friction for surface in contact – wood on wood?

(a) (I) (iii) (L) **(b)** (III) (ii) (K)
(c) (I) (iv) (J) **(d)** (I) (iii) (M)

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(2) What are the coefficients of friction for surface in contact – wood on leather?

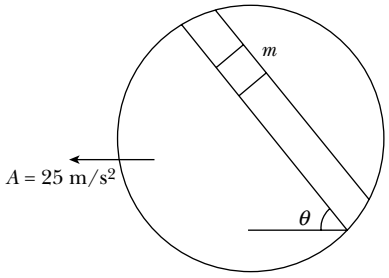
(a) (II) (iii) (J) **(b)** (III) (ii) (L)
(c) (II) (ii) (K) **(d)** (III) (i) (M)

(3) What are the coefficients of friction for surface in contact – steel on steel(hard)?

(a) (II) (iii) (M) **(b)** (I) (i) (K)
(c) (I) (iv) (J) **(d)** (III) (ii) (L)

Integer Type

42. A small block of mass m is placed in a groove carved inside a disc. The disc is placed on smooth horizontal surface and pulled with an acceleration of magnitude 25 m/s^2 as shown in the figure. Find the acceleration of block with respect to disc? Given: sin $\theta = 3/5$, cos $\theta = 4/5$, g = 10 m/s^2 and coefficient of friction between the groove and the block is $\mu = 2/5$.



43. A block moving on an inclined plane making an angle 45° with the horizontal and the coefficient of friction is μ. the force required to just push it up the inclined plane is three times the force required to just prevent it from sliding down. If we define $N = 10\mu$, then N is _____.

ANSWER KEY

Checkpoints

1. (a) zero (because there is no attempt at sliding); (b) 5 N; (c) No; (d) Yes; (e) 8 N

2. 70 N **3.** (a) 0.58; (b) 0.54 **4.** μmg **5.** 37.5

Problems

1. $F = 33$ N **2.** $\mu_k = 0.60$ **3.** (a) $a = 5.75$ m/s^2; (b) $T_1 \approx 17$ N; (c) $T_2 \approx 40$ N

4. (a) $m \approx 1.12$ kg; (b) $m = 3.56$ kg **5.** (a) 6.0 N; (b) 3.6 N; (c) 3.1 N **6.** 0.60

7. $F = 1.6 \times 10^2$ N **8.** (a) $f_k = 11$ N; (b) 0.14 m/s^2

9. 0.35 **10.** $F_N = 2.8 \times 10^2$ N

11. (a) $f_{s,\max} = 1.3 \times 10^2$ N; (b) The block, which is initially at rest, stays at rest since $F < f_{s,\max}$. Thus, it does not move; (c) $f_s = 1.1 \times 10^2$ N; (d) $F_{2,\min} = 46$ N; (e) $F_{2,\min} = 17$ N

12. (b) $F = 2.5 \times 10^7$ N **13.** 49 N **14.** (a) $V_{\min} = \sqrt{2a_r l} = \sqrt{\frac{22}{10}(\mu_2 - \mu_1)gl}$; (b) $t = \frac{v_{\min}}{a_r} = \sqrt{\frac{2ol}{11(\mu_2 - \mu_1)g}}$

15. (a) $\vec{f}_s = (17\text{ N})\hat{i}$; (b) $\vec{f}_s = (20\text{ N})\hat{i}$; (c) $\vec{f}_k = (15\text{ N})\hat{i}$

16. (a) 10.0 m/s; (b) 19.7 m/s **17.** (a) The block does not slide; (b) $\vec{F}_w = -F_N\hat{i} + f\hat{j} = -(12\text{ N})\hat{i} + (5.0\text{ N})\hat{j}$

18. 8.4 N **19.** (a) $a \approx 0.81$m/s^2; (b) $T = 18$ N

20. 9.5° **21.** $\mu_k = 0.37$

22. $\mu_k = 0.79$ **23.** (a) $F_{32} = m_3(a + \mu_k g) = 142$ N; (b) The answer here is the same as in part (a)

24. (a) $f_s = F_{\text{app}} = 2.0$ N; (b) $a = 0.091$ m/s^2

25. 4.9 kg **26.** $a = 2.5$ m/s^2 **27.** (a) $F = 89$ N; (b) $F = \frac{94\text{ N}}{\cos\theta + (0.47)\sin\theta}$; (c) $\mu_s = 25°$; (d) $F = 85$ N

28. $a = 2.6$ m/s^2 **29.** $\theta = 60°$ **30.** $t = 11$ s

31. (a) $\vec{a}_b = -(6.1\text{ m/s})^2\hat{i}$; (b) $\vec{a}_s = -(1.2\text{ m/s})^2\hat{i}$ **32.** 562 N

33. $\frac{D_1}{D_2} \approx 2.0$

34. (a) 3.2×10 km/h; (b) 6.5×10^2 km/h; (c) The result is not reasonable for a terrestrial storm. A category 5 hurricane has speeds on the order of 2.6×10^2 m/s.

35. (a) $D \approx 2 \times 10^4$ N; (b) $|a| = \frac{D}{2m} = \frac{g}{2}\left(\frac{v}{v_t}\right)^2 = 18g$ **36.** 2.3 s

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Practice Questions

Single Correct Choice Type

1. (c)	2. (b)	3. (c)	4. (a)	5. (d)
6. (c)	7. (a)	8. (d)	9. (d)	10. (b)
11. (a)	12. (d)	13. (b)	14. (a)	15. (c)
16. (a)	17. (d)	18. (d)	19. (b)	20. (a)
21. (a)	22. (d)	23. (c)	24. (c)	25. (c)

More than One Correct Choice Type

26. (a), (d)	27. (a), (b)	28. (a), (b), (c)	29. (a), (c)	30. (a), (b), (c), (d)

Linked Comprehension

31. (d)	32. (a)	33. (d)	34. (b)	35. (b)
36. (a)	37. (d)	38. (d)		

Matrix-Match

39. (a) → (p), (q)(r), (s); (b) → (p), (q)(r), (s); (c) → (p), (q)(r), (s); (d) → (p), (q)(r), (s)
40. (1) → (a); (2) → (c); (3) → (b)
41. (1) → (c); (2) → (a); (3) → (d)

Integer Type

42. 10
43. 5

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CHAPTER 7

Circular Motion

7.1 | WHAT IS PHYSICS?

There are many examples of motion on a circular path. However, for a circular motion to be uniform, it has to satisfy the following definition: A uniform circular motion is the motion of an object traveling at a constant (uniform) speed on a circular path (object covers equal distances on circumference in equal intervals of time).

The motion of an object is said to be a uniform circular motion if the object moves in a circular path with a constant speed (object covers equal distances on circumference in equal intervals of time).

A particle is in uniform circular motion if it travels in a circle or along a circular arc at constant (uniform) speed. Although the speed does not vary, *the particle is accelerating* because the velocity changes its direction.

Circular motion, like that of the spinning hammer, is another motion in two dimensions that can be analyzed using Newton's second law of motion. Examples of motion that are approximately circular include the revolution of the Earth around the Sun, a race car zooming along a circular track, an electron moving near the center of a large electromagnet, and a stone tied to the end of a string that is twirled in a circle above one's head. In all of these cases, if the object's speed is constant, we define its motion as **uniform circular motion**.

Not all circular motion is uniform. For example, a roller-coaster cart doing a loop-the-loop slows down near the top of the loop and speeds up near the bottom. Analyzing loop-the-loop motion is more complex than analyzing uniform circular motion. For this reason, we start with the ideal case of uniform circular motion.

As an example of uniform circular motion, Fig. 7-1 shows a model airplane on a guideline. The speed of the plane is the magnitude of the velocity vector $\vec{v}$ and since the speed is constant, the vectors in the drawing have the same magnitude at all points on the circle.

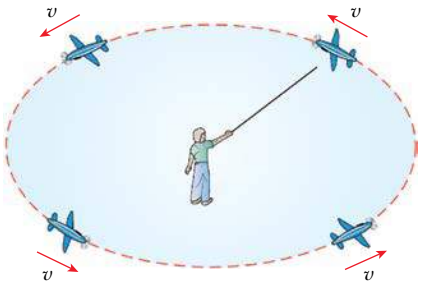

Figure 7-1 The motion of a model airplane flying at a constant speed on a horizontal circular path is an example of uniform circular motion.

Contents

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7.3 Relation between Angular Velocity and Linear Velocity
7.4 Particle in Uniform Circular Motion
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7.6 Dynamics of Uniform Circular Motion
7.7 Dynamics of Non-Uniform Circular Motion
7.8 Centrifugal Force

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7.2 | ANGULAR VARIABLES

Key Concepts

- A uniform circular motion is the motion of an object traveling at a constant (uniform) speed on a circular path (object covers equal distances on circumference in equal intervals of time).
- Angular displacement of a body is the angle in radians (i.e., degrees, revolutions) through which a point or line is rotated in a specified direction about a specified axis.
- The **average angular velocity** of the body in the time interval Δt from t_1 to t_2 is defined as

$$\omega_{avg} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t},$$

in which $\Delta\theta$ is the angular displacement that occurs during the time interval Δt.

- The instantaneous angular velocity ω is the limit of the above ratio as Δt approaches zero. Thus,

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}.$$

- The average angular acceleration of the rotating body in the interval from t_1 to t_2 is defined as

$$\alpha_{avg} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t},$$

in which $\Delta\omega$ is the change in the angular velocity that occurs during the time interval Δt.

- The (instantaneous) angular acceleration α, is the limit of the above ratio as Δt approaches zero. Thus,

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}.$$

Angular Displacement

Angular displacement of a body is defined as the angle in radians (i.e., degrees, revolutions) through which a point or line is rotated in a specified direction about a specified axis, or otherwise, angular displacement is the angle of the movement of a body in a circular path.

Now, let us understand what is meant by rotational motion. When a rigid body is rotating about its own axis, motion ceases to become a particle. It is so due to the reason that in a circular path, velocity and acceleration can change at any time. The rotation of rigid bodies or bodies which remain constant throughout the duration of rotation, over a fixed axis, is called rotational motion. The angle made by the body from its point of rest at any point in the rotational motion is the angular displacement (Fig. 7-2).

For example, when a dancer dances around a pole does one full rotation, the dancer's angular rotation is 360°. On the other hand, the dancer makes half a rotation where the displacement is 180°. It is a vector quantity, which means that it has both magnitude and direction.

It is interesting to note that the displacement of 360°, clockwise direction is different from 360° counterclockwise direction. Unit of angular displacement is radian (rad) or degree (°) and is a dimensionless quantity.

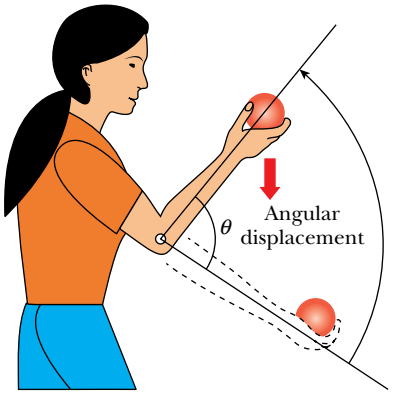

Figure 7-2 The up–down movement of an object in the hand of a girl in circular path defining an angle θ.

MEASURE OF RADIAN

Before we discuss circular motion in detail, a brief description of conversion of degrees to radians and vice versa are to be discussed, which is helpful in calculating the angles in circular motion.

Radian, with unit symbol rad, is the unit of angle in a plane. It is a constant angle.

The angle subtended at the center O of a circle of radius r, by the arc having length same as that of the radius of the circle is called one radian. If R is the radius of the circle and s is the length of the arc, then the angle θ subtended by the arc at the center O is given by (Fig. 7-3).

$$\theta \text{ (in radians)} = \frac{s}{R}$$

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Therefore, 2π radian = 360° or 1 radian = 0.159 rev.

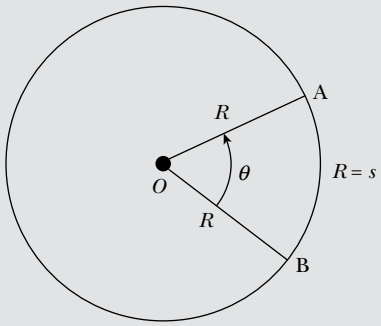

Figure 7-3 Angle θ subtended by the arc of length $s(= R)$, at the center of a circle.

Angular displacement can be measured using the relation

$$\theta = \frac{s}{R} \tag{7-1}$$

where θ is the angular displacement, s is the distance traveled by the body, and R is the radius of the circle along which it is moving. The displacement of object is the distance traveled by it around the circumference of a circle divided by its radius.

Angular Velocity

Suppose that our rotating body is at angular position θ_1 at time t_1 and at angular position θ_2 at time t_2 as in Fig. 7-4. We define the **average angular velocity** of the body in the time interval Δt from t_1 to t_2 to be

$$\omega_{avg} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}, \tag{7-2}$$

in which $\Delta\theta$ is the angular displacement that occurs during the time interval Δt (ω is the lowercase Greek letter omega).

The quantity $\Delta\theta\,(\theta_2 - \theta_1)$ is the angular displacement that occurs during the time interval $\Delta t\,(= t_2 - t_1)$. The body itself is not shown.

The **(instantaneous) angular velocity** ω, with which we shall be most concerned, is the limit of the ratio in Eq. 7-2 as Δt approaches zero. Thus,

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}. \tag{7-3}$$

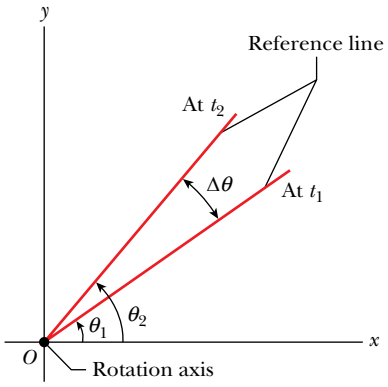

Figure 7-4 The reference line of two rigid bodies is at angular position θ_1 at time t_1 and at angular position θ_2 at a later time t_2.

If we know $\theta(t)$, we can find the angular velocity ω by differentiation.

Equations 7-2 and 7-3 hold not only for the rotating rigid body as a whole but also for *every particle of that body* because the particles are all locked together. The unit of angular velocity is commonly the radian per second (rad/s) or the revolution per second (rev/s). Another measure of angular velocity was used during at least the first three decades of rock: Music was produced by vinyl (phonograph) records that were played on turntables at "33 1/3" or "45 rpm," meaning at "33 1/3" or 45 rev/min.

If a particle moves in translation along an x axis, its linear velocity v is either positive or negative, depending on whether the particle is moving in the positive or negative direction of the axis. Similarly, the angular velocity ω of a rotating rigid body is either positive or negative, depending on whether the body is rotating counterclockwise (positive) or clockwise (negative). ("Clocks are negative" still works.) The magnitude of an angular velocity is called the **angular speed**, which is also represented with ω.

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Angular Acceleration

If the angular velocity of a rotating body is not constant, then the body has an angular acceleration. Let ω_2 and ω_1 be its angular velocities at times t_2 and t_1, respectively. The **average angular acceleration** of the rotating body in the interval from t_1 to t_2 is defined as

$$\alpha_{avg} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}, \tag{7-4}$$

in which $\Delta\omega$ is the change in the angular velocity that occurs during the time interval Δt. The **(instantaneous) angular acceleration** α, with which we shall be most concerned, is the limit of this quantity as Δt approaches zero. Thus,

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}. \tag{7-5}$$

Equations 7-4 and 7-5 also hold for *every particle of that body*. The unit of angular acceleration is commonly the radian per second-squared (rad/s^2) or the revolution per second-squared (rev/s^2) and dimensions T^{-2}.

The direction of angular acceleration is the same as that of angular velocity if the angular velocity increases whereas the direction of angular acceleration is opposed if the angular velocity decreases. The following equations can be proven easily for uniform angular acceleration:

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2, \tag{7-6}$$

$$\omega = \omega_0 + \alpha t, \tag{7-7}$$

$$\omega^2 = \omega_0^2 + 2a\theta, \tag{7-8}$$

where ω_0 and ω are the angular velocities at time $t = 0$ and $t = t$, respectively; also, α and θ represent the angular acceleration and angular displacement, respectively, at time t.

7.3 | RELATION BETWEEN ANGULAR VELOCITY AND LINEAR VELOCITY

Key Concept

- As the velocity is perpendicular to the line joining the center and particle, the angular velocity of particle with respect to the center is

$$\omega = \frac{v \sin\phi}{R}$$

That is,

$$\omega = \frac{v}{R} \quad (\text{as } \phi = 90°)$$

Let us consider a particle B that moves with a speed v along a curve (Fig. 7-5) and an observer is located at A. Let AB represent an imaginary line joining particle B to the observer A. The velocity vector makes an angle ϕ with AB at this instant.

Note that the component, $v \sin\phi$ (component of velocity perpendicular to AB), is the reason for angular displacement. That is, we infer the following:

- If only $v \cos\phi$ exists, we need not turn our heads to keep the particle in our line of sight.
- If only $v \sin\phi$ exists, then BC is the distance traveled by the particle B in time Δt.

$$BC = (v \sin\phi)\,\Delta t.$$

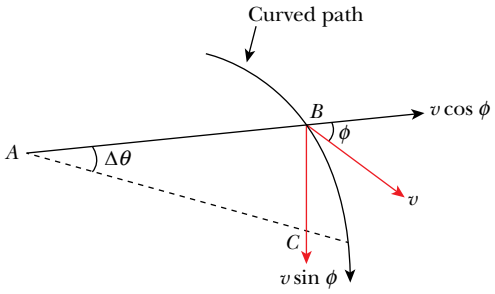

Figure 7-5 Particle B that moves with a speed v along a curved path.

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From triangle ABC, we see that for a very small angle $\Delta\theta$:

$$BC = AB\ (\Delta\theta).$$

Solving this, we get

$$\frac{\Delta\theta}{\Delta t} = \frac{v\sin\phi}{AB}.$$

In the limit $\Delta t \to 0$, $\Delta\theta/\Delta t$ becomes $d\theta/dt$, which is the instantaneous angular velocity. Therefore, we can write as follows:

$$\omega = \frac{\text{Component of velocity perpendicular to line joining}}{\text{Length of line joining the point to origin}}$$

We can derive relation between angular velocity and speed for a circular motion (Fig. 7-6).

As we know that the velocity is perpendicular to the line joining the center and particle, the angular velocity of particle with respect to the center is

$$\omega = \frac{v\sin\phi}{R} \tag{7-9}$$

That is,

$$\omega = \frac{v}{R} \quad (\text{as } \phi = 90°)$$

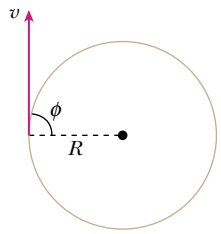

Figure 7-6 A particle that undergoes a circular motion.

CHECKPOINT 1

A stone is projected at a speed v_0 and it makes an angle θ with the horizontal. Find the angular velocity of the projectile at the time when it reaches the same level with respect to the point of projection at the time it touches the ground.

7.4 | PARTICLE IN UNIFORM CIRCULAR MOTION

Key Concepts

- The acceleration associated with uniform circular motion is called **centripetal** (meaning "center seeking") acceleration. The magnitude of this **acceleration** $\vec{a}$ is

$$a = \frac{v^2}{R},$$

where R is the radius of the circle and v is the speed of the particle.

- During this acceleration at constant speed, the particle travels the circumference of the circle (a distance of $2\pi R$) in time

$$T = \frac{2\pi R}{v},$$

where T is called the period of revolution.

A particle is in **uniform circular motion** if it travels around a circle or a circular arc at constant (*uniform*) speed. Although the speed does not vary, *the particle is accelerating* because the velocity changes in direction.

Figure 7-7 shows the relationship between the velocity and acceleration vectors at various stages during uniform circular motion. Both vectors have constant magnitude, but their directions change continuously. The velocity is always directed tangent to the circle in the direction of motion. The acceleration is always directed *radially inward*. Because of this, the acceleration associated with uniform circular motion is called a **centripetal** (meaning "center seeking") **acceleration**. As we prove next, the magnitude of this acceleration $\vec{a}$ is

$$a = \frac{v^2}{R}, \quad \text{(centripetal acceleration)} \tag{7-10}$$

where R is the radius of the circle and v is the speed of the particle.

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In addition, during this acceleration at constant speed, the particle travels the circumference of the circle (a distance of $2\pi R$) in time

$$T = \frac{2\pi R}{v}, \quad \text{(period)} \tag{7-11}$$

T is called the *period of revolution*, or simply the *period*, of the motion. It is, in general, the time for a particle to go around a closed path exactly once.

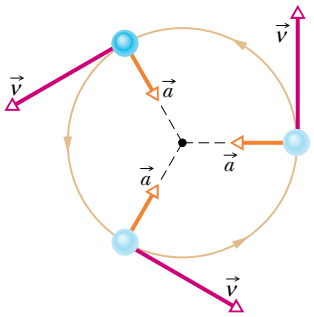

Figure 7-7 Velocity and acceleration vectors for uniform circular motion.

Proof of Eq. 7-10

To find the magnitude and direction of the acceleration for uniform circular motion, we consider Fig. 7-8. In Fig. 7-8*a*, particle p moves at constant speed v around a circle of radius r. At the instant shown, p has coordinates x_p and y_p.

Let us recall that the velocity of a moving particle is always tangent to the particle's path at the particle's position. In Fig. 7-8*a*, that means velocity is perpendicular to a radius r drawn to the particle's position. Then, the angle θ that $\vec{v}$ makes with a vertical at p equals the angle θ that radius r makes with the x axis.

The scalar components of $\vec{v}$ are shown in Fig. 7-8*b*. With them, we can write the velocity $\vec{v}$ as

$$\vec{v} = v_x\hat{\mathrm{i}} + v_y\hat{\mathrm{j}} = (-v\sin\theta)\hat{\mathrm{i}} + (v\cos\theta)\hat{\mathrm{j}}. \tag{7-12}$$

Now, using the right triangle in Fig. 7-8*a*, we can replace $\sin\theta$ with y_p/r and $\cos\theta$ with x_p/r to write

$$\vec{v} = \left(-\frac{vy_p}{r}\right)\hat{\mathrm{i}} + \left(-\frac{vx_p}{r}\right)\hat{\mathrm{j}}. \tag{7-13}$$

To find the acceleration $\vec{a}$ of particle p, we must take the time derivative of this equation. Noting that speed v and radius r do not change with time, we obtain

$$\vec{a} = \frac{d\vec{v}}{dt} = \left(-\frac{v}{r}\frac{dy_p}{dt}\right)\hat{\mathrm{i}} + \left(-\frac{v}{r}\frac{dx_p}{dt}\right)\hat{\mathrm{j}}. \tag{7-14}$$

Now, note that the rate dy_p/dt at which y_p changes is equal to the velocity component v_y.

Similarly, $dx_p/dt = v_x$, and, again from Fig. 7-8*b*, we see that $v_x = -v\sin\theta$ and $v_y = v\cos\theta$. Making these substitutions in Eq. 7-14, we find

$$\vec{a} = \left(-\frac{v^2}{r}\cos\theta\right)\hat{\mathrm{i}} + \left(-\frac{v^2}{r}\sin\theta\right)\hat{\mathrm{j}}. \tag{7-15}$$

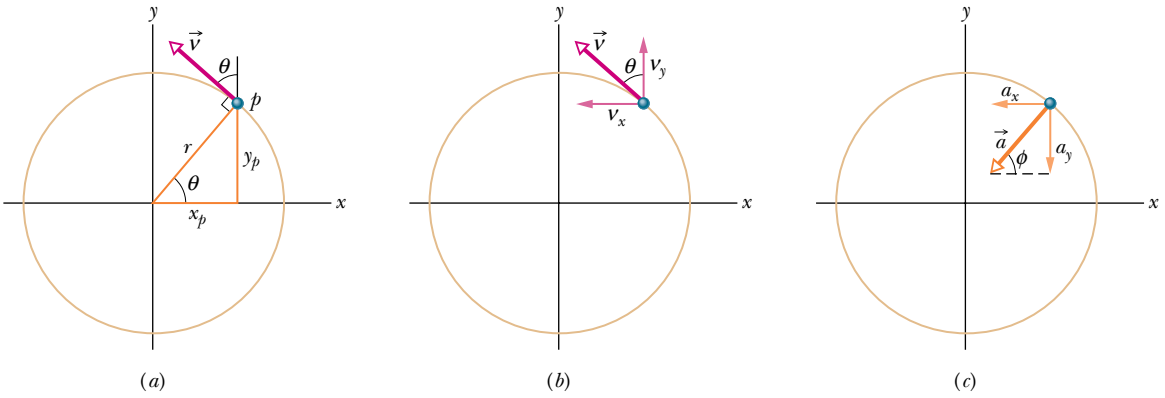

Figure 7-8 Particle p moves in counterclockwise uniform circular motion. (*a*) Its position and velocity $\vec{v}$ at a certain instant. (*b*) Velocity $\vec{v}$. (*c*) Acceleration a.

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This vector and its components are shown in Fig. 7-8*c*. Now, we find

$$a = \sqrt{a_x^2 + a_y^2} = \frac{v^2}{r}\sqrt{(\cos\theta)^2 + (\sin\theta)^2} = \frac{v^2}{r}\sqrt{1} = \frac{v^2}{r}$$

as we wanted to prove. To orient $\vec{a}$, we find the angle ϕ shown in Fig. 7-8*c*:

$$\tan\phi = \frac{a_y}{a_x} = \frac{-(v^2/r)\sin\theta}{-(v^2/r)\cos\theta} = \tan\theta.$$

Thus, $\phi = \theta$, which means that $\vec{a}$ is directed along the radius r of Fig. 7-8*a*, toward the circle's center, as we wanted to prove.

Geometrical Proof of Eq. 7-10

Let us recall that a body that moves in a circular motion (of radius r) at constant speed (v) is accelerated. This acceleration is at right angles to the direction of motion (toward the center of the circle) and of magnitude v^2/r. The direction of acceleration is derived from symmetry.

- If the acceleration pointed out of the plane of the circle, the body would leave the plane of the circle but it does not occur and hence the acceleration is not pointed out of the plane of the circle.
- If the acceleration is pointed in any direction other than the perpendicular (left or right), the body would speed up or slow down but it does not occur.

Now, for the magnitude, as shown in Fig. 7-9, let us consider the distance traveled by the body over a small time increment Δt:

We can calculate the arc length s of the both the distances traveled (Distance = Rate × Time = $v\Delta t$) and using the definition of a radian [Arc = Radius × Angle (in rad) = $r\Delta\theta$]:

$$s = v\Delta t; \quad s = r\Delta\theta.$$

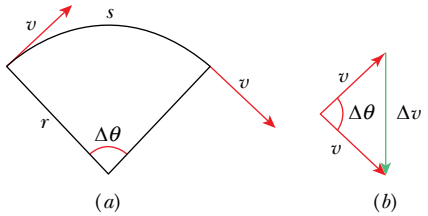

Figure 7-9 Distance traveled by a body over a small time increment Δt.

Therefore,

$$r\Delta\theta = v\Delta t.$$

That is,

$$\frac{\Delta\theta}{\Delta t} = \frac{v}{r}.$$

Therefore,

$$\frac{d\theta}{dt} = \frac{v}{r}.$$

That is,

$$d\theta = \frac{v\,dt}{r}. \tag{7-16}$$

The angular velocity of the object is thus v/r (in radians per unit of time.) Figure 7-9*b* is drawn by placing the tails of the two velocity vectors (v) together. It is to be noted that $\Delta\theta$ is the same in both the diagrams. Now,

$$\sin\left(\frac{\Delta\theta}{2}\right) = \left(\frac{\Delta v/2}{v}\right).$$

That is,

$$\Delta v = 2v\sin\frac{\Delta\theta}{2}.$$

Therefore,

$$\frac{\Delta v}{\Delta\theta} = 2v \times \frac{\sin(\Delta\theta/2)}{\Delta\theta} = v \times \frac{\sin(\Delta\theta/2)}{\Delta\theta/2}.$$

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From this, we get

$$\frac{dv}{d\theta} = \lim_{\Delta\theta\to 0} v\frac{\sin(\Delta\theta/2)}{\Delta\theta/2} = v\lim_{\Delta\theta\to 0}\frac{[\cos(\Delta\theta/2)]\times(1/2)}{1/2} = v. \qquad \text{(using L'Hospital rule)}$$

Therefore,

$$\frac{dv}{dt} = \frac{dv}{d\theta}\times\frac{d\theta}{dt} = v\times\frac{v}{r} = \frac{v^2}{r}$$

If ω is the angular velocity of the body, $v = r\omega$ and we can also write as

$$a = \omega^2 R$$

CHECKPOINT 2

A body moves along a circular path and at constant speed in a horizontal xy plane, with the center at the origin. Assuming the object is located at $x = -2$ m and its velocity is given as $-(4\text{ m/s})\hat{i}$, find the (a) velocity of the body and (b) acceleration of the body at $y = 2$ m.

SAMPLE PROBLEM 7.01

Centripetal acceleration of a car around a curve

Find the magnitude of the centripetal acceleration of a car following a curve of radius 500 m at a speed of 25.0 m/s (about 90 km/h). Compare the acceleration with that due to gravity for this fairly gentle curve taken at highway speed.

KEY IDEA

As the values of v and R are given, the expression $a = v^2/R$ is the most suitable to use.

Calculation: Substituting the values of v and r into $a = v^2/R$ gives

$$a = \frac{v^2}{R} = \frac{(25.0\text{ m/s})^2}{500\text{ m}} = 1.25\text{ m/s}^2$$

On comparing this with the acceleration due to gravity ($g = 9.80\text{ m/s}^2$), we consider the ratio of a/g:

$$\frac{a}{g} = \frac{1.25\text{ m/s}^2}{9.80\text{ m/s}^2} = 0.128.$$

Thus, $a = 0.128g$, which is apparent, particularly if you were not wearing a seat belt.

SAMPLE PROBLEM 7.02

Centripetal acceleration of pilot on horizontal circular turn

IAF fighter pilots have long worried about taking a turn too tightly. As a pilot's body undergoes centripetal acceleration, with the head toward the center of curvature, the blood pressure in the brain decreases, leading to loss of brain function.

There are several warning signs. When the centripetal acceleration is $2g$ or $3g$, the pilot feels heavy. At about $4g$, the pilot's vision switches to black and white and narrows to "tunnel vision." If that acceleration is sustained or increased, vision ceases and, soon after, the pilot is unconscious–a condition known as g-LOC for "g-induced loss of consciousness."

What is the magnitude of the acceleration, in g units, of a pilot whose aircraft enters a horizontal circular turn with a velocity of $\vec{v}\hat{i} = (400\hat{i} + 300\hat{j})$ m/s and 25 s later leaves the turn with a velocity of $\vec{v}\hat{j} = (-400\hat{i} - 300\hat{j})$ m/s? Assume speed is constant.

KEY IDEA

We assume the turn is made with uniform circular motion. Then the pilot's acceleration is centripetal and has magnitude a given by Eq. 7-10 ($a = v^2/R$), where R is

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the circle's radius. Also, the time required to complete a full circle is the period given by Eq. 7-11 ($T = 2\pi R/v$).

Calculation: Because we do not know radius R, let's solve Eq. 7-11 for R and substitute into Eq. 7-10. We find

$$a = \frac{2\pi v}{T}$$

Speed v here is the (constant) magnitude of the velocity during the turning. Let's substitute the components of the initial velocity into:

$$v = \sqrt{v_x^2 + v_y^2}$$

$$v = \sqrt{(400\text{ m/s})^2 + (300\text{ m/s})^2} = 500\text{ m/s}$$

To find the period T of the motion, first note that the final velocity is the reverse of the initial velocity. This means the aircraft leaves on the opposite side of the circle from the initial point and must have completed half a circle in the given 25 s. Thus, a full circle would have taken $T = 50$ s.

Substituting these values into our equation for a, we find

$$a = \frac{2\pi(500\text{ m/s})}{50\text{ s}} = 62.83\text{ m/s}^2 \approx 6.4g \qquad \text{(Answer)}$$

Thus, the pilot may have fainted.

7.5 | PARTICLE IN NON-UNIFORM CIRCULAR MOTION

Key Concepts

- Change in direction is due to the radial acceleration (centripetal acceleration), which is given by

$$\vec{a}_c = \vec{a}_r = \frac{v^2}{r}$$

- The non-uniform circular motion basically involves a change in speed. This change is accounted by the tangential acceleration, which results due to a tangential force and acts along the direction of velocity:

$$a_t = \frac{d|\vec{v}|}{dt} = \frac{d(\omega R)}{dt} = \frac{R d\omega}{dt} = R\alpha$$

- If R_C is radius of curvature of the imaginary circle touching the curve at a point, then

$$R_C = \frac{v^2}{a_r} = \frac{v^2}{a_\perp}$$

The speed of a particle under circular motion is not constant. A change in speed means that unequal length of arc (s) is covered in equal time intervals. It further means that the change in the velocity (v) of the particle is not limited to change in direction as in the case of uniform circular motion.

Radial or centripetal acceleration: Change in direction is due to the radial acceleration (centripetal acceleration), which is given by

$$\vec{a}_c = \vec{a}_r = \frac{v^2}{r}$$

Tangential acceleration: The non-uniform circular motion basically involves a change in speed. This change is accounted by the tangential acceleration, which results due to a tangential force and which acts along the direction of velocity:

$$a_t = \frac{d|\vec{v}|}{dt} = \frac{d(\omega R)}{dt} = \frac{R d\omega}{dt} = R\alpha \qquad \text{(Tangential acceleration)}$$

where α is the angular acceleration along the direction of velocity vector $\vec{v}$. We can recall from previos section that under uniform circular motion the acceleration has only the radial component and no tangential component. The total acceleration vector $\vec{a}$ can be written as the vector sum of the component vectors:

$$\vec{a} = \vec{a}_r + \vec{a}_t$$

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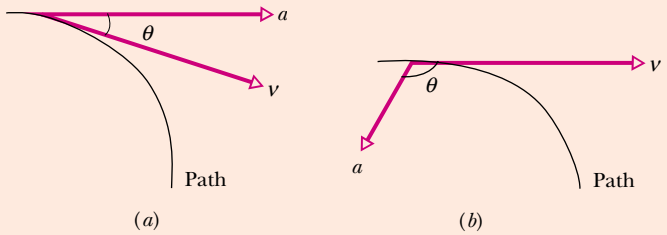
CHECKPOINT 3

Among the two figures shown here, choose the correct one in which the depicted situation is possible.



PROBLEM-SOLVING TACTICS

Tactic 1: Magnitude of acceleration is same as magnitude of rate of change of velocity.

$$|\vec{a}| = \frac{|d\vec{v}|}{dt} = \sqrt{a_t^2 + a_r^2}$$

or

$$\frac{|d\vec{v}|}{dt} = \sqrt{(\alpha R)^2 + (\omega^2 R)^2}$$

Tactic 2: Rate of change of magnitude of velocity is called tangential acceleration:

$$\vec{a}_t = \frac{d|\vec{v}|}{dt}$$

Here, $\vec{a}_t$ is not the magnitude of acceleration. Hence, $|\vec{a}|$ is the magnitude of rate of change of velocity and $\vec{a}_t$ is the rate of change of magnitude of velocity.

SAMPLE PROBLEM 7.03

A speeding truck at constant rate

A truck increases its speed at constant rate of 0.1 m/s² and it passes over a semicircular bridge (Fig. 7-10*a*) of radius 40 m. At the time the car reaches the top of the bridge, its speed is 2 m/s. What are the magnitude and direction of the total acceleration vector at this moment?

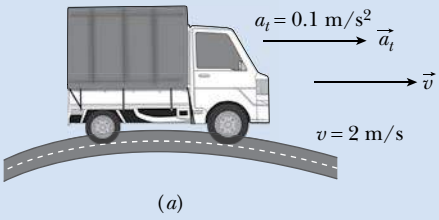


Figure 7-10 (*a*) Truck accelerating over a semicircular bridge.

KEY IDEA

The truck moves along a curved path with variable speed. Thus, the truck has both tangential and radial acceleration. The radial acceleration is given by $a_r = v^2/R$, with $v = 2$ m/s and $R = 40$ m. The radial acceleration vector is directed straight downward, and the tangential acceleration vector has magnitude 0.1 m/s² and it is horizontal.

Calculations: Let us calculate the radial acceleration:

$$a_r = \frac{v^2}{R} = \frac{(2\text{ m/s})^2}{40\text{ m}} = 0.1\text{ m/s}$$

Now,

$$|\vec{a}| = \sqrt{a_t^2 + a_r^2} = \sqrt{2}(0.1)\text{ m/s}^2$$

which is the magnitude of $\vec{a}$. Now, the angle ϕ (Fig. 7-10*b*) between $\vec{a}$ and the horizontal is given by

$$\phi = \tan^{-1}\left(\frac{a_r}{a_t}\right) = 45°$$

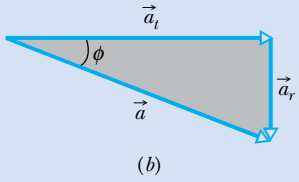


Figure 7-10 (*b*) Free-body diagram of the accelerating truck on semicircular bridge.

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Radius of Curvature

When a particle undergoes any general motion along any arbitrary path, we can define a geometrical quantity called radius of curvature. Let us say a particle going along any arbitrary path passes through point A. When the particle crosses this point, it satisfies centripetal condition of moving on an imaginary circle of radius R_C, then we say radius of curvature of the path at A is R_C (Fig. 7-11).

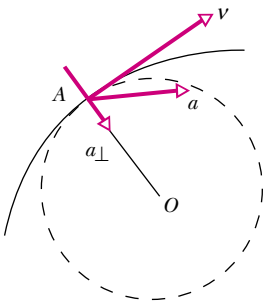

Figure 7-11 Path of a particle moving along an imaginary circle satisfying centripetal condition.

If a_r represents the component of acceleration perpendicular to velocity, then

$$a_r = \frac{v^2}{R_C}$$

where R_C is radius of curvature of the imaginary circle touching the curve at A and

$$R_C = \frac{v^2}{a_r} = \frac{v^2}{a_\perp}$$

Hence, $$a_r = a_\perp$$

In general, R_C at any point on the path can be defined as

$$R_C = \frac{(\text{Speed})^2}{\text{Component of acceleration perpendicular to velocity}}$$

CHECKPOINT 4

If a missile is propelled horizontally with 20 m/s from some height, what is the value of R_C at $t = 2$ s?

7.6 | DYNAMICS OF UNIFORM CIRCULAR MOTION

Key Concepts

- If a particle moves in a circle or a circular arc of radius R at constant speed v, the particle is said to be in uniform circular motion. It then has a centripetal acceleration $\vec{a}$ with magnitude given by

$$a_r = \frac{v^2}{R}.$$

- This acceleration is due to a net centripetal force on the particle, with magnitude given by

$$F = \frac{mv^2}{R},$$

where m is the particle's mass. The vector quantities $\vec{a}$ and $\vec{F}$ are directed toward the center of curvature of the particle's path.

We recall that when a body moves in a circle (or a circular arc) at constant speed v, it is said to be in uniform circular motion. Also recall that the body has a centripetal acceleration (directed toward the center of the circle) of constant magnitude given by Eq. 7-10.

Here are two examples:

1. ***Rounding a curve in a car.*** You are sitting in the center of the rear seat of a car moving at a constant high speed along a flat road. When the driver suddenly turns left, rounding a corner in a circular arc, you slide across the seat toward the right and then jam against the car wall for the rest of the turn. What is going on?

   While the car moves in the circular arc, it is in uniform circular motion; that is, it has an acceleration that is directed toward the center of the circle. By Newton's second law, a force must cause this acceleration. Moreover, the force must also be directed toward the center of the circle. Thus, it is a **centripetal force**, where the adjective indicates the direction. In this example, the centripetal force is a frictional force on the tires from the road; it makes the turn possible.

   If you are to move in uniform circular motion along with the car, there must also be a centripetal force on you. However, apparently the frictional force on you from the seat was not great enough to make you go in

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a circle with the car. Thus, the seat slid beneath you, until the right wall of the car jammed into you. Then its push on you provided the needed centripetal force on you, and you joined the car's uniform circular motion.

2. ***Orbiting Earth.*** This time you are a passenger in the space shuttle *Atlantis.* As it and you orbit Earth, you float through your cabin. What is going on?

Both you and the shuttle are in uniform circular motion and have accelerations directed toward the center of the circle. Again by Newton's second law, centripetal forces must cause these accelerations. This time the centripetal forces are gravitational pulls (the pull on you and the pull on the shuttle) exerted by Earth and directed radially inward, toward the center of Earth.

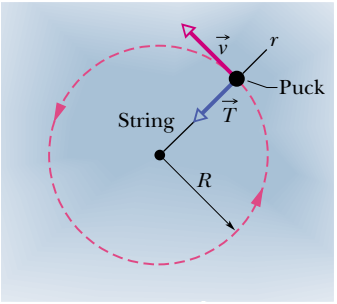
In both car and shuttle you are in uniform circular motion, acted on by a centripetal force—yet your sensations in the two situations are quite different. In the car, jammed up against the wall, you are aware of being compressed by the wall. In the orbiting shuttle, however, you are floating around with no sensation of any force acting on you. Why this difference?

The difference is due to the nature of the two centripetal forces. In the car, the centripetal force is the push on the part of your body touching the car wall. You can sense the compression on that part of your body. In the shuttle, the centripetal force is Earth's gravitational pull on every atom of your body. Thus, there is no compression (or pull) on any one part of your body and no sensation of a force acting on you. (The sensation is said to be one of "weightlessness," but that description is tricky. The pull on you by Earth has certainly not disappeared and, in fact, is only a little less than it would be with you on the ground.)

Another example of a centripetal force is shown in Fig. 7-12. There a hockey puck moves around in a circle at constant speed v while tied to a string looped around a central peg. This time the centripetal force is the radially inward pull on the puck from the string. Without that force, the puck would slide off in a straight line instead of moving in a circle.

Note again that a centripetal force is not a new kind of force. The name merely indicates the direction of the force. It can, in fact, be a frictional force, a gravitational force, the force from a car wall or a string, or any other force. For any situation:

A centripetal force accelerates a body by changing the direction of the body's velocity without changing the body's speed.



The puck moves in uniform circular motion only because of a toward-the-center force.

Figure 7-12 An overhead view of a hockey puck moving with constant speed v in a circular path of radius R on a horizontal frictionless surface. The centripetal force on the puck is $\vec{T}$, the pull from the string, directed inward along the radial axis r extending through the puck.

From Newton's second law and Eq. 7-10 ($a = v^2/R$), we can write the magnitude F of a centripetal force (or a net centripetal force) as

$$F = m\frac{v^2}{R} \quad \text{(magnitude of centripetal force).} \qquad (7\text{-}17)$$

Because the speed v here is constant, the magnitudes of the acceleration and the force are also constant.

However, the directions of the centripetal acceleration and force are not constant; they vary continuously so as to always point toward the center of the circle. For this reason, the force and acceleration vectors are sometimes drawn along a radial axis r that moves with the body and always extends from the center of the circle to the body, as in Fig. 7-12. The positive direction of the axis is radially outward, but the acceleration and force vectors point radially inward.

CHECKPOINT 5

As every amusement park fan knows, a Ferris wheel is a ride consisting of seats mounted on a tall ring that rotates around a horizontal axis. When you ride in a Ferris wheel at constant speed, what are the directions of your acceleration $\vec{a}$ and the normal force $\vec{F}_N$ on you (from the always upright seat) as you pass through (a) the highest point and (b) the lowest point of the ride? (c) How does the magnitude of the acceleration at the highest point compare with that at the lowest point? (d) How do the magnitudes of the normal force compare at those two points?

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PROBLEM-SOLVING TACTICS

Tactic 3: Problems related to circular motion can be solved using the following steps:

1. First, the plane of circular motion needs to be identified.
2. The radius needs to be calculated by locating the center.
3. Draw a free-body diagram and resolve forces along the three following directions:

(i) *In the plane along radial direction:* Add the force assuming radially inward direction as positive. Therefore,

$$\sum F_r = \frac{mv^2}{R}$$

We know that $F = ma$; therefore,

$$F = \frac{mv^2}{R}$$

(ii) *In the plane along tangential direction:*

$\sum F_t$

- $\sum F_t = 0$ (uniform circular motion) constant ω
- $\sum F_t \neq 0$ (Non uniform circular motion) changing ω

(iii) *Perpendicular to the plane of circular motion:* Since plane cannot be accelerating, then $\sum F_1 = 0$. If plane accelerates then actually it will not be circular motion from ground frame.

SAMPLE PROBLEM 7.04

Vertical circular loop, Diavolo

Largely because of riding in cars, you are used to horizontal circular motion. Vertical circular motion would be a novelty. In this sample problem, such motion seems to defy the gravitational force.

In a 1901 circus performance, Allo "Dare Devil" Diavolo introduced the stunt of riding a bicycle in a loop-the-loop (Fig. 7-13*a*). Assuming that the loop is a circle with radius $R = 2.7$ m, what is the least speed v that Diavolo and his bicycle could have at the top of the loop to remain in contact with it there?

KEY IDEA

We can assume that Diavolo and his bicycle travel through the top of the loop as a single particle in uniform circular motion. Thus, at the top, the acceleration $\vec{a}$ of this particle must have the magnitude $a = v^2/R$ given by Eq. 7-10 and be directed downward, toward the center of the circular loop.

Calculations: The forces on the particle when it is at the top of the loop are shown in the free-body diagram of Fig. 7-13*b*. The gravitational force is $\vec{F}_g$ downward along a y axis; so is the normal force $\vec{F}_N$ on the particle from the loop (the loop can push down, not pull up); so also is the

FOREPAUGH & SELLS BROTHERS
ENORMOUS SHOWS UNITED
DIAVOLO
LOOPING THE LOOP
THE EXTREME AND ABSOLUTE LIMIT OF SENSATIONALISM REACHED AT LAST. BEYOND THE TREMENDOUSLY TERRIBLE TEMERITY AND ILLIMITABLE, INIMITABLE INTREPIDITY OF DIAVOLO, NO MAN MAY GO.

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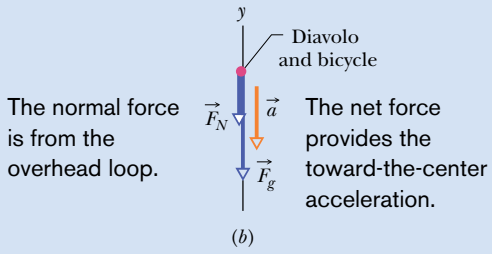
(*a*)



(*b*)

Figure 7-13 (*a*) Contemporary advertisement for Diavolo and (*b*) free-body diagram for the performer at the top of the loop.

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centripetal acceleration of the particle. Thus, Newton's second law for y components ($F_{net,y} = ma_y$) gives us

$$-F_N - F_g = m(-a)$$

and

$$-F_N - mg = m\left(-\frac{v^2}{R}\right). \qquad (7\text{-}18)$$

If the particle has the *least speed* v needed to remain in contact, then it is on the *verge of losing contact* with the loop (falling away from the loop), which means that $F_N = 0$ at the top of the loop (the particle and loop touch but without any normal force). Substituting 0 for F_N in Eq. 7-18, solving for v, and then substituting known values give us

$$v = \sqrt{gR} = \sqrt{(9.8\ \text{m/s}^2)(2.7\ \text{m})} = 5.1\ \text{m/s}. \qquad \text{(Answer)}$$

Comments: Diavolo made certain that his speed at the top of the loop was greater than 5.1 m/s so that he did not lose contact with the loop and fall away from it. Note that this speed requirement is independent of the mass of Diavolo and his bicycle. Had he feasted on, say, pierogis before his performance, he still would have had to exceed only 5.1 m/s to maintain contact as he passed through the top of the loop.

SAMPLE PROBLEM 7.05

A rotor in an amusement park, showing the forces on a rider

Even some seasoned roller-coaster riders blanch at the thought of riding the Rotor, which is essentially a large, hollow cylinder that is rotated rapidly around its central axis (Fig. 7-14). Before the ride begins, a rider enters the cylinder through a door on the side and stands on a floor, up against a canvas-covered wall. The door is closed, and as the cylinder begins to turn, the rider, wall, and floor move in unison. When the rider's speed reaches some predetermined value, the floor abruptly and alarmingly falls away. The rider does not fall with it but instead is pinned to the wall while the cylinder rotates, as if an unseen (and somewhat unfriendly) agent is pressing the body to the wall. Later, the floor is eased back to the rider's feet, the cylinder slows, and the rider sinks a few centimeters to regain footing on the floor. (Some riders consider all this to be fun.)

Suppose that the coefficient of static friction μ_s between the rider's clothing and the canvas is 0.40 and that the cylinder's radius R is 2.1 m.

(a) What minimum speed v must the cylinder and rider have if the rider is not to fall when the floor drops?

KEY IDEAS

1. The gravitational force $\vec{F}_g$ on the rider tends to slide her down the wall, but she does not move because a frictional force from the wall acts upward on her (Fig. 7-14).
2. If she is to be on the verge of sliding down, that upward force must be a *static* frictional force $\vec{f}_s$ at its maximum value $\mu_s F_N$, where F_N is the magnitude of the normal force on her from the cylinder (Fig. 7-14).
3. This normal force is directed horizontally toward the central axis of the cylinder and is the centripetal force that causes the rider to move in a circular path, with centripetal acceleration of magnitude $a = v^2/R$ and directed toward the center of the circle.

We want speed v in that last expression, for the condition that the rider is on the verge of sliding.

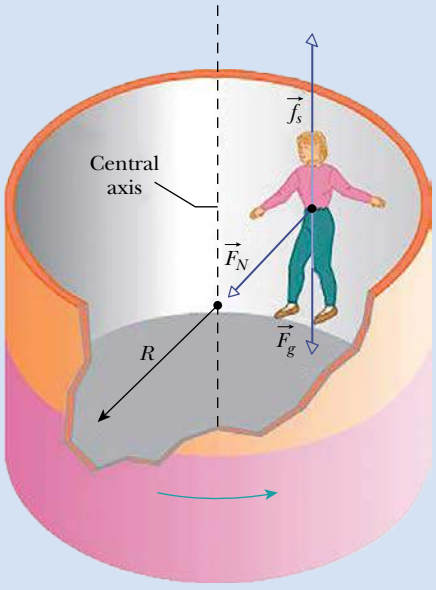

Figure 7-14 A rotor in an amusement park, showing the forces on a rider. The centripetal force is the normal force $\vec{F}_N$ with which the wall pushes inward on the rider.

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Vertical calculations: We first place a vertical y axis through the rider, with the positive direction upward. We can then apply Newton's second law to the rider, writing it for y components ($F_{net,y} = ma_y$) as

$$f_s - mg = m(0),$$

where m is the rider's mass and mg is the magnitude of $\vec{F}_g$. Because the rider is on the verge of sliding, we substitute the maximum value $\mu_s F_N$ for f_s in this equation, getting

$$\mu_s F_N - mg = 0$$

or

$$F_N = \frac{mg}{\mu_s}. \qquad (7\text{-}19)$$

Radial calculations: Next, we place a radial r axis through the rider, with the positive direction outward. We can then write Newton's second law for components along that axis as

$$F_N = \frac{mg}{\mu_s}. \qquad (7\text{-}20)$$

Substituting Eq. 7-19 for F_N and then solving for v, we find

$$v = \sqrt{\frac{gR}{\mu_s}} = \sqrt{\frac{(9.8\ \text{m/s}^2)(2.1\ \text{m})}{0.40}}$$
$$= 7.17\ \text{m/s} \approx 7.2\ \text{m/s} \qquad \text{(Answer)}$$

Note that the result is independent of the rider's mass; it holds for anyone riding the Rotor, from a child to a sumo wrestler, which is why no one has to "weigh in" to ride the Rotor.

(b) If the rider's mass is 49 kg, what is the magnitude of the centripetal force on her?

Combining results: According to Eq. 7-20,

$$F_N = \frac{v^2}{R} = (49\ \text{kg})\frac{(7.17\ \text{m/s})^2}{2.1\ \text{m}}$$
$$\approx 1200\ \text{N} \qquad \text{(Answer)}$$

Learn: Although this force is directed toward the central axis, the rider has an overwhelming sensation that the force pinning her against the wall is directed radially outward. Her sensation stems from the fact that she is in a non-inertial frame (she and it are accelerating). As measured from such frames, forces can be illusionary. The illusion is part of the Rotor's attraction.

SAMPLE PROBLEM 7.06

Car in flat circular turn

Upside-down racing: A modern race car is designed so that the passing air pushes down on it, allowing the car to travel much faster through a flat turn in a Grand Prix without friction failing. This downward push is called *negative lift.* Can a race car have so much negative lift that it could be driven upside down on a long ceiling, as done fictionally by a sedan in the first *Men in Black* movie?

Figure 7-15*a* represents a Grand Prix race car of mass $m = 600$ kg as it travels on a flat track in a circular arc of radius $R = 100$ m. Because of the shape of the car and the wings on it, the passing air exerts a negative lift $\vec{F}_L$

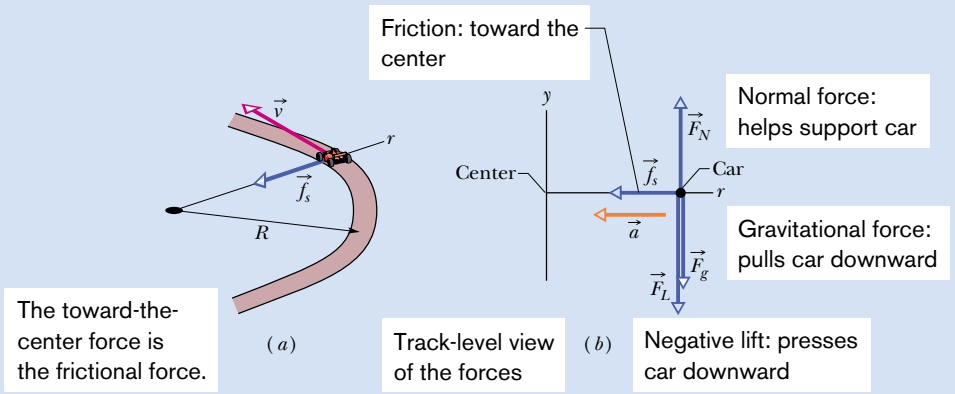

Figure 7-15 (*a*) A race car moves around a flat curved track at constant speed v. The frictional force $\vec{f}_s$ provides the necessary centripetal force along a radial axis r. (*b*) A free-body diagram (not to scale) for the car, in the vertical plane containing r.

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downward on the car. The coefficient of static friction between the tires and the track is 0.75. (Assume that the forces on the four tires are identical.)

(a) If the car is on the verge of sliding out of the turn when its speed is 28.6 m/s, what is the magnitude of the negative lift $\vec{F}_L$ acting downward on the car?

KEY IDEAS

1. A centripetal force must act on the car because the car is moving around a circular arc; that force must be directed toward the center of curvature of the arc (here, that is horizontally).
2. The only horizontal force acting on the car is a frictional force on the tires from the road. So, the required centripetal force is a frictional force.
3. Because the car is not sliding, the frictional force must be a *static* frictional force $\vec{f}_s$ (Fig. 7-15*a*).
4. Because the car is on the verge of sliding, the magnitude f_s is equal to the maximum value $f_{s,\max} = \mu_s F_N$, where F_N is the magnitude of the normal force $\vec{F}_N$ acting on the car from the track.

Radial calculations: The frictional force $\vec{f}_s$ is shown in the free-body diagram of Fig. 7-15*b*. It is in the negative direction of a radial axis r that always extends from the center of curvature through the car as the car moves. The force produces a centripetal acceleration of magnitude v^2/R. We can relate the force and acceleration by writing Newton's second law for components along the r axis ($F_{\text{net},r} = ma_r$) as

$$-f_s = m\left(-\frac{v^2}{R}\right). \tag{7-21}$$

Substituting $f_{s,\max} = \mu_s F_N$ for f_s leads us to

$$\mu_s F_N = m\left(\frac{v^2}{R}\right). \tag{7-22}$$

Vertical calculations: Next, let's consider the vertical forces on the car. The normal force $\vec{F}_N$ is directed up, in the positive direction of the y axis in Fig. 7-15*b*. The gravitational force $\vec{F}_g = m\vec{g}$ and the negative lift $\vec{F}_L$ are directed down. The acceleration of the car along the y axis is zero. Thus we can write Newton's second law for components along the y axis ($F_{\text{net},y} = ma_y$) as

$$F_N - mg - F_L = 0,$$

or

$$F_N = mg + F_L. \tag{7-23}$$

Combining results: Now we can combine our results along the two axes by substituting Eq. 7-23 for F_N in Eq. 7-22. Doing so and then solving for F_L lead to

$$\begin{aligned} F_L &= m\left(\frac{v^2}{\mu_s R} - g\right) \\ &= (600\ \text{kg})\left(\frac{(28.6\ \text{m/s})^2}{(0.75)(100\ \text{m})} - 9.8\ \text{m/s}^2\right) \\ &= 663.7\ \text{N} \approx 660\ \text{N}. \end{aligned} \tag{Answer}$$

(b) The magnitude F_L of the negative lift on a car depends on the square of the car's speed v^2, just as the drag force does. Thus, the negative lift on the car here is greater when the car travels faster, as it does on a straight section of track. What is the magnitude of the negative lift for a speed of 90 m/s?

KEY IDEA

F_L is proportional to v^2.

Calculations: Thus we can write a ratio of the negative lift $F_{L,90}$ at $v = 90$ m/s to our result for the negative lift F_L at $v = 28.6$ m/s as

$$\frac{F_{L,90}}{F_L} = \frac{(90\ \text{m/s})^2}{(28.6\ \text{m/s})^2}.$$

Substituting our known negative lift of $F_L = 663.7$ N and solving for $F_{L,90}$ give us

$$F_{L,90} = 6572\ \text{N} \approx 6600\ \text{N}. \tag{Answer}$$

Upside-down racing: The gravitational force is, of course, the force to beat if there is a chance of racing upside down:

$$\begin{aligned} F_g &= mg = (600\ \text{kg})\ (9.8\ \text{m/s}^2) \\ &= 5880\ \text{N}. \end{aligned}$$

With the car upside down, the negative lift is an *upward* force of 6600 N, which exceeds the downward 5880 N. Thus, the car could run on a long ceiling *provided* that it moves at about 90 m/s (= 324 km/h = 201 mi/h). However, moving that fast while right side up on a horizontal track is dangerous enough, so you are not likely to see upside-down racing except in the movies.

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SAMPLE PROBLEM 7.07

Circular motion of a ball attached to string

A ball of mass 0.5 kg is attached to a light string of length 2 m. The ball is whirled on a horizontal smooth surface in a circle of radius 2 m as shown in Fig. 7-16*a*. If the string can withstand tension up to 100 N, what is the maximum speed at which the ball can be whirled.

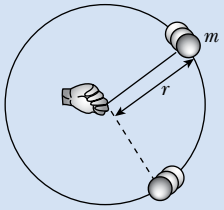

Fig. 7-16 (*a*) Ball attached to light string and whirled along a circular path.

KEY IDEA

A stronger cord can whirl the ball faster before the string breaks. Also more massive the ball, probability of the string breaking at same speed is also very high (Fig. 7-16*b*).

Calculations: Only radial force is acting on the ball, so we need to write only one equation, $F\sum F_r = mv_{max}^2/r$.

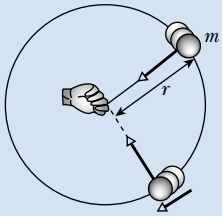

Fig. 7-16 (*b*) Vector representation.

Furthermore, the maximum speed the ball can attain corresponds to the maximum tension that the string can withstand:

$$T_{max} = \frac{mv_{max}^2}{r} \qquad (7\text{-}24)$$

Solve for v

$$v_{max} = \sqrt{\frac{T_{max}r}{m}} = \sqrt{\frac{(100\text{ N})(2\text{ m})}{0.5\text{ kg}}} = 20\text{ m/s}$$

Learn: Equation 7-24 shows that increase in mass or speed increases the chances of breaking the cord as we had conceptualized in the beginning.

SAMPLE PROBLEM 7.08

Car in banked circular turn

This problem is quite challenging in setting up but takes only a few lines of algebra to solve. We deal with not only uniformly circular motion but also a ramp. However, we will not need a tilted coordinate system as with other ramps. Instead we can take a freeze-frame of the motion and work with simply horizontal and vertical axes. As always in this chapter, the starting point will be to apply Newton's second law, but that will require us to identify the force component that is responsible for the uniform circular motion.

Curved portions of highways are always banked (tilted) to prevent cars from sliding off the highway. When a highway is dry, the frictional force between the tires and the road surface may be enough to prevent sliding. When the highway is wet, however, the frictional force may be negligible, and banking is then essential. Figure 7-17*a* represents a car of mass m as it moves at a constant speed v of 20 m/s around a banked circular track of radius $R = 190$ m. (It is a normal car, rather than a race car, which means that any vertical force from the passing air is negligible.) If the frictional force from the track is negligible, what bank angle θ prevents sliding?

KEY IDEAS

Here the track is banked so as to tilt the normal force $\vec{F}_N$ on the car toward the center of the circle (Fig. 7-17*b*). Thus, $\vec{F}_N$ now has a centripetal component of magnitude F_{Nr}, directed inward along a radial axis r. We want to find the value of the bank angle θ such that this centripetal

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component keeps the car on the circular track without need of friction.

Radial calculation: As Fig. 7-17*b* shows (and as you should verify), the angle that force $\vec{F}_N$ makes with the vertical is equal to the bank angle θ of the track. Thus, the radial component F_{Nr} is equal to $F_N \sin\theta$. We can now write Newton's second law for components along the r axis ($F_{\text{net},r} = ma_r$) as

$$-F_N \sin\theta = m\left(-\frac{v^2}{R}\right). \qquad (7\text{-}25)$$

We cannot solve this equation for the value of θ because it also contains the unknowns F_N and m.

Vertical calculations: We next consider the forces and acceleration along the y axis in Fig. 7-17*b*. The vertical component of the normal force is $F_{Ny} = F_N \cos\theta$, the gravitational force $\vec{F}_g$ on the car has the magnitude mg, and the acceleration of the car along the y axis is zero. Thus we can write Newton's second law for components along the y axis ($F_{\text{net},y} = ma_y$) as

$$F_N \cos\theta - mg = m(0),$$

from which

$$F_N \cos\theta = mg. \qquad (7\text{-}26)$$

Combining results: Equation 7-26 also contains the unknowns F_N and m, but note that dividing Eq. 7-25 by Eq. 7-26 neatly eliminates both those unknowns. Doing so, replacing $(\sin\theta)/(\cos\theta)$ with $\tan\theta$, and solving for θ then yield

$$\theta = \tan^{-1}\frac{v^2}{gR}$$

$$= \tan^{-1}\frac{(20\text{ m/s})^2}{(9.8\text{ m/s}^2)(190\text{ m})} = 12°. \qquad \text{(Answer)}$$

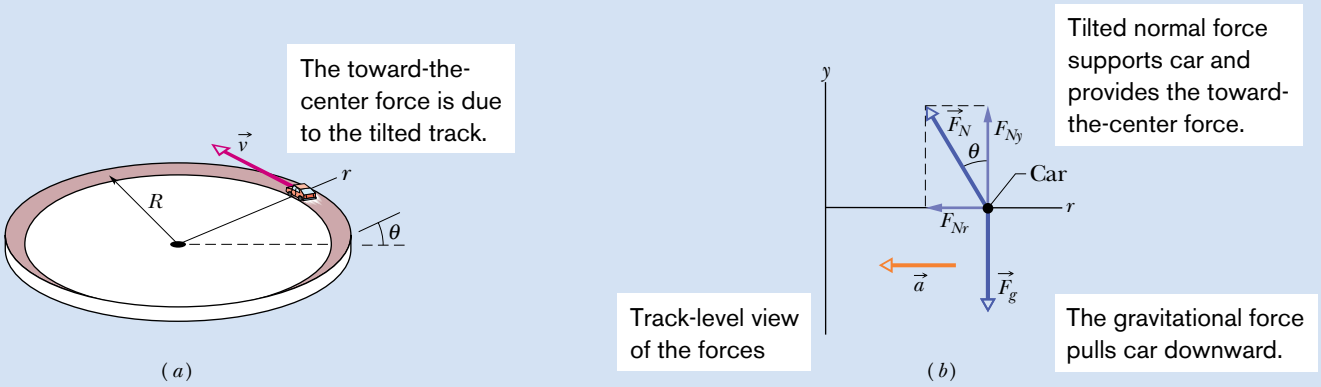

Figure 7-17 (*a*) A car moves around a curved banked road at constant speed v. The bank angle is exaggerated for clarity. (*b*) A free-body diagram for the car, assuming that friction between tires and road is zero and that the car lacks negative lift. The radially inward component F_{Nr} of the normal force (along radial axis r) provides the necessary centripetal force and radial acceleration.

SAMPLE PROBLEM 7.09

Circular motion of particle tied to a roof and a wall

Particle A is suspended by a string tied at point O from the roof of a room. The same particle is also tied to one of the walls of that room, at point B (Fig. 7-18*a*). Find the tension in OA before and after AB is cut.

KEY IDEAS

Before cutting the rope it is a simple equilibrium problem. After cutting the rope the particle will move on a

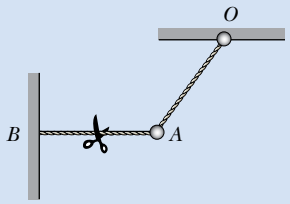

Fig. 7-18 (*a*) Particle suspended from the roof and tied to wall of room by means of a string.

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circular path. Our analysis for both situations has to be different.

Calculations: Case 1: Consider the free-body diagram of the particle before cutting (Fig. 7-18*b*). Resolving forces vertically and horizontally and applying equilibrium model

$$T_1 \cos\theta - mg = 0 \qquad (7\text{-}27)$$

$$T_1 \sin\theta - T_2 = 0 \qquad (7\text{-}28)$$

From Eq. 7-27, we have

$$T_1 = \frac{mg}{\cos\theta} \qquad \text{(Answer)}$$

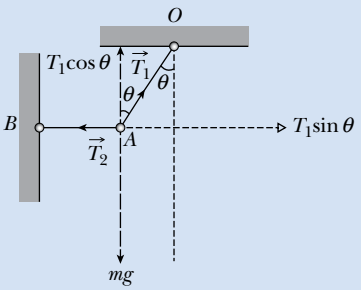

Fig. 7-18 (*b*) Free-body diagram.

Case 2: Making new free-body diagram after string is cut (Fig. 7-18*c*). Tension in the rope can change instantaneously. We cannot resolve forces in vertical and horizontal direction because acceleration of the particle is unknown. We can resolve forces along the length of rope and perpendicular to the rope. We know that radial acceleration of the particle will be zero as just after cutting the rope, particle remains at rest.

Learn: Writing Newton's second law along the string

$$T_1' - mg\cos\theta = m\frac{v^2}{l}$$

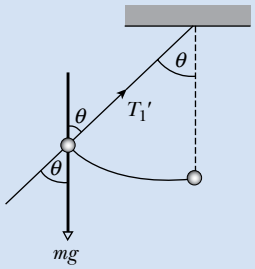

Fig. 7-18 (*c*) Free-body diagrams.

As $v = 0$ after cutting

$$T_1' - mg\cos\theta = 0 \qquad \text{(Answer)}$$

In Case of: In case we want to write Newton's second law in vertical direction, how do we go about it?

Calculations: It is easy to find that tangential acceleration $a_t = g\sin\theta$ and we know $a_c = 0$ as centripetal acceleration is zero just after cutting. Taking component of forces and acceleration in vertical direction

$$mg - T_1'\cos\theta = ma_t\sin\theta$$

$$T_1'\cos\theta = mg(1-\sin^2\theta)$$

thus $$T_1' = mg\cos\theta$$

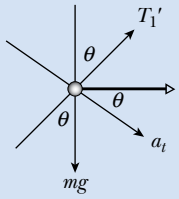

Fig. 7-18 (*d*) Free body diagram showing tangential acceleration.

7.7 | DYNAMICS OF NON-UNIFORM CIRCULAR MOTION

Key Concept

- The net force acting on the particle should have both radial and a tangential components and thus the total force exerted on the particle is

$$\sum\vec{F} = \sum\vec{F}_r + \sum\vec{F}_t.$$

Previously, we learnt that particle moving with varying speed on a circular path has a tangential component of acceleration, in addition to the radial component. Therefore, the net force acting on the particle should have

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both radial and a tangential components. The total force exerted on the particle is $\sum \vec{F} = \sum \vec{F}_r + \sum \vec{F}_t$ as shown in Fig. 7-19. The vector $\sum \vec{F}_r$ is directed toward the center and provides the radial acceleration which changes the direction of velocity. The vector $\sum \vec{F}_t$, tangent to the circle, provides the tangential acceleration, which causes change in the particle's speed with time.

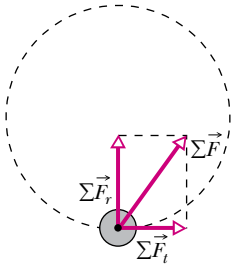

Figure 7-19 Forces acting on a particle moving with varying speed on a circular path.

SAMPLE PROBLEM 7.10

Ball attached to string in a vertical circle

A small ball of mass m is attached to a string of length R and moves in a vertical circle with a speed v m/s about a fixed point O as illustrated in Fig. 7-20. Find the tension in the string, when it makes an angle θ with the vertical, in terms of v, θ and g.

KEY IDEAS

1. The speed of the ball is not uniform in this example as the gravitational force has a tangential component, at almost all points on the path.
2. It is not a uniform circular motion.

Calculations: Making the free-body diagram, we see that the only forces acting on the ball are the weight and the tension exerted by the cord. Tension is always acting radially inward. We need to resolve mg into a tangential component $mg \sin\theta$ and a radial component $mg \cos\theta$.

Apply Newton's second law to the ball in the tangential direction:

$$mg \sin\theta = ma_t$$
$$a_t = g \sin\theta$$

Apply Newton's second law to the forces acting on the ball in the radial direction:

$$T - mg\cos\theta = \frac{mv^2}{R}$$
$$T = mg\left(\frac{v^2}{Rg} + \cos\theta\right)$$

Learn: It is important to note that v in the expression above varies for different positions as there is net tangential force acting on the particle.

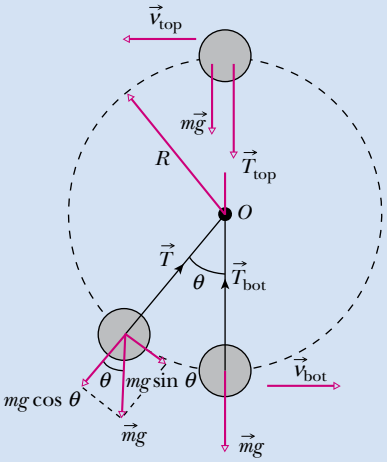

Figure 7-20 Ball of mass m suspended at center by a string undergoing circular motion.

Case 1: What speed would the ball have at the top of the circle if the tension in the cord becomes zero at this point?

Keeping tension equal to zero in the above expression, we get

$$0 = mg\left(\frac{v_{top}^2}{Rg} - 1\right) \quad \Rightarrow \quad v_{top} = \sqrt{gR}$$

Case 2: What if the ball is given an initial velocity such that the speed at the top is less than this value?

In this case, the ball will not reach the top of the circle, as the tension in the string becomes zero at some lower point. The ball becomes a projectile as only force acting is gravity and ball has some oblique velocity.

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7.8 | CENTRIFUGAL FORCE

Key Concepts

- If an object moving in a circle and experiences an outward force, then this pseudo force is called centrifugal force, which depends on the mass of the object, the distance from the center of the circle and the speed of rotation.
- Centrifugal force is expressed as
$$F_C = \frac{mv^2}{r}$$
where, m is the mass of object, v is the speed of rotation and r is the radius (distance from center of circle).

Consider a car traveling on a highway at a high speed and approaching a curved road. As the car takes the right turn, a passenger slides to the left and hits the door. What causes him to move towards the door?

This phenomenon can be explained without using fictitious force. Before the car takes the turn, the passenger is moving in a straight line path. As the car enters the turn and travels a curved path, the passenger is still moving along the original straight line path due to inertia. If a sufficiently large force, which satisfies the centripetal condition acts on the passenger toward the center of curvature, he can move in a curved path along with the car. However, the frictional force between the seat and car is not larger enough to satisfy centripetal condition, so the seat follows a curved path and the passenger continues in the original straight line path. Therefore, from reference frame of an observer in the car, the passenger slides to the right relative to the seat. Finally, he encounters the door, which provides a sufficient force which satisfies the centripetal condition, allowing him to follow the same curved path as the car.

This can also be explained by using concept of fictitious force and using reference frame attached to the car. Since car is accelerating towards right, an observer sitting in car must apply a fictitious force toward left (Newton's second law). The force acting toward the left pushes the passenger away from the center of the circular path. This fictitious force is called the "centrifugal force," This fictitious force is due to the centripetal acceleration associated with the changing direction of the car's velocity vector.

If an object moving in a circle and experiences an outward force, then this pseudo force is called **centrifugal force**, which depends on the mass of the object, the distance from the center of the circle and the speed of rotation.

If the object has larger mass, the force of the movement and the speed of the object are higher. If the distance is far from the center of the circle, then the force of the movement is more. Centrifugal force is expressed as

$$F_C = \frac{mv^2}{r}$$

where F_C is the centrifugal force, m is the mass, v is the speed and r is the radius.

CHECKPOINT 6

Consider a car is moving on circular path of radius R with speed v. A block kept inside the frictionless surface of car is touching the wall. Write Newton's second law from the reference frame attached to observer at ground and car.

REVIEW AND SUMMARY

Uniform Circular Motion If a particle moves in a circle or a circular arc of radius R at constant speed v, the particle is said to be in **uniform circular motion.** It then has a **centripetal acceleration** $\vec{a}$ with magnitude given by

$$a = \frac{v^2}{R}. \tag{7-10}$$

This acceleration is due to a net **centripetal force** on the particle, with magnitude given by

$$F = \frac{mv^2}{R}, \tag{7-17}$$

where m is the particle's mass. The vector quantities $\vec{a}$ and $\vec{F}$ are directed toward the center of curvature of the particle's

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path. A particle can move in circular motion only if a net centripetal force acts on it.

Angular Variables For a circular motion to be uniform, it has to satisfy the following definition: A uniform circular motion is the motion of an object traveling at a constant (uniform) speed on a circular path (object covers equal distances on circumference in equal intervals of time).

If an object's speed is constant, we define its motion as **uniform circular motion**.

Angular displacement of a body is defined as the angle in radians (i.e., degrees, revolutions) through which a point or line is rotated in a specified direction about a specified axis, or otherwise, angular displacement is the angle of the movement of a body in a circular path.

We define the **average angular velocity** of the body in the time interval Δt from t_1 to t_2 to be

$$\omega_{\text{avg}} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}, \tag{7-2}$$

in which $\Delta\theta$ is the angular displacement that occurs during Δt (ω is the lowercase Greek letter omega).

The **average angular acceleration** of the rotating body in the interval from t_1 to t_2 is defined as

$$\alpha_{\text{avg}} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}, \tag{7-4}$$

in which $\Delta\omega$ is the change in the angular velocity that occurs during the time interval Δt.

Relation between Angular Velocity and Linear Velocity As the velocity is perpendicular to the line joining the center and particle, the angular velocity of particle with respect to the center is

$$\omega = \frac{v\sin\phi}{R} \tag{7-9}$$

That is, $$\omega = \frac{v}{R} \quad (\text{as } \phi = 90°)$$

Particle in Uniform Circular Motion The acceleration associated with uniform circular motion is called a **centripetal** (meaning "center seeking") **acceleration**. As we prove next, the magnitude of this acceleration $\vec{a}$ is

$$a = \frac{v^2}{r}, \quad \text{(centripetal acceleration)} \tag{7-10}$$

where r is the radius of the circle and v is the speed of the particle.

During this acceleration at constant speed, the particle travels the circumference of the circle (a distance of $2\pi r$) in time

$$T = \frac{2\pi r}{v}, \quad \text{(period)} \tag{7-11}$$

Particle in Non-Uniform Circular Motion Radial or centripetal acceleration: Change in direction is due to the radial acceleration (centripetal acceleration), which is given by

$$\vec{a}_c = \vec{a}_r = \frac{v^2}{r}.$$

Tangential Acceleration: The non-uniform circular motion basically involves a change in speed. This change is accounted by the tangential acceleration, which results due to a tangential force and which acts along the direction of velocity:

$$a_t = \frac{d|\vec{v}|}{dt} = \frac{d(\omega R)}{dt} = \frac{Rd\omega}{dt} = R\alpha.$$

If R_C is radius of curvature of the imaginary circle touching the curve at a point, then

$$R_C = \frac{v^2}{a_r} = \frac{v^2}{a_\perp}.$$

Radius of Curvature In general, R_C at any point on the path can be defined as

$$R_C = \frac{(\text{Speed})^2}{\text{Component of acceleration perpendicular to velocity}}.$$

Dynamics of Non-uniform Circular Motion The net force acting on the particle should have both radial and a tangential components and thus the total force exerted on the particle is

$$\sum\vec{F} = \sum\vec{F}_r + \sum\vec{F}_t.$$

Centrifugal Force If an object moving in a circle and experiences an outward force, then this pseudo force is called **centrifugal force**, which depends on the mass of the object, the distance from the center of the circle and the speed of rotation.

Centrifugal force is expressed as

$$F_C = \frac{mv^2}{r}.$$

where F_C is the centrifugal force, m is the mass, v is the speed and r is the radius.

PROBLEMS

1. A police officer in hot pursuit drives her car through a circular turn of radius 300 m with a constant speed of 75.0 km/h. Her mass is 55.0 kg. What are (a) the magnitude and (b) the angle (relative to vertical) of the *net* force of the officer on the car seat? (*Hint:* Consider both horizontal and vertical forces.)

2. A circular-motion addict of mass 80 kg rides a Ferris wheel around in a vertical circle of radius 12 m at a constant speed of 5.5 m/s. (a) What is the period of the motion? What is the magnitude of the normal force on the addict from the seat when both go through (b) the highest point of the circular path and (c) the lowest point?

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3. A roller-coaster car has a mass of 1300 kg when fully loaded with passengers. As the car passes over the top of a circular hill of radius 20 m, its speed is not changing. At the top of the hill, what are the (a) magnitude F_N and (b) direction (up or down) of the normal force on the car from the track if the car's speed is $v = 11$ m/s? What are (c) F_N and (d) the direction if $v = 14$ m/s?

4. In Fig. 7-21, a car is driven at constant speed over a circular hill and then into a circular valley with the same radius. At the top of the hill, the normal force on the driver from the car seat is 0. The driver's mass is 80.0 kg. What is the magnitude of the normal force on the driver from the seat when the car passes through the bottom of the valley?

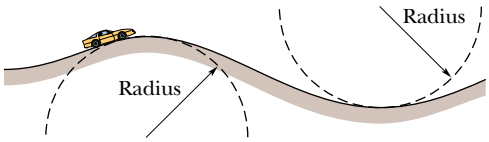

Figure 7-21 Problem 4.

5. An 85.0 kg passenger is made to move along a circular path of radius $r = 3.50$ m in uniform circular motion. (a) Figure 7-22*a* is a plot of the required magnitude F of the net centripetal force for a range of possible values of the passenger's speed v. What is the plot's slope at $v = 8.30$ m/s? (b) Figure 7-22*b* is a plot of F for a range of possible values of T, the period of the motion. What is the plot's slope at $T = 2.50$ s?

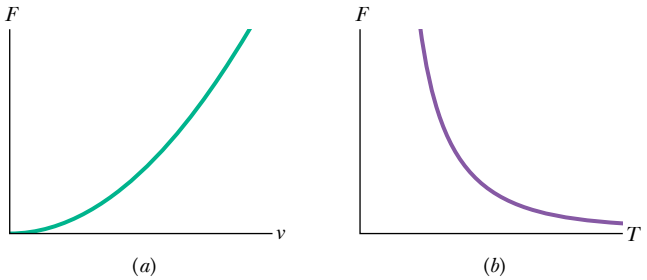

Figure 7-22 Problem 5.

6. An airplane is flying in a horizontal circle at a speed of 600 km/h (Fig. 7-23). If its wings are tilted at $\theta = 40°$ to the horizontal, what is the radius of the circle in which the plane is flying? Assume that the required force is provided entirely by an "aerodynamic lift" that is perpendicular to the wing surface.

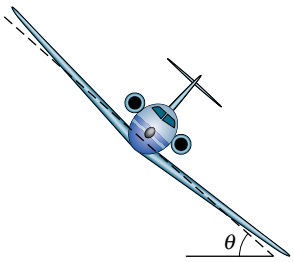

Figure 7-23 Problem 6.

7. An amusement park ride consists of a car moving in a vertical circle on the end of a rigid boom of negligible mass. The combined weight of the car and riders is 6.0 kN, and the circle's radius is 10 m. At the top of the circle, what are the (a) magnitude F_B and (b) direction (up or down) of the force on the car from the boom if the car's speed is $v = 5.0$ m/s? What are (c) F_B and (d) the direction if $v = 12$ m/s?

8. An old streetcar rounds a flat corner of radius 10.5 m, at 16 km/h. What angle with the vertical will be made by the loosely hanging hand straps?

9. A bolt is threaded onto one end of a thin horizontal rod, and the rod is then rotated horizontally about its other end. An engineer monitors the motion by flashing a strobe lamp onto the rod and bolt, adjusting the strobe rate until the bolt appears to be in the same eight places during each full rotation of the rod (Fig. 7-24). The strobe rate is 2000 flashes per second; the bolt has mass 33 g and is at radius 4.0 cm. What is the magnitude of the force on the bolt from the rod?

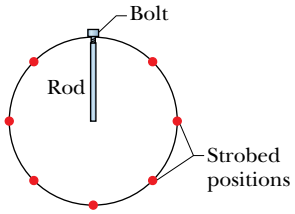

Figure 7-24 Problem 9.

10. A banked circular highway curve is designed for traffic moving at 65 km/h. The radius of the curve is 200 m. Traffic is moving along the highway at 40 km/h on a rainy day. What is the minimum coefficient of friction between tires and road that will allow cars to take the turn without sliding off the road? (Assume the cars do not have negative lift.)

11. A puck of mass $m = 1.50$ kg slides in a circle of radius $r = 25.0$ cm on a frictionless table while attached to a hanging cylinder of mass $M = 2.50$ kg by means of a cord that extends through a hole in the table (Fig. 7-25). What speed keeps the cylinder at rest?

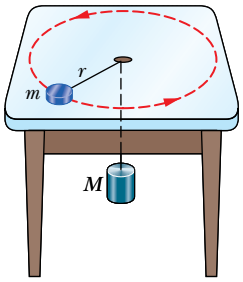

Figure 7-25 Problem 11.

12. *Brake or turn?* Figure 7-26 depicts an overhead view of a car's path as the car travels toward a wall. Assume that the driver begins to brake the car when the distance to the wall is $d = 107$ m, and take the car's mass as $m = 1400$ kg, its initial speed as $v_0 = 35$ m/s, and the coefficient of static friction as $\mu_s = 0.50$. Assume that the car's weight is distributed evenly on the four wheels, even during braking. (a) What magnitude of static friction is needed (between tires and road) to stop the car just as it reaches the wall? (b) What is the maximum

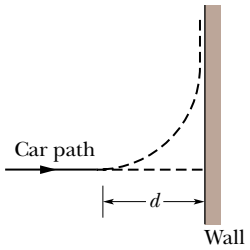

Figure 7-26 Problem 12.

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possible static friction $f_{s,\max}$? (c) If the coefficient of kinetic friction between the (sliding) tires and the road is $\mu_k = 0.40$, at what speed will the car hit the wall? To avoid the crash, a driver could elect to turn the car so that it just barely misses the wall, as shown in the figure. (d) What magnitude of frictional force would be required to keep the car in a circular path of radius d and at the given speed v_0, so that the car moves in a quarter circle and then parallel to the wall? (e) Is the required force less than $f_{s,\max}$ so that a circular path is possible?

13. In Fig. 7-27, a 1.34 kg ball is connected by means of two massless strings, each of length L = 1.70 m, to a vertical, rotating rod. The strings are tied to the rod with separation d = 1.70 m and are taut. The tension in the upper string is 35 N. What are the (a) tension in the lower string, (b) magnitude of the net force $\vec{F}_{net}$ on the ball, and (c) speed of the ball? (d) What is the direction of $\vec{F}_{net}$?

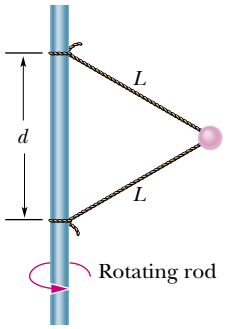

Figure 7-27 Problem 13.

14. A cylindrical bucket filled with water is whirled around in a vertical circle of radius r. What can be the minimum speed at the top of the path if water does not fall out from the bucket? If it continues with this speed, what is the normal contact force exerted by the bucket on water at the lowest point of the path?

15. A metal ring of mass m and radius R is placed on a smooth horizontal table and is set rotating about its own axis in such a way that each part of the ring moves with a speed v. Find the tension in the ring.

16. Imagine the standard kilogram is located on Earth's equator, where it moves in a circle of radius 6.40×10^6 m (Earth's radius) at a constant speed of 465 m/s due to Earth's rotation. (a) What is the magnitude of the centripetal force on the standard kilogram during the rotation? Imagine that the standard kilogram hangs from a spring balance at that location and assume that it would weigh exactly 9.80 N if Earth did not rotate. (b) What is the reading on the spring balance; that is, what is the magnitude of the spring balance from the standard kilogram.

17. A car is rounding a flat curve of radius R = 220 m at the curve's maximum design speed v = 94.0 km/h. What is the magnitude of the net force on the seat cushion from a passenger with mass m = 85.0 kg?

18. A motorcycle has to move with a constant speed on an over-bridge which is in the form of a circular arc of radius R and has a total length L. Suppose the motorcycle starts from the highest point. (a) What can its maximum velocity be for which the contact with the road is not broken at the highest point? (b) If the motorcycle goes at speed $1/\sqrt{2}$ times the maximum found in part (a), where will it lose the contact with the road? (c) What maximum uniform speed can it maintain on the bridge if it does not lose contact anywhere on the bridge?

19. A car moving at a speed of 36 km/hr is taking a turn on a circular road of radius 50 m. A small wooden plate is kept on the seat with its plane perpendicular to the radius of the circular road (Fig. 7-28). A small block of mass 100 g is kept on the seat which rests against the plate. The friction coefficient between the block and the plate is $\mu = 0.58$. (a) Find the normal contact force exerted by the plate on the block. (b) The plate is slowly turned so that the angle between the normal to the plate and the radius of the road slowly increases. Find the angle at which the block will just start sliding on the plate.

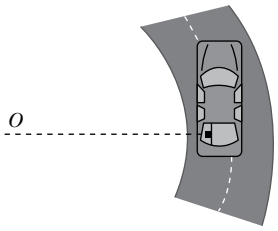

Figure 7-28 Problem 19.

20. A block of mass m moves on a horizontal circle against the wall of a cylindrical room of radius R. The floor of the room on which the block moves is smooth but the friction coefficient between the wall and the block is μ. The block is given an initial speed v_0. As a function of the speed v write (a) the normal force by the wall on the block, (b) the frictional force by the wall and (c) the tangential acceleration of the block. (d) Integrate the tangential acceleration $[(dv/dt) = v(dv/ds)]$ = to obtain the speed of the block after one revolution.

21. A track consists of two circular parts ABC and CDE of equal radius 100 m and joined smoothly as shown in Fig. 7-29. Each part subtends a right angle at its center. A cycle weighing 100 kg together with the rider travels at a constant speed of 18 km/h on the track. (a) Find the normal contact force by the road on the cycle when it is at B and at D. (b) Find the force of friction exerted by the track on the tires when the cycle is at B, C and D. (c) Find the normal force between the road and the cycle just before and just after the cycle crosses C. (d) What should be the minimum friction coefficient between the road and the tire, which will ensure that the cyclist can move with constant speed? ($g = 10$ m/s^2)

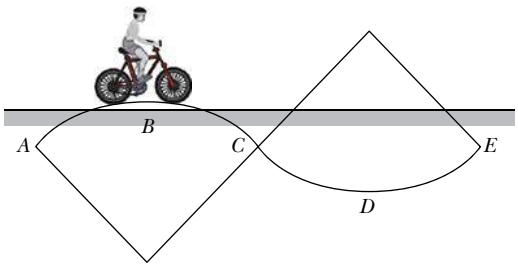

Figure 7-29 Problem 21.

22. A table with smooth horizontal surface is turning at an angular speed ω about its axis. A groove is made on the surface along a radius and a particle is gently placed inside the groove at a distance a from the center. Find the speed of the particle as its distance from the center becomes L.

23. The bob of a simple pendulum of length 1m has mass 100 g and a speed of 1.4 m/s when the string makes an

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angle of 0.20 radian with the vertical. Find the tension at this instant. You can use $\cos\theta \approx 1 - \theta^2/2$ and $\sin\theta = \theta$ for small θ.

24. A block of mass m is kept on a horizontal ruler. The friction coefficient between the ruler and the block is m. The ruler is fixed at one end and the block is at a distance L from the fixed end. The ruler is rotated about the fixed end in the horizontal plane through the fixed end. (a) What can the maximum angular speed be for which the block does not slip? (b) If the angular speed of the ruler is uniformly increased from zero at an angular acceleration a, at what angular speed will the block slip?

25. A table with smooth horizontal surface is fixed in a cabin that rotates with a uniform angular velocity ω in a circular path of radius R (Fig. 7-30). A smooth groove AB of length L ($\ll R$) is made on the surface of the table. The groove makes an angle θ with the radius OA of the circle in which the cabin rotates. A small particle is kept at the point A in the groove and is released to move along AB. Find the time taken by the particle to reach the point B.

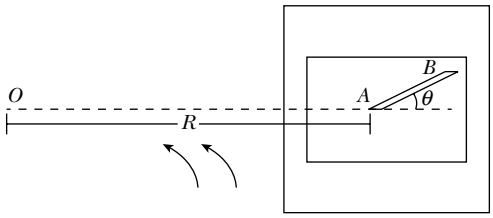

Figure 7-30 Problem 25.

26. A car goes on a horizontal circular road of radius R, the speed increasing at a constant rate $dv/dt = a$. The friction coefficient between the road and the tire is μ. Find the speed at which the car will skid.

27. A table with smooth horizontal surface is placed in a cabin which moves in a circle of a large radius R (Fig. 7-31). A smooth pulley of small radius is fastened to the table. Two masses m and $2m$ placed on the table are connected through a string going over the pulley. Initially the masses are held by a person with the strings along the outward radius and then the system is released from rest (with respect to the cabin). Find the magnitude of the initial acceleration of the masses as seen from the cabin and the tension in the string.

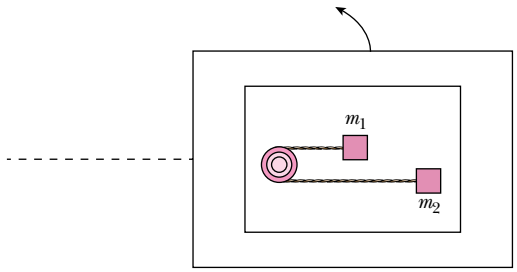

Figure 7-31 Problem 27.

28. Figure 7-32 shows a *conical pendulum*, in which the bob (the small object at the lower end of the cord) moves in a horizontal circle at constant speed. (The cord sweeps out a cone as the bob rotates.) The bob has a mass of 0.040 kg, the string has length $L = 0.90$ m and negligible mass, and the bob follows a circular path of circumference 0.94 m. What are (a) the tension in the string and (b) the period of the motion?

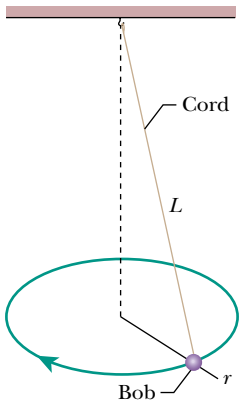

Figure 7-32 Problem 28.

29. In Fig. 7-33, a stuntman drives a car (without negative lift) over the top of a hill, the cross section of which can be approximated by a circle of radius $R = 250$ m. What is the greatest speed at which he can drive without the car leaving the road at the top of the hill?

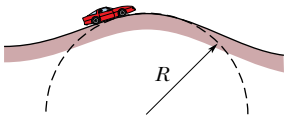

Figure 7-33 Problem 29.

30. A car weighing 10.7 kN and traveling at 13.4 m/s without negative lift attempts to round an unbanked curve with a radius of 61.0 m. (a) What magnitude of the frictional force on the tires is required to keep the car on its circular path? (b) If the coefficient of static friction between the tires and the road is 0.350, is the attempt at taking the curve successful?

31. A certain string can withstand a maximum tension of 40 N without breaking. A child ties a 0.37 kg stone to one end and, holding the other end, whirls the stone in a vertical circle of radius 0.91 m, slowly increasing the speed until the string breaks. (a) Where is the stone on its path when the string breaks? (b) What is the speed of the stone as the string breaks?

32. When a small 2.0 g coin is placed at a radius of 5.0 cm on a horizontal turntable that makes three full revolutions in 3.14 s, the coin does not slip. What are (a) the coin's speed, the (b) magnitude and (c) direction (radially inward or outward) of the coin's acceleration, and the (d) magnitude and (e) direction (inward or outward) of the frictional force on the coin? The coin is on the verge of slipping if it is placed at a radius of 10 cm. (f) What is the coefficient of static friction between coin and turntable?

33. A child places a picnic basket on the outer rim of a merry-go-round that has a radius of 4.6 m and revolves once every 30 s. (a) What is the speed of a point on that rim? (b) What is the lowest value of the coefficient of

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static friction between basket and merry-go-round that allows the basket to stay on the ride?

34. In the Bohr model of the hydrogen atom an electron is pictured rotating in a circle (with a radius of 0.5×10^{-10} m) about the positive nucleus of the atom. The centripetal force is furnished by the electric attraction of the positive nucleus for the negative electron. How large is this force if the electron is moving with a speed of 2.3×10^6 m/s? (The mass of an electron is 9×10^{-31} kg.)

35. As indicated in Fig. 7-34 a plane flying at constant speed is banked at angle θ in order to fly in a horizontal circle of radius r. The aerodynamic lift force acts generally upward at right angles to the plane's wings and fuselage. This lift force corresponds to the tension provided by the string in a conical pendulum, or the normal force of a banked road. (a) Obtain the equation for the required banking angle θ in terms of v, r and g. (b) What is the required angle for $v = 60$ m/s (216 km/h) and $r = 1.0$ km?

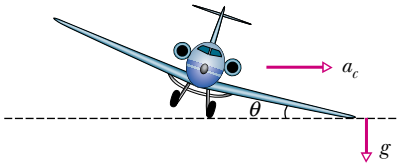

Figure 7-34 Problem 35.

36. A boy on a bicycle pedals around a circle of 22 m radius at a speed of 10 m/s. The combined mass of the boy and the bicycle combined mass of the boy and the bicycle is 80 kg. (a) What is the centripetal force exerted by the pavement on the bicycle? (b) What is the upward force exerted by the pavement on the bicycle ? See Fig. 7-35.

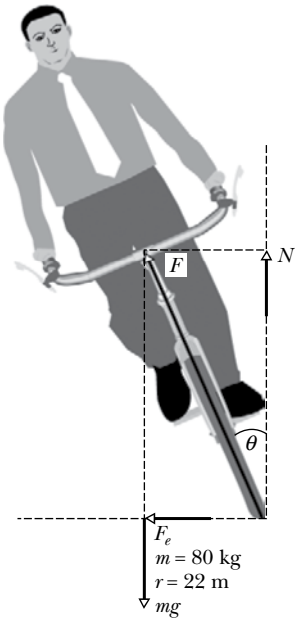

Figure 7-35 Problem 36.

37. At the equator, the effective value of g is smaller than at the poles. One reason for this is the centripetal acceleration due to the Earth's rotation. The magnitude of the centripetal acceleration must be subtracted from the magnitude of the acceleration due purely to gravity in order to obtain the effective value of g. (a) Calculate the fractional diminution of g at the equator. Express your result as a percentage. (b) How short would the Earth's period of rotation have to be in order for objects at the equator to the "weightless" (that is, in order for the effective value of g to be zero) ? (c) How would be the period found in part (b) compare with that of a satellite skimming the surface of an airless Earth?

38. A particle is to slide along a horizontal circular path on the inside of a funnel. The surface of the funnel is frictionless. How fast must the particle be moving (in terms of r and θ) if it is to execute this motion ?

39. The designer of a roller coaster wishes the riders to experience "Weightlessness" as they round the top of one hill. How fast must the car be going if the radius of curvature at the hilltop is 20 m?

40. The angular acceleration of the toppling pole shown in Fig. 7-36 is given by $a = k \sin \theta$, where θ is the angle between the axis of the pole and the vertical, and k is a constant. The pole starts from rest at $\theta = 0$. Find (a) the tangential and (b) the centripetal acceleration of the upper end of the pole in terms of k, θ and (the length of the pole).

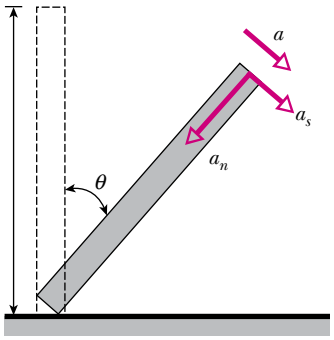

Figure 7-36 Problem 40.

41. A wet open umbrella is held up-right as shown in Fig. 7-37*a* and is twirled about the handle at a uniform rate of 21 rev in 44s. If the rim of the umbrella is a circle 1 m in diameter, and the height of the rim above the floor is 1.5 m. The drops of water spun off the rim hit the floor at a distance x from the axis. Find x.

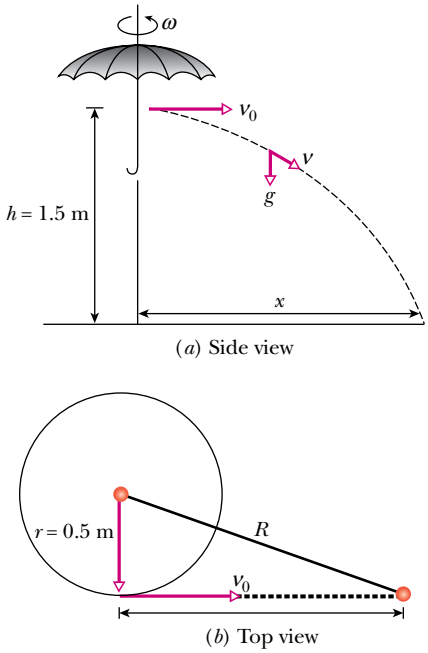

Figure 7-37 Problem 41.

42. A particle whose mass is 2 kg moves with a speed of 44 m/s on a curved path. The resultant force acting on the particle at a particular point of the curve is 30 N at 60° to the tangent to the curve, as shown in Fig. 7-38. At the

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point, find (a) the radius of curvature of the curve and (b) the tangential acceleration of the particle.

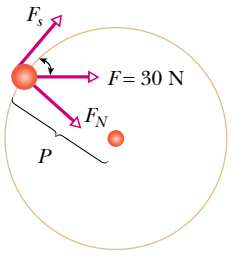

Figure 7-38 Problem 42.

43. A bug is crawling with constant speed v along the spoke of a bicycle wheel, of radius a, while the bicycle moves down the road with constant speed v. Find the accelerations of the bug, as observed by a man standing beside the road, along the perpendicular to the spoke of the wheel.

44. At the instant $\theta = \theta_1$ in the Fig. 7-39, the boy's center of mass G has speed v_G. Determine the rate of increase in his speed and the tension in each of the two supporting cords of the swing at this instant. The boy has a weight W. Neglect his size and the mass of the seat and cords. Given: m = 20 kg; $\theta_1 = 53°$; $v_G = 3$ m/s; $l = 2$ m.

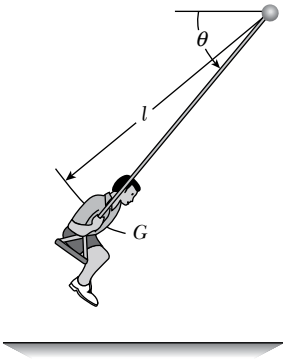

Figure 7-39 Problem 44.

45. If the crest of the hill has a radius of curvature p (Fig. 7-40), determine the maximum constant speed at which the car can travel over it without leaving the surface of the road. Neglect the size of the car in the calculation. The car has mass m. Given: $p = 40m$; $m = 1700$ kg; $g = 10$ m/s^2.

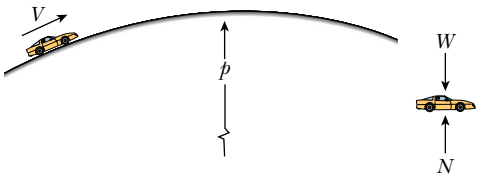

Figure 7-40 Problem 45.

46. The airplane, traveling at constant speed v is executing a horizontal turn (Fig. 7-41). If the plane is banked at angle θ when the pilot experiences only a normal force on the seat of the plane, determine the radius of curvature p of the turn. Also, what is the normal force of the seat on the pilot if he has mass M? (Units used: kN = 10^3 N) Given: $v = 50$ m/s; $\theta = 15°$; $M = 70$ kg; $g = 9.815$ m/s^2.

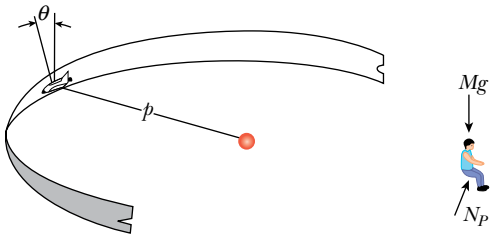

Figure 7-41 Problem 46.

47. The man has weight W and lies against the cushion for which the coefficient of static friction is μ_s, (Fig. 7-42). Determine the resultant normal and frictional forces the cushion exerts on him if, due to rotation about the z axis, he has constant speed v. Neglect the size of the man. Given: $W = 150$ lb; $\mu_s = 0.5$; v = 20 ft/s; $\theta = 60°$; $d = 8$ ft; $g = 32$ ft/s^2.

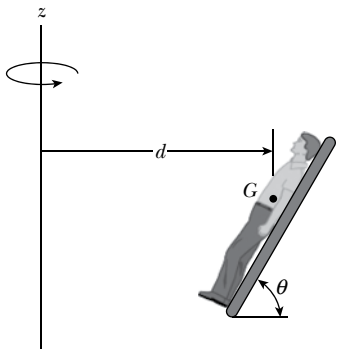

Figure 7-42 Problem 47.

48. A collar having a mass M and negligible size slides over the surface of a horizontal circular rod for which the coefficient of kinetic friction is μ_k, (Fig. 7-43). If the collar is given a speed v_1 and then released at $\theta = 0°$, determine how far, d, it slides on the rod before coming to rest. Given: $M = 0.75$ kg; $r = 100$ mm; $\mu_k = 0.3$; $g = 9.81$ m/s^2; $v_1 = 4$ m/s.

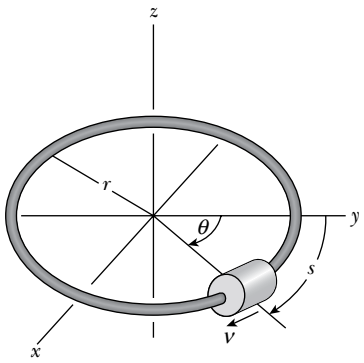

Figure 7-43 Problem 48.

49. The smooth block B, having mass M, is attached to the vertex A of the right circular cone using a light cord. The cone is rotating at a constant angular rate about the z axis such that the block attains speed v (Fig. 7-44). At this speed, determine the tension in the cord and the reaction which the cone exerts on the block. Neglect the size of the block. Given: $M = 0.2$ kg; $v = 0.5$ m/s; $a = 300$ mm; $b = 400$ mm; $c = 200$ mm; $g = 9.81$ m/s^2.

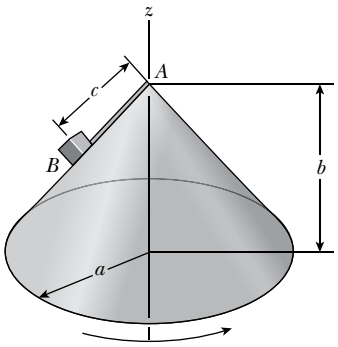

Figure 7-44 Problem 49.

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50. The pendulum bob B of mass M is released from rest when $\theta = 0°$ (Figure 7-45). Determine the initial tension in the cord and also at the instant the bob reaches point D, $\theta = \theta_1$. Neglect the size of the bob. Given: $M = 3$ kg; $\theta_1 = 45°$; $L = 2$ m; $g = 9.81$ m/s^2.

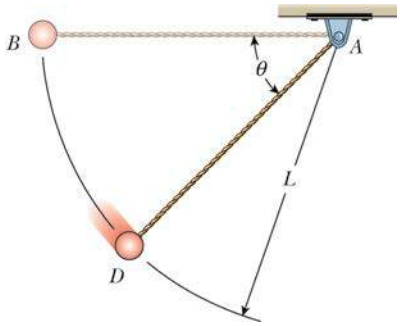

Figure 7-45 Problem 50.

51. The spool S of mass M fits loosely on the inclined rod for which the coefficient of static friction is μ_s. If the spool is located a distance d from A, determine the maximum constant speed the spool can have so that it does not slip up the rod (Fig. 7-46). Given: $M = 2$ kg; $e = 3$; $\mu_k = 0.2$ $f = 4$; $d = 0.25$ m; $g = 9.81$ m/s^2.

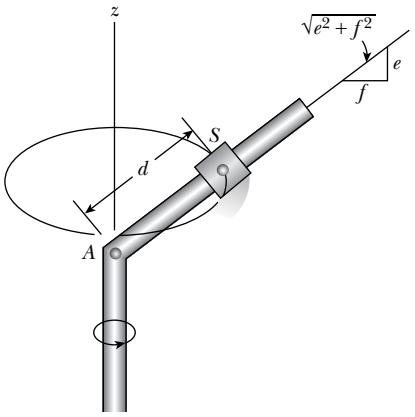

Figure 7-46 Problem 51.

52. The man has mass M and sits a distance d from the center of the rotating platform (Fig. 7-47). Due to the rotation, his speed is increased from rest by the rate v'. If the coefficient of static friction between his clothes and the platform is μ_s, determine the time required to cause him to slip. Given: $M = 80$ kg; $\mu_e = 0.3$; $d = 3$ m; $D = 10$ m; $v' = 0.4$ m/s^2; $g = 9.81$ m/s^2.

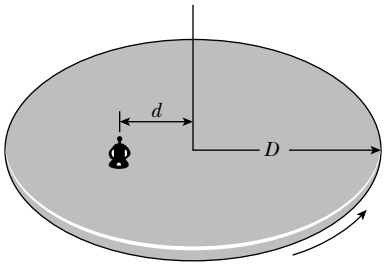

Figure 7-47 Problem 52.

53. The block has weight W and it is free to move along the smooth slot in the rotating disk. The spring has stiffness k and an unstretched length δ (Fig. 7-48). Determine the force of the spring on the block and the tangential component of force which the slot exerts on the side of the block is at rest with respect to the disk and is traveling with constant speed v. Given: $W = 2$ lb; $k = 2.5$ lb/ft; $d = 1.25$ ft; $v = 12$ ft/s.

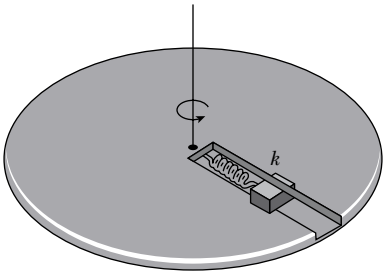

Figure 7-48 Problem 53.

54. A small coin is placed on a flat, horizontal turntable. The turntable is observed to make three revolutions in 3.14 s. (a) What is the speed of the coin when it rides without slipping at a distance of 5.0 cm from the center of the turntable? (b) What is the acceleration (magnitude and direction) of the coin? (c) What are the magnitude and direction of the frictional force acting on the coin if the coin has a mass of 2.0 g? (d) What is the coefficient of static friction between the coin and the turntable if the coin is observed to slide off the turntable when it is more than 10 cm from the center of the turntable?

55. A chain of mass m, radius r (mass uniformly distributed) is kept on a cone (Fig. 7-49) with semi-vertex angle θ. If the chain moves by an angular velocity ω, then find the tension in the chain due to rotation.

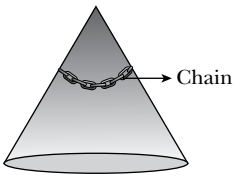

Figure 7-49 Problem 55.

56. A hemispherical bowl of radius R is rotated about its axis of symmetry which is kept vertical. A small block is kept at a position where the radius makes an angle θ with the vertical. The block rotates with the bowl without slipping. The friction coefficient between the block and the bowl surface is m. Find the range of angular speed for which the block will not slip.

57. A cat dozes on a stationary merry-go-round, at a radius of 5.4 m from the center of the ride. Then the operator turns on the ride and brings it up to its proper turning rate of one complete rotation every 6.0 s. What is the least coefficient of static friction between the cat and the merry-go-round that will allow the cat to stay in place, without sliding?

58. A small block of mass m is placed on the rotating conical surface with semi-vertex angle θ at the radius $r = 0.2$ m shown in Fig. 7-50. Determine the maximum angular velocity ω of the cone about the vertical axis for which the block will not slip. Friction between block and surface is 0.8.

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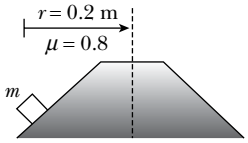

Figure 7-50 Problem 58.

59. The two balls of mass m_A = 10 kg and m_B = 15 kg are connected by an elastic string and supported on a turntable as shown in Fig. 7-51. When the turn is at rest, the tension in the string is T = 100 N and the balls exert this same force on each of the stops. What forces will they exert on the stops when the turntable is rotating uniformly about vertical axis OO' at 60 rpm?

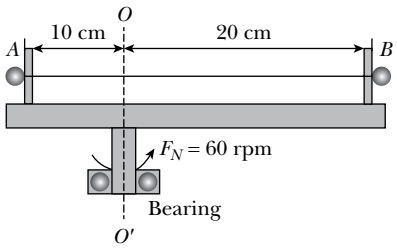

Figure 7-51 Problem 59.

60. An elastic cord having an unstretched length L, stiffness k, and mass per unit length l is stretched around the drum of radius r ($2\pi > L$). Determine the angular velocity of the cord due to the rotation of the drum, which will allow the cord to loosen its contact with the drum. See Fig. 7-52.

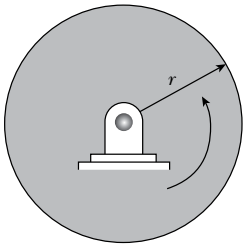

Figure 7-52 Problem 60.

61. A small 0.25 kg collar C may slide on a semicircular rod which is made to rotate about the vertical axis yy' at a constant rate of 7.5 rad/s (Fig. 7-53). Determine the values of θ for which the collar will not slide on the rod, assuming no friction between the collar and the rod.

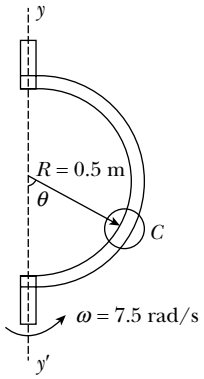

Figure 7-53 Problem 61.

62. Block A has a mass of M_A = 15 kg and B has mass of M_B = 45 kg. They are on a rotating surface and connected by a string as shown in Fig. 7-54. Determine the value of ω at which radial sliding will occur. The coefficient of friction between blocks surface is 0.25.

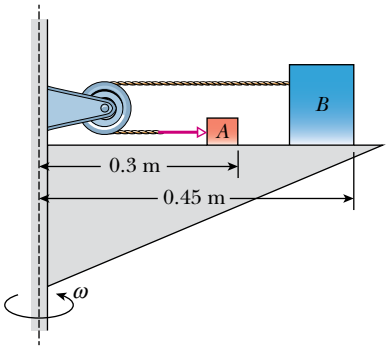

Figure 7-54 Problem 62.

63. A package of mass m is placed inside a drum that rotates in the vertical plane at the constant angular speed ω = 1.36 rad/s. If the package reaches the position θ = 45° before slipping, determine the coefficient of friction between the package and drum. See Fig. 7-55.

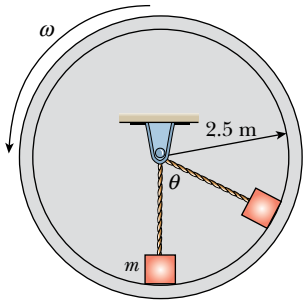

Figure 7-55 Problem 63.

64. The mass at C is attached to the vertical pole AB by two wires. The assembly is rotating about AB at the constant angular speed ω. If the force in wire BC is twice the force in AC, determine the value of ω. See Fig. 7-56.

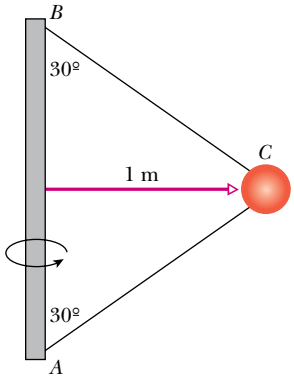

Figure 7-56 Problem 64.

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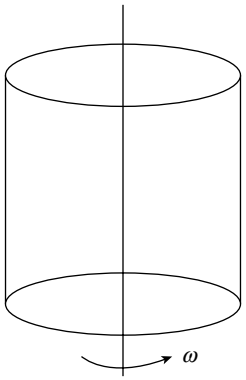
PRACTICE QUESTIONS

Single Correct Choice Type

1. If a particle is revolving in a circle with increasing its speed uniformly, which of the following is constant?
(a) Centripetal acceleration
(b) Tangential acceleration
(c) Angular acceleration
(d) None of these

2. A particle of mass 2 kg is moving along a circular path of radius 1 m. If its angular speed is 2π rad/s, the centripetal force on it is
(a) 4π N **(b)** 8π N
(c) $4\pi^4$ N **(d)** $8\pi^2$ N

3. An insect of mass $m = 3$ kg is inside a vertical drum of radius 2 m that is rotation with an angular velocity of 5 rad/s. The minimum coefficient of friction required for the insect for not falling is



(a) 0.5 **(b)** 0.4
(c) 0.2 **(d)** None of these

4. The speed of revolution of a particle going around a circle is doubled and its angular speed is halved, what happens to the centripetal acceleration?
(a) Becomes 4 times **(b)** Doubled
(c) Halved **(d)** Remains unchanged

5. A particle is moving on a circular path of 10 m radius. At any instant of time, its speed is 5 m/s and the speed is increasing at a rate of 2 m/s^2. At this instant, the magnitude of the net acceleration is
(a) 3.2 m/s^2 **(b)** 2 m/s^2
(c) 2.5 m/s^2 **(d)** 4.3 m/s^2

6. A coin, which is placed on a rotation turn-table, slips when it is placed at a distance of 9 cm from the center. If the angular velocity of the turn-table is tripled, it just slips if its distance from the center is
(a) 27 cm **(b)** 9 cm
(c) 3 cm **(d)** 1 cm

7. A particle moves in a uniform circular motion. Choose the wrong statement:
(a) The particle moves with constant speed.
(b) The acceleration is always normal to the velocity.
(c) The particle moves with uniform acceleration.
(d) The particle moves with variable velocity.

8. A bob of mass m, suspended by a string of length l_1 is given a minimum velocity required to complete a full circle in the vertical plane. At the highest point, it collides elastically with another bob of mass $2m$ suspended by a string of length l_2, which is initially at rest. Both strings are massless and inextensible. If the second bob, after collision acquires the minimum speed required to complete a full circle in the vertical plane, the ratio l_2/l_1 is $(5/4)n$. The value of n is
(a) 3 **(b)** 6
(c) 9 **(d)** 5

9. The direction of the angular velocity vector is along
(a) the tangent to the circular path.
(b) the inward radius.
(c) the outward radius.
(d) the axis of rotation.

10. A stone is tied with a string and is rotated in a circle horizontally. When the string suddenly breaks, the stone moves
(a) tangential to the motion. **(b)** away from the center.
(c) towards the center. **(d)** none of these.

11. The maximum tension that an inextensible ring of radius 1 m and mass density 0.1 kg/m can bear is 40 N. The maximum angular velocity with which it can be rotated in a circular path is
(a) 20 rad/s **(b)** 18 rad/s
(c) 16 rad/s **(d)** 15 rad/s

12. Two particles of equal masses are revolving in circular paths of radii r_1 and r_2, respectively, with the same speed. The ratio of their centripetal forces is
(a) $\frac{r_2}{r_1}$ **(b)** $\sqrt{\frac{r_2}{r_1}}$
(c) $\left(\frac{r_1}{r_2}\right)^2$ **(d)** $\left(\frac{r_2}{r_1}\right)^2$

13. The angular speed of a fly-wheel, which is making 120 rev/min, is
(a) 2π rad/s **(b)** $4\pi^2$ rad/s
(c) π rad/s **(d)** 4π rad/s

14. A mass of 2 kg is whirled in a horizontal circle by means of a string at an initial speed of 5 rev/min. On keeping the radius constant, the tension in the string is doubled. The new speed is nearly
(a) $\frac{5}{\sqrt{2}}$ rev/min **(b)** 10 rev/min
(c) $10\sqrt{2}$ rev/min **(d)** $5\sqrt{2}$ rev/min

15. A stone of mass of 16 kg is attached to a string 144 m long and is whirled in a horizontal circle. The maximum tension the string can withstand is 16 N. The maximum velocity of revolution, which can be given to the stone without breaking it, is
(a) 20 m/s **(b)** 16 m/s
(c) 14 m/s **(d)** 12 m/s

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16. A particle moves with constant speed v along a circular path of radius r and completes the circle in time T. The acceleration of the particle is

(a) $\frac{2\pi v}{T}$ **(b)** $\frac{2\pi r}{T}$

(c) $\frac{2\pi r^2}{T}$ **(d)** $\frac{2\pi v^2}{T}$

17. A stone is attached to a rope of length $l = 80$ cm is rotated with a speed of 240 rpm. At the moment when the velocity is directed vertically upwards, the rope breaks. To what height does the stone rise further?

(a) 1.2 m **(b)** 41.2 m
(c) 20.6 m **(d)** 24.9 m

18. If the wheel of a cycle, whose radius is 0.4 m, completes 1 revolution in 1 s, then the acceleration of the cycle is

(a) 0.4π m/s^2 **(b)** 0.8π m/s^2
(c) $0.4\pi^2$ m/s^2 **(d)** $1.6\pi^2$ m/s^2

19. A stone is tied to one end of a string. On holding the other end, the string is whirled in a horizontal plane with progressively increasing speed. It breaks at some speed because

(a) the gravitational forces of the Earth is greater than the tension in string.
(b) the required centripetal force is greater than the tension sustained by the string.
(c) the required centripetal force is less than the tension in the string.
(d) the centripetal force is greater than the weight of the stone.

20. A stone of mass 250 g, which is attached at the end of a string of length 1.25 m is whirled in a horizontal circle at a speed of 5 m/s. What is the tension in the string?

(a) 2.5 N **(b)** 5 N
(c) 6 N **(d)** 8 N

21. A tube of length L is filled completely with an incompressible liquid of mass M and closed at both the ends. The tube is then rotated in a horizontal plane about one of its ends with a uniform angular velocity ω. The force exerted by the liquid at the outer end is

(a) $\frac{ML\omega^2}{2}$ **(b)** $ML\omega^2$

(c) $\frac{ML^2\omega^2}{2}$ **(d)** $\frac{ML\omega^2}{4}$

22. A van is moving with speed of 108 km/h. on level road where coefficient of friction between tires and road 0.5. For the safe driving of van, the minimum radius of curvature of the road is ($g = 10$ m/s^2)

(a) 80 m **(b)** 40 m
(c) 180 m **(d)** 20 m

23. A weightless thread can bear tension up to 3.7 kg weight. A stone of mass 500 g is tied at its one end and revolved in a vertical circular path of radius 4 m. If $g = 10$ m/s^2, then the maximum angular velocity of the stone is (rad/s) is

(a) 3 **(b)** 4
(c) 5 **(d)** 6

24. A wheel is subjected to uniform angular acceleration about its axis. Initially, its angular velocity is zero. In the first 2 s, it rotates through θ_1 and in next 2 s, it rotates through θ_2. The ratio θ_2/θ_1 is

(a) 1 **(b)** 2
(c) 3 **(d)** 4

25. A wheel of diameter 20 cm is rotating at 600 rpm. The linear velocity of particle at its rim is

(a) 6.28 cm/s **(b)** 62.8 cm/s
(c) 0.628 cm/s **(d)** 628.4 cm/s

26. A wheel rotates with a constant angular velocity of 600 rpm. What is the angle through which the wheel rotates in 1 s?

(a) 5π rad **(b)** 20π rad
(c) 15π rad **(d)** 10π rad

27. When an aeroplane is taking a turn in a horizontal plane

(a) it remains horizontal.
(b) it inclines inward.
(c) it inclines outward.
(d) its wings becomes vertical.

28. An electric fan has blades of length 30 cm as measured from the axis of rotation. If the fan is rotating at 1200 rpm, then the acceleration of a point on the tip of the blade is (Given: $\pi^2 = 10$)

(a) 1600 m/s^2 **(b)** 3200 m/s^2
(c) 4800 m/s^2 **(d)** 6000 m/s^2

29. An electron revolves around the nucleolus the radius of the circular orbit is r to double the kinetic energy of electron its orbit radius of

(a) $\sqrt{2}r$ **(b)** $-\sqrt{2}r$
(c) $\sqrt{3}r$ **(d)** $-\sqrt{3}r$

30. Angle between the centripetal acceleration and radius vector is

(a) 90° **(b)** 180°
(c) 0° **(d)** 45°

31. The angular velocity of an hour hand of a watch is equal to

(a) $\frac{\pi}{43,200}$ rad/s **(b)** $\frac{\pi}{21,600}$ rad/s

(c) $\frac{\pi}{30}$ rad/s **(d)** $\frac{\pi}{1800}$ rad/s

32. At a curved path of the road, the road bed is raised a little on the side away from the center of the curved path. The slope of the road bed is given by

(a) $\tan\theta = \frac{r}{gv^2}$ **(b)** $\tan\theta = \frac{rg}{v^2}$

(c) $\tan\theta = \frac{v^2 g}{r}$ **(d)** $\tan\theta = \frac{v^2}{rg}$

33. Centripetal force in vector form can be expressed as

(a) $\vec{F} = \frac{mv^2}{r}$ **(b)** $\vec{F} = \frac{mv^2}{r}\vec{r}$

(c) $\vec{F} = \frac{mv^2}{r^2}\vec{r}$ **(d)** $\vec{F} = -\frac{mv^2}{r}\vec{r}$

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34. For a particle performing a uniform circular motion, the acceleration is
- **(a)** constant in direction.
- **(b)** constant in magnitude but not in direction.
- **(c)** constant in magnitude and direction.
- **(d)** constant neither in magnitude nor in direction.

35. If a particle moves with uniform speed, then its tangential acceleration is
- **(a)** zero. **(b)** constant.
- **(c)** infinite. **(d)** none of these.

36. If a stone of mass m is rotated in a vertical circular path of radius 1 m, the critical velocity is
- **(a)** 6.32 m/s **(b)** 3.13 m/s
- **(c)** 9.48 m/s **(d)** 12.64 m/s

37. If T_1 and T_2 are the periods of a simple pendulum and a conical pendulum, respectively, of the same length, then
- **(a)** $T_1 = T_2$ **(b)** $T_1 > T_2$
- **(c)** $T_1 < T_2$ **(d)** $T_1 = \frac{T_2}{2}$

38. In a tension of a string is 6.4 N. Load at the lower end of a string is 0.1 kg the length of string is 6 m, then its angular velocity is ($g = 10$ m/s^2)
- **(a)** 4 rad/s **(b)** 3 rad/s
- **(c)** 2 rad/s **(d)** 1 rad/s

39. In a vertical circle of radius r at what point in its path, a particle has a tension equal to zero?
- **(a)** Highest point.
- **(b)** Lowest point.
- **(c)** Any point.
- **(d)** An horizontal point.

40. In an atom, two electrons move round the nucleus in circular orbits of radii R and $4R$, respectively, the ratio of time taken by them to complete 1 rev is
- **(a)** $\frac{1}{4}$ **(b)** 4
- **(c)** 8 **(d)** $\frac{1}{8}$

More than One Correct Choice Type

41. A constant force $\vec{F}$ is applied on a particle of mass m and finds that the particle moves in a circle of radius r with a uniform speed v as seen from an inertial frame of reference.
- **(a)** This is not possible.
- **(b)** There are other forces on the particle.
- **(c)** The resultant of the other forces is mv^2/r towards the center.
- **(d)** The resultant of the other forces varies in magnitude as well as in direction.

42. A cart moves with a constant speed along a horizontal circular path. From the cart, a particle is thrown up vertically with respect to the cart. The particle will
- **(a)** land outside the circular path.
- **(b)** land somewhere on the circular path.
- **(c)** follow a parabolic path.
- **(d)** follow an elliptical path.

43. A car of mass M is moving on a horizontal circular path of radius r. At an instant, its speed is v and is increasing at a rate a.
- **(a)** The acceleration of the car is towards the center of the path.
- **(b)** The magnitude of the frictional force on the car is greater than mv^2/r.
- **(c)** The friction coefficient between the ground and the car is not less than a/g.
- **(d)** The friction coefficient between the ground and the car is $\mu = \tan^{-1}(v^2/rg)$.

44. A car moves on a circular road, describing equal angles about the center in equal intervals of time. Which of the following statements about the velocity of car are not true?
- **(a)** Velocity is constant.
- **(b)** Magnitude of velocity is constant but the direction changes.
- **(c)** Both magnitude and direction of velocity change.
- **(d)** Velocity is directed towards the center of circle.

45. A particle moves in a circle of radius 20 cm. Its linear speed is given by $v = 2t$, where t is in s and v in m/s. Then
- **(a)** the radial acceleration at $t = 2$ s is 80 m/s^2.
- **(b)** the tangential acceleration at $t = 2$ s is 2 m/s^2.
- **(c)** the net acceleration at $t = 2$ s is greater than 80 m/s^2.
- **(d)** Tangential acceleration remains constant in magnitude.

46. An object follows a curved path. The following quantity/quantities may remain constant during the motion:
- **(a)** Speed
- **(b)** Velocity
- **(c)** Acceleration
- **(d)** Magnitude of acceleration

Linked Comprehension

Paragraph for Questions 47 and 48: When a cyclist turns on a circular path, the necessary centripetal force is provided by friction between the tires and the road. If centripetal force is not provided by friction, then for the vehicle to move on circular path, the track is banked.

47. The correct angle of banking for a curved smooth road of radius 120 m for a speed of 108 km/h ($g = 10$ m/s^2) is
- **(a)** 30° **(b)** 37°
- **(c)** 45° **(d)** 60°

48. If the sped of a vehicle is doubled, then for safety of vehicle
- **(a)** the angle of banking must be doubled.
- **(b)** the angle to banking must be four times.
- **(c)** the tangent of angle of banking must be doubled.
- **(d)** the tangent of angle of banking must be increased to four times.

Matrix-Match

49. A particle of mass m is moving on a circular path of constant r such that its centripetal acceleration a_C is varying

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with time t as $a_C = k^2rt^2$, when k is a constant. Then, match the columns of the following data:

Column I	Column II
(a) Centripetal force	**(p)** mkr
(b) Tangential force	**(q)** mk^2r^2t
(c) Power delivered by centripetal force	**(r)** mk^2rt^2
(d) Power delivered by tangential force	**(s)** Zero

Directions for Questions 50—51: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

50. In the given table, Column I shows the different values of masses of the body, Column II shows the different values of angular speed of the body and Column III shows the radius of circle in which the body moves.

Column I	Column II	Column III
(I) Mass = 0.035 kg	**(i)** Angular speed = 2.09 rad/s	**(J)** Radius = 0.75 m
(II) Mass = 0.056 kg	**(ii)** Angular speed = 2.11 rad/s	**(K)** Radius = 0.85 m
(III) Mass = 0.06 kg	**(iii)** Angular speed = 1.9 rad/s	**(L)** Radius = 0.72 m
(IV) Mass = 0.029 kg	**(iv)** Angular speed = 2.17 rad/s	**(M)** Radius = 0.80 m

(1) Which combination has 0.13 N as the centripetal force?
(a) (I) (iii) (L) **(b)** (IV) (i) (M)
(c) (II) (iv) (K) **(d)** (I) (i) (K)

(2) Which combination has 0.199 N as the centripetal force?
(a) (III) (ii) (L) **(b)** (II) (ii) (M)
(c) (II) (iii) (K) **(d)** (I) (i) (M)

(3) Which combination has 0.041 N as the centripetal force?
(a) (IV) (iii) (J) **(b)** (I) (i) (K)
(c) (III) (iii) (L) **(d)** (I) (ii) (L)

51. In the given table Column I shows the values of linear velocity, Column II shows the values of angular velocity and Column III shows the radius of the circle in which circular (angular) motion takes place.

Column I	Column II	Column III
(I) Linear velocity = 6 m/s	**(i)** Angular velocity = 4 rad/s	**(J)** Radius = 3 m
(II) Linear velocity = 10.15 m/s	**(ii)** Angular velocity = 2.11 rad/s	**(K)** Radius = 2 m
(III) Linear velocity = 3 m/s	**(iii)** Angular velocity = 3 rad/s	**(L)** Radius = 4 m
(IV) Linear velocity = 16 m/s	**(iv)** Angular velocity = 3.38 rad/s	**(M)** Radius = 4 m

(1) Which combination describes the conditions when body starts with zero initial linear velocity and accelerates with 2 m/s^2 for 3 s?
(a) (II)(iv)(L) **(b)** (III)(i)(M)
(c) (I)(iii)(K) **(d)** (IV)(ii)(J)

(2) Which combination describes the conditions when body starts with zero initial angular velocity and accelerates with 2 rad/s^2 for 3 s?
(a) (IV)(i)(L) **(b)** (III)(iv)(J)
(c) (I)(ii)(M) **(d)** (II)(iii)(K)

(3) Which combination describes the conditions when body is displaced by 28 m in 4 s and accelerates with 2 m/s^2?
(a) (IV)(iii)(K) **(b)** (I)(ii)(L)
(c) (III)(i)(M) **(d)** (II)(iv)(J)

Integer Type

52. A particle of mass m is observed from an inertial frame of reference and is found to move in a circle of radius r with a uniform speed v. The centrifugal force on it is _____.

53. A manufacturer of CD drives claims that the player can spin the disc as frequently as 1200 revolutions per minute. If the spinning is at this rate, what is the speed (in m/s) of the outer row of data on the disc; this row is located 5.6 cm from the center of the disc?

ANSWER KEY

Checkpoints

1. $\frac{g}{2v_0}\cos\theta$ **2.** (a) $-(4\text{ m/s})\hat{i}$; (b) $-(8\text{ m/s}^2)\hat{j}$

3. (a) This situation is not possible as perpendicular component of acceleration is away from the curvature; (b) This is possible but speed will be decreasing.

4. $80\sqrt{2}$ **5.** mv^2/R

Problems

1. (a) 545 N; (b) 8.40° **2.** (a) 14 s; (b) 5.8×10^2 N; (c) 9.9×10^2 N

3. (a) 4.9×10^3 N; (b) upward; (c) 0 N; (d) No direction since the normal force is zero.

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4. 1.57×10^3 N **5.** (a) 403 N m/s; (b) -1.50×10^3 N/s **6.** $R \approx 3.4 \times 10^3$ m

7. (a) 4.5 kN; (b) Upward; (c) 2.8 kN; (d) Downward

8. 11° **9.** 3.3×10^3 N. **10.** 0.103 **11.** 2.02 m/s

12. (a) 8.0×10^3 N; (b) 6.9 × 10 N; (c) 20 m/s; (d) 1.6 × 10 N; (e) No circular path is possible

13. (a) 8.74 N; (b) 37.9 N; (c) 6.45 m/s; (d) Leftward (radially inward) **14.** $\sqrt{gr}2Mg$ **15.** $mv^2/2\pi R$

16. (a) 0.0338 N; (b) 9.77 N **17.** 874 N

18. (a) $\sqrt{Rg}$; (b) a distance $(\pi R)/3$ along the bridge from the highest point; (c) $\sqrt{gR\cos\left(\frac{L}{2R}\right)}$

19. (a) 0.2 N; (b) 30° **20.** (a) $\frac{mv^2}{R}$; (b) $\frac{\mu mv^2}{R}$; (c) $\frac{-\mu v}{R^2}$; (d) $v_0 e^{-2\pi\mu}$

21. (a) 975 N, 1025 N; (b) 0, 707 N, 0; (c) 682 N, 732 N; (d) 1.037 N

22. $v = \omega\sqrt{L^2 - a^2}$ **23.** 1.16 N **24.** (a) $\sqrt{\frac{\mu g}{L}}$; (b) $\left[\left(\frac{\mu g}{L}\right)^2 - \alpha^2\right]^{1/4}$ **25.** $\sqrt{\frac{2L}{\omega^2 R\cos\theta}}$

26. $[(\mu^2 g^2 - a^2)R^2]^{1/4}$ **27.** $\frac{\omega^2 R}{3}, \frac{4}{3}m\omega^2 R$ **28.** (a) 0.40 N; (b) 1.9 s **29.** 178 km/h

30. (a) 3.21 × 103 N; (b) 3.75×10^3 N **31.** (a) Lowest position; (b) 9.5 m/s

32. (a) 0.30 m/s; (b) 1.8 m/s^2; (d) 3.6 × 10–3 N; (e) Same direction as acceleration; (f) 0.37

33. (a) 0.96 m/s; (b) 0.021 **34.** 9.5×10^{-8} N **35.** (a) $\tan\theta = v^2/rg$; (b) 19.7°

36. (a) 364 N; (b) 784 N **37.** (a) 0.344%; (b) 84.4 min; (c) Same **38.** $v = (rg\cot\theta)^{1/2}$ **39.** 14 m/s

40. (a) $lk\sin\theta$; (b) $2kl(1 - \cos\theta)$ **41.** 0.97 m **42.** (a) 149 m; (b) 7.5 m/s^2

43. (a) $a_r = \frac{RV^2}{a^2}$; (b) $a_\theta = \frac{2vV}{a}$ **44.** 6 m/s^2, 125 N **45.** 20 m/s

46. $\rho = 68.3$ m, $N_p = 2.654$ kN **47.** $N = 276.714$ lb, $f = 13.444$ lb **48.** 0.581 m

49. $T = 1.82$ N, $N_B = 0.844$ N **50.** $T = 0$, $T_D = 104.1$ N **51.** 1.48 m/s **52.** 7.394 s

53. $F_s = 3.419$ lb, $F_t = 0$ **54.** (a) 30 cm/s; (b) 180 cm/s^2, radially inward **55.** $\frac{m(g\cot\theta + \omega^2 r)}{2\pi}$

56. $\left(\frac{g(\sin\theta - \mu\cos\theta)}{R\sin\theta(\cos\theta + \mu\sin\theta)}\right)^{1/2}$ to $\left(\frac{g(\sin\theta + \mu\cos\theta)}{R\sin\theta(\cos\theta - \mu\sin\theta)}\right)^{1/2}$ **57.** 0.60

58. $\frac{g(\sin\theta - \mu\cos\theta)}{R\sin\theta(\cos\theta + \mu\sin\theta)}$ **59.** 78.87 N **60.** $\frac{k(2\pi r - L)}{R^2\lambda}$ **61.** $\cos^{-1}\left(\frac{9}{25}\right)$ **62.** 3.08 rad/s

63. $\frac{g\sin\theta}{g\cos\theta + \omega^2 r}$ **64.** $(g\sqrt{3})^{1/2}$

Practice Questions

Single Correct Choice Type

1. (b)	**2.** (d)	**3.** (c)	**4.** (d)	**5.** (a)
6. (d)	**7.** (c)	**8.** (c)	**9.** (d)	**10.** (a)
11. (a)	**12.** (a)	**13.** (d)	**14.** (d)	**15.** (d)
16. (a)	**17.** (c)	**18.** (d)	**19.** (b)	**20.** (b)
21. (b)	**22.** (c)	**23.** (b)	**24.** (c)	**25.** (d)
26. (b)	**27.** (b)	**28.** (c)	**29.** (a)	**30.** (b)
31. (b)	**32.** (d)	**33.** (c)	**34.** (b)	**35.** (a)
36. (b)	**37.** (b)	**38.** (b)	**39.** (a)	**40.** (d)

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More than One Correct Choice Type

41. (b), (d) **42.** (a), (b) **43.** (b), (c) **44.** (a), (d), (c) **45.** (a), (b), (c), (d)
46. (a), (c), (d)

Linked Comprehension

47. (b) **48.** (d)

Matrix-Match

49. (a) → (r); (b) → (p); (c) → (s); (d) → (q)
50. (1) → (d); (2) → (b); (3) → (a)
51. (1) → (c); (2) → (a); (3) → (d)

Integer Type

52. 0 **53.** 7.0

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CHAPTER 8

Work, Power, and Energy

8.1 | WHAT IS PHYSICS?

One of the fundamental goals of physics is to investigate something that everyone talks about: energy. The topic is obviously important. Indeed, our civilization is based on acquiring and effectively using energy.

For example, everyone knows that any type of motion requires energy: Flying across the Pacific Ocean requires it. Lifting material to the top floor of an office building or to an orbiting space station requires it. Throwing a fastball requires it. We spend a tremendous amount of money to acquire and use energy. Wars have been started because of energy resources. Wars have been ended because of a sudden, overpowering use of energy by one side. Everyone knows many examples of energy and its use, but what does the term *energy* really mean?

The term *energy* is so broad that a clear definition is difficult to write. Technically, energy is a scalar quantity associated with the state (or condition) of one or more objects. However, this definition is too vague to be of help to us now.

A looser definition might at least get us started. Energy is a number that we associate with a system of one or more objects. If a force changes one of the objects by, say, making it move, then the energy number changes. After countless experiments, scientists and engineers realized that if the scheme by which we assign energy numbers is planned carefully, the numbers can be used to predict the outcomes of experiments and, even more important, to build machines, such as flying machines. This success is based on a wonderful property of our universe: Energy can be transformed from one type to another and transferred from one object to another, but the total amount is always the same (energy is *conserved*). No exception to this *principle of energy conservation* has ever been found.

Money. Think of the many types of energy as being numbers representing money in many types of bank accounts. Rules have been made about what such money numbers mean and how they can be changed. You can transfer money numbers from one account to another or from one system to another, perhaps electronically with nothing material actually moving. However, the total amount (the total of all the money numbers) can always be accounted for: It is always conserved.

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8.3 Work
8.4 Calculation of Work for Uniform Force
8.5 Work Done by the Gravitational Force
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8.7 Work Done by a General Variable (Nonuniform) Force
8.8 Validity of Work–Kinetic Energy Theorem in Inertial Reference Frames
8.9 Potential Energy
8.10 Work and Potential Energy
8.11 Path Independence of Conservative Forces
8.12 Determining Potential Energy Values
8.13 Work–Mechanical Energy Theorem
8.14 Conservation of Mechanical Energy
8.15 Work Done on a System by an External Force
8.16 Conservation of Energy
8.17 Power
8.18 Relation Between Conservative Force and Potential Energy
8.19 Vertical Circular Motion

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In this chapter, we focus on two types of energy—*kinetic energy and potential energy*—and on only one type in which energy can be transferred—*work*. We also examine a few other types of energy and how the principle of energy conservation can be written as equations to be solved.

8.2 | KINETIC ENERGY

Key Concept

- The kinetic energy K associated with the motion of a particle of mass m and speed v, where v is well below the speed of light, is

$$K = \frac{1}{2}mv^2 \quad \text{(kinetic energy)}.$$

Kinetic energy K is energy associated with the *state of motion* of an object. The faster the object moves, the greater is its kinetic energy. When the object is stationary, its kinetic energy is zero.

For an object of mass m whose speed v is well below the speed of light,

$$K = \frac{1}{2}mv^2 \quad \text{(kinetic energy)}. \tag{8-1}$$

For example, a 3.0 kg duck flying past us at 2.0 m/s has a kinetic energy of 6.0 kg · m²/s²; that is, we associate that number with the duck's motion.

The SI unit of kinetic energy (and all types of energy) is the **joule** (J), named for James Prescott Joule, an English scientist of the 1800s, and defined as

$$1 \text{ joule} = 1 \text{ J} = 1 \text{ kg} \cdot \text{m}^2/\text{s}^2. \tag{8-2}$$

Thus, the flying duck has a kinetic energy of 6.0 J.

SAMPLE PROBLEM 8.01

Kinetic energy, train crash

In 1896 in Waco, Texas, William Crush parked two locomotives at opposite ends of a 6.4-km-long track, fired them up, tied their throttles open, and then allowed them to crash head-on at full speed (Fig. 8-1) in front of 30,000 spectators. Hundreds of people were hurt by flying debris; several were killed. Assuming each locomotive weighed 1.2×10^6 N and its acceleration was a constant 0.26 m/s², what was the total kinetic energy of the two locomotives just before the collision?

KEY IDEAS

(1) We need to find the kinetic energy of each locomotive with Eq. 8-1, but that means we need each locomotive's speed just before the collision and its mass. (2) Because we can assume each locomotive had constant acceleration, we can use the equations in Table 2-1 (Chapter 2) to find its speed v just before the collision.

Calculations: We choose Eq. 2-16 (Chapter 2) because we know values for all the variables except v:

$$v^2 = v_0^2 + 2a(x - x_0).$$

With $v_0 = 0$ and $x - x_0 = 3.2 \times 10^3$ m (half the initial separation), this yields

$$v^2 = 0 + 2(0.26 \text{ m/s}^2)(3.2 \times 10^3 \text{ m}),$$

or

$$v = 40.8 \text{ m/s} = 147 \text{ km/h}.$$

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We can find the mass of each locomotive by dividing its given weight by g:

$$m = \frac{1.2 \times 10^6 \text{ N}}{9.8 \text{ m/s}^2} = 1.22 \times 10^5 \text{ kg}.$$

Now, using Eq. 8-1, we find the total kinetic energy of the two locomotives just before the collision as

$$K = 2\left(\frac{1}{2}mv^2\right) = 1.22 \times 10^5 \text{ kg})(40.8 \text{ m/s})^2$$
$$= 2.0 \times 10^8 \text{ J}. \qquad \text{(Answer)}$$

This collision was like an exploding bomb.

Courtesy Library of Congress

Figure 8-1 The aftermath of an 1896 crash of two locomotives.

8.3 | WORK

Key Concept

- Work W is energy transferred to or from an object via a force acting on the object. Energy transferred to the object is positive work, and from the object, negative work.

If you accelerate an object to a greater speed by applying a force to the object, you increase the kinetic energy K ($= 1/2\ mv^2$) of the object. Similarly, if you decelerate the object to a lesser speed by applying a force, you decrease the kinetic energy of the object. We account for these changes in kinetic energy by saying that your force has transferred energy *to* the object from yourself or *from* the object to yourself. In such a transfer of energy via a force, **work** W is said to be *done on the object by the force*. More formally, we define work as follows:

Work W is energy transferred to or from an object by means of a force acting on the object. Energy transferred to the object is positive work, and energy transferred from the object is negative work.

"Work," then, is transferred energy; "doing work" is the act of transferring the energy. Work has the same units as energy and is a scalar quantity.

The term *transfer* can be misleading. It does not mean that anything material flows into or out of the object; that is, the transfer is not like a flow of water. Rather, it is like the electronic transfer of money between two bank accounts: The number in one account goes up while the number in the other account goes down, with nothing material passing between the two accounts.

Note that we are not concerned here with the common meaning of the word "work," which implies that *any* physical or mental labor is work. For example, if you push hard against a wall, you tire because of the continuously repeated muscle contractions that are required, and you are, in the common sense, working. However, such effort does not cause an energy transfer to or from the wall and thus is not work done on the wall as defined here.

To avoid confusion in this chapter, we shall use the symbol W only for work and shall represent a weight with its equivalent mg.

Here, it is important to note that a particle may be subjected to many forces. It may have some instantaneous velocity which causes very small displacement $d\vec{s}$ in very small time. The work done by $\vec{F}_1$ is $dW = \vec{F}_1 \cdot d\vec{s}$. Here, force $\vec{F}_1$ may not be the cause of small displacement $d\vec{s}$. Work is a relative quantity because displacement is relative. (Remember force is not dependent on choice of reference frame.)

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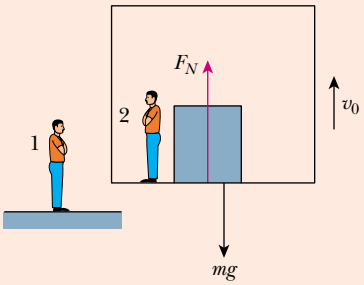
CHECKPOINT 1

A lift is going up with constant velocity v_0. Calculate the work done by normal force F_N from the reference frame of ground and lift in time t.



8.4 | CALCULATION OF WORK FOR UNIFORM FORCE

Key Concepts

- The work done on a particle by a constant force $\vec{F}$ during displacement $\vec{d}$ is

$$W = Fd\cos\phi = \vec{F}\cdot\vec{d} \quad \text{(work, constant force)},$$

  in which ϕ is the constant angle between the directions of $\vec{F}$ and $\vec{d}$.
- Only the component of $\vec{F}$ that is along the displacement $\vec{d}$ can do work on the object.
- When two or more forces act on an object, their net work is the sum of the individual works done by the forces, which is also equal to the work that would be done on the object by the net force $\vec{F}_{net}$ of those forces.
- For a particle, a change ΔK in the kinetic energy equals the net work W done on the particle:

$$\Delta K = K_f - K_i = W \quad \text{(work–kinetic energy theorem)},$$

  in which K_i is the initial kinetic energy of the particle and K_f is the kinetic energy after the work is done. The equation rearranged gives us

$$K_f = K_i + W.$$

Work done is given by

$$dW = \vec{F}\cdot d\vec{s}$$

The total work done is given by

$$dW = \int \vec{F}\cdot d\vec{s}$$

For uniform force, F can be taken out of integral. Hence, $W = \vec{F}\int d\vec{s}$

Thus, $W = \vec{F}\cdot\vec{s}$ (*this is true only for uniform forces*)

$$W = F(s\cos\theta)$$

where $s\cos\theta$ is component of displacement along force. Alternately, we can write

$$W = (F\cos\theta)s$$

where $F\cos\theta$ is component of force along displacement.

Relation Between Work and Kinetic Energy for Uniform Forces

Finding an Expression for Work

Let us consider a bead that can slide along a frictionless wire that is stretched along a horizontal x axis (Fig. 8-2). A constant force $\vec{F}$, directed at an angle ϕ to the wire, accelerates the bead along the wire. We can relate the force and the acceleration with Newton's second law, written for components along the x axis:

$$F_x = ma_x, \tag{8-3}$$

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where m is the bead's mass. As the bead moves through a displacement $\vec{d}$, the force changes the bead's velocity from an initial value $\vec{v}_0$ to some other value $\vec{v}$. Because the force is constant, we know that the acceleration is also constant. Thus, we can use Eq. 2-16 to write, for components along the x axis,

$$v^2 = v_0^2 + 2a_x d. \tag{8-4}$$

Solving this equation for a_x, substituting into Eq. 8-3, and rearranging then give us

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = F_x d. \tag{8-5}$$

The first term is the kinetic energy K_f of the bead at the end of the displacement d, and the second term is the kinetic energy K_i of the bead at the start. Thus, the left side of Eq. 8-5 tells us the kinetic energy has been changed by the force, and the right side tells us the change is equal to $F_x d$. Therefore, the work W done on the bead by the force (the energy transfer due to the force) is

$$W = F_x d. \tag{8-6}$$

If we know values for F_x and d, we can use this equation to calculate the work W.

To calculate the work a force does on an object as the object moves through some displacement, we use only the force component along the object's displacement. The force component perpendicular to the displacement does zero work.

From Fig. 8-2, we see that we can write F_x as $F \cos \phi$, where ϕ is the angle between the directions of the displacement $\vec{d}$ and the force $\vec{F}$. Thus,

$$W = Fd \cos \phi \quad \text{(work done by a constant force)}. \tag{8-7}$$

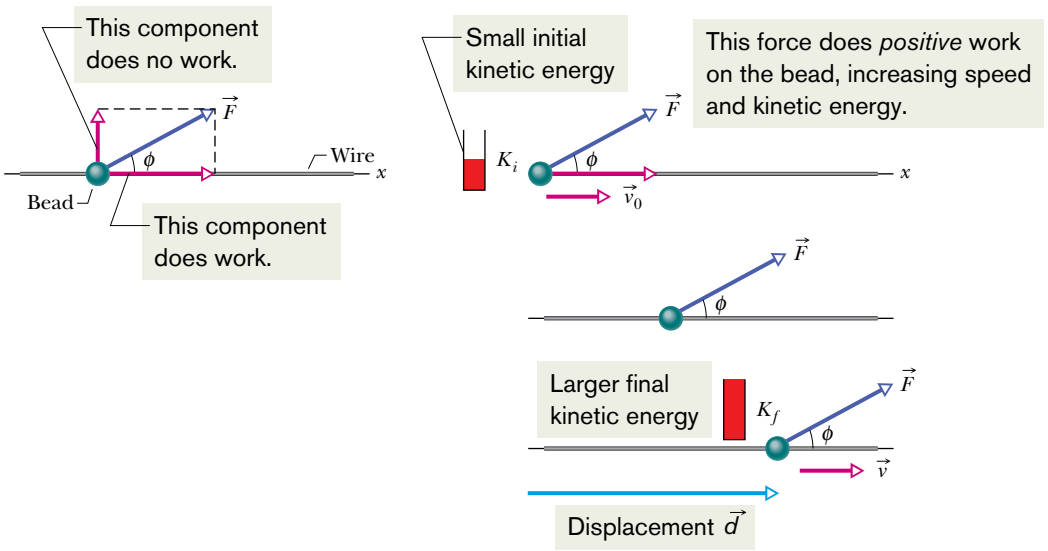

Figure 8-2 A constant force $\vec{F}$ directed at angle ϕ to the displacement $\vec{d}$ of a bead on a wire accelerates the bead along the wire, changing the velocity of the bead from $\vec{v}_0$ to $\vec{v}$. A "kinetic energy gauge" indicates the resulting change in the kinetic energy of the bead, from the value K_i to the value K_f.

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We can use the definition of the scaler (dot) product (Eq. 3-20) to write

$$W = \vec{F} \cdot \vec{d} \quad \text{(work done by a constant force)}, \tag{8-8}$$

where F is the magnitude of $\vec{F}$. (You may wish to review the discussion of scaler products in Section 3.3.) Equation 8-8 is especially useful for calculating the work when $\vec{F}$ and $\vec{d}$ are given in unit-vector notation.

Cautions. There are two restrictions to using Eqs. 8-6 through 8-8 to calculate work done on an object by a force. First, the force must be a *constant force*; that is, it must not change in magnitude or direction as the object moves. (Later, we shall discuss what to do with a *variable force* that changes in magnitude.) Second, the object must be *particle-like*. This means that the object must be *rigid*; all parts of it must move together, in the same direction. In this chapter we consider only particle-like objects, such as the bed and its occupant being pushed in Fig. 8-3.

Signs for Work. The work done on an object by a force can be either positive work or negative work. For example, if angle ϕ in Eq. 8-7 is less than 90°, then cos ϕ is positive and thus so is the work. However, if ϕ is greater than 90° (up to 180°), then cos ϕ is negative and thus so is the work. (Can you see that the work is zero when $\phi = 90°$?) These results lead to a simple rule. To find the sign of the work done by a force, consider the force vector component that is parallel to the displacement:

A force does positive work when it has a vector component in the same direction as the displacement, and it does negative work when it has a vector component in the opposite direction. It does zero work when it has no such vector component.

Units for Work. Work has the SI unit of the joule, the same as kinetic energy. However, from Eqs. 8-6 and 8-7 we can see that an equivalent unit is the newton meter (N · m). The corresponding unit in the British system is the foot-pound (ft · lb). Extending Eq. 8-2, we have

$$1 \text{ J} = 1 \text{ kg} \cdot \text{m}^2/\text{s}^2 = 1 \text{ N} \cdot \text{m} = 0.738 \text{ ft} \cdot \text{lb}. \tag{8-9}$$

Net Work. When two or more forces act on an object, the **net work** done on the object is the sum of the works done by the individual forces. We can calculate the net work in two ways. (1) We can find the work done by each force and then sum those works. (2) Alternatively, we can first find the net force $\vec{F}_{net}$ of those forces. Then we can use Eq. 8-7, substituting the magnitude F_{net} for F and also the angle between the directions of $\vec{F}_{net}$ and $\vec{d}$ for ϕ. Similarly, we can use Eq. 8-8 with $\vec{F}_{net}$ substituted for $\vec{F}$.

SAMPLE PROBLEM 8.02

Work done on box by applied force

Figure 8-3 shows four situations in which the same box is pulled by an applied force $\vec{F}$ up a frictionless ramp through (and then past) the same vertical distance h. In each situation, the force has a magnitude of 10 N. In situations (*b*) and (*d*), the force is directed along the plane; in situations (*a*) and (*c*), it is directed at an angle $\phi = 37°$ to the plane, as shown. Rank the situations according to the work done on the box in the vertical distance h by the applied force. Also, discuss whether answer depends on the initial speed of box or the presence of other forces.

KEY IDEAS

Situations (*b*) and (*d*) will involve more work than (*a*) and (*c*), respectively, because though displacement in (*b*) and (*a*) is same and similarly displacement in (*d*) and (*a*) is also same, but in (*b*) and (*d*) force is acting along displacement. Thus, scalar product of force and displacement will be larger.

Calculations: Now, we compare magnitude of displacement in (*b*) and (*d*); we can see easily that displacement is larger for (*d*). Thus, by similar argument, we can prove that displacement of (*c*) is larger than (*a*).

Hence, $\quad W_d > W_b, \quad \text{and} \quad W_c > W_a$

Numerical calculations also reveal that our arguments are correct. We find the relation between W_b and W_c.

In order to derive the relations for work done, we first need to calculate the displacement, that is, the length of

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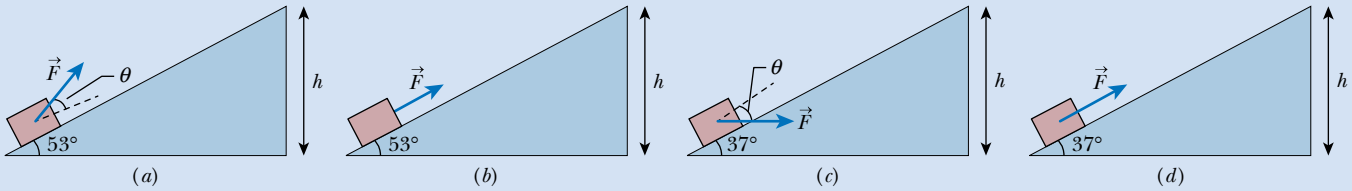

Figure 8-3 Boxes being pulled on inclines by applying forces in different directions.

the ramp in terms of vertical distance h. Also, in situations the force (F) needs to be resolved along the direction of displacement to determine the work done.

$$W_d = Fh\,\text{cosec}\,37° = \frac{5Fh}{3}$$

$$W_b = Fh\,\text{cosec}\,53° = \frac{5Fh}{4}$$

$$W_c = (F\cos 37°)(h\,\text{cosec}\,37°) = \frac{4Fh}{3}$$

$$W_a = (F\cos 37°)(h\,\text{cosec}\,53°) = Fh$$

Hence, $W_d > W_c > W_b > W_a$ (Answer)

SAMPLE PROBLEM 8.03

Work done on an object by a force specified time interval

A force $\vec{F} = (3.00\text{ N})\hat{i} + (7.00\text{ N})\hat{j} + (7.00\text{ N})\hat{k}$ acts on a 2.00 kg mobile object that moves from an initial position of $\vec{r}_i = (3.00\text{ m})\hat{i} - (2.00\text{ m})\hat{j} + (5.00\text{ m})\hat{k}$ to a final position of $\vec{r}_f = +(5.00\text{ m})\hat{i} + (4.00\text{ m})\hat{j} + (7.00\text{ m})\hat{k}$ in 4.00 s. Find the work done on the object by the force in the 4.00 s interval.

Calculations: We know that displacement vector is equal to change in position vector. Thus,

$$\vec{S} = \vec{r}_f - \vec{r}_i = (2\vec{i} + 6\vec{j} + 2\vec{k})\text{ m}$$

Thus, the work done by force $\vec{F}$ is given by

$$W = \vec{F}\cdot\vec{s}$$

$$W = (3\hat{i} + 7\hat{j} + 7\hat{k})$$

$$= (6 + 42 + 14)\text{ J}$$

$$= 62\text{ J}$$ (Answer)

Work–Kinetic Energy Theorem

Equation 8-5 relates the change in kinetic energy of the bead (from an initial $K_i = 1/2\ mv_0^2$ to a later $K_f = 1/2\ mv^2$) to the work W ($= F_x d$) done on the bead. For such particle-like objects, we can generalize that equation. Let ΔK be the change in the kinetic energy of the object, and let W be the net work done on it. Then

$$\Delta K = K_f - K_i = W, \tag{8-10}$$

which says that

$$\begin{pmatrix}\text{change in the kinetic}\\ \text{energy of a particle}\end{pmatrix} = \begin{pmatrix}\text{net work done on}\\ \text{the particle}\end{pmatrix}.$$

We can also write

$$K_f = K_i + W, \tag{8-11}$$

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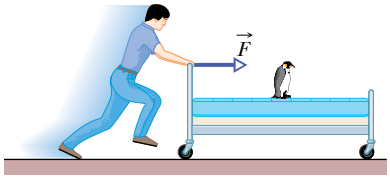

Figure 8-4 A contestant in a bed race. We can approximate the bed and its occupant as being a particle for the purpose of calculating the work done on them by the force applied by the contestant.

which says that

$$\begin{pmatrix}\text{kinetic energy after}\\ \text{the net work is done}\end{pmatrix}=\begin{pmatrix}\text{kinetic energy}\\ \text{before the net work}\end{pmatrix}+\begin{pmatrix}\text{the net}\\ \text{work done}\end{pmatrix}.$$

These statements are known traditionally as the **work–kinetic energy theorem** for particles. They hold for both positive and negative work: If the net work done on a particle is positive, then the particle's kinetic energy increases by the amount of the work. If the net work done is negative, then the particle's kinetic energy decreases by the amount of the work.

For example, if the kinetic energy of a particle is initially 5 J and there is a net transfer of 2 J to the particle (positive net work), the final kinetic energy is 7 J. If, instead, there is a net transfer of 2 J from the particle (negative net work), the final kinetic energy is 3 J.

CHECKPOINT 2

A particle moves along an x axis. Does the kinetic energy of the particle increase, decrease, or remain the same if the particle's velocity changes (a) from –3 m/s to –2 m/s and (b) from –2 m/s to 2 m/s? (c) In each situation, is the work done on the particle positive, negative, or zero?

SAMPLE PROBLEM 8.04

Work done by two constant forces, industrial spies

Figure 8-5*a* shows two industrial spies sliding an initially stationary 225 kg floor safe a displacement $\vec{d}$ of magnitude 8.50 m. The push $\vec{F}_1$ of spy 001 is 12.0 N at an angle of 30.0° downward from the horizontal; the pull $\vec{F}_2$ of spy 002 is 10.0 N at 40.0° above the horizontal. The magnitudes and directions of these forces do not change as the safe moves, and the floor and safe make frictionless contact.

(a) What is the net work done on the safe by forces $\vec{F}_1$ and $\vec{F}_2$ during the displacement $\vec{d}$?

KEY IDEAS

(1) The net work W done on the safe by the two forces is the sum of the works they do individually. (2) Because we can treat the safe as a particle and the forces are constant in both magnitude and direction, we can use either Eq. 8-7 ($W = Fd\cos\phi$) or Eq. 8-8 ($W = \vec{F}\cdot\vec{d}$) to calculate those works. Let's choose Eq. 8-7.

Calculations: From Eq. 8-7 and the free-body diagram for the safe in Fig. 8-5*b*, the work done by $\vec{F}_1$ is

$$W_1 = F_1 d\cos\phi_1 = (12.0\text{ N})(8.50\text{ m})(\cos 30.0°)$$
$$= 88.33\text{ J},$$

and the work done by $\vec{F}_2$ is

$$W_2 = F_2 d\cos\phi_2 = (10.0\text{ N})(8.50\text{ m})(\cos 40.0°)$$
$$= 65.11\text{ J}.$$

Thus, the net work W is

$$W = W_1 + W_2 = 88.33\text{ J} + 65.11\text{ J}$$
$$= 153.4\text{ J} \approx 153\text{ J}. \qquad \text{(Answer)}$$

During the 8.50 m displacement, therefore, the spies transfer 153 J of energy to the kinetic energy of the safe.

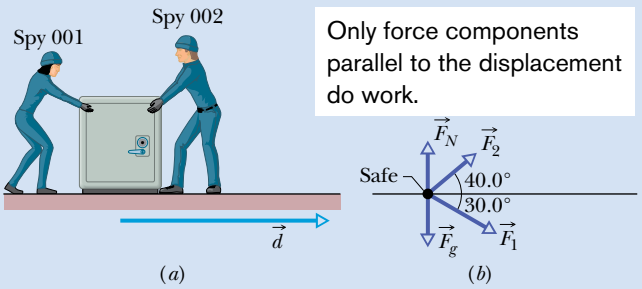

Figure 8-5 (*a*) Two spies move a floor safe through a displacement $\vec{d}$ (*b*) A free-body diagram for the safe.

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(b) During the displacement, what is the work W_g done on the safe by the gravitational force $\vec{F}_g$ and what is the work W_N done on the safe by the normal force $\vec{F}_N$ from the floor?

KEY IDEA

Because these forces are constant in both magnitude and direction, we can find the work they do with Eq. 8-7.

Calculations: Thus, with mg as the magnitude of the gravitational force, we write

$$W_g = mgd\cos 90° = mgd(0) = 0 \qquad \text{(Answer)}$$

And $$W_N = F_N d\cos 90° = F_N d(0) = 0. \qquad \text{(Answer)}$$

We should have known this result. Because these forces are perpendicular to the displacement of the safe, they do zero work on the safe and do not transfer any energy to or from it.

(c) The safe is initially stationary. What is its speed v_f at the end of the 8.50 m displacement?

KEY IDEA

The speed of the safe changes because its kinetic energy is changed when energy is transferred to it by $\vec{F}_1$ and $\vec{F}_2$.

Calculations: We relate the speed to the work done by combining Eqs. 8-10 (the work–kinetic energy theorem) and 8-1 (the definition of kinetic energy):

$$W = K_f - K_t = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2.$$

The initial speed v_i is zero, and we now know that the work done is 153.4 J. Solving for v_f and then substituting known data, we find that

$$v_f = \sqrt{\frac{2W}{m}} = \sqrt{\frac{2(153.4\text{ J})}{225\text{ kg}}}$$
$$= 1.17\text{ m/s}. \qquad \text{(Answer)}$$

SAMPLE PROBLEM 8.05

Work done by a constant force in unit-vector notation

During a storm, a crate of crepe is sliding across a slick, oily parking lot through a displacement $\vec{d} = (-3.0\text{ m})\hat{i}$ while a steady wind pushes against the crate with a force $\vec{F} = (2.0\text{ N})\hat{i} + (-6.0\text{ N})\hat{j}$. The situation and coordinate axes are shown in Fig. 8-6.

(a) How much work does this force do on the crate during the displacement?

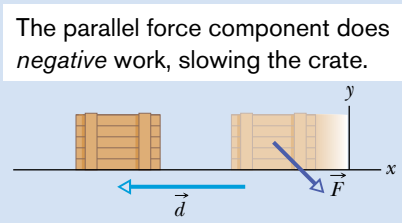

Figure 8-6 Force $\vec{F}$ slows a crate during displacement $\vec{d}$.

KEY IDEAS

Because we can treat the crate as a particle and because the wind force is constant ("steady") in both magnitude and direction during the displacement, we can use either Eq. 8-7 ($W = Fd\cos\phi$) or Eq. 8-8 ($W = \vec{F}\cdot\vec{d}$) to calculate the work. Since we know $\vec{F}$ and $\vec{d}$ in unit-vector notation, we choose Eq. 8-8.

Calculations: We write

$$W = \vec{F}\cdot\vec{d} = [(2.0\text{ N})\hat{i} + (-6.0\text{ N})\hat{j}]\cdot[(-3.0\text{ m})\hat{i}].$$

Of the possible unit-vector dot products, only $\hat{i}\cdot\hat{i}$, $\hat{j}\cdot\hat{j}$, and $\hat{k}\cdot\hat{k}$ are nonzero. Here we obtain

$$W = (2.0\text{ N})(-3.0\text{ m})\hat{i}\cdot\hat{i} + (-6.0\text{ N})(-3.0\text{ m})\hat{j}\cdot\hat{i}$$
$$= (-6.0\text{ J})(1) + 0 = -6.0\text{ J}. \qquad \text{(Answer)}$$

Thus, the force does a negative 6.0 J of work on the crate, transferring 6.0 J of energy from the kinetic energy of the crate.

(b) If the crate has a kinetic energy of 10 J at the beginning of displacement $\vec{d}$, what is its kinetic energy at the end of $\vec{d}$?

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KEY IDEA

Because the force does negative work on the crate, it reduces the crate's kinetic energy.

Calculation: Using the work–kinetic energy theorem in the form of Eq. 8-11, we have

$$K_f = K_i + W = 10\text{ J} + (-6.0\text{ J}) = 4.0\text{ J}. \quad \text{(Answer)}$$

Less kinetic energy means that the crate has been slowed.

8.5 | WORK DONE BY THE GRAVITATIONAL FORCE

Key Concepts

- The work W_g done by the gravitational force $\vec{F}_g$ on a particle-like object of mass m as the object moves through a displacement $\vec{d}$ is given by

$$W_g = mgd\cos\phi,$$

in which ϕ is the angle between $\vec{F}_g$ and $\vec{d}$.

- The work W_a done by an applied force as a particle-like object is either lifted or lowered is related to the work W_g done by the gravitational force and the change ΔK in the object's kinetic energy by

$$\Delta K = K_f - K_i = W_a + W_g.$$

If $K_f = K_i$, then the equation reduces to

$$W_a = -W_g,$$

which tells us that the applied force transfers as much energy to the object as the gravitational force transfers from it.

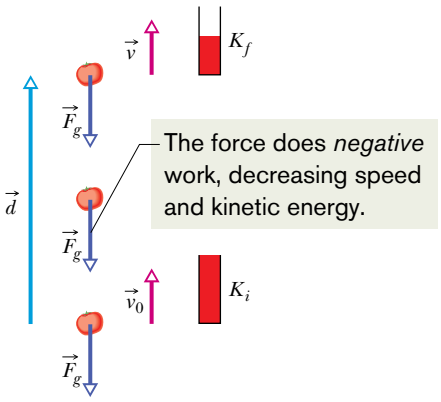

Figure 8-7 Because the gravitational force $\vec{F}_g$ acts on it, a particle-like tomato of mass m thrown upward slows from velocity $\vec{v}_0$ to velocity $\vec{v}$ during displacement $\vec{d}$. A kinetic energy gauge indicates the resulting change in the kinetic energy of the tomato, from $K_i (= 1/2\, mv_0^2)$ to $K_f (= 1/2\, mv^2)$.

We next examine the work done on an object by the gravitational force acting on it. Figure 8-7 shows a particle-like tomato of mass m that is thrown upward with initial speed v_0 and thus with initial kinetic energy $K_i = 1/2\, mv_0^2$. As the tomato rises, it is slowed by a gravitational force $\vec{F}_g$; that is, the tomato's kinetic energy decreases because $\vec{F}_g$ does work on the tomato as it rises. Because we can treat the tomato as a particle, we can use Eq. 8-7 ($W = Fd\cos\phi$) to express the work done during a displacement $\vec{d}$. For the force magnitude F, we use mg as the magnitude of $\vec{F}_g$. Thus, the work W_g done by the gravitational force $\vec{F}_g$ is

$$W_g = mgd\cos\phi \quad \text{(work done by gravitational force)}. \tag{8-12}$$

For a rising object, force $\vec{F}_g$ is directed opposite the displacement $\vec{d}$, as indicated in Fig. 8-7. Thus, $\phi = 180°$ and

$$W_g = mgd\cos 180° = mgd(-1) = -mgd. \tag{8-13}$$

The minus sign tells us that during the object's rise, the gravitational force acting on the object transfers energy in the amount mgd from the kinetic energy of the object. This is consistent with the slowing of the object as it rises.

After the object has reached its maximum height and is falling back down, the angle ϕ between force $\vec{F}_g$ and displacement $\vec{d}$ is zero. Thus,

$$W_g = mgd\cos 0° = mgd(+1) = +mgd. \tag{8-14}$$

The plus sign tells us that the gravitational force now transfers energy in the amount mgd to the kinetic energy of the falling object (it speeds up, of course). This is consistent with the speeding up of the object if falls.

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(Actually, energy transfers associated with lifting and lowering an object involve the full object – Earth system.)

Work Done in Lifting and Lowering an Object

Now suppose we lift a particle-like object by applying a vertical force $\vec{F}$ to it. During the upward displacement, our applied force does positive work W_a on the object while the gravitational force does negative work W_g on it. Our applied force tends to transfer energy to the object while the gravitational force tends to transfer energy from it. By Eq. 8-10, the change ΔK in the kinetic energy of the object due to these two energy transfers is

$$\Delta K = K_f - K_i = W_a + W_g, \tag{8-15}$$

in which K_f is the kinetic energy at the end of the displacement and K_i is that at the start of the displacement. This equation also applies if we lower the object, but then the gravitational force tends to transfer energy *to* the object while our force tends to transfer energy *from* it.

If an object is stationary before and after a lift (as when you lift a book from the floor to a shelf), then K_f and K_i are both zero, and Eq. 8-15 reduces to

$$W_a + W_g = 0$$

or

$$W_a = -W_g. \tag{8-16}$$

Note that we get the same result if K_f and K_i are not zero but are still equal. Either way, the result means that the work done by the applied force is the negative of the work done by the gravitational force; that is, the applied force transfers the same amount of energy to the object as the gravitational force transfers from the object. Using Eq. 8-12, we can rewrite Eq. 8-16 as

$$W_a = -mgd \cos\phi \quad \text{(work done in lifting and lowering; } K_f = K_i\text{)}, \tag{8-17}$$

with ϕ being the angle between $\vec{F}_g$ and $\vec{d}$. If the displacement is vertically upward (Fig. 8-8*a*), then $\phi = 180°$ and the work done by the applied force equals mgd. If the displacement is vertically downward (Fig. 8-8*b*), then $\phi = 0°$ and the work done by the applied force equals $-mgd$.

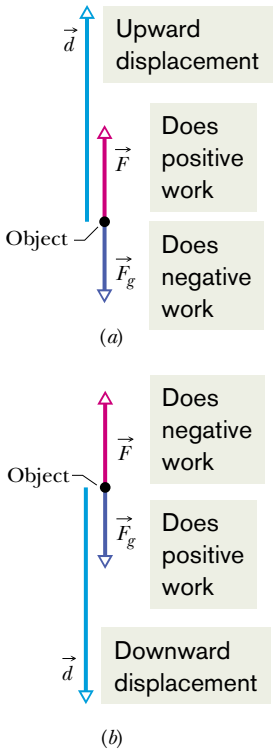

Figure 8-8 (*a*) An applied force $\vec{F}$ lifts an object. The object's displacement $\vec{d}$ makes an angle $\phi = 180°$ with the gravitational force $\vec{F}_g$ on the object. The applied force does positive work on the object. (*b*) An applied force $\vec{F}$ lowers an object. The displacement $\vec{d}$ of the object makes an angle $\phi = 0°$ with the gravitational force $\vec{F}_g$. The applied force does negative work on the object.

Equations 8-16 and 8-17 apply to any situation in which an object is lifted or lowered, with the object stationary before and after the lift.They are independent of the magnitude of the force used. For example, if you lift a mug from the floor to over your head, your force on the mug varies considerably during the lift. Still, because the mug is stationary before and after the lift, the work your force does on the mug is given by Eqs. 8-16 and 8-17, where, in Eq. 8-17, mg is the weight of the mug and d is the distance you lift it.

Here is a simple example where we calculate work done by uniform forces acting on an object. We take a small block of mass m from A to B along a fixed smooth spherical bowl by applying horizontal force F on it. Let us say that arc AB subtends angle θ on O (Fig. 8-9). The horizontal component of displacement is $R(\sin\theta)$ and vertical component of displacement is $R(1 - \cos\theta)$.

Then work done by gravity W_g can be very easily written as product of force and vertical component of displacement. Since they are opposite to each other, our answer has negative sign:

$$W_g = -mgR(1 - \cos\theta)$$

Similarly, the work done by force (F) is

$$W_f = FR(\sin\theta)$$

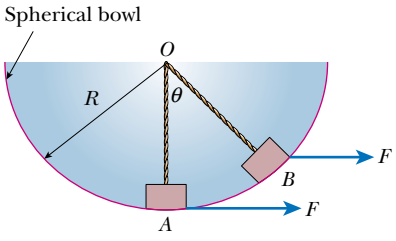

Figure 8-9 Application of force on a block of mass m along a spherical bowl.

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SAMPLE PROBLEM 8.06

Work in pulling a sleigh up a snowy slope

In this problem an object is pulled along a ramp but the object starts and ends at rest and thus has no overall change in its kinetic energy (that is important). Figure 8-10*a* shows the situation. A rope pulls a 200 kg sleigh (which you may know) up a slope at incline angle $\theta = 30°$, through distance $d = 20$ m. The sleigh and its contents have a total mass of 200 kg. The snowy slope is so slippery that we take it to be frictionless. How much work is done by each force acting on the sleigh?

KEY IDEAS

(1) During the motion, the forces are constant in magnitude and direction and thus we can calculate the work done by each with Eq. 8-7 ($W = Fd \cos \phi$) in which ϕ is the angle between the force and the displacement. We reach the same result with Eq. 8-8 ($W = \vec{F} \cdot \vec{d}$) in which we take a dot product of the force vector and displacement vector. (2) We can relate the net work done by the forces to the change in kinetic energy (or lack of a change, as here) with the work–kinetic energy theorem of Eq. 8-10 ($\Delta K = W$).

Calculations: The first thing to do with most physics problems involving forces is to draw a free-body diagram to organize our thoughts. For the sleigh, Fig. 8-10*b* is our free-body diagram, showing the gravitational force $\vec{F}_g$, the force $\vec{T}$ from the rope, and the normal force $\vec{F}_N$ from the slope.

Work W_N by the normal force. Let's start with this easy calculation. The normal force is perpendicular to the slope and thus also to the sleigh's displacement.Thus the normal force does not affect the sleigh's motion and does zero work. To be more formal, we can apply Eq. 8-7 to write

$$W_N = F_N d \cos 90° = 0. \qquad \text{(Answer)}$$

Work W_g by the gravitational force. We can find the work done by the gravitational force in either of two ways (you pick the more appealing way). From an earlier discussion about ramps (Sample Problem 5.06 and Fig. 5-23), we know that the component of the gravitational force along the slope has magnitude $mg \sin \theta$ and is directed down the slope. Thus the magnitude is

$$F_{gx} = mg \sin \theta = (200 \text{ kg})(9.8 \text{ m/s}^2) \sin 30° = 980 \text{ N}.$$

The angle ϕ between the displacement and this force component is 180°. So we can apply Eq. 8-7 to write

$$W_g = F_{gx} d \cos 180° = (980 \text{ N})(20 \text{ m})(-1) = -1.96 \times 10^4 \text{ J}. \qquad \text{(Answer)}$$

The negative result means that the gravitational force removes energy from the sleigh.

The second (equivalent) way to get this result is to use the full gravitational force $\vec{F}_g$ instead of a component. The angle between $\vec{F}_g$ and $\vec{d}$ is 120° (add the incline angle 30° to 90°). So, Eq. 8-7 gives us

$$W_g = F_g d \cos 120° = mgd \cos 120° = (200 \text{ kg})(9.8 \text{ m/s}^2)(20 \text{ m}) \cos 120° = 1.96 \times 10^4 \text{ J}. \qquad \text{(Answer)}$$

Work W_T by the rope's force. We have two ways of calculating this work. The quickest way is to use the work–kinetic energy theorem of Eq. 8-10 ($\Delta K = W$), where the net work W done by the forces is $W_N + W_g + W_T$ and the change ΔK in the kinetic energy is just zero (because the initial and final kinetic energies are the same—namely, zero). So, Eq. 8-10 gives us

$$0 = W_N + W_g + W_T = 0 - 1.96 \times 10^4 \text{ J} + W_T$$

and

$$W_T = 1.96 \times 10^4 \text{ J}. \qquad \text{(Answer)}$$

Instead of doing this, we can apply Newton's second law for motion along the x axis to find the magnitude F_T of the rope's force. Assuming that the acceleration

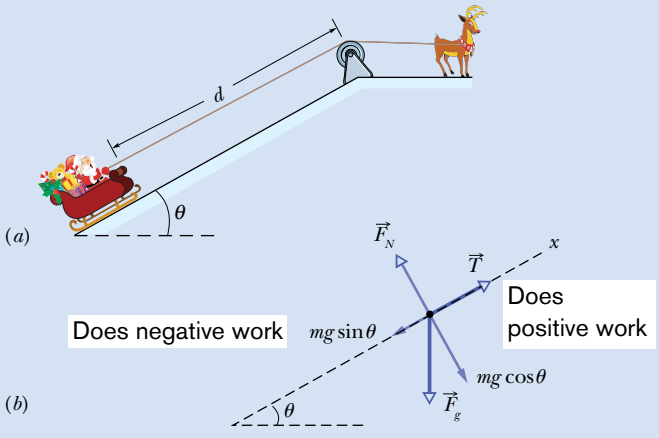

Figure 8-10 (*a*) A sleigh is pulled up a snowy slope. (*b*) The freebody diagram for the sleigh.

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along the slope is zero (except for the brief starting and stopping), we can write

$$F_{\text{net},x} = ma_x,$$
$$F_T - mg \sin 30° = m(0),$$

to find

$$F_T = mg \sin 30°.$$

This is the magnitude. Because the force and the displacement are both up the slope, the angle between those two vectors is zero. So, we can now write Eq. 8-7 to find the work done by the rope's force:

$$W_T = F_T d \cos 0° = (mg \sin 30°) d \cos 0°$$
$$= (200 \text{ kg})(9.8 \text{ m/s}^2)(\sin 30°)(20 \text{ m}) \cos 0°$$
$$= 1.96 \times 10^4 \text{ J}. \quad \text{(Answer)}$$

SAMPLE PROBLEM 8.07

Work done on an accelerating elevator cab

An elevator cab of mass m = 500 kg is descending with speed $v_i = 4.0$ m/s when its supporting cable begins to slip, allowing it to fall with constant acceleration $\vec{a} = \vec{g}/5$ (Fig. 8-11*a*).

(a) During the fall through a distance d = 12 m, what is the work W_g done on the cab by the gravitational force $\vec{F}_g$?

KEY IDEA

We can treat the cab as a particle and thus use Eq. 8-12 ($W_g = mgd \cos \phi$) to find the work W_g.

Calculation: From Fig. 8-11*b*, we see that the angle between the directions of $\vec{F}_g$ and the cab's displacement $\vec{d}$ is 0°. So,

$$W_g = mgd \cos 0° = (500 \text{ kg})(9.8 \text{ m/s}^2)(12 \text{ m})(1)$$
$$= 5.88 \times 10^4 \text{ J} \approx 59 \text{ kJ}. \quad \text{(Answer)}$$

(b) During the 12 m fall, what is the work W_T done on the cab by the upward pull $\vec{T}$ of the elevator cable?

KEY IDEA

We can calculate work W_T with Eq. 8-7 ($W = Fd \cos \phi$) by first writing $F_{\text{net},y} = ma_y$ for the components in Fig. 8-11*b*.

Calculations: We get

$$T - F_g = ma. \quad (8\text{-}18)$$

Solving for T, substituting mg for F_g, and then substituting the result in Eq. 8-7, we obtain

$$W_T = Td \cos \phi = m(a + g)d \cos \phi. \quad (8\text{-}19)$$

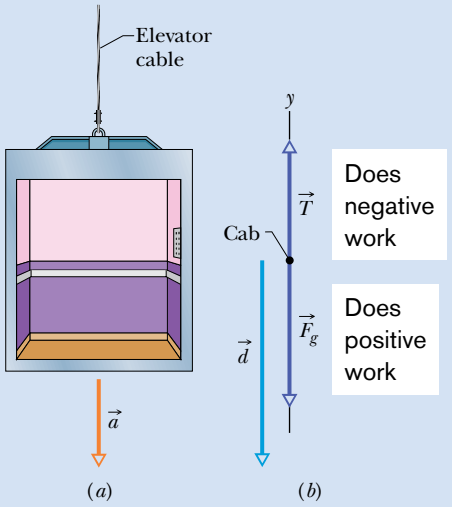

Figure 8-11 An elevator cab, descending with speed v_i, suddenly begins to accelerate downward. (*a*) It moves through a displacement $\vec{d}$ with constant acceleration $\vec{a} = g/5$. (*b*) A freebody diagram for the cab, displacement included.

Next, substituting $-g/5$ for the (downward) acceleration a and then 180° for the angle ϕ between the directions of forces $\vec{T}$ and $m\vec{g}$, we find

$$W_T = m\left(-\frac{g}{5} + g\right) d \cos \phi = \frac{4}{5} mgd \cos \phi$$
$$= \frac{4}{5}(500 \text{ kg})(9.8 \text{ m/s}^2)(12 \text{ m}) \cos 180°$$
$$= -4.70 \times 10^4 \text{ j} \approx -47 \text{ kJ}. \quad \text{(Answer)}$$

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Caution: Note that W_T is not simply the negative of W_g because the cab accelerates during the fall. Thus, Eq. 8-16 (which assumes that the initial and final kinetic energies are equal) does not apply here.

(c) What is the net work W done on the cab during the fall?

Calculation: The net work is the sum of the works done by the forces acting on the cab:

$$W = W_g + W_T = 5.88 \times 10^4 \text{ J} - 4.70 \times 10^4 \text{ J}$$
$$= 1.18 \times 10^4 \text{ J} \approx 12 \text{ kJ}. \quad \text{(Answer)}$$

(d) What is the cab's kinetic energy at the end of the 12 m fall?

KEY IDEA

The kinetic energy changes *because* of the net work done on the cab, according to Eq. 8-11 ($K_f = K_i + W$).

Calculation: From Eq. 8-1, we write the initial kinetic energy as $K_i = 1/2mv_i^2$. We then write Eq. 8-11 as

$$K_f = K_i + W = \frac{1}{2}mv_i^2 + W$$
$$= \frac{1}{2}(500 \text{ kg})(4.0 \text{ m/s})^2 + 1.18 \times 10^4 \text{ J}$$
$$= 1.58 \times 10^4 \text{ J} \approx 16 \text{ kJ}. \quad \text{(Answer)}$$

8.6 | WORK DONE BY A SPRING FORCE

Key Concepts

- The force $\vec{F}_s$ from a spring is

$$\vec{F}_s = -k\vec{d} \quad \text{(Hooke's law)},$$

where $\vec{d}$ is the displacement of the spring's free end from its position when the spring is in its relaxed state (neither compressed nor extended), and k is the spring constant (a measure of the spring's stiffness). If an x axis lies along the spring, with the origin at the location of the spring's free end when the spring is in its relaxed state, we can write

$$F_x = -kx \quad \text{(Hooke's law)}.$$

- A spring force is thus a variable force: It varies with the displacement of the spring's free end.
- If an object is attached to the spring's free end, the work W_s done on the object by the spring force when the object is moved from an initial position x_i to a final position x_f is

$$W_s = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2.$$

If $x_i = 0$ and $x_f = x$, then the equation becomes

$$W_s = -\frac{1}{2}kx^2.$$

We next want to examine the work done on a particle-like object by a particular type of *variable force*—namely, a **spring force**, the force from a spring. Many forces in nature have the same mathematical form as the spring force. Thus, by examining this one force, you can gain an understanding of many others.

The Spring Force

Figure 8-12*a* shows a spring in its **relaxed state**—that is, neither compressed nor extended. One end is fixed, and a particle-like object—a block, say—is attached to the other, free end. If we stretch the spring by pulling the block to the right as in Fig. 8-12*b*, the spring pulls on the block toward the left. (Because a spring force acts to restore the relaxed state, it is sometimes said to be a *restoring force*.) If we compress the spring by pushing the block to the left as in Fig. 8-12*c*, the spring now pushes on the block toward the right.

To a good approximation for many springs, the force $\vec{F}_s$ from a spring is proportional to the displacement $\vec{d}$ of the free end from its position when the spring is in the relaxed state. The *spring force* is given by

$$\vec{F}_s = -k\vec{d} \quad \text{(Hooke's law)}, \tag{8-20}$$

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which is known as **Hooke's law** after Robert Hooke, an English scientist of the late 1600s. The minus sign in Eq. 8-20 indicates that the direction of the spring force is always opposite the direction of the displacement of the spring's free end. The constant k is called the **spring constant** (or **force constant**) and is a measure of the stiffness of the spring. The larger k is, the stiffer the spring; that is, the larger k is, the stronger the spring's pull or push for a given displacement. The SI unit for k is the newton per meter.

In Fig. 8-12 an x axis has been placed parallel to the length of the spring, with the origin ($x = 0$) at the position of the free end when the spring is in its relaxed state. For this common arrangement, we can write Eq. 8-20 as

$$F_x = -kx \quad \text{(Hooke's law)}, \tag{8-21}$$

where we have changed the subscript. If x is positive (the spring is stretched toward the right on the x axis), then F_x is negative (it is a pull toward the left). If x is negative (the spring is compressed toward the left), then F_x is positive (it is a push toward the right). Note that a spring force is a *variable force* because it is a function of x, the position of the free end. Thus F_x can be symbolized as $F(x)$. Also note that Hooke's law is a *linear* relationship between F_x and x.

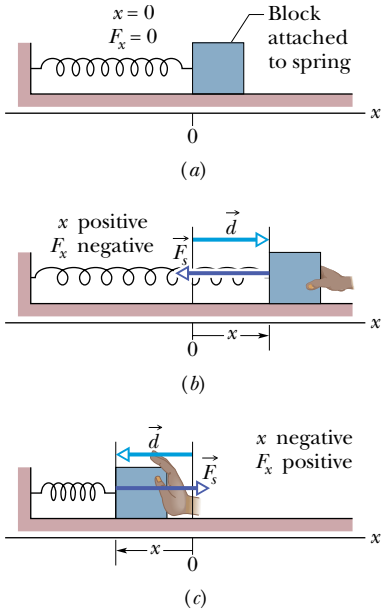

Figure 8-12 (a) A spring in its relaxed state. The origin of an x axis has been placed at the end of the spring that is attached to a block. (b) The block is displaced by $\vec{d}$, and the spring is stretched by a positive amount x. Note the restoring force $\vec{F}_s$ exerted by the spring. (c) The spring is compressed by a negative amount x. Again, note the restoring force.

The Work Done by a Spring Force

To find the work done by the spring force as the block in Fig. 8-12a moves, let us make two simplifying assumptions about the spring. (1) It is *massless*; that is, its mass is negligible relative to the block's mass. (2) It is an *ideal spring*; that is, it obeys Hooke's law exactly. Let us also assume that the contact between the block and the floor is frictionless and that the block is particle-like.

We give the block a rightward jerk to get it moving and then leave it alone. As the block moves rightward, the spring force F_x does work on the block, decreasing the kinetic energy and slowing the block. However, we *cannot* find this work by using Eq. 8-7 ($W = Fd \cos \phi$) because there is no one value of F to plug into that equation—the value of F increases as the block stretches the spring.

There is a neat way around this problem. (1) We break up the block's displacement into tiny segments that are so small that we can neglect the variation in F in each segment. (2) Then in each segment, the force has (approximately) a single value and thus we *can* use Eq. 8-7 to find the work in that segment. (3) Then we add up the work results for all the segments to get the total work. Well, that is our intent, but we don't really want to spend the next several days adding up a great many results and, besides, they would be only approximations. Instead, let's make the segments *infinitesimal* so that the error in each work result goes to zero. And then let's add up all the results by integration instead of by hand. Through the ease of calculus, we can do all this in minutes instead of days.

Let the block's initial position be x_i and its later position be x_f. Then divide the distance between those two positions into many segments, each of tiny length Δx. Label these segments, starting from x_i, as segments 1, 2, and so on. As the block moves through a segment, the spring force hardly varies because the segment is so short that x hardly varies. Thus, we can approximate the force magnitude as being constant within the segment. Label these magnitudes as F_{x1} in segment 1, F_{x2} in segment 2, and so on.

With the force now constant in each segment, we *can* find the work done within each segment by using Eq. 8-7. Here $\phi = 180°$, and so $\cos \phi = -1$. Then the work done is $-F_{x1}\,\Delta x$ in segment 1, $-F_{x2}\,\Delta x$ in segment 2, and so on. The net work W_s done by the spring, from x_i to x_f, is the sum of all these works:

$$W_s = \sum -F_{xj}\,\Delta x, \tag{8-22}$$

where j labels the segments. In the limit as Δx goes to zero, Eq. 8-22 becomes

$$W_s = \int_{x_i}^{x_f} -F_x\,dx. \tag{8-23}$$

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From Eq. 8-21, the force magnitude F_x is kx. Thus, substitution leads to

$$W_s = \int_{x_i}^{x_f} -kx\, dx = -k\int_{x_i}^{x_f} x\, dx$$
$$= \left(-\frac{1}{2}k\right)[x^2]_{x_i}^{x_f} = \left(-\frac{1}{2}k\right)(x_f^2 - x_i^2). \quad (8\text{-}24)$$

Multiplied out, this yields

$$W_s = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2 \quad \text{(work by a spring force)}. \quad (8\text{-}25)$$

This work W_s done by the spring force can have a positive or negative value, depending on whether the *net* transfer of energy is to or from the block as the block moves from x_i to x_f.

Caution: The final position x_f appears in the *second* term on the right side of Eq. 8-25. Therefore, Eq. 8-25 tells us:

Work W_s is positive if the block ends up closer to the relaxed position ($x = 0$) than it was initially. It is negative if the block ends up farther away from $x = 0$. It is zero if the block ends up at the same distance from $x = 0$.

If $x_i = 0$ and if we call the final position x, then Eq. 8-25 becomes

$$W_s = \frac{1}{2}kx^2 \quad \text{(work by a spring force)}. \quad (8\text{-}26)$$

The Work Done by an Applied Force

Now suppose that we displace the block along the x axis while continuing to apply a force $\vec{F}_a$ to it. During the displacement, our applied force does work W_a on the block while the spring force does work W_s. By Eq. 8-10, the change ΔK in the kinetic energy of the block due to these two energy transfers is

$$\Delta K = K_f - K_i = W_a + W_s, \quad (8\text{-}27)$$

in which K_f is the kinetic energy at the end of the displacement and K_i is that at the start of the displacement. If the block is stationary before and after the displacement, then K_f and K_i are both zero and Eq. 8-27 reduces to

$$W_a = -W_s. \quad (8\text{-}28)$$

If a block that is attached to a spring is stationary before and after a displacement, then the work done on it by the applied force displacing it is the negative of the work done on it by the spring force.

Caution: If the block is not stationary before and after the displacement, then this statement is *not* true.

CHECKPOINT 3

For three situations, the initial and final positions, respectively, along the x axis for the block in Fig. 8-12 are (a) –3 cm, 2 cm; (b) 2 cm, 3 cm; and (c) –2 cm, 2 cm. In each situation, is the work done by the spring force on the block positive, negative, or zero?

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SAMPLE PROBLEM 8.08

Work done by a spring to change kinetic energy

When a spring does work on an object, we *cannot* find the work by simply multiplying the spring force by the object's displacement. The reason is that there is no one value for the force—it changes. However, we can split the displacement up into an infinite number of tiny parts and then approximate the force in each as being constant. Integration sums the work done in all those parts. Here we use the generic result of the integration.

In Fig. 8-13, a canister of mass $m = 0.40$ kg slides across a horizontal frictionless counter with speed $v = 0.50$ m/s. It then runs into and compresses a spring of spring constant $k = 750$ N/m. When the canister is momentarily stopped by the spring, by what distance d is the spring compressed?

KEY IDEAS

1. The work W_s done on the canister by the spring force is related to the requested distance d by Eq. 8-26 ($W_s = -1/2\ kx^2$), with d replacing x.
2. The work W_s is also related to the kinetic energy of the canister by Eq. 8-10 ($K_f - K_i = W$).
3. The canister's kinetic energy has an initial value of $K = 1/2\ mv^2$ and a value of zero when the canister is momentarily at rest.

Calculations: Putting the first two of these ideas together, we write the work–kinetic energy theorem for the canister as

$$K_f - K_i = -\frac{1}{2}kd^2.$$

Substituting according to the third key idea gives us this expression:

$$0 - \frac{1}{2}mv^2 = -\frac{1}{2}kd^2.$$

Simplifying, solving for d, and substituting known data then give us

$$d = v\sqrt{\frac{m}{k}} = (0.50\ \text{m/s})\sqrt{\frac{0.40\ \text{kg}}{750\ \text{N/m}}}$$
$$= 1.2 \times 10^{-2}\ \text{m} = 1.2\ \text{cm}. \qquad \text{(Answer)}$$

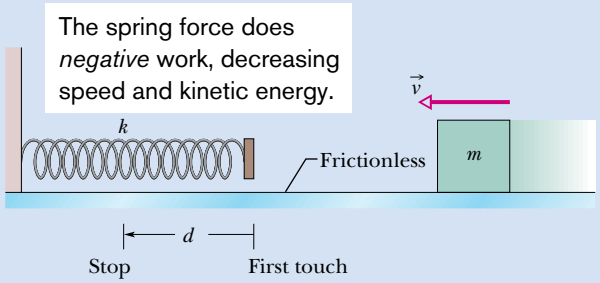

Figure 8-13 A canister moves toward a spring.

8.7 | WORK DONE BY A GENERAL VARIABLE (NONUNIFORM) FORCE

Key Concepts

- When the force $\vec{F}$ on a particle-like object depends on the position of the object, the work done by $\vec{F}$ on the object while the object moves from an initial position r_i with coordinates (x_i, y_i, z_i) to a final position r_f with coordinates (x_f, y_f, z_f) must be found by integrating the force. If we assume that component F_x may depend on x but not on y or z, component F_y may depend on y but not on x or z, and component F_z may depend on z but not on x or y, then the work is

$$W = \int_{x_i}^{x_f} F_x dx + \int_{y_i}^{y_f} F_y dy + \int_{z_i}^{z_f} F_z dz.$$

- If $\vec{F}$ has only an x component, then this reduces to

$$W = \int_{x_i}^{x_f} F(x)\, dx.$$

One-Dimensional Analysis

Let us return to the situation of Fig. 8-2 but now consider the force to be in the positive direction of the x axis and the force magnitude to vary with position x. Thus, as the bead (particle) moves, the magnitude $F(x)$ of the force

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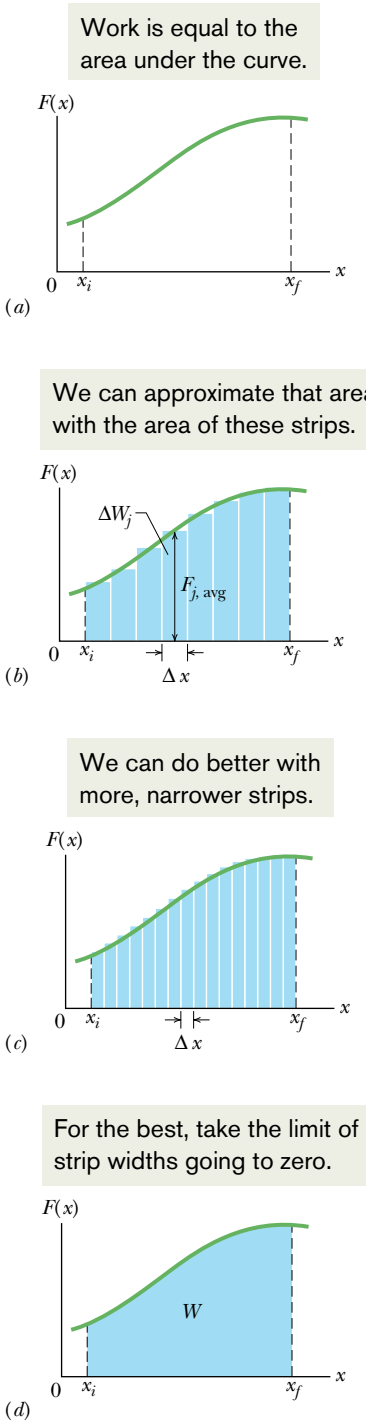

Figure 8-14 (*a*) A one-dimensional force $\vec{F}(x)$ plotted against the displacement x of a particle on which it acts. The particle moves from x_i to x_f. (*b*) Same as (*a*) but with the area under the curve divided into narrow strips. (*c*) Same as (*b*) but with the area divided into narrower strips. (*d*) The limiting case. The work done by the force is given by Eq. 8-32 and is represented by the shaded area between the curve and the x axis and between x_i and x_f.

doing work on it changes. Only the magnitude of this variable force (or nonuniform force) changes, not its direction, and the magnitude at any position does not change with time.

Figure 8-14*a* shows a plot of such a *one-dimensional variable force*. We want an expression for the work done on the particle by this force as the particle moves from an initial point x_i to a final point x_f. However, we *cannot* use Eq. 8-7 ($W = Fd\cos\phi$) because it applies only for a constant force $\vec{F}$. Here, again, we shall use calculus. We divide the area under the curve of Fig. 8-14*a* into a number of narrow strips of width Δx (Fig. 8-14*b*). We choose Δx small enough to permit us to take the force $F(x)$ as being reasonably constant over that interval. We let $F_{j,\text{avg}}$ be the average value of $F(x)$ within the jth interval. Then in Fig. 8-14*b*, $F_{j,\text{avg}}$ is the height of the jth strip.

With $F_{j,\text{avg}}$ considered constant, the increment (small amount) of work ΔW_j done by the force in the jth interval is now approximately given by Eq. 8-7 and is

$$\Delta W_j = F_{j,\text{avg}}\,\Delta x. \tag{8-29}$$

In Fig. 8-14*b*, ΔW_j is then equal to the area of the jth rectangular, shaded strip.

To approximate the total work W done by the force as the particle moves from x_i to x_f, we add the areas of all the strips between x_i and x_f in Fig. 8-14*b*:

$$W = \sum \Delta W_j = \sum F_{j,\text{avg}}\Delta x. \tag{8-30}$$

Equation 8-30 is an approximation because the broken "skyline" formed by the tops of the rectangular strips in Fig. 8-14*b* only approximates the actual curve of $F(x)$.

We can make the approximation better by reducing the strip width Δx and using more strips (Fig. 8-14*c*). In the limit, we let the strip width approach zero; the number of strips then becomes infinitely large and we have, as an exact result,

$$W = \lim_{\Delta x \to 0} \sum F_{j,\text{avg}}\Delta x. \tag{8-31}$$

This limit is exactly what we mean by the integral of the function $F(x)$ between the limits x_i and x_f. Thus, Eq. 8-31 becomes

$$W = \int_{x_i}^{x_f} F(x)\,dx \quad \text{(work: variable force)}. \tag{8-32}$$

If we know the function $F(x)$, we can substitute it into Eq. 8-32, introduce the proper limits of integration, carry out the integration, and thus find the work. (Appendix E contains a list of common integrals.) Geometrically, the work is equal to the area between the $F(x)$ curve and the x axis, between the limits x_i and x_f (shaded in Fig. 8-14*d*).

Three-Dimensional Analysis

Consider now a particle that is acted on by a three-dimensional force

$$\vec{F} = F_x\hat{\mathrm{i}} + F_y\hat{\mathrm{j}} + F_z\hat{\mathrm{k}}, \tag{8-33}$$

in which the components F_x, F_y, and F_z can depend on the position of the particle; that is, they can be functions of that position.

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However, we make three simplifications: F_x may depend on x but not on y or z, F_y may depend on y but not on x or z, and F_z may depend on z but not on x or y. Now let the particle move through an incremental displacement

$$d\vec{r} = dx\hat{\mathrm{i}} + dy\hat{\mathrm{j}} + dz\hat{\mathrm{k}}. \tag{8-34}$$

The increment of work dW done on the particle by $\vec{F}$ during the displacement $d\vec{r}$ is, by Eq. 8-8,

$$dw = \vec{F} \cdot d\vec{r} = F_x\,dx + F_y\,dy + F_z\,dz \tag{8-35}$$

The work W done by $\vec{F}$ while the particle moves from an initial position r_i having coordinates (x_i, y_i, z_i) to a final position r_f having coordinates (x_f, y_f, z_f) is then

$$W = \int_{r_i}^{x_f} dW = \int_{x_i}^{x_f} F_x dx + \int_{y_i}^{y_f} F_y dy + \int_{z_i}^{z_f} F_z dz. \tag{8-36}$$

If $\vec{F}$ has only an x component, then the y and z terms in Eq. 8-36 are zero and the equation reduces to Eq. 8-32.

Work–Kinetic Energy Theorem for a Variable Force

Equation 8-32 gives the work done by a variable force on a particle in a one-dimensional situation. Let us now make certain that the work is equal to the change in kinetic energy, as the work–kinetic energy theorem states.

Consider a particle of mass m, moving along an x axis and acted on by a net force $F(x)$ that is directed along that axis. The work done on the particle by this force as the particle moves from position x_i to position x_f is given by Eq. 8-32 as

$$W = \int_{x_i}^{x_f} F(x)\,dx = \int_{x_i}^{x_f} ma\,dx, \tag{8-37}$$

in which we use Newton's second law to replace $F(x)$ with ma. We can write the quantity $ma\,dx$ in Eq. 8-37 as

$$ma\,dx = m\frac{dv}{dt}dx. \tag{8-38}$$

From the chain rule of calculus, we have

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v, \tag{8-39}$$

and Eq. 8-38 becomes

$$ma\,dx = m\frac{dv}{dx}v\,dx = mv\,dv. \tag{8-40}$$

Substituting Eq. 8-40 into Eq. 8-37 yields

$$\begin{aligned} W &= \int_{v_i}^{v_f} mv\,dv = m\int_{v_i}^{v_f} v\,dv \\ &= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2. \end{aligned} \tag{8-41}$$

Note that when we change the variable from x to v we are required to express the limits on the integral in terms of the new variable. Note also that because the mass m is a constant, we are able to move it outside the integral.

Recognizing the terms on the right side of Eq. 8-41 as kinetic energies allows us to write this equation as

$$W = K_f - K_i = \Delta K,$$

which is the work kinetic energy theorem.

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SAMPLE PROBLEM 8.09

Work calculated by graphical integration

In Fig. 8-15*b*, an 8.0 kg block slides along a frictionless floor as a force acts on it, starting at $x_1 = 0$ and ending at $x_3 = 6.5$ m. As the block moves, the magnitude and direction of the force varies according to the graph shown in Fig. 8-15*a*. For example, from $x = 0$ to $x = 1$ m, the force is positive (in the positive direction of the x axis) and increases in magnitude from 0 to 40 N. And from $x = 4$ m to $x = 5$ m, the force is negative and increases in magnitude from 0 to 20 N.

(Note that this latter value is displayed as –20 N.) The block's kinetic energy at x_1 is $K_1 = 280$ J. What is the block's speed at $x_1 = 0$, $x_2 = 4.0$ m, and $x_3 = 6.5$ m?

KEY IDEAS

(1) At any point, we can relate the speed of the block to its kinetic energy with Eq. 8-1 ($K = 1/2mv^2$). (2) We can relate the kinetic energy K_f at a later point to the initial kinetic K_i and the work W done on the block by using the work–kinetic energy theorem of Eq. 8-10 ($K_f - K_i = W$). (3) We can calculate the work W done by a variable force $F(x)$ by integrating the force versus position x. Equation 8-32 tells us that

$$W = \int_{x_i}^{x_f} F(x)\, dx.$$

We do not have a function $F(x)$ to carry out the integration, but we do have a graph of $F(x)$ where we can integrate by finding the area between the plotted line and the x axis. Where the plot is above the axis, the work (which is equal to the area) is positive. Where it is below the axis, the work is negative.

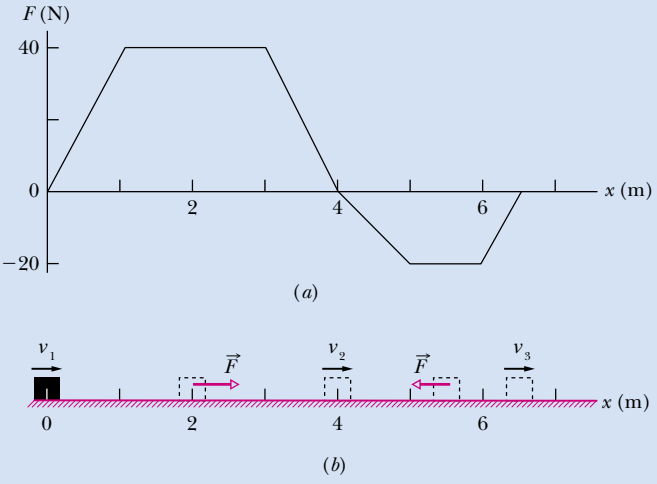

Figure 8-15 (*a*) A graph indicating the magnitude and direction of a variable force that acts on a block as it moves along an x axis on a floor. (*b*) The location of the block at several times.

Calculations: The requested speed at $x = 0$ is easy because we already know the kinetic energy. So, we just plug the kinetic energy into the formula for kinetic energy:

$$K_1 = \frac{1}{2}mv_1^2,\ 280\text{ J} = \frac{1}{2}(8.0\text{ kg})v_1^2,$$

and then

$$v_1 = 8.37\text{ m/s} \approx 8.4\text{ m/s}. \qquad \text{(Answer)}$$

As the block moves from $x = 0$ to $x = 4.0$ m, the plot in Figure 8-15*a* is above the x axis, which means that positive work is being done on the block. We split the area under the plot into a triangle at the left, a rectangle in the center, and a triangle at the right. Their total area is

$$\frac{1}{2}(40\text{ N})(1\text{ m}) + (40\text{ N})(2\text{ m}) + \frac{1}{2}(40\text{ N})(1\text{ m}) = 120\text{ N}\cdot\text{m} = 120\text{ J}.$$

This means that between $x = 0$ and $x = 4.0$ m, the force does 120 J of work on the block, increasing the kinetic energy and speed of the block. So, when the block reaches $x = 4.0$ m, the work–kinetic energy theorem tells us that the kinetic energy is

$$K_2 = K_1 + W = 280\text{ J} + 120\text{ J} = 400\text{ J}.$$

Again using the definition of kinetic energy, we find

$$K_2 = \frac{1}{2}mv_2^2,\ 400\text{ J} = \frac{1}{2}(8.0\text{ kg})v_2^2,$$

and then

$$v_2 = 10\text{ m/s}. \qquad \text{(Answer)}$$

This is the block's greatest speed because from $x = 4.0$ m to $x = 6.5$ m the force is negative, meaning that it opposes the block's motion, doing negative work on the block and thus decreasing the kinetic energy and speed. In that range, the area between the plot and the x axis is

$$\frac{1}{2}(20\text{ N})(1\text{ m}) + (20\text{ N})(1\text{ m}) + \frac{1}{2}(20\text{ N})(0.5\text{ m}) = 35\text{ N}\cdot\text{m} = 35\text{ J}.$$

This means that the work done by the force in that range is –35 J. At $x = 4.0$ m, the block's $K = 400$ J. At $x = 6.5$ m,

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the work–kinetic energy theorem tells us that its kinetic energy is

$$K_3 = K_2 + W = 400\ \text{J} - 35\ \text{J} = 365\ \text{J}.$$

Again using the definition of kinetic energy, we find

$$K_3 = \frac{1}{2}mv_3^2,\ 365\ \text{J} = \frac{1}{2}(8.0\ \text{kg})v_3^2,$$

and then

$$v_3 = 9.55\ \text{m/s} \approx 9.6\ \text{m/s}. \qquad \text{(Answer)}$$

The block is still moving in the positive direction of the x axis, a bit faster than initially.

SAMPLE PROBLEM 8.10

Work, two-dimensional integration

When the force on an object depends on the position of the object, we *cannot* find the work done by it on the object by simply multiplying the force by the displacement. The reason is that there is no one value for the force—it changes. So, we must find the work in tiny little displacements and then add up all the work results. We effectively say, "Yes, the force varies over any given tiny little displacement, but the variation is so small we can approximate the force as being constant during the displacement." Sure, it is not precise, but if we make the displacements infinitesimal, then our error becomes infinitesimal and the result becomes precise. But, to add an infinite number of work contributions by hand would take us forever, longer than a semester. So, we add them up via an integration, which allows us to do all this in minutes (much less than a semester).

Force $\vec{F} = (3x^2\ \text{N})\hat{\text{i}} + (4\ \text{N})\hat{\text{j}}$, with x in meters, acts on a particle, changing only the kinetic energy of the particle. How much work is done on the particle as it moves from coordinates (2 m, 3 m) to (3 m, 0 m)? Does the speed of the particle increase, decrease, or remain the same?

KEY IDEA

The force is a variable force because its x component depends on the value of x. Thus, we cannot use Eqs. 8-7 and 8-8 to find the work done. Instead, we must use Eq. 8-36 to integrate the force.

Calculation: We set up two integrals,one along each axis:

$$W = \int_2^3 3x^2 dx + \int_3^0 4\ dy = 3\int_2^3 x^2\ dx + 4\int_3^0 dy$$
$$= 3\left[\frac{1}{2}x^3\right]_2^3 + 4[y]_3^0 = [3^3 - 2^3] + 4[0-3]$$
$$= 7.0\ \text{J}. \qquad \text{(Answer)}$$

The positive result means that energy is transferred to the particle by force $\vec{F}$. Thus, the kinetic energy of the particle increases and, because $K = 1/2mv^2$, its speed must also increase. If the work had come out negative, the kinetic energy and speed would have decreased.

8.8 | VALIDITY OF WORK–KINETIC ENERGY THEOREM IN INERTIAL REFERENCE FRAMES

Key Concept

- For a given particle, two observers in two different inertial frames observe the same force but different displacements.

Newton's second law of motion holds only in inertial reference frames. Another reference frame, which is moving with a constant velocity relative to any inertial reference frame, is also considered an inertial reference frame. Many physical quantities when observed from different inertial frames, always give the same magnitude. For example, mass, time, length, acceleration, and force. Other quantities such as displacement, velocity, and hence kinetic energy have different values when measured in different inertial frames.

Two observers in different inertial frames observe the same change in velocity vector in a given period of a moving particle. Thus, they measure the same acceleration for a moving particle. However, the change in kinetic energy for both observers will be different. The same observers in different inertial frames, however, are measuring different velocities, that is, they have measured different values for the displacement of a particle in a given period.

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However, interestingly, they measured the same values for the forces acting on the particle as force is independent of reference frame. *To conclude, two observers in two different inertial frames observe the same force but different displacements for a given particle.* Thus, both observers measure different values for the work done on the particle.

Following example clarifies the above discussion and we shall see how work–kinetic energy theorem is valid in different inertial frames.

SAMPLE PROBLEM 8.11

Pushing a box on a moving railway cart

A worker on a railway cart is pushing a box. The cart is moving at a constant speed of 20 m/s by some external agent. The box has a mass of 25 kg, and is being pushed forward over a distance of 2 m on the cart by the worker at constant acceleration, increasing its speed from 0 to 2 m/s relative to car. (a) What will be the value of change in kinetic energy and work done by worker as calculated by Tenzing, who is standing on the cart.

KEY IDEA

Figure 8-16*a* shows the starting and finishing positions according to Tenzing who is riding on the cart.

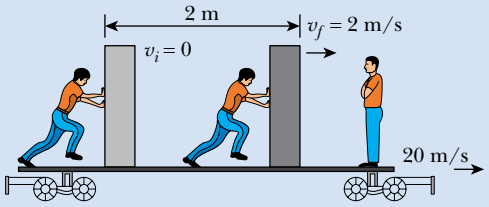

Figure 8-16 (*a*) A worker pushing a box on a moving railway cart being observed from the cart by Tenzing.

Calculations: The change in kinetic energy (ΔK) as calculated by Tenzing is given by

$$\Delta K = K_f - K_i = \frac{1}{2}(25\ \text{kg})(2\ \text{m/s}^2)^2 - 0 = 50\ \text{J}$$

because the initial velocity of the box is zero ($v_i = 0$) and its final velocity is 2 m/s ($v_f = 2$ m/s).*

The assumed constant acceleration of the box can be found from equation, which gives for $v_i = 0$, $v_f = 2$ m/s.

$$a = \frac{v_f^2 - v_i^2}{2s} = \frac{(2\ \text{m/s})^2 - 0}{2(2\ \text{m})} = 1\ \text{m/s}^2$$

This acceleration is due to the force applied by the worker. We can find net force applied by the worker

$$F = ma = (25\ \text{kg})(1\ \text{m/s}^2) = 25\ \text{N}$$

As the box moves through a displacement of $s = 2$ m, the work done on the box by this force is

$$W = Fs = (25\ \text{N})(2\ \text{m}) = 50\ \text{J} \qquad \text{(Answer)}$$

Reasoning: The observer on the train finds that $W = \Delta K$ and thus the work–energy theorem is valid.

(b) How does Girish standing on the ground interpret a similar measurement?

KEY IDEA

Figure 8-16*b* shows the starting and finishing positions according to Girish who is standing on the ground.

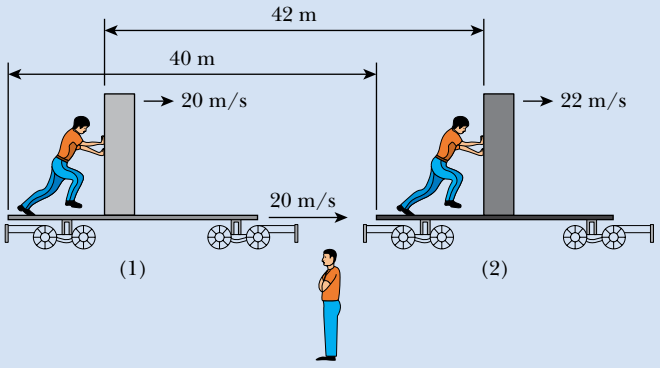

Figure 8-16 (*b*) A worker pushing a box on a moving railway cart being observed from ground by Girish.

Calculation: (We are using prime symbols to represent the measurements of the ground-based observer.) When the box is at rest on the railway cart, it is moving forward with a velocity of $v_i' = 20$ m/s according to Girish. After the box is pushed, the ground-based observer concludes

**Note:* Remember here that the observe is also riding on the cart, so in determining the kinetic energy, we have to consider velocity from Tenzing's reference frame.

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its speed to be $v'_f = 22$ m/s. Girish determines the change in kinetic energy to be

$$\Delta K' = K_f - K_i = \frac{1}{2}mv'^2_f - \frac{1}{2}mv'^2_i$$
$$= \frac{1}{2}(25\text{ kg})(22\text{ m/s})^2 - \frac{1}{2}(25\text{ kg})(20\text{ m/s})^2$$
$$= 1050\text{ J}$$

As expected, this is different from what the observer on the train, Tenzing, measured for the change in kinetic energy ($\Delta K = 50$ J).

Let us calculate the work done on the box according to the ground-based observer Girish. The total displacement of the box also depends on the observer's reference frame, as Fig. 8-16*b* shows. To Girish, the force is exerted over a larger distance of 42 m because at a speed of 20 m/s, the train travels 40.0 m in 2 s, and the total displacement s' of the box in this time is

$$40\text{ m} + 2\text{ m} = 42\text{ m}$$

Force, on the other hand, is independent of reference frame; thus, for Girish, $F' = F = 25$ N. So, Girish, concludes that the work done by worker is

$$W' = F's' = (25\text{ N})(42\text{ m}) = 1050\text{ J} \qquad \text{(Answer)}$$

Reasoning: The work–energy theorem also holds for the ground-based observer. Even though the two observers find different values for displacements and velocities and thus calculate different values of work and kinetic energy but each finally concludes that work done is equal to change in kinetic energy.

Work–Kinetic Energy Theorem for a System of Particles

While verifying work–kinetic energy theorem for a system of many particles, we must remember that work done by all the forces (external and internal) must be considered. Otherwise, this equation should be used separately on individual particles. (Although the total work done by static friction, tension, and normal contact force, that is, by action and reaction on a system add up to zero.)

SAMPLE PROBLEM 8.12

Two boxes and work–kinetic energy theorem for a system of particles

Two blocks are kept over one another as shown in Fig. 8-17*a* Ground is smooth and friction coefficient between blocks is 0.5. Lower block is pulled by a uniform horizontal force of 15 N for 2 m. Find final speed of blocks. Apply work–kinetic energy theorem on the blocks separately and verify that work done by friction between blocks adds to zero.

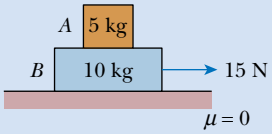

Figure 8-17 (*a*) Two block kept over one another and lower block pulled by a uniform horizontal force.

Calculation: The speed for individual blocks may be calculated from free-body diagram as shown in Fig. 8-17*b*.

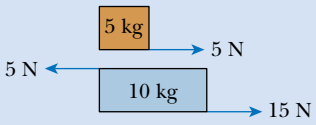

Figure 8-17 (*b*) Free-body diagrams of blocks.

$$W_A = 5\text{ N} \times 2\text{ m}$$
$$W_B = 15\text{ N} \times 2\text{ m} + (-5\text{ N}) \times 2\text{ m}$$
$$\sum W = 30\text{ J} = K_{sys} = \frac{1}{2} \times 10v^2 + \frac{1}{2} \times 5v^2$$

Solving, we get $v = 2$ m/s.

We know that the work done by static friction is zero because action–reaction is in opposite direction but displacement of contact points is the same. Thus, $\vec{f}_A d\vec{s}_A + \vec{f}_B d\vec{s}_B$ because $d\vec{s}_A + d\vec{s}_B$ but $\vec{f}_A + \vec{f}_B$. Thus,

$$15 \times 2 = \frac{1}{2} \times 10v^2 + \frac{1}{2} \times 5v^2$$
$$v = 2\text{ m/s} \qquad \text{(Answer)}$$

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8.9 | POTENTIAL ENERGY

Key Concept

- A force is a conservative force if the net work it does on a particle moving around any closed path, from an initial point and then back to that point, is zero. Equivalently, a force is conservative if the net work it does on a particle moving between two points does not depend on the path taken by the particle. The gravitational force and the spring force are conservative forces; the kinetic frictional force is a nonconservative force.

One job of physics is to identify the different types of energy in the world, especially those that are of common importance. One general type of energy is **potential energy** U. Technically, potential energy is energy that can be associated with the configuration (arrangement) of a system of objects that exert forces on one another.

This is a pretty formal definition of something that is actually familiar to you. An example might help better than the definition: A bungee cord jumper plunges from a staging platform (Fig. 8-18). The system of objects consists of Earth and the jumper. The force between the objects is the gravitational force. The configuration of the system changes (the separation between the jumper and Earth decreases—that is, of course, the thrill of the jump). We can account for the jumper's motion and increase in kinetic energy by defining a **gravitational potential energy** U. This is the energy associated with the state of separation between two objects that attract each other by the gravitational force, here the jumper and Earth.

When the jumper begins to stretch the bungee cord near the end of the plunge, the system of objects consists of the cord and the jumper. The force between the objects is an elastic (spring-like) force. The configuration of the system changes (the cord stretches). We can account for the jumper's decrease in kinetic energy and the cord's increase in length by defining an **elastic potential energy** U. This is the energy associated with the state of compression or extension of an elastic object, here the bungee cord.

Physics determines how the potential energy of a system can be calculated so that energy might be stored or put to use. For example, before any particular bungee cord jumper takes the plunge, someone (probably a mechanical engineer) must determine the correct cord to be used by calculating the gravitational and elastic potential energies that can be expected. Then the jump is only thrilling and not fatal.

Rough Guides/Greg Roden/Getty Images, Inc.

Figure 8-18 The kinetic energy of a bungee cord jumper increases during the free fall, and then the cord begins to stretch, slowing the jumper.

8.10 | WORK AND POTENTIAL ENERGY

Key Concept

- Potential energy is energy that is associated with the configuration of a system in which a conservative force acts. When the conservative force does work W on a particle within the system, the change ΔU in the potential energy of the system is

$$\Delta U = -W.$$

In the topic of kinetic energy and its related topics that we discussed in this chapter, the relation between work and a change in kinetic energy. Here we discuss the relation between work and a change in potential energy.

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Let us throw a tomato upward (Fig. 8-19). We already know that as the tomato rises, the work W_g done on the tomato by the gravitational force is negative because the force transfers energy *from* the kinetic energy of the tomato. We can now finish the story by saying that this energy is transferred by the gravitational force *to* the gravitational potential energy of the tomato–Earth system.

The tomato slows, stops, and then begins to fall back down because of the gravitational force. During the fall, the transfer is reversed: The work W_g done on the tomato by the gravitational force is now positive—that force transfers energy *from* the gravitational potential energy of the tomato–Earth system *to* the kinetic energy of the tomato.

For either rise or fall, the change ΔU in gravitational potential energy is defined as being equal to the negative of the work done on the tomato by the gravitational force. Using the general symbol W for work, we write this as

$$\Delta U = -W. \tag{8-42}$$

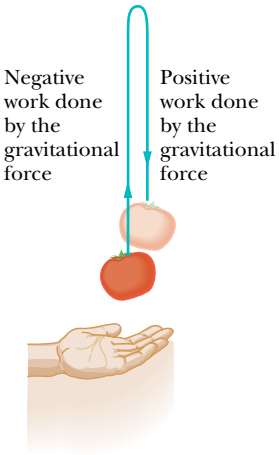

Figure 8-19 A tomato is thrown upward. As it rises, the gravitational force does negative work on it, decreasing its kinetic energy. As the tomato descends, the gravitational force does positive work on it, increasing its kinetic energy.

This equation also applies to a block–spring system, as in Fig. 8-20. If we abruptly shove the block to send it moving rightward, the spring force acts leftward and thus does negative work on the block, transferring energy from the kinetic energy of the block to the elastic potential energy of the spring–block system. The block slows and eventually stops, and then begins to move leftward because the spring force is still leftward. The transfer of energy is then reversed—it is from potential energy of the spring–block system to kinetic energy of the block.

Conservative and Nonconservative Forces

Let us list the key elements of the two situations we just discussed:

1. The *system* consists of two or more objects.
2. A *force* acts between a particle-like object (tomato or block) in the system and the rest of the system.
3. When the system configuration changes, the force does *work* (call it W_1) on the particle-like object, transferring energy between the kinetic energy K of the object and some other type of energy of the system.
4. When the configuration change is reversed, the force reverses the energy transfer, doing work W_2 in the process.

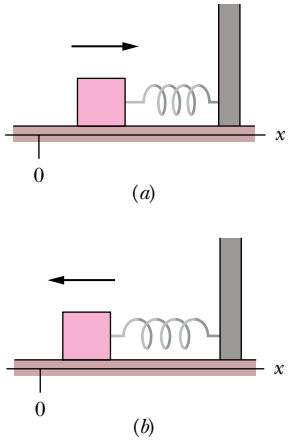

Figure 8-20 A block, attached to a spring and initially at rest at $x = 0$, is set in motion toward the right. (*a*) As the block moves rightward (as indicated by the arrow), the spring force does negative work on it. (*b*) Then, as the block moves back toward $x = 0$, the spring force does positive work on it.

In a situation in which $W_1 = -W_2$ is always true, the other type of energy is a potential energy and the force is said to be a **conservative force**. As you might suspect, the gravitational force and the spring force are both conservative (since otherwise we could not have spoken of gravitational potential energy and elastic potential energy, as we did previously).

A force that is not conservative is called a **nonconservative force**. The kinetic frictional force and drag force are nonconservative. For an example, let us send a block sliding across a floor that is not frictionless. During the sliding, a kinetic frictional force from the floor slows the block by transferring energy from its kinetic energy to a type of energy called *thermal energy* (which has to do with the random motions of atoms and molecules). We know from experiment that this energy transfer cannot be reversed (thermal energy cannot be transferred back to kinetic energy of the block by the kinetic frictional force). Thus, although we have a system (made up of the block and the floor), a force that acts between parts of the system, and a transfer of energy by the force, the force is not conservative. Therefore, thermal energy is not a potential energy.

When only conservative forces act on a particle-like object, we can greatly simplify otherwise difficult problems involving motion of the object. Let's next develop a test for identifying conservative forces, which will provide one means for simplifying such problems.

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8.11 | PATH INDEPENDENCE OF CONSERVATIVE FORCES

Key Concept

- For a particle moving between two points, the work done by a conservative force is independent of the path taken by the particle

$$W_{ab,1} = W_{ab,2}$$

The primary test for determining whether a force is conservative or nonconservative is this: Let the force act on a particle that moves along any *closed path*, beginning at some initial position and eventually returning to that position (so that the particle makes a *round trip* beginning and ending at the initial position). The force is conservative only if the total energy it transfers to and from the particle during the round trip along this and any other closed path is zero. In other words:

> The net work done by a conservative force on a particle moving around any closed path is zero.

We know from experiment that the gravitational force passes this *closedpath test*. An example is the tossed tomato of Fig. 8-19. The tomato leaves the launch point with speed v_0 and kinetic energy $1/2mv_0^2$. The gravitational force acting on the tomato slows it, stops it, and then causes it to fall back down. When the tomato returns to the launch point, it again has speed v_0 and kinetic energy $1/2mv_0^2$. Thus, the gravitational force transfers as much energy *from* the tomato during the ascent as it transfers *to* the tomato during the descent back to the launch point. The net work done on the tomato by the gravitational force during the round trip is zero.

An important result of the closed-path test is that:

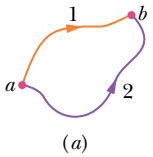
> The work done by a conservative force on a particle moving between two points does not depend on the path taken by the particle.

For example, suppose that a particle moves from point a to point b in Fig. 8-21a along either path 1 or path 2. If only a conservative force acts on the particle, then the work done on the particle is the same along the two paths. In symbols, we can write this result as

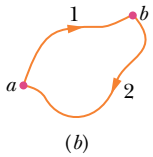
$$W_{ab,1} = W_{ab,2}, \tag{8-43}$$

where the subscript ab indicates the initial and final points, respectively, and the subscripts 1 and 2 indicate the path.

This result is powerful because it allows us to simplify difficult problems when only a conservative force is involved. Suppose you need to calculate the work done by a conservative force along a given path between two points, and the calculation is difficult or even impossible without additional information. You can find the work by substituting some other path between those two points for which the calculation is easier and possible.



(a) The force is conservative. Any choice of path between the points gives the same amount of work.



(b) And a round trip gives a total work of zero.

Figure 8-21 (a) As a conservative force acts on it, a particle can move from point a to point b along either path 1 or path 2. (b) The particle moves in a round trip, from point a to point b along path 1 and then back to point a along path 2.

Proof of Eq. 8-43

Figure 8-21b shows an arbitrary round trip for a particle that is acted upon by a single force. The particle moves from an initial point a to point b along path 1 and then back to point a along path 2. The force does work on the

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particle as the particle moves along each path. Without worrying about where positive work is done and where negative work is done, let us just represent the work done from a to b along path 1 as $W_{ab,1}$ and the work done from b back to a along path 2 as $W_{ab,2}$. If the force is conservative, then the net work done during the round trip must be zero:

$$W_{ab,1} + W_{ab,2} = 0,$$

and thus

$$W_{ab,1} = -W_{ab,2}. \tag{8-44}$$

In words, the work done along the outward path must be the negative of the work done along the path back.

Let us now consider the work $W_{ab,2}$ done on the particle by the force when the particle moves from a to b along path 2, as indicated in Fig. 8-21*a*. If the force is conservative, that work is the negative of $W_{ab,2}$:

$$W_{ab,2} = -W_{ab,2}. \tag{8-45}$$

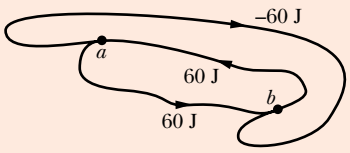
Substituting $W_{ab,2}$ for $-W_{ab,2}$ in Eq. 8-44, we obtain

$$W_{ab,1} = W_{ab,2},$$

which is what we set out to prove.

CHECKPOINT 4

The figure shows three paths connecting points a and b. A single force $\vec{F}$ does the indicated work on a particle moving along each path in the indicated direction. On the basis of this information, is force $\vec{F}$ conservative?



SAMPLE PROBLEM 8.13

Equivalent paths for calculating work, slippery cheese

The main lesson of this sample problem is this: It is perfectly all right to choose an easy path instead of a hard path. Figure 8-22*a* shows a 2.0 kg block of slippery cheese that slides along a frictionless track from point a to point b. The cheese travels through a total distance of 2.0 m along the track, and a net vertical distance of 0.80 m. How much work is done on the cheese by the gravitational force during the slide?

KEY IDEAS

(1) We *cannot* calculate the work by using Eq. 8-12 ($W_g = mgd \cos\phi$). The reason is that the angle ϕ between the directions of the gravitational force $\vec{F}_g$ and the displacement $\vec{d}$ varies along the track in an unknown way. (Even if we did know the shape of the track and could calculate ϕ along it, the calculation could be very difficult.) (2) Because $\vec{F}_g$ is a conservative force, we can find the work by choosing some other path between a and b—one that makes the calculation easy.

Calculations: Let us choose the dashed path in Fig. 8-22*b*; it consists of two straight segments. Along the horizontal segment, the angle ϕ is a constant 90°. Even though we do not know the displacement along that horizontal segment, Eq. 8-12 tells us that the work W_h done there is

$$W_h = mgd \cos 90° = 0.$$

Along the vertical segment, the displacement d is 0.80 m and, with $\vec{F}_g$ and $\vec{d}$ both downward, the angle ϕ is

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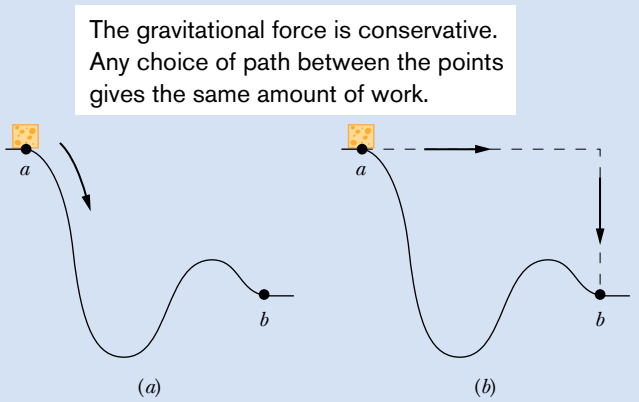

Figure 8-22 (*a*) A block of cheese slides along a frictionless track from point *a* to point *b*. (*b*) Finding the work done on the cheese by the gravitational force is easier along the dashed path than along the actual path taken by the cheese; the result is the same for both paths.

a constant 0°. Thus, Eq. 8-12 gives us, for the work W_v done along the vertical part of the dashed path,

$$\begin{aligned} W_v &= mgd \cos 90° \\ &= (2.0 \text{ kg})(9.8 \text{ m/s}^2)(0.80 \text{ m})(1) \\ &= 15.7 \text{ J}. \end{aligned}$$

The total work done on the cheese by $\vec{F}_g$ as the cheese moves from point *a* to point *b* along the dashed path is then

$$W = W_h + W_v = 0 + 15.7 \text{ J} \approx 16 \text{ J}. \qquad \text{(Answer)}$$

This is also the work done as the cheese slides along the track from *a* to *b*.

8.12 | DETERMINING POTENTIAL ENERGY VALUES

Key Concept

- If the particle moves from point x_i to point x_f, the change in the potential energy of the system is

$$\Delta U = -\int_{x_i}^{x_f} F(x)\, dx.$$

- The potential energy associated with a system consisting of Earth and a nearby particle is gravitational potential energy. If the particle moves from height y_i to height y_f, the change in the gravitational potential energy of the particle–Earth system is

$$\Delta U = mg(y_f - y_i) = mg\,\Delta y.$$

- If the reference point of the particle is set as $y_i = 0$ and the corresponding gravitational potential energy of the system is set as $U_i = 0$, then the gravitational potential energy U when the particle is at any height y is

$$U(y) = mgy.$$

- Elastic potential energy is the energy associated with the state of compression or extension of an elastic object. For a spring that exerts a spring force $F = -kx$ when its free end has displacement x, the elastic potential energy is

$$U(x) = \frac{1}{2}kx^2.$$

- The reference configuration has the spring at its relaxed length, at which $x = 0$ and $U = 0$.

Here, we find equations that give the value of the two types of potential energy discussed in this chapter: Gravitational potential energy and elastic potential energy. However, first we must find a general relation between a conservative force and the associated potential energy.

Consider a particle-like object that is part of a system in which a conservative force $\vec{F}$ acts. When that force does work W on the object, the change ΔU in the potential energy associated with the system is the negative of the work done. We wrote this fact as Eq. 8-42 ($\Delta U = -W$). For the most general case, in which the force may vary with position, we may write the work W as in Eq. 8-32:

$$W = \int_{x_i}^{x_f} F(x)\, dx. \qquad (8\text{-}46)$$

This equation gives the work done by the force when the object moves from point x_i to point x_f, changing the configuration of the system. (Because the force is conservative, the work is the same for all paths between those two points.)

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Substituting Eq. 8-46 into Eq. 8-42, we find that the change in potential energy due to the change in configuration is, in general notation,

$$\Delta U = -\int_{x_i}^{x_f} F(x)\, dx. \tag{8-47}$$

Gravitational Potential Energy

We first consider a particle with mass m moving vertically along a y axis (the positive direction is upward). As the particle moves from point y_i to point y_f, the gravitational force $\vec{F}_g$ does work on it.To find the corresponding change in the gravitational potential energy of the particle–Earth system, we use Eq. 8-47 with two changes: (1) We integrate along the y axis instead of the x axis, because the gravitational force acts vertically. (2) We substitute $-mg$ for the force symbol F, because $\vec{F}_g$ has the magnitude mg and is directed down the y axis. We then have

$$\Delta U = -\int_{y_i}^{y_f} (-mg)\, dy = mg\int_{y_i}^{y_f} dy = mg\big[y\big]_{y_i}^{y_f},$$

which yields

$$\Delta U = mg(y_f - y_i) = mg\,\Delta y. \tag{8-48}$$

Only *changes* ΔU in gravitational potential energy (or any other type of potential energy) are physically meaningful. However, to simplify a calculation or a discussion, we sometimes would like to say that a certain gravitational potential value U is associated with a certain particle–Earth system when the particle is at a certain height y. To do so, we rewrite Eq. 8-48 as

$$U - U_i = mg\,(y - y_i). \tag{8-49}$$

Then we take U_i to be the gravitational potential energy of the system when it is in a **reference configuration** in which the particle is at a **reference point** y_i. Usually we take $U_i = 0$ and $y_i = 0$. Doing this changes Eq. 8-49 to

$$U(y) = mgy \quad \text{(gravitational potential energy)}. \tag{8-50}$$

This equation tells us:

The gravitational potential energy associated with a particle–Earth system depends only on the vertical position y (or height) of the particle relative to the reference position $y = 0$, not on the horizontal position.

Elastic Potential Energy

We next consider the block–spring system shown in Fig. 8-20, with the block moving on the end of a spring of spring constant k. As the block moves from point x_i to point x_f, the spring force $F_x = -kx$ does work on the block. To find the corresponding change in the elastic potential energy of the block–spring system, we substitute $-kx$ for $F(x)$ in Eq. 8-47. We then have

$$\Delta U = -\int_{y_i}^{y_f} (-kx)\, dx = k\int_{x_i}^{x_f} x\, dx = \frac{1}{2}k\big[x^2\big]_{x_i}^{x_f},$$

or

$$\Delta U = \frac{1}{2}kx_f^2 - \frac{1}{2}kx_f^2. \tag{8-51}$$

To associate a potential energy value U with the block at position x, we choose the reference configuration to be when the spring is at its relaxed length and the block is at $x_i = 0$. Then the elastic potential energy U_i is 0, and Eq. 8-51 becomes

$$U - 0 = \frac{1}{2}kx^2 - 0,$$

which gives us

$$U(x) = \frac{1}{2}kx^2 \quad \text{(elastic potential energy)}. \tag{8-52}$$

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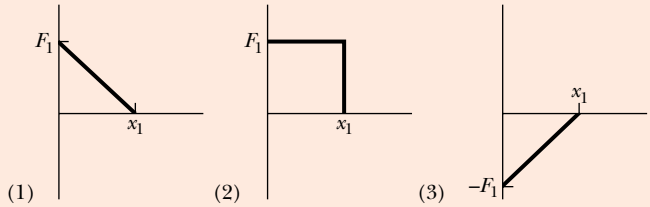
CHECKPOINT 5

A particle is to move along an x axis from $x = 0$ to x_1 while a conservative force, directed along the x axis, acts on the particle. The figure shows three situations in which the x component of that force varies with x. The force has the same maximum magnitude F_1 in all three situations. Rank the situations according to the change in the associated potential energy during the particle's motion, most positive first.



SAMPLE PROBLEM 8.14

Choosing reference level for gravitational potential energy, sloth

Here is an example with this lesson plan: Generally you can choose any level to be the reference level, but once chosen, be consistent. A 2.0 kg sloth hangs 5.0 m above the ground (Fig. 8-23).

(a) What is the gravitational potential energy U of the sloth–Earth system if we take the reference point $y = 0$ to be (1) at the ground, (2) at a balcony floor that is 3.0 m above the ground, (3) at the limb, and (4) 1.0 m above the limb? Take the gravitational potential energy to be zero at $y = 0$.

KEY IDEA

Once we have chosen the reference point for $y = 0$, we can calculate the gravitational potential energy U of the system *relative to that reference point* with Eq. 8-50.

Calculations: For choice (1), the sloth is at $y = 5.0$ m, and

$$U = mgy = (2.0\text{ kg})(9.8\text{ m/s}^2)(5.0\text{ m}) = 98\text{ J}. \quad \text{(Answer)}$$

For the other choices, the values of U are

(2) $U = mgy = mg(2.0\text{ m}) = 39\text{ J}$,

(3) $U = mgy = mg(0) = 0\text{ J}$,

(4) $U = mgy = mg(-1.0\text{ m})$

$$= -19.6\text{ J} \approx -20\text{ J}. \quad \text{(Answer)}$$

(b) The sloth drops to the ground. For each choice of reference point, what is the change ΔU in the potential energy of the sloth–Earth system due to the fall?

KEY IDEA

The *change* in potential energy does not depend on the choice of the reference point for $y = 0$; instead, it depends on the change in height Δy.

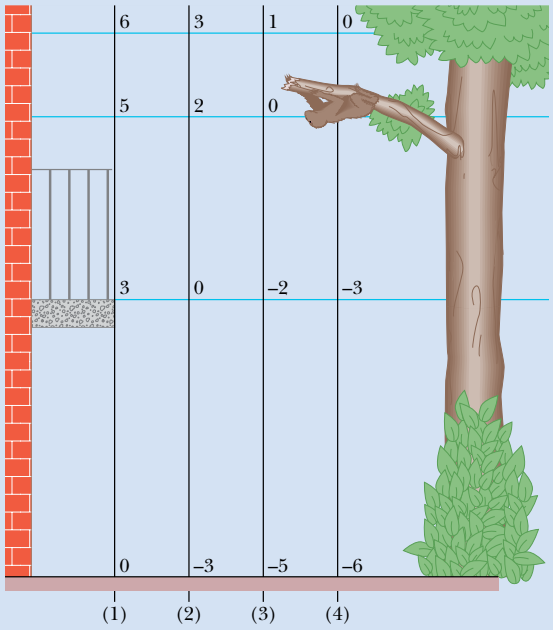


Figure 8-23 Four choices of reference point $y = 0$. Each y axis is marked in units of meters.The choice affects the value of the potential energy U of the sloth–Earth system. However, it does not affect the change ΔU in potential energy of the system if the sloth moves by, say, falling.

Calculation: For all four situations, we have the same $\Delta y = -5.0$ m. Thus, for (1) to (4), Eq. 8-48 tells us that

$$\Delta U = mg\,\Delta y = (2.0\text{ kg})(9.8\text{ m/s}^2)(-5.0\text{ m})$$

$$= -98\text{ J}. \quad \text{(Answer)}$$

8.13 | WORK–MECHANICAL ENERGY THEOREM

Key Concept

- Effect of a force can either be written as work or as potential energy.

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We have learnt from work–kinetic energy theorem that if some forces act on a body, then the sum of the work done by individual forces that are acting on it is equal to the change in kinetic energy.

$$W_1 + W_2 + \cdots + W_n = K_f - K_i$$

If some of them are conservative and others are non-conservative, then for conservative forces, we can write the potential energy as follows:

$$\sum W_c + \sum W_{nc} = K_f - K_i$$
$$\sum[-(U_f + U_i)] + \sum W_{nc} = K_f - K_i$$
$$\sum W_{nc} = K_f - K_i + \sum[-U_f - U_i]$$

The term on RHS is often called **mechanical energy**. To conclude, effect of a force can either be written as work on LHS or it can come as potential energy on RHS.

8.14 | CONSERVATION OF MECHANICAL ENERGY

Key Concepts

- The mechanical energy E_{mec} of a system is the sum of its kinetic energy K and potential energy U:

$$E_{mec} = K + U.$$

- An isolated system is one in which no external force causes energy changes. If only conservative forces do work within an isolated system, then the mechanical energy E_{mec} of the system cannot change. This principle of conservation of mechanical energy is written as

$$K_2 + U_2 = K_1 + U_1,$$

in which the subscripts refer to different instants during an energy transfer process. This conservation principle can also be written as

$$\Delta E_{mec} = \Delta K + \Delta U = 0.$$

The **mechanical energy** E_{mec} of a system is the sum of its potential energy U and the kinetic energy K of the objects within it:

$$E_{mec} = K + U \quad \text{(mechanical energy)}. \tag{8-53}$$

In this section, we examine what happens to this mechanical energy when only conservative forces cause energy transfers within the system—that is, when frictional and drag forces do not act on the objects in the system. Also, we shall assume that the system is *isolated* from its environment; that is, no *external force* from an object outside the system causes energy changes inside the system.

When a conservative force does work W on an object within the system, that force transfers energy between kinetic energy K of the object and potential energy U of the system. From Eq. 8-10, the change ΔK in kinetic energy is

$$\Delta K = W \tag{8-54}$$

and from Eq. 8-42, the change ΔU in potential energy is

$$\Delta U = -W \tag{8-55}$$

Combining Eqs. 8-54 and 8-55, we find that

$$\Delta K = -\Delta U. \tag{8-56}$$

In words, one of these energies increases exactly as much as the other decreases.

We can rewrite Eq. 8-56 as

$$K_2 - K_1 = -(U_2 - U_1), \tag{8-57}$$

©AP/Wide World Photos

In olden days, a person would be tossed via a blanket to be able to see farther over the flat terrain. Nowadays, it is done just for fun. During the ascent of the person in the photograph, energy is transferred from kinetic energy to gravitational potential energy. The maximum height is reached when that transfer is complete. Then the transfer is reversed during the fall.

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where the subscripts refer to two different instants and thus to two different arrangements of the objects in the system. Rearranging Eq. 8-57 yields

$$K_2 + U_2 = K_1 - U_1 \quad \text{(conservation of mechanical energy)}. \tag{8-58}$$

In words, this equation says:

$$\begin{pmatrix}\text{the sum of } K \text{ and } U \text{ for}\\ \text{any state of a system}\end{pmatrix} = \begin{pmatrix}\text{the sum of } K \text{ and } U \text{ for}\\ \text{any other state of the system}\end{pmatrix},$$

when the system is isolated and only conservative forces act on the objects in the system. In other words:

In an isolated system where only conservative forces cause energy changes, the kinetic energy and potential energy can change, but their sum, the mechanical energy E_{mec} of the system, cannot change.

This result is called the **principle of conservation of mechanical energy**. (Now you can see where *conservative* forces got their name.) With the aid of Eq. 8-56, we can write this principle in one more form, as

$$\Delta E_{mec} = \Delta K + \Delta U = 0. \tag{8-59}$$

The principle of conservation of mechanical energy allows us to solve problems that would be quite difficult to solve using only Newton's laws:

When the mechanical energy of a system is conserved, we can relate the sum of kinetic energy and potential energy at one instant to that at another instant *without considering the intermediate motion* and *without finding the work done by the forces involved.*

Figure 8-24 shows an example in which the principle of conservation of mechanical energy can be applied: As a pendulum swings, the energy of the pendulum–Earth system is transferred back and forth between kinetic energy K

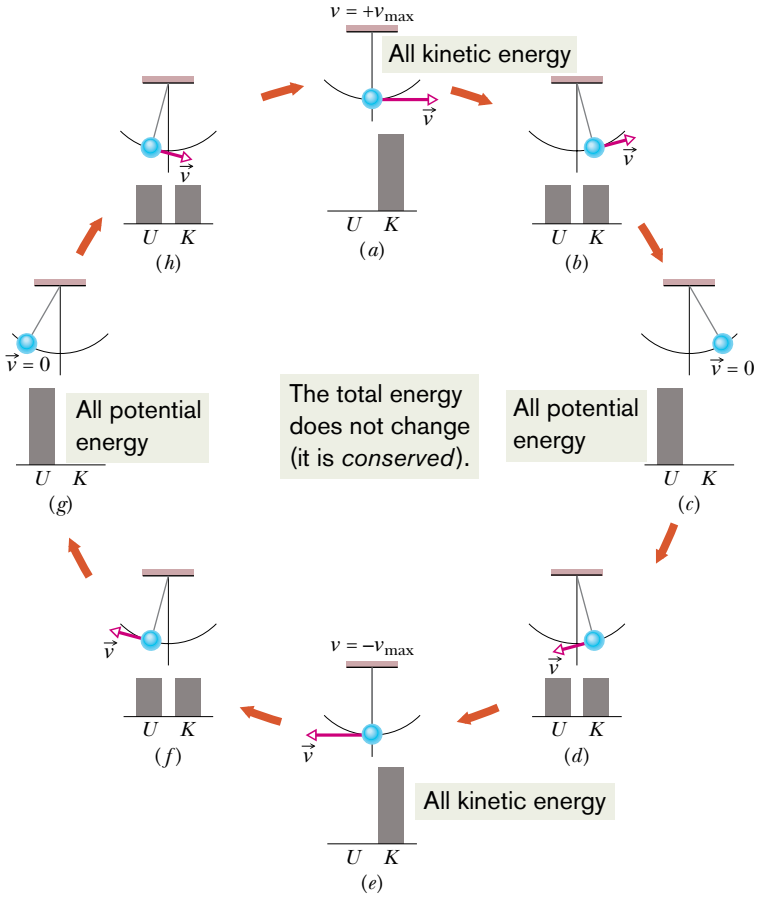

Figure 8-24 A pendulum, with its mass concentrated in a bob at the lower end, swings back and forth. One full cycle of the motion is shown. During the cycle the values of the potential and kinetic energies of the pendulum–Earth system vary as the bob rises and falls, but the mechanical energy E_{mec} of the system remains constant. The energy E_{mec} can be described as continuously shifting between the kinetic and potential forms. In stages (*a*) and (*e*), all the energy is kinetic energy. The bob then has its greatest speed and is at its lowest point. In stages (*c*) and (*g*), all the energy is potential energy. The bob then has zero speed and is at its highest point. In stages (*b*), (*d*), (*f*), and (*h*), half the energy is kinetic energy and half is potential energy. If the swinging involved a frictional force at the point where the pendulum is attached to the ceiling, or a drag force due to the air, then E_{mec} would not be conserved, and eventually the pendulum would stop.

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and gravitational potential energy U, with the sum $K + U$ being constant. If we know the gravitational potential energy when the pendulum bob is at its highest point (Fig. 8-24*c*), Eq. 8-58 gives us the kinetic energy of the bob at the lowest point (Fig. 8-24*e*).

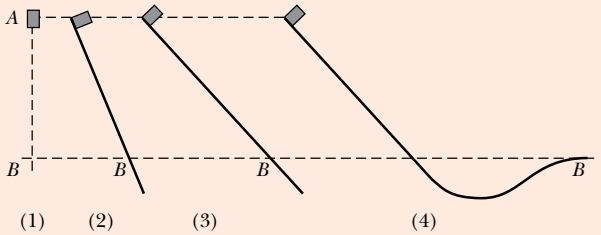
For example, let us choose the lowest point as the reference point, with the gravitational potential energy $U_2 = 0$. Suppose then that the potential energy at the highest point is $U_1 = 20$ J relative to the reference point. Because the bob momentarily stops at its highest point, the kinetic energy there is $K_1 = 0$. Putting these values into Eq. 8-58 gives us the kinetic energy K_2 at the lowest point:

$$K_2 + 0 = 0 + 20 \text{ J} \quad \text{or} \quad K_2 = 20 \text{ J}$$

Note that we get this result without considering the motion between the highest and lowest points (such as in Fig. 8-24*d*) and without finding the work done by any forces involved in the motion.

CHECKPOINT 6

The figure shows four situations—one in which an initially stationary block is dropped and three in which the block is allowed to slide down frictionless ramps. (a) Rank the situations according to the kinetic energy of the block at point *B*, greatest first. (b) Rank them according to the speed of the block at point *B*, greatest first.



SAMPLE PROBLEM 8.15

Conservation of mechanical energy, water slide

The huge advantage of using the conservation of energy instead of Newton's laws of motion is that we can jump from the initial state to the final state without considering all the intermediate motion. Here is an example. In Fig. 8-25, a child of mass m is released from rest at the top of a water slide, at height $h = 8.5$ m above the bottom of the slide. Assuming that the slide is frictionless because of the water on it, find the child's speed at the bottom of the slide.

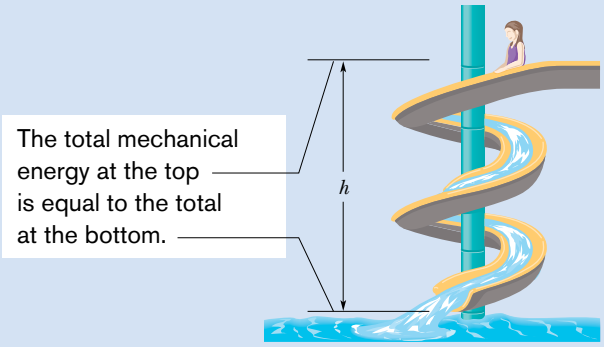


Figure 8-25 A child slides down a water slide as she descends a height h.

KEY IDEAS

(1) We cannot find her speed at the bottom by using her acceleration along the slide as we might have in earlier chapters because we do not know the slope (angle) of the slide. However, because that speed is related to her kinetic energy, perhaps we can use the principle of conservation of mechanical energy to get the speed. Then we would not need to know the slope. (2) Mechanical energy is conserved in a system *if* the system is isolated and *if* only conservative forces cause energy transfers within it. Let's check.

Forces: Two forces act on the child. The *gravitational force*, a conservative force, does work on her. The *normal force* on her from the slide does no work because its direction at any point during the descent is always perpendicular to the direction in which the child moves.

System: Because the only force doing work on the child is the gravitational force, we choose the child–Earth system as our system, which we can take to be isolated.

Thus, we have only a conservative force doing work in an isolated system, so we *can* use the principle of conservation of mechanical energy.

Calculations: Let the mechanical energy be $E_{\text{mec},t}$ when the child is at the top of the slide and $E_{\text{mec},b}$ when she is at the bottom.Then the conservation principle tells us

$$E_{\text{mec},b} = E_{\text{mec},t}. \qquad (8\text{-}60)$$

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To show both kinds of mechanical energy, we have

$$K_b + U_b = K_t + U_t, \tag{8-61}$$

or

$$\frac{1}{2}mv_b^2 + mgy_b = \frac{1}{2}mv_t^2 + mgy_t.$$

Dividing by m and rearranging yield

$$v_b^2 = v_t^2 + 2g(y_t - y_b).$$

Putting $v_t = 0$ and $y_t - y_b = h$ leads to

$$v_b = \sqrt{2gh} = \sqrt{(2)(9.8\ \text{m/s}^2)(8.5\ \text{m})}$$
$$= 13\ \text{m/s}. \qquad \text{(Answer)}$$

This is the same speed that the child would reach if she fell 8.5 m vertically. On an actual slide, some frictional forces would act and the child would not be moving quite so fast.

Comments: Although this problem is hard to solve directly with Newton's laws, using conservation of mechanical energy makes the solution much easier. However, if we were asked to find the time taken for the child to reach the bottom of the slide, energy methods would be of no use; we would need to know the shape of the slide, and we would have a difficult problem.

8.15 | WORK DONE ON A SYSTEM BY AN EXTERNAL FORCE

Key Concepts

- Work W is energy transferred to or from a system by means of an external force acting on the system.
- When more than one force acts on a system, their net work is the transferred energy.
- When friction is not involved, the work done on the system and the change ΔE_{mec} in the mechanical energy of the system are equal:

$$W = \Delta E_{mec} = \Delta K + \Delta U.$$

- When a kinetic frictional force acts within the system, then the thermal energy E_{th} of the system changes. (This energy is associated with the random motion of atoms and molecules in the system.) The work done on the system is then

$$W = \Delta E_{mec} + \Delta E_{th}.$$

- The change ΔE_{th} is related to the magnitude f_k of the frictional force and the magnitude d of the displacement caused by the external force by

$$\Delta E_{th} = f_k d.$$

In kinetic energy and related topics that we discussed in this chapter, we defined work as being energy transferred to or from an object by means of a force acting on the object.We can now extend that definition to an external force acting on a system of objects.

Work is energy transferred to or from a system by means of an external force acting on that system.

Figure 8-26*a* represents positive work (a transfer of energy *to* a system), and Fig. 8-26*b* represents negative work (a transfer of energy *from* a system). When more than one force acts on a system, their *net work* is the energy transferred to or from the system.

These transfers are like transfers of money to and from a bank account. If a system consists of a single particle or particle-like object, as in kinetic energy and work related topics, the work done on the system by a force can change only the kinetic energy of the system.The energy statement for such transfers is the work–kinetic energy theorem of Eq. 8-10 ($\Delta K = W$); that is, a single particle has only one energy account, called kinetic energy. External forces can transfer energy into or out of that account. If a system is more complicated, however, an external force can change other forms of energy (such as potential energy); that is, a more complicated system can have multiple energy accounts.

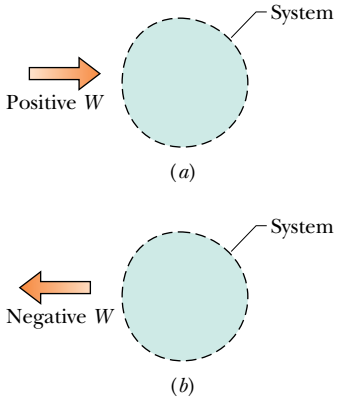

Figure 8-26 (*a*) Positive work W done on an arbitrary system means a transfer of energy to the system. (*b*) Negative work W means a transfer of energy from the system.

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Let us find energy statements for such systems by examining two basic situations, one that does not involve friction and one that does.

No Friction Involved

To compete in a bowling-ball-hurling contest, you first squat and cup your hands under the ball on the floor. Then you rapidly straighten up while also pulling your hands up sharply, launching the ball upward at about face level. During your upward motion, your applied force on the ball obviously does work; that is, it is an external force that transfers energy, but to what system?

To answer, we check to see which energies change. There is a change ΔK in the ball's kinetic energy and, because the ball and Earth become more separated, there is a change ΔU in the gravitational potential energy of the ball–Earth system. To include both changes, we need to consider the ball–Earth system. Then your force is an external force doing work on that system, and the work is

$$W = \Delta K + \Delta U, \tag{8-62}$$

or $$W = \Delta E_{mec} \quad \text{(work done on system, no friction involved)}, \tag{8-63}$$

where $= \Delta E_{mec}$ is the change in the mechanical energy of the system. These two equations, which are represented in Fig. 8-27, are equivalent energy statements for work done on a system by an external force when friction is not involved.

Your lifting force transfers energy to kinetic energy and potential energy.

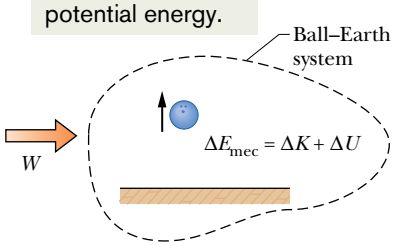

Figure 8-27 Positive work W is done on a system of a bowling ball and Earth, causing a change ΔE_{mec} in the mechanical energy of the system, a change ΔK in the ball's kinetic energy, and a change ΔU in the system's gravitational potential energy.

Friction Involved

We next consider the example in Fig. 8-28*a*. A constant horizontal force $\vec{F}$ pulls a block along an x axis and through a displacement of magnitude d, increasing the block's velocity from $\vec{v}_0$ to $\vec{v}$. During the motion, a constant kinetic frictional force $\vec{f}_k$ from the floor acts on the block. Let us first choose the block as our system and apply Newton's second law to it. We can write that law for components along the x axis ($F_{net,x} = ma_x$) as

$$F - f_k = ma. \tag{8-64}$$

Because the forces are constant, the acceleration $\vec{a}$ is also constant. Thus, we can use Eq. 2-16 to write

$$v^2 = v_0^2 + 2ad.$$

Solving this equation for a, substituting the result into Eq. 8-64, and rearranging then give us

$$Fd = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 + f_k d \tag{8-65}$$

The applied force supplies energy. The frictional force transfers some of it to thermal energy.

So, the work done by the applied force goes into kinetic energy and also thermal energy.

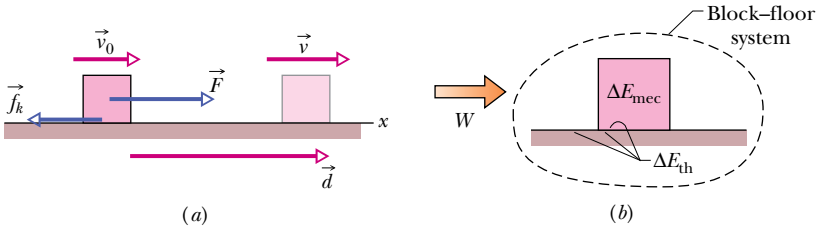

Figure 8-28 (*a*) A block is pulled across a floor by force $\vec{F}$ while a kinetic frictional force $\vec{f}_k$ opposes the motion. The block has velocity $\vec{v}_0$ at the start of a displacement $\vec{d}$ and velocity $\vec{v}$ at the end of the displacement. (*b*) Positive work W is done on the block–floor system by force $\vec{F}$, resulting in a change ΔE_{mec} in the block's mechanical energy and a change ΔE_{th} in the thermal energy of the block and floor.

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or, because $1/2mv^2 - 1/2mv_0^2 = \Delta K$ for the block,

$$Fd = \Delta K + f_k d. \tag{8-66}$$

In a more general situation (say, one in which the block is moving up a ramp), there can be a change in potential energy. To include such a possible change, we generalize Eq. 8-66 by writing

$$Fd = \Delta E_{mec} + f_k d. \tag{8-67}$$

By experiment, we find that the block and the portion of the floor along which it slides become warmer as the block slides. As we shall discuss in Chapter 18, the temperature of an object is related to the object's thermal energy E_{th} (the energy associated with the random motion of the atoms and molecules in the object). Here, the thermal energy of the block and floor increases because (1) there is friction between them and (2) there is sliding. Recall that friction is due to the cold-welding between two surfaces. As the block slides over the floor, the sliding causes repeated tearing and re-forming of the welds between the block and the floor, which makes the block and floor warmer. Thus, the sliding increases their thermal energy E_{th}.

Through experiment, we find that the increase ΔE_{th} in thermal energy is equal to the product of the magnitudes f_k and d:

$$\Delta E_{th} = f_k d \quad \text{(increase in thermal energy by sliding)}. \tag{8-68}$$

Thus, we can rewrite Eq. 8-67 as

$$Fd = \Delta E_{mec} + \Delta E_{th}. \tag{8-69}$$

Fd is the work W done by the external force $\vec{F}$ (the energy transferred by the force), but on which system is the work done (where are the energy transfers made)? To answer, we check to see which energies change. The block's mechanical energy changes, and the thermal energies of the block and floor also change.Therefore, the work done by force $\vec{F}$ is done on the block–floor system.That work is

$$W = \Delta E_{mec} + \Delta E_{th}. \quad \text{(work done on system, friction involved)}. \tag{8-70}$$

This equation, which is represented in Fig. 8-28*b*, is the energy statement for the work done on a system by an external force when friction is involved.

CHECKPOINT 7

In three trials, a block is pushed by a horizontal applied force across a floor that is not frictionless, as in Fig. 8-28*a*. The magnitudes F of the applied force and the results of the pushing on the block's speed are given in the table. In all three trials, the block is pushed through the same distance d. Rank the three trials according to the change in the thermal energy of the block and floor that occurs in that distance d, greatest first.

Trial	F	Result on Block's Speed
a	5.0 N	decreases
b	7.0 N	remains constant
c	8.0 N	increases

SAMPLE PROBLEM 8.16

Work, friction, change in thermal energy, cabbage heads

A food shipper pushes a wood crate of cabbage heads (total mass m = 14 kg) across a concrete floor with a constant horizontal force $\vec{F}$ of magnitude 40 N. In a straight-line displacement of magnitude d = 0.50 m, the speed of the crate decreases from v_0 = 0.60 m/s to v = 0.20 m/s.

(a) How much work is done by force $\vec{F}$, and on what system does it do the work?

KEY IDEA

Because the applied force $\vec{F}$ is constant, we can calculate the work it does by using Eq. 8-7 ($W = Fd \cos\phi$).

Calculation: Substituting given data, including the fact that Force $\vec{F}$ and displacement $\vec{d}$ are in the same direction, we find

$$W = Fd \cos\phi = (40\text{ N})(0.50\text{ m}) \cos 0°$$
$$= 20\text{ J}. \quad \text{(Answer)}$$

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Reasoning: To determine the system on which the work is done, let's check which energies change. Because the crate's speed changes, there is certainly a change ΔK in the crate's kinetic energy. Is there friction between the floor and the crate, and thus a change in thermal energy? Note that $\vec{F}$ and the crate's velocity have the same direction.Thus, if there is no friction, then $\vec{F}$ should be accelerating the crate to a *greater* speed. However, the crate is *slowing*, so there must be friction and a change ΔE_{th} in thermal energy of the crate and the floor. Therefore, the system on which the work is done is the crate–floor system, because both energy changes occur in that system.

(b) What is the increase ΔE_{th} in the thermal energy of the crate and floor?

KEY IDEA

We can relate ΔE_{th} to the work W done by $\vec{F}$ with the energy statement of Eq. 8-70 for a system that involves friction:

$$W = \Delta E_{mec} + \Delta E_{th}. \qquad (8\text{-}71)$$

Calculation: We know the value of W from (a). The change ΔE_{mec} in the crate's mechanical energy is just the change in its kinetic energy because no potential energy changes occur, so we have

$$\Delta E_{mec} = \Delta K = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2.$$

Substituting this into Eq. 8-71 and solving for ΔE_{th}, we find

$$\Delta E_{th} = W - \left(\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2\right) = W - \frac{1}{2}m(v^2 - v_0^2)$$

$$= 20\,\text{J} - \frac{1}{2}(14\text{ kg})[(0.20\text{ m/s})^2 - (0.60\text{ m/s})^2]$$

$$= 22.2\,\text{J} \approx 22\,\text{J}. \qquad \text{(Answer)}$$

Without further experiments, we cannot say how much of this thermal energy ends up in the crate and how much in the floor.We simply know the total amount.

8.16 | CONSERVATION OF ENERGY

Key Concepts

- The total energy E of a system (the sum of its mechanical energy and its internal energies, including thermal energy) can change only by amounts of energy that are transferred to or from the system. This experimental fact is known as the law of conservation of energy.
- If work W is done on the system, then

$$W = \Delta E = \Delta E_{mec} + \Delta E_{th} + \Delta E_{int}.$$

If the system is isolated ($W = 0$), this gives

$$\Delta E_{mec} + \Delta E_{th} + \Delta E_{int} = 0$$

and $$\Delta E_{mec,2} = \Delta E_{mec,1} - \Delta E_{th} - \Delta E_{int},$$

where the subscripts 1 and 2 refer to two different instants.

- The power due to a force is the *rate* at which that force transfers energy. If an amount of energy ΔE is transferred by a force in an amount of time Δt, the average power of the force is

$$P_{avg} = \frac{\Delta E}{\Delta t}.$$

- The instantaneous power due to a force is

$$P = \frac{dE}{dt}.$$

On a graph of energy E versus time t, the power is the slope of the plot at any given time.

We now have discussed several situations in which energy is transferred to or from objects and systems, much like money is transferred between accounts. In each situation we assume that the energy that was involved could always be accounted for; that is, energy could not magically appear or disappear. In more formal language, we assumed (correctly) that energy obeys a law called the **law of conservation of energy**, which is concerned with the

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total energy E of a system. That total is the sum of the system's mechanical energy, thermal energy, and any type of *internal energy* in addition to thermal energy. (We have not yet discussed other types of internal energy.) The law states that

The total energy E of a system can change only by amounts of energy that are transferred to or from the system.

The only type of energy transfer that we have considered is work W done on a system by an external force. Thus, for us at this point, this law states that

$$W = \Delta E = \Delta E_{mec} + \Delta E_{th} + \Delta E_{int}, \tag{8-72}$$

where ΔE_{mec} is any change in the mechanical energy of the system, ΔE_{th} is any change in the thermal energy of the system, and ΔE_{int} is any change in any other type of internal energy of the system. Included in ΔE_{mec} are changes ΔK in kinetic energy and changes ΔU in potential energy (elastic, gravitational, or any other type we might find).

This law of conservation of energy is *not* something we have derived from basic physics principles. Rather, it is a law based on countless experiments. Scientists and engineers have never found an exception to it. Energy simply cannot magically appear or disappear.

Isolated System

If a system is isolated from its environment, there can be no energy transfers to or from it. For that case, the law of conservation of energy states:

The total energy E of an isolated system cannot change.

Many energy transfers may be going on *within* an isolated system—between, say, kinetic energy and a potential energy or between kinetic energy and thermal energy. However, the total of all the types of energy in the system cannot change. Here again, energy cannot magically appear or disappear.

We can use the rock climber in Fig. 8-29 as an example, approximating him, his gear, and Earth as an isolated system. As he rappels down the rock face, changing the configuration of the system, he needs to control the transfer of energy from the gravitational potential energy of the system. (That energy cannot just disappear.) Some of it is transferred to his kinetic energy. However, he obviously does not want very much transferred to that type or he will be moving too quickly, so he has wrapped the rope around metal rings to produce friction between the rope and the rings as he moves down. The sliding of the rings on the rope then transfers the gravitational potential energy of the system to thermal energy of the rings and rope in a way that he can control. The total energy of the climber–gear–Earth system (the total of its gravitational potential energy, kinetic energy, and thermal energy) does not change during his descent.

For an isolated system, the law of conservation of energy can be written in two ways. First, by setting $W = 0$ in Eq. 8-72, we get

$$\Delta E_{mec} + \Delta E_{th} + \Delta E_{int} = 0 \quad \text{(isolated system)}. \tag{8-73}$$

We can also let $\Delta E_{mec} = \Delta E_{mec,2} - \Delta E_{mec,1}$, where the subscripts 1 and 2 refer to two different instants—say, before and after a certain process has occurred. Then Eq. 8-73 becomes

$$\Delta E_{mec,2} = \Delta E_{mec,1} - \Delta E_{th} - \Delta E_{int}. \tag{8-74}$$

Tyler Stableford/The Image Bank/Getty Images

Figure 8-29 To descend, the rock climber must transfer energy from the gravitational potential energy of a system consisting of him, his gear, and Earth. He has wrapped the rope around metal rings so that the rope rubs against the rings.This allows most of the transferred energy to go to the thermal energy of the rope and rings rather than to his kinetic energy.

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Equation 8-74 tells us:

> In an isolated system, we can relate the total energy at one instant to the total energy at another instant *without considering the energies at intermediate times.*

This fact can be a very powerful tool in solving problems about isolated systems when you need to relate energies of a system before and after a certain process occurs in the system.

In Section 8.15, we discussed a special situation for isolated systems—namely, the situation in which nonconservative forces (such as a kinetic frictional force) do not act within them. In that special situation, ΔE_{th} and ΔE_{int} are both zero, and so Eq. 8-74 reduces to Eq. 8-59. In other words, the mechanical energy of an isolated system is conserved when nonconservative forces do not act in it.

External Forces and Internal Energy Transfers

An external force can change the kinetic energy or potential energy of an object without doing work on the object—that is, without transferring energy to the object. Instead, the force is responsible for transfers of energy from one type to another inside the object.

Figure 8-30 shows an example. An initially stationary ice-skater pushes away from a railing and then slides over the ice (Figs. 8-30*a* and *b*). Her kinetic energy increases because of an external force $\vec{F}$ on her from the rail. However, that force does not transfer energy from the rail to her. Thus, the force does no work on her. Rather, her kinetic energy increases as a result of internal transfers from the biochemical energy in her muscles.

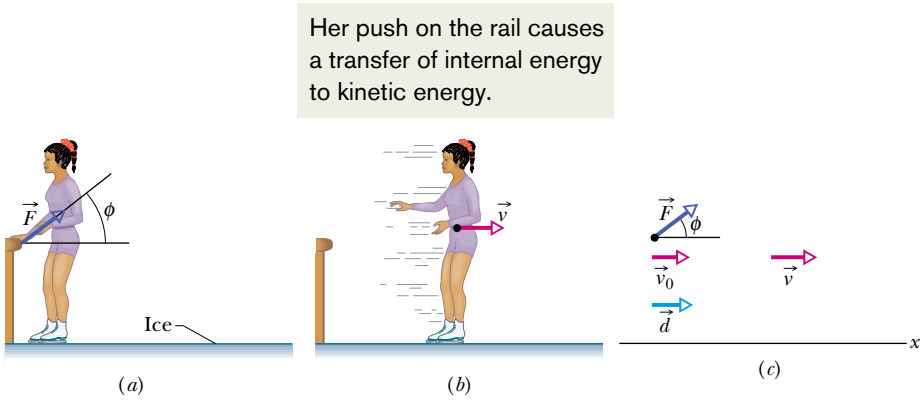

Figure 8-30 (*a*) As a skater pushes herself away from a railing, the force on her from the railing is $\vec{F}$. (*b*) After the skater leaves the railing, she has velocity $\vec{v}$. (*c*) External force $\vec{F}$ acts on the skater, at angle ϕ with a horizontal x axis. When the skater goes through displacement $\vec{d}$, her velocity is changed from $\vec{v}_0$ (= 0) to $\vec{v}$ by the horizontal component of $\vec{F}$.

Figure 8-31 shows another example. An engine increases the speed of a car with four-wheel drive (all four wheels are made to turn by the engine). During the acceleration, the engine causes the tires to push backward on the road surface. This push produces frictional forces $\vec{f}$ that act on each tire in the forward direction. The net external force $\vec{F}$ from the road, which is the sum of these frictional forces, accelerates the car, increasing its kinetic energy. However, $\vec{F}$ does not transfer energy from the road to the car and so does no work on the car. Rather, the car's kinetic energy increases as a result of internal transfers from the energy stored in the fuel.

In situations like these two, we can sometimes relate the external force $\vec{F}$ on an object to the change in the object's mechanical energy if we can simplify the situation. Consider the ice-skater example. During her push through distance d in Fig. 8-30*c*, we can simplify by assuming that the acceleration is constant, her speed changing from $v_0 = 0$ to v. (That is, we assume $\vec{F}$ has

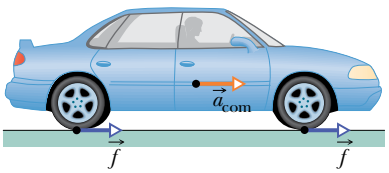

Figure 8-31 A vehicle accelerates to the right using four-wheel drive. The road exerts four frictional forces (two of them shown) on the bottom surfaces of the tires. Taken together, these four forces make up the net external force $\vec{F}$ acting on the car.

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constant magnitude F and angle ϕ.) After the push, we can simplify the skater as being a particle and neglect the fact that the exertions of her muscles have increased the thermal energy in her muscles and changed other physiological features. Then we can apply Eq. 8-5 ($1/2mv^2 - 1/2mv_0^2 = F_x d$) to write

$$K - K_0 = (F \cos \phi)d,$$

or

$$\Delta K = Fd \cos \phi. \tag{8-75}$$

If the situation also involves a change in the elevation of an object, we can include the change ΔU in gravitational potential energy by writing

$$\Delta U + \Delta K = Fd \cos \phi. \tag{8-76}$$

The force on the right side of this equation does no work on the object but is still responsible for the changes in energy shown on the left side.

SAMPLE PROBLEM 8.17

Lots of energies at an amusement park water slide

Figure 8-32 shows a water-slide ride in which a glider is shot by a spring along a water-drenched (frictionless) track that takes the glider from a horizontal section down to ground level. As the glider then moves along the ground-level track, it is gradually brought to rest by friction. The total mass of the glider and its rider is $m = 200$ kg, the initial compression of the spring is $d = 5.00$ m, the spring constant is $k = 3.20 \times 10^3$ N/m, the initial height is $h = 35.0$ m, and the coefficient of kinetic friction along the ground-level track is $\mu_k = 0.800$. Through what distance L does the glider slide along the ground-level track until it stops?

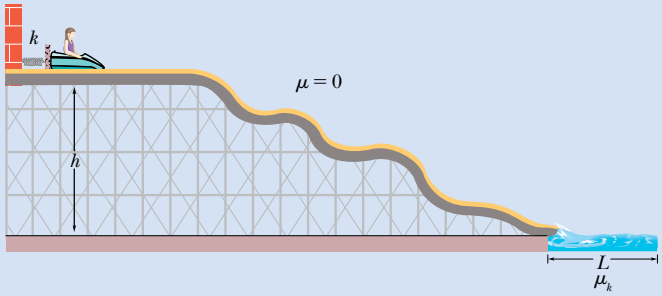

Figure 8-32 A spring-loaded amusement park water slide.

KEY IDEAS

First of all we need to examine all the forces and then determine what our system should be. Only then can we decide what equation to write. Do we have an isolated system (our equation would be for the conservation of energy) or a system on which an external force does work (our equation would relate that work to the system's change in energy)?

Forces: The normal force on the glider from the track does no work on the glider because the direction of this force is always perpendicular to the direction of the glider's displacement. The gravitational force does work on the glider, and because the force is conservative we can associate a potential energy with it. As the spring pushes on the glider to get it moving, a spring force does work on it, transferring energy from the elastic potential energy of the compressed spring to kinetic energy of the glider. The spring force also pushes against a rigid wall. Because there is friction between the glider and the ground-level track, the sliding of the glider along that track section increases their thermal energies.

System: Let's take the system to contain all the interacting bodies: glider, track, spring, Earth, and wall. Then, because all the force interactions are *within* the system, the system is *isolated* and thus its total energy cannot change. So, the equation we should use is not that of some external force doing work on the system. Rather, it is a conservation of energy.We write this in the form of Eq. 8-74:

$$E_{mec,2} = E_{mec,1} - \Delta E_{th}. \tag{8-77}$$

This is like a money equation: The final money is equal to the initial money *minus* the amount stolen away by a thief. Here, the final mechanical energy is equal to the initial mechanical energy *minus* the amount stolen away by friction. None has magically appeared or disappeared.

Calculations: Now that we have an equation, let's find distance L. Let subscript 1 correspond to the initial state of the glider (when it is still on the compressed spring) and subscript 2 correspond to the final state of the glider (when it has come to rest on the ground-level track).

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For both states, the mechanical energy of the system is the sum of any potential energy and any kinetic energy.

We have two types of potential energy: the elastic potential energy ($U_e = 1/2kx^2$) associated with the compressed spring and the gravitational potential energy ($U_g = mgy$) associated with the glider's elevation. For the latter, let's take ground level as the reference level. That means that the glider is initially at height $y = h$ and finally at height $y = 0$.

In the initial state, with the glider stationary and elevated and the spring compressed, the energy is

$$E_{mec,1} = K_1 + U_{e1} + U_{g1} = 0 + \frac{1}{2}kd^2 + mgh. \quad (8\text{-}78)$$

In the final state, with the spring now in its relaxed state and the glider again stationary but no longer elevated, the final mechanical energy of the system is

$$E_{mec,2} = K_2 + U_{e2} + U_{g2} = 0 + 0 + 0. \quad (8\text{-}79)$$

Let's next go after the change ΔE_{th} of the thermal energy of the glider and ground-level track. From Eq. 8-68, we can substitute for ΔE_{th} with $f_k L$ (the product of the frictional force magnitude and the distance of rubbing). From Eq. 6-2, we know that $f_k = \mu_k F_N$, where F_N is the normal force. Because the glider moves horizontally through the region with friction, the magnitude of F_N is equal to mg (the upward force matches the downward force). So, the friction's theft from the mechanical energy amounts to

$$\Delta E_{th} = \mu_k mgL. \quad (8\text{-}80)$$

(By the way, without further experiments, we *cannot* say how much of this thermal energy ends up in the glider and how much in the track. We simply know the total amount.) Substituting Eqs. 8-78 through 8-79 into Eq. 8-77, we find

$$0 = \frac{1}{2}kd^2 + mgh - \mu_k mgL, \quad (8\text{-}81)$$

and

$$L = \frac{kd^2}{2\mu_k mg} + \frac{h}{\mu_k} = \frac{(3.20\times10^3\ \text{N/m})(5.00\ \text{m})^2}{2(0.800)(200\ \text{kg})(9.8\ \text{m/s}^2)} + \frac{35\ \text{m}}{0.800} = 69.3\ \text{m} \quad \text{(Answer)}$$

Finally, note how algebraically simple our solution is. By carefully defining a system and realizing that we have an isolated system, we get to use the law of the conservation of energy. That means we can relate the initial and final states of the system with no consideration of the intermediate states. In particular, we did not need to consider the glider as it slides over the uneven track. If we had, instead, applied Newton's second law to the motion, we would have had to know the details of the track and would have faced a far more difficult calculation.

8.17 | POWER

Key Concepts

- The power due to a force is the *rate* at which that force does work on an object.
- If the force does work W during a time interval Δt, the average power due to the force over that time interval is

$$P_{avg} = \frac{W}{\Delta t}.$$

- Instantaneous power is the instantaneous rate of doing work:

$$P = \frac{dW}{dt}.$$

- For a force $\vec{F}$ at an angle ϕ to the direction of travel of the instantaneous velocity $\vec{v}_1$, the instantaneous power is

$$P = Fv\cos\phi = \vec{F}\cdot\vec{v}.$$

Power

The time rate at which work is done by a force is said to be the **power** due to the force. If a force does an amount of work W in an amount of time Δt, the **average power** due to the force during that time interval is

$$P_{avg} = \frac{W}{\Delta t} \quad \text{(average power)}. \quad (8\text{-}82)$$

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The **instantaneous power** P is the instantaneous time rate of doing work, which we can write as

$$P = \frac{dW}{dt} \quad \text{(instantaneous power)}. \tag{8-83}$$

Suppose we know the work $W(t)$ done by a force as a function of time. Then to get the instantaneous power P at, say, time $t = 3.0$ s during the work, we would first take the time derivative of $W(t)$ and then evaluate the result for $t = 3.0$ s.

The SI unit of power is the joule per second.This unit is used so often that it has a special name, the **watt** (W), after James Watt, who greatly improved the rate at which steam engines could do work. In the British system, the unit of power is the foot-pound per second. Often the horsepower is used. These are related by

$$1 \text{ watt} = 1 \text{ W} = 1 \text{ J/s} = 0.738 \text{ ft} \cdot \text{lb/s} \tag{8-84}$$

and

$$1 \text{ horsepower} = 1 \text{ hp} = 550 \text{ ft} \cdot \text{l/s} = 746 \text{ W}. \tag{8-85}$$

Inspection of Eq. 8-82 shows that work can be expressed as power multiplied by time, as in the common unit kilowatt-hour. Thus,

$$\begin{aligned} 1 \text{ kilowatt-hour} &= 1 \text{ kW} \cdot \text{h} = (10^3 \text{ W})(3600 \text{ s}) \\ &= 3.60 \times 10^6 \text{ J} = 3.60 \text{ MJ}. \end{aligned} \tag{8-86}$$

Perhaps because they appear on our utility bills, the watt and the kilowatt-hour have become identified as electrical units.They can be used equally well as units for other examples of power and energy. Thus, if you pick up a book from the floor and put it on a tabletop, you are free to report the work that you have done as, say, 4×10^6 kW · h (or more conveniently as 4 mW · h).

We can also express the rate at which a force does work on a particle (or particle-like object) in terms of that force and the particle's velocity. For a particle that is moving along a straight line (say, an x axis) and is acted on by a constant force $\vec{F}$ directed at some angle ϕ to that line, Eq. 8-83 becomes

$$P = \frac{dW}{dt} = \frac{F \cos\phi \, dx}{dt} = F \cos\phi \left(\frac{dx}{dt}\right),$$

or

$$P = Fv \cos \phi. \tag{8-87}$$

Reorganizing the right side of Eq. 8-87 as the dot product $\vec{F} \cdot \vec{v}$, we may also write the equation as

$$P = \vec{F} \cdot \vec{v} \quad \text{(instantaneous power)}. \tag{8-88}$$

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Figure 8-33 The power due to the truck's applied force on the trailing load is the rate at which that force does work on the load.

For example, the truck in Fig. 8-33 exerts a force $\vec{F}$ on the trailing load, which has velocity $\vec{v}$ at some instant. The instantaneous power due to $\vec{F}$ is the rate at which $\vec{F}$ does work on the load at that instant and is given by Eqs. 8-87 and 8-88. Saying that this power is "the power of the truck" is often acceptable, but keep in mind what is meant: Power is the rate at which the applied *force* does work.

CHECKPOINT 8

A block moves with uniform circular motion because a cord tied to the block is anchored at the center of a circle. Is the power due to the force on the block from the cord positive, negative, or zero?

SAMPLE PROBLEM 8.18

Power, force, and velocity

Here we calculate an instantaneous work—that is, the rate at which work is being done at any given instant rather than averaged over a time interval. Figure 8-34 shows constant forces $\vec{F}_1$ and $\vec{F}_2$ acting on a box as the box slides rightward across a frictionless floor. Force $\vec{F}_1$ is horizontal, with magnitude 2.0 N; force $\vec{F}_2$ is angled

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upward by 60° to the floor and has magnitude 4.0 N. The speed v of the box at a certain instant is 3.0 m/s. What is the power due to each force acting on the box at that instant, and what is the net power? Is the net power changing at that instant?

KEY IDEAS

We want an instantaneous power, not an average power over a time period. Also, we know the box's velocity (rather than the work done on it).

Calculation: We use Eq. 8-87 for each force. For force $\vec{F}_1$, at angle $\phi_1 = 180°$ to velocity $\vec{v}$, we have

$$P_1 = F_1 v \cos\phi_1 = (2.0\text{ N})(3.0\text{ m/s})\cos 180°$$
$$= -6.0\text{ W}. \quad \text{(Answer)}$$

This negative result tells us that force $\vec{F}_1$ is transferring energy *from* the box at the rate of 6.0 J/s.

For force $\vec{F}_2$, at angle $\phi_2 = 60°$ to velocity $\vec{v}$, we have

$$P_2 = F_2 v \cos\phi_2 = (4.0\text{ N})(3.0\text{ m/s})\cos 60°$$
$$= 6.0\text{ W}. \quad \text{(Answer)}$$

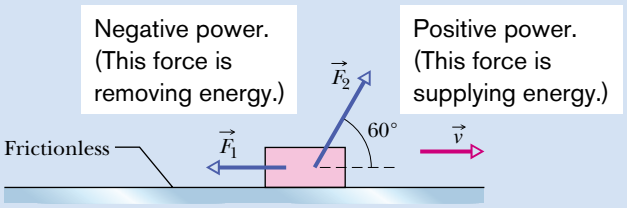

Figure 8-34 Two forces $\vec{F}_1$ and $\vec{F}_2$ act on a box that slides rightward across a frictionless floor. The velocity of the box is $\vec{v}$.

This positive result tells us that force $\vec{F}_2$ is transferring energy *to* the box at the rate of 6.0 J/s.

The net power is the sum of the individual powers (complete with their algebraic signs):

$$P_{net} = P_1 + P_2$$
$$= -60\text{ W} + 6.0\text{ W} = 0, \quad \text{(Answer)}$$

which tells us that the net rate of transfer of energy to or from the box is zero. Thus, the kinetic energy ($K = 1/2\ mv^2$) of the box is not changing, and so the speed of the box will remain at 3.0 m/s. With neither the forces $\vec{F}_1$ and $\vec{F}_2$ nor the velocity $\vec{v}$ changing, we see from Eq. 8-88 that P_1 and P_2 are constant and thus so is P_{net}.

SAMPLE PROBLEM 8.19

Frictional force, moving truck

A loaded truck of mass 3000 kg moves on a level road at a constant speed of 6.000 m/s. The frictional force on the truck from the road is 1000 N. Assume that air drag is negligible.

(a) How much work is done by the truck engine in 10.00 min?

KEY IDEA

The power generated by the engine is

$$P = Fv = (1000\text{ N})(6.000\text{ m/s}) = 6.000 \times 10^3\text{ W}$$

Calculations: The work done in 10 minutes is then

$$W = Pt = (6.000 \times 10^3\text{ W})(10\text{ min})(60\text{ s/min})$$
$$= 3.600 \times 10^6\text{ J}.$$

(b) After 10.00 min, the truck enters a hilly region whose inclination is 30° and continues to move with the same speed for another 10.00 min. What is the total work done by the engine during that period against the gravitational force and the frictional force?

KEY IDEA

When the truck starts to climb the hill, the power needed to overcome the gravitational pull of the Earth is

$$P_1 = F_g v = (mg\sin\theta)v$$
$$= (3000\text{ kg})(9.8\text{ m/s}^2)\sin 30°\ (6.000\text{ m/s})\ 8.820$$
$$\times 10^4\text{ W}$$

Calculations: Similarly, the power needed to overcome friction is

$$P_2 = (\mu mg\cos\theta)v = (f\cos\theta)v$$
$$= (1000\text{ N})\cos 30°\ (6.000\text{ m/s}) = 5.196 \times 10^3\text{ W}$$

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Therefore, the total work done in 10 minutes is

$$W' = (P_1 + P_2)t$$
$$= (8.820 \times 10\ \text{W} + 5.196 \times 10^3\ \text{W})(10\ \text{min})(60\ \text{s/min})$$
$$= 5.604 \times 10\ \text{J}$$

or about 56 MJ.

(c) What is the total work done by the engine in the full 20 min?

KEY IDEA

Adding up the results from (a) and (b), we can find the total work done by engine in 20 minutes.

Calculations: The total work done by the engine in 20 minutes is

$$W_{\text{tot}} = W + W' = 3.600 \times 10\ \text{J} + 5.604 \times 10\ \text{J}$$
$$= 5.964 \times 10\ \text{J}$$

or about 60 MJ.

8.18 | RELATION BETWEEN CONSERVATIVE FORCE AND POTENTIAL ENERGY

Key Concepts

- If we know the potential energy function $U(x)$ for a system in which a one-dimensional force $F(x)$ acts on a particle, we can find the force as
$$F(x) = -\frac{dU(x)}{dx}.$$
- If $U(x)$ is given on a graph, then at any value of x, the force $F(x)$ is the negative of the slope of the curve there and the kinetic energy of the particle is given by
$$K(x) = E_{\text{mec}} - U(x),$$
where E_{mec} is the mechanical energy of the system.
- A turning point is a point x at which the particle reverses its motion (there, $K = 0$).
- The particle is in equilibrium at points where the slope of the $U(x)$ curve is zero (there, $F(x) = 0$).

Once again we consider a particle that is part of a system in which a conservative force acts. This time suppose that the particle is constrained to move along an x axis while the conservative force does work on it. We want to plot the potential energy $U(x)$ that is associated with that force and the work that it does, and then we want to consider how we can relate the plot back to the force and to the kinetic energy of the particle. However, before we discuss such plots, we need one more relationship between the force and the potential energy.

Finding the Force Analytically

Equation 8-47 tells us how to find the change ΔU in potential energy between two points in a one-dimensional situation if we know the force $F(x)$. Now we want to go the other way; that is, we know the potential energy function $U(x)$ and want to find the force.

For one-dimensional motion, the work W done by a force that acts on a particle as the particle moves through a distance Δx is $F(x)\,\Delta x$. We can then write Eq. 8-42 as

$$\Delta U(x) = -W = -F(x)\,\Delta x. \tag{8-89}$$

Solving for $F(x)$ and passing to the differential limit yield

$$F(x) = -\frac{dU(x)}{dx} \quad \text{(one-dimensional motion)}, \tag{8-90}$$

which is the relation we sought.

We can check this result by putting $U(x) = (1/2)kx^2$, which is the elastic potential energy function for a spring force. Equation 8-90 then yields, as expected, $F(x) = -kx$, which is Hooke's law. Similarly, we can substitute $U(x) = mgx$, which is the gravitational potential energy function for a particle–Earth system, with a particle of mass m at height x above Earth's surface. Equation 8-90 then yields $F = -mg$, which is the gravitational force on the particle.

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The Potential Energy Curve

Figure 8-35*a* is a plot of a potential energy function $U(x)$ for a system in which a particle is in one-dimensional motion while a conservative force $F(x)$ does work on it.We can easily find $F(x)$ by (graphically) taking the slope of the $U(x)$ curve at various points. (Equation 8-90 tells us that $F(x)$ is the negative of the slope of the $U(x)$ curve.) Figure 8-35*b* is a plot of $F(x)$ found in this way.

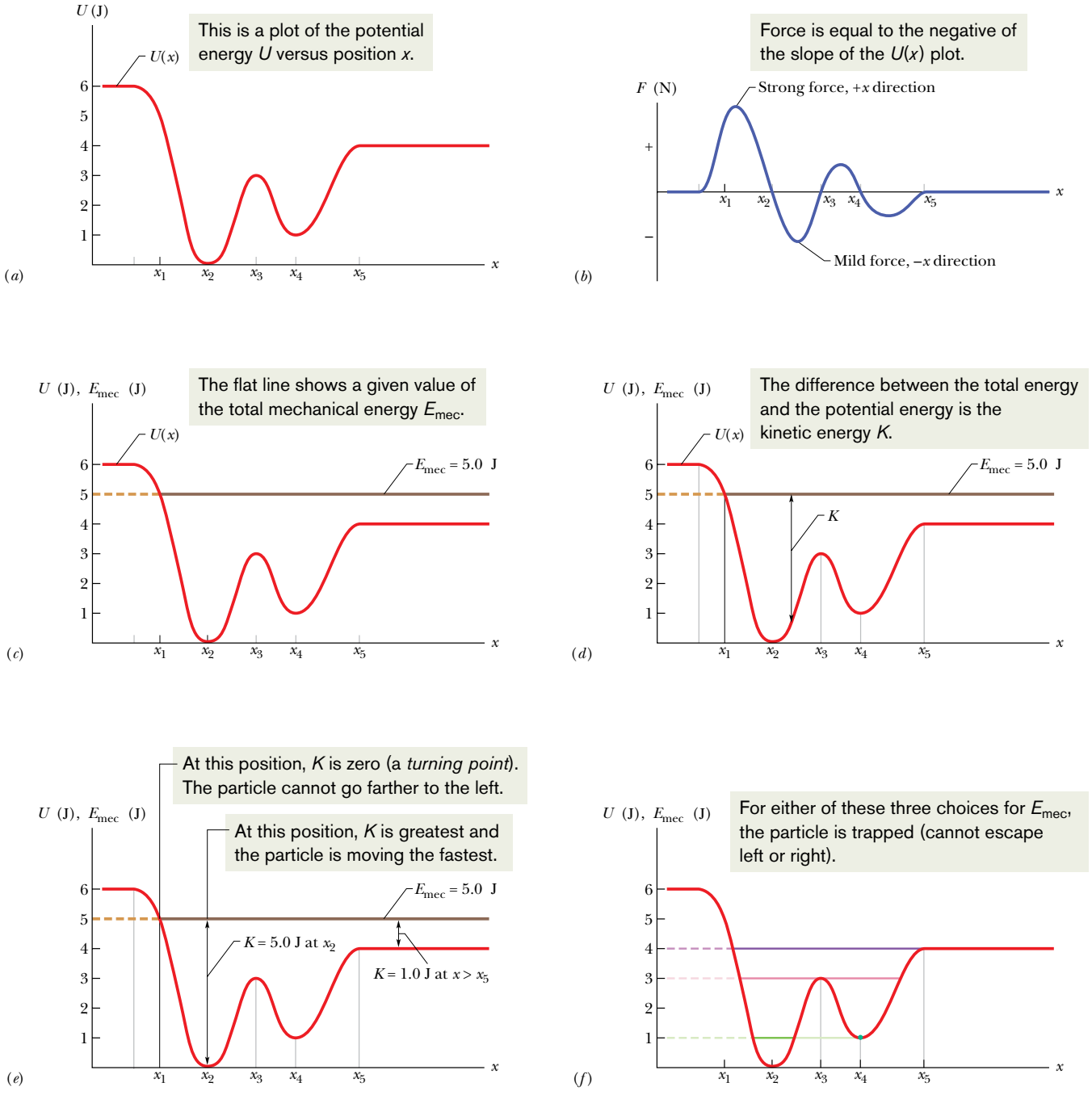
U (J)
U(x)
This is a plot of the potential energy U versus position x.
(a)
F (N)
Force is equal to the negative of the slope of the U(x) plot.
Strong force, +x direction
Mild force, –x direction
(b)
U (J), E_mec (J)
The flat line shows a given value of the total mechanical energy E_mec.
E_mec = 5.0 J
(c)
The difference between the total energy and the potential energy is the kinetic energy K.
K
(d)
At this position, K is zero (a turning point). The particle cannot go farther to the left.
At this position, K is greatest and the particle is moving the fastest.
K = 5.0 J at x2
K = 1.0 J at x > x5
(e)
For either of these three choices for E_mec, the particle is trapped (cannot escape left or right).
(f)

Figure 8-35 (*a*) A plot of $U(x)$, the potential energy function of a system containing a particle confined to move along an x axis. There is no friction, so mechanical energy is conserved. (*b*) A plot of the force $F(x)$ acting on the particle, derived from the potential energy plot by taking its slope at various points. (*c*)–(*e*) How to determine the kinetic energy. (*f*) The $U(x)$ plot of (*a*) with three possible values of E_{mec} shown.

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Turning Points

In the absence of a nonconservative force, the mechanical energy E of a system has a constant value given by

$$U(x) + K(x) = E_{mec}. \tag{8-91}$$

Here $K(x)$ is the *kinetic energy function* of a particle in the system (this $K(x)$ gives the kinetic energy as a function of the particle's location x). We may rewrite Eq. 8-91 as

$$K(x) = E_{mec} - U(x). \tag{8-92}$$

Suppose that E_{mec} (which has a constant value, remember) happens to be 5.0 J. It would be represented in Fig. 8-35*c* by a horizontal line that runs through the value 5.0 J on the energy axis. (It is, in fact, shown there.)

Equation 8-92 and Fig. 8-35*d* tell us how to determine the kinetic energy K for any location x of the particle: On the $U(x)$ curve, find U for that location x and then subtract U from E_{mec}. In Fig. 8-35*e* for example, if the particle is at any point to the right of x_5, then $K = 1.0$ J.The value of K is greatest (5.0 J) when the particle is at x_2 and least (0 J) when the particle is at x_1.

Since K can never be negative (because v^2 is always positive), the particle can never move to the left of x_1, where $E_{mec} - U$ is negative. Instead, as the particle moves toward x_1 from x_2, K decreases (the particle slows) until $K = 0$ at x_1 (the particle stops there).

Note that when the particle reaches x_1, the force on the particle, given by Eq. 8-63, is positive (because the slope dU/dx is negative). This means that the particle does not remain at x_1 but instead begins to move to the right, opposite its earlier motion. Hence x_1 is a **turning point,** a place where $K = 0$ (because $U = E$) and the particle changes direction. There is no turning point (where $K = 0$) on the right side of the graph. When the particle heads to the right, it will continue indefinitely.

Equilibrium Points

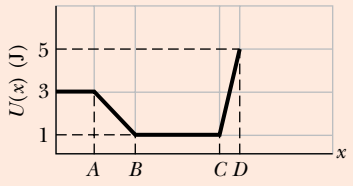
Figure 8-35*f* shows three different values for E_{mec} superposed on the plot of the potential energy function $U(x)$ of Fig. 8-35*a*. Let us see how they change the situation. If $E_{mec} = 4.0$ J (purple line), the turning point shifts from x_1 to a point between x_1 and x_2. Also, at any point to the right of x_5, the system's mechanical energy is equal to its potential energy; thus, the particle has no kinetic energy and (by Eq. 8-63) no force acts on it, and so it must be stationary. A particle at such a position is said to be in **neutral equilibrium**. (A marble placed on a horizontal tabletop is in that state.)

If $E_{mec} = 3.0$ J (pink line), there are two turning points: One is between x_1 and x_2, and the other is between x_4 and x_5. In addition, x_3 is a point at which $K = 0$. If the particle is located exactly there, the force on it is also zero, and the particle remains stationary. However, if it is displaced even slightly in either direction, a nonzero force pushes it farther in the same direction, and the particle continues to move. A particle at such a position is said to be in **unstable equilibrium**. (A marble balanced on top of a bowling ball is an example.)

Next consider the particle's behavior if $E_{mec} = 1.0$ J (green line). If we place it at x_4, it is stuck there. It cannot move left or right on its own because to do so would require a negative kinetic energy. If we push it slightly left or right, a restoring force appears that moves it back to x_4.A particle at such a position is said to be in **stable equilibrium**. (A marble placed at the bottom of a hemispherical bowl is an example.) If we place the particle in the cup-like *potential well* centered at x_2, it is between two turning points. It can still move somewhat, but only partway to x_1 or x_3.

CHECKPOINT 9

The figure gives the potential energy function $U(x)$ for a system in which a particle is in one-dimensional motion. (a) Rank regions AB, BC, and CD according to the magnitude of the force on the particle, greatest first. (b) What is the direction of the force when the particle is in region AB?



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SAMPLE PROBLEM 8.20

Reading a potential energy graph

A 2.00 kg particle moves along an x axis in one-dimensional motion while a conservative force along that axis acts on it. The potential energy $U(x)$ associated with the force is plotted in Fig. 8-36*a*. That is, if the particle were placed at any position between $x = 0$ and $x = 7.00$ m, it would have the plotted value of U. At $x = 6.5$ m, the particle has velocity $\vec{v}_0 = (-4.00 \text{ m/s})\hat{i}$.

(a) From Fig. 8-36*a*, determine the particle's speed at $x_1 = 4.5$ m.

KEY IDEA

(1) The particle's kinetic energy is given by Eq. 8-1 ($K = 1/2mv^2$). (2) Because only a conservative force acts on the particle, the mechanical energy $E_{mec} (= K + U)$ is conserved as the particle moves. (3) Therefore, on a plot of $U(x)$ such as Fig. 8-36*a*, the kinetic energy is equal to the difference between E_{mec} and U.

Calculations: At $x = 6.5$ m, the particle has kinetic energy

$$K_0 = \frac{1}{2}mv_0^2 = \frac{1}{2}(2.00 \text{ kg})(4.00 \text{ m/s})^2 = 16.0 \text{ J}.$$

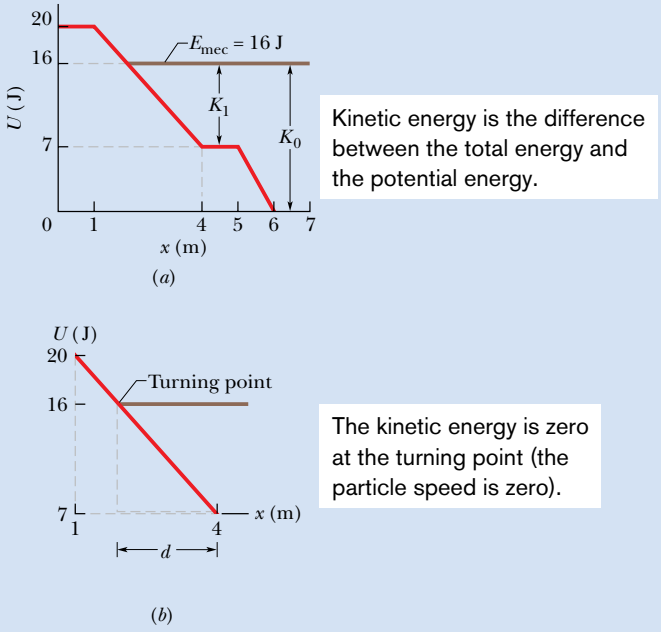

Figure 8-36 (*a*) A plot of potential energy U versus position x. (*b*) A section of the plot used to find where the particle turns around.

Because the potential energy there is $U = 0$, the mechanical energy is

$$E_{mec} = K_0 + U_0 = 16.0 \text{ J} + 0 = 16.0 \text{ J}.$$

This value for E_{mec} is plotted as a horizontal line in Fig. 8-36*a*. From that figure we see that at $x = 4.5$ m, the potential energy is $U_1 = 7.0$ J. The kinetic energy K_1 is the difference between E_{mec} and U_1:

$$K_1 = E_{mec} - U_1 = 16.0 \text{ J} - 7.0 \text{ J} = 9.0 \text{ J}.$$

Because $K_1 = 1/2mv_1^2$, we find

$$v_1 = 3.0 \text{ m/s}. \qquad \text{(Answer)}$$

(b) Where is the particle's turning point located?

KEY IDEA

The turning point is where the force momentarily stops and then reverses the particle's motion. That is, it is where the particle momentarily has $v = 0$ and thus $K = 0$.

Calculations: Because K is the difference between E_{mec} and U, we want the point in Fig. 8-36*a* where the plot of U rises to meet the horizontal line of E_{mec}, as shown in Fig. 8-36*b*. Because the plot of U is a straight line in Fig. 8-36*b*, we can draw nested right triangles as shown and then write the proportionality of distances

$$\frac{16 - 7.0}{d} = \frac{20 - 7.0}{4.0 - 1.0}$$

which gives us $d = 2.08$ m. Thus, the turning point is at

$$x = 4.0 \text{ m} - d = 1.9 \text{ m}. \qquad \text{(Answer)}$$

(c) Evaluate the force acting on the particle when it is in the region $1.9 \text{ m} < x < 4.0 \text{ m}$.

KEY IDEA

The force is given by Eq. 8-90 ($F(x) = -dU(x)/dx$): The force is equal to the negative of the slope on a graph of $U(x)$.

Calculations: For the graph of Fig. 8-36*b*, we see that for the range $1.0 \text{ m} < x < 4.0 \text{ m}$ the force is

$$F = -\frac{20 \text{ J} - 7.0 \text{ J}}{1.0 \text{ m} - 4.0 \text{ m}} = 4.3 \text{ N}. \qquad \text{(Answer)}$$

Thus, the force has magnitude 4.3 N and is in the positive direction of the x axis. This result is consistent with the fact that the initially leftward-moving particle is stopped by the force and then sent rightward.

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8.19 | VERTICAL CIRCULAR MOTION

Key Concepts

- The minimum speed with which a particle of mass m in a vertical circle reaches point B (see Fig. 8-37) is
$$v_0 = \sqrt{2gR}$$
- Equation valid for complete circle (i.e., for $0 < \theta < \pi/2$) is
$$F_N = mg\cos\theta + \frac{mv^2}{R}$$

Motion of Particle Kept on Fixed Smooth Sphere

A particle of mass m is kept in fixed smooth sphere of radius R and projected horizontally from the bottom point A with speed v_0 as shown in Fig. 8-37. Let us find the minimum speed so that it can reach the point B.

Since we have to determine the minimum speed which will provide it just the sufficient kinetic energy so that it comes to rest at B, thus $K = 0$ at B. Taking gravitational potential energy at A to be zero, we can write using work–energy theorem.

$$(K_f - K_i) + (U_f - U_i)$$

$$\left(0 - \frac{1}{2}mv_0^2\right) + (mgR - 0) = 0$$

Hence, $$v_0 = \sqrt{2gR}$$

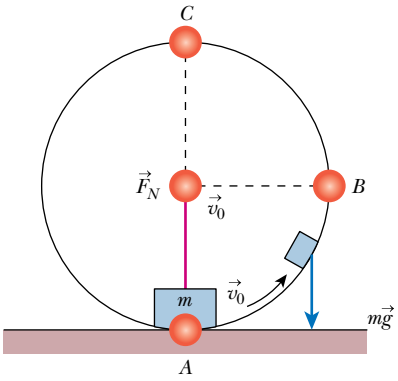

Figure 8-37 A particle of is kept in a fixed smooth sphere of radius and is undergoing circular motion in a vertical plane.

Now, let us find the speed required to reach C. If we, again, use work–energy theorem as above, we get

$$\left(0 - \frac{1}{2}mv_0^2\right) + [mg(2R) - 0] = 0$$

Solving, we get

$$v_0 = \sqrt{4gR} \qquad (8\text{-}93)$$

We show that this answer is wrong because the particle may leave contact with the sphere before reaching C. We must find the equation describing variation of normal contact force. It is good time to review Sample Problem 7.07.

As the particle is undergoing circular motion, we can write centripetal condition at any θ with vertical. Making free-body diagram as shown in Fig. 8-38.

$$F_N - mg\cos\theta = \frac{mv^2}{R}$$

where F_N is the normal contact force and $mg\cos\theta$ represents the normal component of weight mg at that point. Solving for F_N, we get

$$F_N = mg\cos\theta + \frac{mv^2}{R} \qquad (8\text{-}94)$$

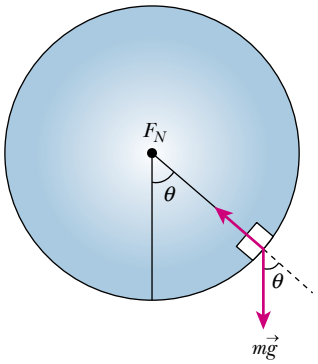

Figure 8-38 Free-body diagram of a particle undergoing circular motion inside a fixed sphere.

This equation is valid for the complete circle, that is, for values of θ from 0 to 2π. It is important to note that for $0 < \theta < \pi/2$, F_N never becomes zero as both $mg\cos\theta$ and mv^2/R are positive. Hence, it is in contact and it has circular motion till it crosses B. To find speed, we have to write the work–kinetic energy theorem when particle is at angle θ with vertical. Work done by gravity should be equal to change in kinetic energy.

$$-mg[R(1 - \cos\theta)] = \frac{mv^2}{2} - \frac{mv_0^2}{2}$$

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where v is the final velocity and v_0 is the initial velocity. Solving, we get

$$v^2 = v_0^2 - 2gR(1-\cos\theta)$$
$$= v_0^2 - 2gR + 2gR\cos\theta$$

We can put this value of v^2 in Eq. 8-94. Thus,

$$F_N = mg\cos\theta + \frac{m(v_0^2 - 2gR + 2gR\cos\theta)}{R}$$
$$= \frac{m}{R}[v_0^2 - 2gR + 3\cos\theta gR]$$

Now, we have a direct relation between normal F_N and angle θ. We can see that if we want to find minimum value of speed to reach point B, there is no need to check the equation; what then matters is the kinetic energy. As normal does not become zero for

$$0 < \theta < \frac{\pi}{2}$$

On comparing, we get

$$v^2 = v_0^2 - 2gR + 2gR\cos\theta$$

$$F_N = \frac{m}{R}[v_0^2 - 2gR + 3\cos\theta\ gR]$$

We can see that for positive $\cos\theta$, F_N is greater than v. The region AB ends to values of θ from 0 to $\pi/2$ and will have positive $\cos\theta$. Thus, in the region AB when speed becomes zero, normal will have positive value. So, for the region AB, we should check speed. On the other hand, when $\cos\theta$ has negative values, F_N has smaller values than v. The region BC corresponds to values of θ from $\pi/2$ to π and will give negative values of $\cos\theta$. Thus, in the region BC, when F_N becomes zero, speed is non-zero so for the region BC we should check value of F_N rather than v. Now, we will discuss the motion of particle after being projected with different speeds (v_0).

Case I: If $v_0 = \sqrt{2gR}$, it just reaches B and executes an oscillatory motion from A to B then return to A, continues to D and so on.

Case II: If $v_0 < \sqrt{2gR}$, the body does not reach B but its velocity becomes zero before it reaches B and it executes oscillatory motion from right extreme to left extreme (Fig. 8-39).

Case III: If $v_0 > \sqrt{2gR}$. When we provide velocity larger than $\sqrt{2gR}$, the particle will cross B and now major concern is of normal contact force as it will become zero before speed becomes zero. This is why minimum velocity to reach C is not $\sqrt{4gR}$ as we are considering speed and not normal contact force; whereas to reach C it is necessary that "F_N" does not become zero. Corresponding to point C, θ is equal to π. We are checking for "π" as $\cos\pi$ has a maximum negative value (–1). If N is not zero at this point, then for all θ, the normal will never be zero.

Thus, for the minimum speed to reach point C, we keep $\theta = \pi$ and $N = 0$:

$$0 = v^2 - 2gR + 3gR(-1)$$

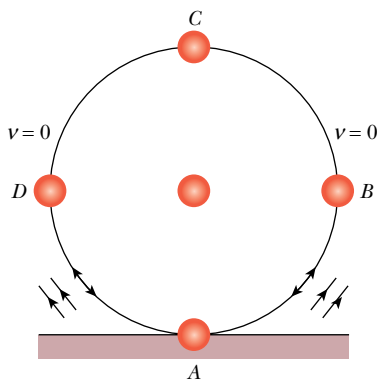

Figure 8-39 Case when particle does not have sufficient energy to reach point B.

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Solving, we get

$$v_0 = 5gR$$

So, we find that the minimum speed required to reach point C is $5gR$ and not $v_0 = \sqrt{4gR}$ as in Eq. 8-88.

Case IV: If $v_0 > \sqrt{5gR}$. The body will freely move in a circle as F_N will never be zero.

Case V: If $v_0 = \sqrt{5gR}$, the body will just complete the circle. We can also find speed at point C:

$$v^2 = 5gR - 2gR - 2gR$$

Solving, we get $\sqrt{vgR}$. We have already solved that at point C, $F_N = 0$ for $\sqrt{5gR}$.

Case VI: If $\sqrt{5gR} > v > \sqrt{2gR}$

It will leave circular motion between points B and C and start a projectile motion (Fig. 8-40).

For the above mentioned value, the normal becomes zero somewhere between points B and C. At this point, velocity is non-zero. So, the particle leaves circular motion and it becomes a projectile.

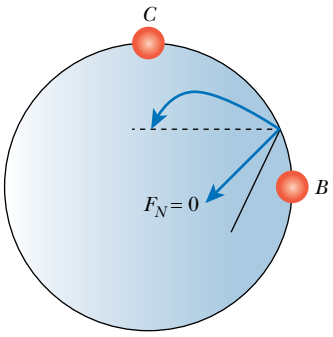

Figure 8-40 Case when particle leaves circular motion between B and C and starts projectile motion.

Motion of Particle Attached to a Rigid Rod

Consider a mass tied by a light rod about O as in Fig. 8-41. Here, the problem is simply of energy because rod can both pull and push so we can have any value of tension with any sign.

Different results are given below considering only energy.

Case I: If $v_0 < \sqrt{2gR}$, the body shows motion similar to a pendulum.

Case II: If $v_0 = \sqrt{2gR}$, the body reaches point B and comes back.

Case III: If $\sqrt{4gR} > v_0 > \sqrt{2gR}$, the body continues to move in a circular motion as tension of a rod can go negative, which is allowed as the rod instead of pulling the body, it pushes the body.

$$A \to P \to Q \to P \to A \to P' \to Q'$$

Case IV: If $v_0 = \sqrt{4gR}$, the body stops at the top.

Case V: If $v_0 > \sqrt{4gR}$, the body will freely move in a circle.

These cases are illustrated in Fig. 8-42.

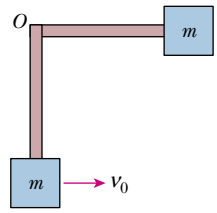

Figure 8-41 A particle attached to a rigid rod in a vertical plane.

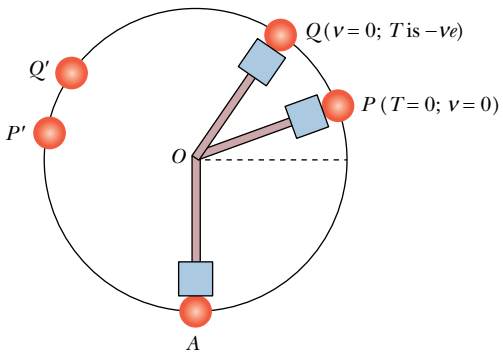

Figure 8-42 Various cases for a particle attached to a rigid rod in vertical plane.

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REVIEW AND SUMMARY

Kinetic Energy The **kinetic energy** K associated with the motion of a particle of mass m and speed v, where v is well below the speed of light, is

$$K = \frac{1}{2}mv^2 \quad \text{(kinetic energy)}. \tag{8-1}$$

Work Work W is energy transferred to or from an object via a force acting on the object. Energy transferred to the object is positive work, and from the object, negative work.

Work Done by a Constant Force The work done on a particle by a constant force $\vec{F}$ during displacement $\vec{d}$ is

$$W = Fd\cos\phi = \vec{F}\cdot\vec{d} \quad \text{(work, constant force)}, \tag{8-7, 8-8}$$

in which ϕ is the constant angle between the directions of $\vec{F}$ and $\vec{d}$. Only the component of $\vec{F}$ that is along the displacement $\vec{d}$ can do work on the object. When two or more forces act on an object, their **net work** is the sum of the individual works done by the forces, which is also equal to the work that would be done on the object by the net force $\vec{F}_{\text{net}}$ of those forces.

Work and Kinetic Energy For a particle, a change ΔK in the kinetic energy equals the net work W done on the particle:

$$\Delta K = K_f - K_i = W \quad \text{(work–kinetic energy theorem)}, \tag{8-10}$$

in which K_i is the initial kinetic energy of the particle and K_f is the kinetic energy after the work is done. Equation 8-10 rearranged gives us

$$K_f = K_i + W. \tag{8-11}$$

Work Done by the Gravitational Force The work W_g done by the gravitational force $\vec{F}_g$ on a particle-like object of mass m as the object moves through a displacement $\vec{d}$ is given by

$$W_g = mgd\cos\phi, \tag{8-12}$$

in which ϕ is the angle between $\vec{F}_g$ and $\vec{d}$.

Work Done in Lifting and Lowering an Object The work W_a done by an applied force as a particle-like object is either lifted or lowered is related to the work W_g done by the gravitational force and the change ΔK in the object's kinetic energy by

$$\Delta K = K_f - K_i = W_a + W_g. \tag{8-15}$$

If $K_f = K_i$, then Eq. 8-15 reduces to

$$W_a = -W_g, \tag{8-16}$$

which tells us that the applied force transfers as much energy to the object as the gravitational force transfers from it.

Spring Force The force $\vec{F}_s$ from a spring is

$$\vec{F}_s = -k\vec{d} \quad \text{(Hooke's law)}, \tag{8-20}$$

where $\vec{d}$ is the displacement of the spring's free end from its position when the spring is in its **relaxed state** (neither compressed nor extended), and k is the **spring constant** (a measure of the spring's stiffness). If an x axis lies along the spring, with the origin at the location of the spring's free end when the spring is in its relaxed state, Eq. 8-20 can be written as

$$F_x = -kx \quad \text{(Hooke's law)}. \tag{8-21}$$

A spring force is thus a variable force: It varies with the displacement of the spring's free end.

Work Done by a Spring Force If an object is attached to the spring's free end, the work W_s done on the object by the spring force when the object is moved from an initial position x_i to a final position x_f is

$$W_s = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2. \tag{8-25}$$

If $x_i = 0$ and $x_f = x$, then Eq. 8-25 becomes

$$W_s = -\frac{1}{2}kx^2. \tag{8-26}$$

Work Done by a Variable Force When the force $\vec{F}$ on a particlelike object depends on the position of the object, the work done by $\vec{F}$ on the object while the object moves from an initial position r_i with coordinates (x_i, y_i, z_i) to a final position r_f with coordinates (x_f, y_f, z_f) must be found by integrating the force. If we assume that component F_x may depend on x but not on y or z, component F_y may depend on y but not on x or z, and component F_z may depend on z but not on x or y, then the work is

$$W = \int_{x_i}^{x_f} F_x\,dx + \int_{y_i}^{y_f} F_y\,dy + \int_{z_i}^{z_f} F_z dz. \tag{8-36}$$

If $\vec{F}$ has only an x component, then Eq. 8-36 reduces to

$$W = \int_{x_i}^{x_f} F(x)\,dx. \tag{8-32}$$

Conservative Forces A force is a **conservative force** if the net work it does on a particle moving around any closed path, from an initial point and then back to that point, is zero. Equivalently, a force is conservative if the net work it does on a particle moving between two points does not depend on the path taken by the particle. The gravitational force and the spring force are conservative forces; the kinetic frictional force is a **nonconservative force**.

Potential Energy A **potential energy** is energy that is associated with the configuration of a system in which a conservative force acts. When the conservative force does work W on a particle within the system, the change ΔU in the potential energy of the system is

$$\Delta U = -W. \tag{8-42}$$

If the particle moves from point x_i to point x_f, the change in the potential energy of the system is

$$\Delta U = -\int_{x_i}^{x_f} F(x)\,dx. \tag{8-47}$$

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Gravitational Potential Energy The potential energy associated with a system consisting of Earth and a nearby particle is **gravitational potential energy**. If the particle moves from height y_i to height y_f, the change in the gravitational potential energy of the particle–Earth system is

$$\Delta U = mg(y_f - y_i) = mg\,\Delta y. \tag{8-48}$$

If the **reference point** of the particle is set as $y_i = 0$ and the corresponding gravitational potential energy of the system is set as $U_i = 0$, then the gravitational potential energy U when the particle is at any height y is

$$U(y) = mgy. \tag{8-50}$$

Elastic Potential Energy Elastic potential energy is the energy associated with the state of compression or extension of an elastic object. For a spring that exerts a spring force $F = -kx$ when its free end has displacement x, the elastic potential energy is

$$U(x) = \frac{1}{2}kx^2. \tag{8-52}$$

The **reference configuration** has the spring at its relaxed length, at which $x = 0$ and $U = 0$.

Mechanical Energy The **mechanical energy** E_{mec} of a system is the sum of its kinetic energy K and potential energy U:

$$E_{mec} = K + U. \tag{8-53}$$

An *isolated system* is one in which no *external force* causes energy changes. If only conservative forces do work within an isolated system, then the mechanical energy E_{mec} of the system cannot change. This **principle of conservation of mechanical energy** is written as

$$K_2 + U_2 = K_1 + U_1, \tag{8-58}$$

in which the subscripts refer to different instants during an energy transfer process. This conservation principle can also be written as

$$\Delta E_{mec} = \Delta K + \Delta U = 0. \tag{8-59}$$

Work Done on a System by an External Force Work W is energy transferred to or from a system by means of an external force acting on the system. When more than one force acts on a system, their *net work* is the transferred energy. When friction is not involved, the work done on the system and the change ΔE_{mec} in the mechanical energy of the system are equal:

$$W = \Delta E_{mec} = \Delta K + \Delta U. \tag{8-62, 8-63}$$

When a kinetic frictional force acts within the system, then the thermal energy E_{th} of the system changes. (This energy is associated with the random motion of atoms and molecules in the system.) The work done on the system is then

$$W = \Delta E_{mec} + \Delta E_{th}. \tag{8-70}$$

The change ΔE_{th} is related to the magnitude f_k of the frictional force and the magnitude d of the displacement caused by the external force by

$$\Delta E_{th} = f_k d. \tag{8-68}$$

Conservation of Energy The **total energy** E of a system (the sum of its mechanical energy and its internal energies, including thermal energy) can change only by amounts of energy that are transferred to or from the system.This experimental fact is known as the **law of conservation of energy**. If work W is done on the system, then

$$W = \Delta E = \Delta E_{mec} + \Delta E_{th} + \Delta E_{int}. \tag{8-72}$$

If the system is isolated ($W = 0$), this gives

$$\Delta E_{mec} + \Delta E_{th} + \Delta E_{int} = 0 \tag{8-73}$$

and

$$\Delta E_{mec,2} = \Delta E_{mec,1} - \Delta E_{th} - \Delta E_{int}, \tag{8-74}$$

where the subscripts 1 and 2 refer to two different instants.

Power The time rate at which work is done by a force is said to be the power due to the force. If the force does work W during a time interval Δt, the *average power* due to the force over that time interval is

$$P_{avg} = \frac{W}{\Delta t}. \tag{8-82}$$

Instantaneous power is the instantaneous rate of doing work:

$$P = \frac{dW}{dt}. \tag{8-83}$$

For a force $\vec{F}$ at an angle ϕ to the direction of travel of the instantaneous velocity $\vec{v}$, the instantaneous power is

$$P = Fv\cos\phi = \vec{F}\cdot\vec{v}. \tag{8-87, 8-88}$$

Potential Energy Curves If we know the potential energy function $U(x)$ for a system in which a one-dimensional force $F(x)$ acts on a particle, we can find the force as

$$F(x) = -\frac{dU(x)}{dx}. \tag{8-90}$$

If $U(x)$ is given on a graph, then at any value of x, the force $F(x)$ is the negative of the slope of the curve there and the kinetic energy of the particle is given by

$$K(x) = E_{mec} - U(x), \tag{8-92}$$

where E_{mec} is the mechanical energy of the system.A **turning point** is a point x at which the particle reverses its motion (there, $K = 0$). The particle is in **equilibrium** at points where the slope of the $U(x)$ curve is zero (there, $F(x) = 0$).

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PROBLEMS

1. When accelerated along a straight line at 2.8×10^{15} m/s^2 in a machine, an electron (mass $m = 9.1 \times 10^{-31}$ kg) has an initial speed of 1.4×10^7 m/s and travels 5.8 cm. Find (a) the final speed of the electron and (b) the increase in its kinetic energy.

2. If a Saturn V rocket with an Apollo spacecraft attached had a combined mass of 2.9×10^5 kg and reached a speed of 11.2 km/s, how much kinetic energy would it then have?

3. A force $\vec{F}_a$ is applied to a bead as the bead is moved along a straight wire through displacement +5.0 cm. The magnitude of $\vec{F}_a$ is set at a certain value, but the angle ϕ between $\vec{F}_a$ and the bead's displacement can be chosen. Figure 8-43 gives the work W done by $\vec{F}_a$ on the bead for a range of ϕ values; $W_0 = 25$ J. How much work is done by $\vec{F}_a$ if ϕ is (a) 64° and (b) 147°?

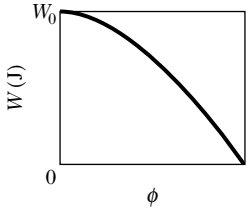

Figure 8-43 Problem 3.

4. A father racing his son has half the kinetic energy of the son, who has half the mass of the father. The father speeds up by 1.0 m/s and then has the same kinetic energy as the son. What are the original speeds of (a) the father and (b) the son?

5. A bead with mass 1.8×10^{-2} kg is moving along a wire in the positive direction of an x axis. Beginning at time $t = 0$, when the bead passes through $x = 0$ with speed 12 m/s, a constant force acts on the bead. Figure 8-44 indicates the bead's position at these four times: $t_0 = 0$, $t_1 = 1.0$ s, $t_2 = 2.0$ s, and $t_3 = 3.0$ s. The bead momentarily stops at $t = 3.0$ s. What is the kinetic energy of the bead at $t = 10$ s?

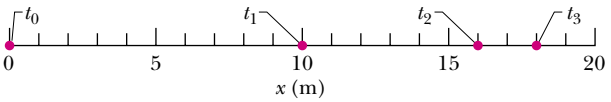

Figure 8-44 Problem 5.

6. A 3.0 kg body is at rest on a frictionless horizontal air track when a constant horizontal force $\vec{F}$ acting in the positive direction of an x axis along the track is applied to the body. A stroboscopic graph of the position of the body as it slides to the right is shown in Fig. 8-45. The force $\vec{F}$ is applied to the body at $t = 0$, and the graph records the position of the body at 0.50 s intervals. How much work is done on the body by the applied force $\vec{F}$ between $t = 0$ and $t = 2.0$ s?

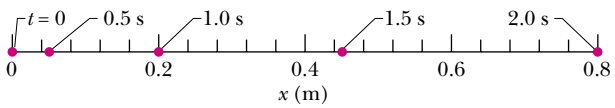

Figure 8-45 Problem 6.

7. A ice block floating in a river is pushed through a displacement $\vec{d} = (20\text{ m})\hat{i} - (16\text{ m})\hat{j}$ along a straight embankment by rushing water, which exerts a force $\vec{F} = (210\text{ N})\hat{i} - (150\text{ N})\hat{j}$ on the block. How much work does the force do on the block during the displacement?

8. The only force acting on a 2.0 kg canister that is moving in an xy plane has a magnitude of 5.0 N. The canister initially has a velocity of 4.0 m/s in the positive x direction and some time later has a velocity of 6.0 m/s in the positive y direction. How much work is done on the canister by the 5.0 N force during this time?

9. A coin slides over a frictionless plane and across an xy coordinate system from the origin to a point with xy coordinates (3.0 m, 4.0 m) while a constant force acts on it. The force has magnitude 2.5 N and is directed at a counterclockwise angle of 100° from the positive direction of the x axis. How much work is done by the force on the coin during the displacement?

10. A particle travels through a three-dimensional displacement given by $\vec{d} = (5.00\hat{i} - 3.00\hat{j} + 4.00\hat{k})$ m. If a force of magnitude 22.0 N and with fixed orientation does work on the particle, find the angle between the force and the displacement if the change in the particle's kinetic energy is (a) 45.0 J and (b) –45.0 J.

11. A can of bolts and nuts is pushed 2.00 m along an x axis by a broom along the greasy (frictionless) floor of a car repair shop in a version of shuffleboard. Figure 8-46 gives the work W done on the can by the constant horizontal force from the broom, versus the can's position x. The scale of the figure's vertical axis is set by $W_s = 6.0$ J. (a) What is the magnitude of that force? (b) If the can had an initial kinetic energy of 3.00 J, moving in the positive direction of the x axis, what is its kinetic energy at the end of the 2.00 m?

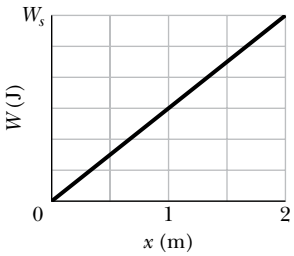

Figure 8-46 Problem 11.

12. A sledge and its rider, with a total mass of 85 kg, emerge from a downhill track onto a horizontal straight track with an initial speed of 37 m/s. If a force slows them to a stop at a constant rate of 2.0 m/s^2, (a) what magnitude F is required for the force, (b) what distance d do they travel while slowing, and (c) what work W is done on them by the force? What are (d) F, (e) d, and (f) W if they, instead, slow at 4.0 m/s^2?

13. Figure 8-47 shows an overhead view of three horizontal forces acting on a cargo canister that was initially stationary but now moves across a frictionless floor. The force magnitudes are $F_1 = 3.00$ N, $F_2 = 4.00$ N, and $F_3 = 9.00$ N, and the indicated angles

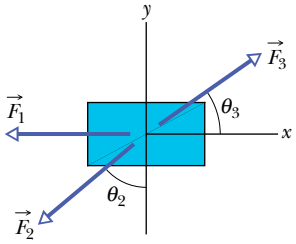

Figure 8-47 Problem 13.

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are $\theta_2 = 50.0°$ and $\theta_3 = 35.0°$. What is the net work done on the canister by the three forces during the first 4.00 m of displacement?

14. A 7.0 kg object is moving in the positive direction of an x axis. When it passes through $x = 0$, a constant force directed along the axis begins to act on it. Figure 8-48 gives its kinetic energy K versus position x as it moves from $x = 0$ to $x = 5.0$ m; $K_0 = 30.0$ J. The force continues to act. What is v when the object moves back through $x = -3.0$ m?

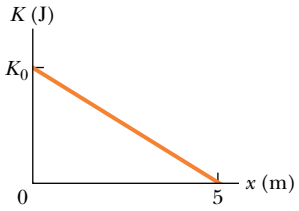

Figure 8-48 Problem 14.

15. A military helicopter lifts a 75 kg flood survivor 16 m vertically from the river by a rope. If the acceleration of the survivor is $g/10$, how much work is done on the survivor by (a) the force from the helicopter and (b) the gravitational force on her? Just before she reaches the helicopter, what are her (c) kinetic energy and (d) speed?

16. In Fig. 8-49, a block of ice slides down a frictionless ramp at angle $\theta = 50°$ while an ice worker pulls on the block (via a rope) with a force $\vec{F}_r$ that has a magnitude of 50 N and is directed up the ramp. As the block slides through distance $d = 0.50$ m along the ramp, its kinetic energy increases by 80 J. How much greater would its kinetic energy have been if the rope had not been attached to the block?

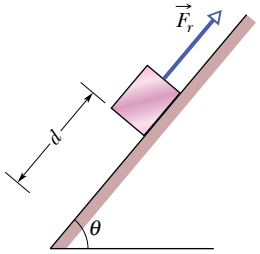

Figure 8-49 Problem 16.

17. A block is sent up a frictionless ramp along which an x axis extends upward. Figure 8-50 gives the kinetic energy of the block as a function of position x; the scale of the figure's vertical axis is set by $K_s = 50.0$ J. If the block's initial speed is 5.00 m/s, what is the normal force on the block?

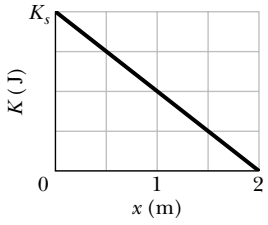

Figure 8-50 Problem 17.

18. A cord is used to vertically lower an initially stationary block of mass M at a constant downward acceleration of $g/4$. When the block has fallen a distance d, find (a) the work done by the cord's force on the block, (b) the work done by the gravitational force on the block, (c) the kinetic energy of the block, and (d) the speed of the block.

19. A cave rescue team lifts an injured person directly upward and out of a sinkhole by means of a motor-driven cable. The lift is performed in three stages, each requiring a vertical distance of 12.0 m: (a) the initially stationary spelunker is accelerated to a speed of 5.00 m/s; (b) he is then lifted at the constant speed of 5.00 m/s; (c) finally he is decelerated to zero speed. How much work is done on the 85.0 kg rescuee by the force lifting him during each stage?

20. In Fig. 8-51, a constant force $\vec{F}_a$ of magnitude 82.0 N is applied to a 3.00 kg shoe box at angle $\phi = 53.0°$, causing the box to move up a frictionless ramp at constant speed. How much work is done on the box by $\vec{F}_a$ when the box has moved through vertical distance $h = 0.150$ m?

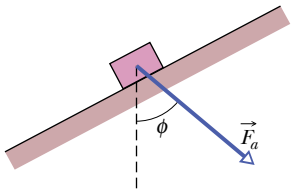

Figure 8-51 Problem 20.

21. In Fig. 8-52, a horizontal force $\vec{F}_a$ of magnitude 23.0 N is applied to a 3.00 kg psychology book as the book slides a distance $d = 0.580$ m up a friction-less ramp at angle $\theta = 30.0°$. (a) During the displacement, what is the net work done on the book by $\vec{F}_a$, the gravitational force on the book, and the normal force on the book? (b) If the book has zero kinetic energy at the start of the displacement, what is its speed at the end of the displacement?

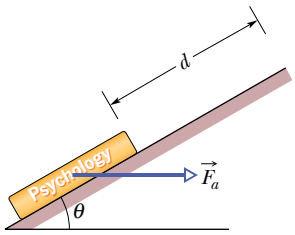

Figure 8-52 Problem 21.

22. In Fig. 8-53, a 0.250 kg block of cheese lies on the floor of a 900 kg elevator cab that is being pulled upward by a cable through distance $d_1 = 2.40$ m and then through distance $d_2 = 10.5$ m. (a) Through d_1, if the normal force on the block from the floor has constant magnitude $F_N = 3.00$ N, how much work is done on the cab by the force from the cable? (b) Through d_2, if the work done on the cab by the (constant) force from the cable is 92.61 kJ, what is the magnitude of F_N?

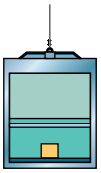

Figure 8-53 Problem 22.

23. A spring of spring constant 5.0×10^3 N/m is stretched initially by 5.0 cm from the unstretched position. What is the work required to stretch it further by another 5.0 cm?

24. A spring and block are in the arrangement of Fig. 8-12. When the block is pulled out to $x = +4.0$ cm, we must apply

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a force of magnitude 360 N to hold it there. We pull the block to $x = 11$ cm and then release it. How much work does the spring do on the block as the block moves from $x_i = +5.0$ cm to (a) $x = +3.0$ cm, (b) $x = -3.0$ cm, (c) $x = -5.0$ cm, and (d) $x = -9.0$ cm?

25. In the arrangement of Fig. 8-12, we gradually pull the block from $x = 0$ to $x = +3.0$ cm, where it is stationary. Figure 8-54 gives the work that our force does on the block. The scale of the figure's vertical axis is set by $W_s = 1.0$ J. We then pull the block out to $x = +5.0$ cm and release it from rest. How much work does the spring do on the block when the block moves from $x_i = +5.0$ cm to (a) $x = +4.0$ cm, (b) $x = -2.0$ cm, and (c) $x = -5.0$ cm?

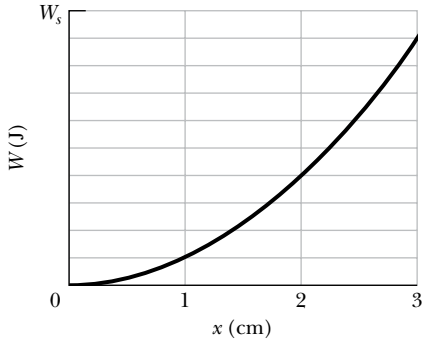

Figure 8-54 Problem 25.

26. In Fig. 8-12*a*, a block of mass m lies on a horizontal frictionless surface and is attached to one end of a horizontal spring (spring constant k) whose other end is fixed. The block is initially at rest at the position where the spring is unstretched ($x = 0$) when a constant horizontal force $\vec{F}$ in the positive direction of the x axis is applied to it. A plot of the resulting kinetic energy of the block versus its position x is shown in Fig. 8-55. The scale of the vertical axis is set by $K_s = 6.0$ J. (a) What is the magnitude of $\vec{F}$? (b) What is the value of k?

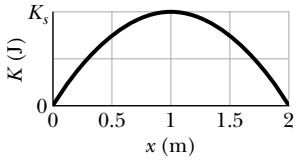

Figure 8-55 Problem 26.

27. As a 2.5 kg body moves in the positive direction along an x axis, a single force acts on it. The force is given by $F_x = -6x$ N, with x in meters. The velocity at $x = 3.5$ m is 8.5 m/s. (a) Find the velocity of the body at $x = 4.5$ m. (b) Find the positive value of x at which the body has a velocity of 5.5 m/s.

28. Figure 8-56 gives spring force F_x versus position x for the spring–block arrangement of Fig. 8-12. The scale is set by $F_s = 160.0$ N. We release the block at $x = 12$ cm. How much work does the spring do on the block when the block moves from $x_i = +8.0$ cm to (a) $x = +5.0$ cm, (b) $x = -5.0$ cm, (c) $x = -8.0$ cm, and (d) $x = -10.0$ cm?

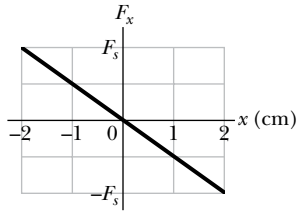

Figure 8-56 Problem 28.

29. The block in Fig. 8-12*a* lies on a horizontal frictionless surface, and the spring constant is 50 N/m. Initially, the spring is at its relaxed length and the block is stationary at position $x = 0$. Then an applied force with a constant magnitude of 3.0 N pulls the block in the positive direction of the x axis, stretching the spring until the block stops. When that stopping point is reached, what are (a) the position of the block, (b) the work that has been done on the block by the applied force, and (c) the work that has been done on the block by the spring force? During the block's displacement, what are (d) the block's position when its kinetic energy is maximum and (e) the value of that maximum kinetic energy?

30. The force on a particle is directed along an x axis and given by $F = F_0(x/x_0 - 1)$. Find the work done by the force in moving the particle from $x = 0$ to $x = 2x_0$ by (a) plotting $F(x)$ and measuring the work from the graph and (b) integrating $F(x)$.

31. A 2.5 kg block moves in a straight line on a horizontal frictionless surface under the influence of a force that varies with position as shown in Fig. 8-57. The scale of the figure's vertical axis is set by $F_s = 10.0$ N. How much work is done by the force as the block moves from the origin to $x = 8.0$ m?

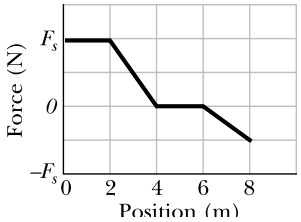

Figure 8-57 Problem 31.

32. Figure 8-58 gives the acceleration of a 2.00 kg particle as an applied Force $\vec{F}_a$ moves it from rest along an x axis from $x = 0$ to $x = 9.0$ m. The scale of the figure's vertical axis is set by $a_s = 6.0$ m/s^2. How much work has the force done on the particle when the particle reaches (a) $x = 4.0$ m, (b) $x = 7.0$ m, and (c) $x = 9.0$ m? What is the particle's speed and direction of travel when it reaches (d) $x = 4.0$ m, (e) $x = 7.0$ m, and (f) $x = 9.0$ m?

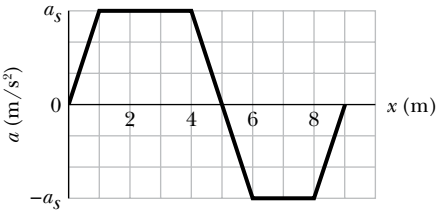

Figure 8-58 Problem 32.

33. A 1.0 kg block is initially at rest on a horizontal frictionless surface when a horizontal force along an x axis is applied

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to the block. The force is given by $\vec{F}(x) = (2.5 - x^2)\hat{i}$ N, where x is in meters and the initial position of the block is $x = 0$. (a) What is the kinetic energy of the block as it passes through $x = 2.0$ m? (b) What is the maximum kinetic energy of the block between $x = 0$ and $x = 2.0$ m?

34. A particle of mass 0.020 kg moves along a curve with velocity $5.0\hat{i} + 18\hat{k}$ m/s. After some time, the velocity changes to $9.0\hat{i} + 22\hat{j}$ m/s due to the action of a single force. Find the work done on the particle during this interval of time.

35. A can of cookies is made to move along an x axis from $x = 0.25$ m to $x = 2.25$ m by a force with a magnitude given by $F = \exp(-4x^2)$, with x in meters and F in newtons. (Here exp is the exponential function.) How much work is done on the can by the force?

36. Only one force is acting on a 2.8 kg particle-like object whose position is given by $x = 4.0t - 5.0t^2 + 2.0t^3$, with x in meters and t in seconds. What is the work done by the force from $t = 0$ s to $t = 6.0$ s?

37. Figure 8-59 shows a cord attached to a cart that can slide along a frictionless horizontal rail aligned along an x axis. The left end of the cord is pulled over a pulley, of negligible mass and friction and at cord height $h = 1.25$ m, so the cart slides from $x_1 = 3.00$ m to $x_2 = 1.00$ m. During the move, the tension in the cord is a constant 28.0 N. What is the change in the kinetic energy of the cart during the move?

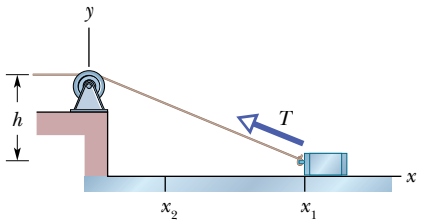

Figure 8-59 Problem 37.

38. A force of 5.0 N acts on a 15 kg body initially at rest. Compute the work done by the force in (a) the first, (b) the second, and (c) the third seconds and (d) the instantaneous power due to the force at the end of the third second.

39. A skier is pulled by a towrope up a frictionless ski slope that makes an angle of 12° with the horizontal. The rope moves parallel to the slope with a constant speed of 1.0 m/s. The force of the rope does 880 J of work on the skier as the skier moves a distance of 7.0 m up the incline. (a) If the rope moved with a constant speed of 2.0 m/s, how much work would the force of the rope do on the skier as the skier moved a distance of 8.0 m up the incline? At what rate is the force of the rope doing work on the skier when the rope moves with a speed of (b) 1.0 m/s and (c) 2.0 m/s?

40. Across a horizontal floor, a 102 kg block is pulled at a constant speed of 5.5 m/s by an applied force of 125 N directed 38° above the horizontal. Calculate the rate at which the force does work on the block.

41. The loaded cab of an elevator has a mass of 5.0×10^3 kg and moves 210 m up the shaft in 23 s at constant speed. At what average rate does the force from the cable do work on the cab?

42. A machine carries a 4.0 kg package from an initial position of $\vec{d}_i = (0.50\text{ m})\hat{i} + (0.75\text{ m})\hat{j} + (0.20\text{ m})\hat{k}$ at $t = 0$ to a final position of $\vec{d}_f = (7.50\text{ m})\hat{i} + (12.0\text{ m})\hat{j} + (7.20\text{ m})\hat{k}$ at $t = 12$ s. The constant force applied by the machine on the package is $\vec{F} = (2.00\text{ N})\hat{i} + (4.00\text{ N})\hat{j} + (6.00\text{ N})\hat{k}$. For that displacement, find (a) the work done on the package by the machine's force and (b) the average power of the machine's force on the package.

43. A fully loaded, slow-moving freight elevator has a cab with a total mass of 1200 kg, which is required to travel upward 54 m in 3.0 min, starting and ending at rest. The elevator's counterweight has a mass of only 950 kg, and so the elevator motor must help. What average power is required of the force the motor exerts on the cab via the cable?

44. (a) At a certain instant, a particle-like object is acted on by a force $\vec{F} = (4.0\text{ N})\hat{i} - (2.0\text{ N})\hat{j} + (9.0\text{ N})\hat{k}$ while the object's velocity is $\vec{v} = -(2.0\text{ m/s})\hat{i} + (4.0\text{ m/s})\hat{k}$. What is the instantaneous rate at which the force does work on the object? (b) At some other time, the velocity consists of only a y component. If the force is unchanged and the instantaneous power is –15 W, what is the velocity of the object?

45. A force $\vec{F} = (3.00\text{ N})\hat{i} + (7.00\text{ N})\hat{j} + (7.00\text{ N})\hat{k}$ acts on a 2.00 kg mobile object that moves from an initial position of $\vec{d}_i = (3.00\text{ m})\hat{i} - (2.00\text{ m})\hat{j} + (5.00\text{ m})\hat{k}$ to a final position of $\vec{d}_f = -(5.00\text{ m})\hat{i} + (4.00\text{ m})\hat{j} + (7.00\text{ m})\hat{k}$ in 4.00 s. Find (a) the work done on the object by the force in the 4.00 s interval, (b) the average power due to the force during that interval, and (c) the angle between vectors $\vec{d}_i$ and $\vec{d}_f$.

46. A funny car accelerates from rest through a measured track distance in time T with the engine operating at a constant power P. If the track crew can increase the engine power by a differential amount dP, what is the change in the time required for the run?

47. Adam stretches a spring by some length. John stretches the same spring later by three times the length stretched by Adam. Find the ratio of the stored energy in the first stretch to that in the second stretch.

48. You drop a 2.00 kg book to a friend who stands on the ground at distance $D = 10.0$ m below. Your friend's outstretched hands are at distance $d = 1.50$ m above the ground (Fig. 8-60). (a) How much work W_g does the gravitational force do on the book as it drops to her hands? (b) What is the change ΔU in the gravitational potential energy of the book–Earth system during the drop? If the gravitational potential energy U of that system is taken to be zero at ground level, what is U (c) when the book is released, and (d) when it reaches her hands? Now take U to be 100 J at ground level and again find (e) W_g, (f) ΔU, (g) U at the release point, and (h) U at her hands.

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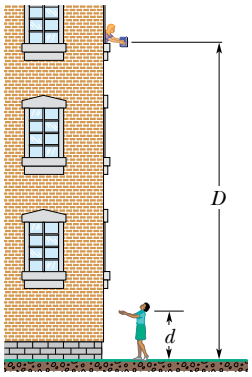

Figure 8-60 Problems 48.

49. Figure 8-61 shows a ball with mass $m = 0.382$ kg attached to the end of a thin rod with length $L = 0.498$ m and negligible mass. The other end of the rod is pivoted so that the ball can move in a vertical circle. The rod is held horizontally as shown and then given enough of a downward push to cause the ball to swing down and around and just reach the vertically up position, with zero speed there. How much work is done on the ball by the gravitational force from the initial point to (a) the lowest point, (b) the highest point, and (c) the point on the right level with the initial point? If the gravitational potential energy of the ball–Earth system is taken to be zero at the initial point, what is it when the ball reaches (d) the lowest point, (e) the highest point, and (f) the point on the right level with the initial point? (g) Suppose the rod were pushed harder so that the ball passed through the highest point with a nonzero speed. Would ΔU_g from the lowest point to the highest point then be greater than, less than, or the same as it was when the ball stopped at the highest point?

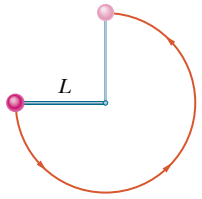

Figure 8-61 Problems 49.

50. In Fig. 8-62, a 2.00 g ice flake is released from the edge of a hemispherical bowl whose radius r is 22.0 cm. The flake–bowl contact is frictionless. (a) How much work is done on the flake by the gravitational force during the flake's descent to the bottom of the bowl? (b) What is the change in the potential energy of the flake–Earth system during that descent? (c) If that potential energy is taken to be zero at the bottom of the bowl, what is its value when the flake is released? (d) If, instead, the potential energy is taken to be zero at the release point, what is its value when the flake reaches the bottom of the bowl? (e) If the mass of the flake were doubled, would the magnitudes of the answers to (a) through (d) increase, decrease, or remain the same?

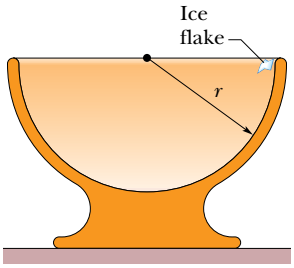

Figure 8-62 Problems 50.

51. In Fig. 8-63, a small block of mass $m = 0.032$ kg can slide along the frictionless loop-the-loop, with loop radius $R = 10$ cm. The block is released from rest at point P, at height $h = 5.0R$ above the bottom of the loop. How much work does the gravitational force do on the block as the block travels from point P to (a) point Q and (b) the top of the loop? If the gravitational potential energy of the block–Earth system is taken to be zero at the bottom of the loop, what is that potential energy when the block is (c) at point P, (d) at point Q, and (e) at the top of the loop? (f) If, instead of merely being released, the block is given some initial speed downward along the track, do the answers to (a) through (e) increase, decrease, or remain the same?

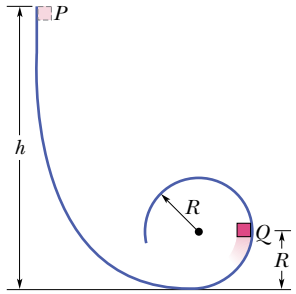

Figure 8-63 Problems 51 and 58.

52. Figure 8-64 shows a thin rod, of length $L = 2.00$ m and negligible mass, that can pivot about one end to rotate in a vertical circle. A ball of mass $m = 5.00$ kg is attached to the other end. The rod is pulled aside to angle $\theta_0 = 30.0°$ and released with initial velocity $\vec{v}_0 = 0$. As the ball descends to its lowest point, (a) how much work does the gravitational force do on it and (b) what is the change in the gravitational potential energy of the ball–Earth system? (c) If the gravitational potential energy is taken to be zero at the lowest point, what is its value just as the ball is released? (d) Do the magnitudes of the answers to (a) through (c) increase, decrease, or remain the same if angle θ_0 is increased?

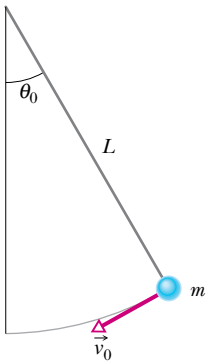

Figure 8-64 Problems 52 and 59.

53. A 1.50 kg snowball is fired from a cliff 11.5 m high. The snowball's initial velocity is 16.0 m/s, directed 41.0° above the horizontal. (a) How much work is done on the snowball by the gravitational force during its flight to the flat ground below the cliff? (b) What is the change in the

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gravitational potential energy of the snowball–Earth system during the flight? (c) If that gravitational potential energy is taken to be zero at the height of the cliff, what is its value when the snowball reaches the ground?

54. (a) In Problem 53, using energy techniques rather than the techniques of Chapter 4, find the speed of the snowball as it reaches the ground below the cliff. What is that speed (b) if the launch angle is changed to 41.0° *below* the horizontal and (c) if the mass is changed to 3.00 kg?

55. A 5.0 g marble is fired vertically upward using a spring gun. The spring must be compressed 8.0 cm if the marble is to just reach a target 20 m above the marble's position on the compressed spring. (a) What is the change ΔU_g in the gravitational potential energy of the marble–Earth system during the 20 m ascent? (b) What is the change ΔU_s in the elastic potential energy of the spring during its launch of the marble? (c) What is the spring constant of the spring?

56. In Fig. 8-65, a runaway truck with failed brakes is moving downgrade at 130 km/h just before the driver steers the truck up a frictionless emergency escape ramp with an inclination of $\theta = 15°$. The truck's mass is 1.2×10^4 kg. (a) What minimum length L must the ramp have if the truck is to stop (momentarily) along it? (Assume the truck is a particle, and justify that assumption.) Does the minimum length L increase, decrease, or remain the same if (b) the truck's mass is decreased and (c) its speed is decreased?

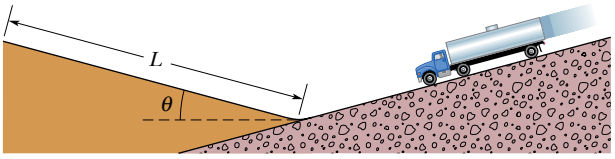

Figure 8-65 Problem 56.

57. A 700 g block is released from rest at height h_0 above a vertical spring with spring constant $k = 450$ N/m and negligible mass. The block sticks to the spring and momentarily stops after compressing the spring 19.0 cm. How much work is done (a) by the block on the spring and (b) by the spring on the block? (c) What is the value of h_0? (d) If the block were released from height $2.00h_0$ above the spring, what would be the maximum compression of the spring?

58. In Problem 51, what are the magnitudes of (a) the horizontal component and (b) the vertical component of the *net* force acting on the block at point Q? (c) At what height h should the block be released from rest so that it is on the verge of losing contact with the track at the top of the loop? (*On the verge of losing contact* means that the normal force on the block from the track has just then become zero.) (d) Graph the magnitude of the normal force on the block at the top of the loop versus initial height h, for the range $h = 0$ to $h = 6R$.

59. (a) In Problem 52, what is the speed of the ball at the lowest point? (b) Does the speed increase, decrease, or remain the same if the mass is increased?

60. A 1.0 kg block moving at 8.0 m/s strikes a spring fixed at one end to a wall and compresses the spring by 0.40 m, where its speed gets reduced to 2.0 m/s. After this event, the spring is mounted upright by fixing its bottom end to a floor, and a stone of mass 2.0 kg is placed on it; the spring is now compressed by 0.50 m from its rest length. The system is then released. How far above the rest-length point does the stone rise?

61. A pendulum consists of a 2.0 kg stone swinging on a 4.5 m string of negligible mass. The stone has a speed of 8.0 m/s when it passes its lowest point. (a) What is the speed when the string is at 60° to the vertical? (b) What is the greatest angle with the vertical that the string will reach during the stone's motion? (c) If the potential energy of the pendulum–Earth system is taken to be zero at the stone's lowest point, what is the total mechanical energy of the system?

62. A 70 kg skier starts from rest at height $H = 22$ m above the end of a ski-jump ramp (Fig. 8-66) and leaves the ramp at angle $\theta = 28°$. Neglect the effects of air resistance and assume the ramp is frictionless. (a) What is the maximum height h of his jump above the end of the ramp? (b) If he increased his weight by putting on a backpack, would h then be greater, less, or the same?

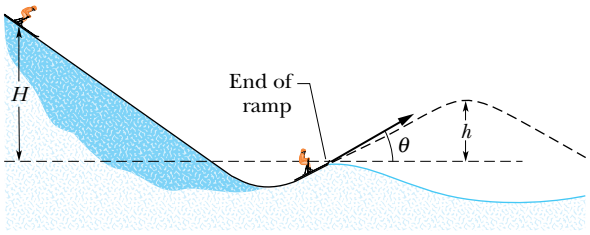

Figure 8-66 Problem 62.

63. The string in Fig. 8-67 is $L = 120$ cm long, has a ball attached to one end, and is fixed at its other end. The distance d from the fixed end to a fixed peg at point P is 75.0 cm. When the initially stationary ball is released with the string horizontal as shown, it will swing along the dashed arc. What is its speed when it reaches (a) its lowest point and (b) its highest point after the string catches on the peg?

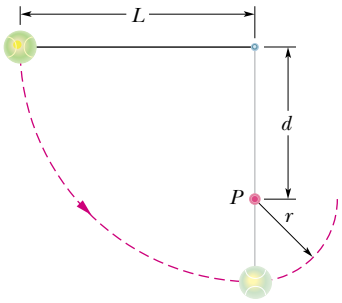

Figure 8-67 Problem 63.

64. A block of mass $m = 2.0$ kg is dropped from height $h = 50$ cm onto a spring of spring constant $k = 1960$ N/m (Fig. 8-68). Find the maximum distance the spring is compressed.

65. At $t = 0$ a 1.0 kg ball is thrown from a tall tower with $\vec{v} = (18 \text{ m/s})\hat{i} + (24 \text{ m/s})\hat{j}$. What is ΔU of the ball–Earth system between $t = 0$ and $t = 6.0$ s (still free fall)?

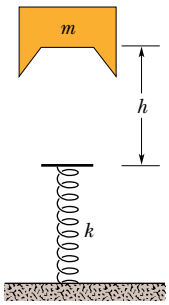

Figure 8-68 Problem 64.

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66. A conservative force, $\vec{F} = (6.0x - 12)\hat{i}$ N, where x is in meters, acts on a particle moving along an x axis. The potential energy U associated with this force is assigned a value of 27 J at $x = 0$. (a) Write an expression for U as a function of x, with U in joules and x in meters. (b) What is the maximum positive potential energy? At what (c) negative value and (d) positive value of x is the potential energy equal to zero?

67. Figure 8-69*a* applies to the spring in a cork gun (Fig. 8-69*b*); it shows the spring force as a function of the stretch or compression of the spring. The spring is compressed by 5.5 cm and used to propel a 4.2 g cork from the gun. (a) What is the speed of the cork if it is released as the spring passes through its relaxed position? (b) Suppose, instead, that the cork sticks to the spring and stretches it 1.5 cm before separation occurs. What now is the speed of the cork at the time of release?

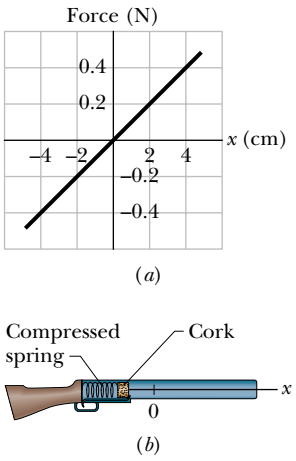

Figure 8-69 Problem 67.

68. A 4.00 kg block moves on a horizontal, frictionless surface and collides with a spring of spring constant k that is fixed to a wall. When the block momentarily stops, the spring has been compressed by 0.20 m. After rebounding, the block has a speed of 1.00 m/s. Next, the spring is put on an inclined surface with its lower end fixed in place (Fig. 8-70). The same block is now released on the incline at a distance of 5.00 m from the spring's free end. When the block momentarily stops, the spring has been compressed by 0.30 m. (a) What is the coefficient of kinetic friction between the block and the incline? (b) How far does the block then move up the incline from the stopping point?

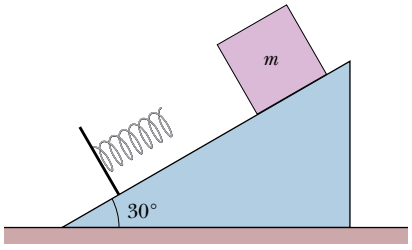

Figure 8-70 Problems 68 and 73.

69. A 2.0 kg breadbox on a frictionless incline of angle $\theta = 40°$ is connected, by a cord that runs over a pulley, to a light spring of spring constant $k = 105$ N/m, as shown in Fig. 8-71. The box is released from rest when the spring is unstretched. Assume that the pulley is massless and frictionless. (a) What is the speed of the box when it has moved 10 cm down the incline? (b) How far down the incline from its point of release does the box slide before momentarily stopping, and what are the (c) magnitude and (d) direction (up or down the incline) of the box's acceleration at the instant the box momentarily stops?

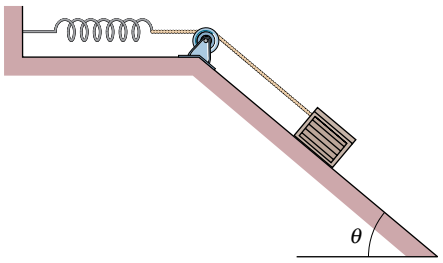

Figure 8-71 Problem 69.

70. As shown in Fig. 8-72, the right end of a spring is fixed to a wall. A 1.00 kg block is then pushed against the free end so that the spring is compressed by 0.25 m. After the block is released, it slides along a horizontal floor and (after leaving the spring) up an incline; both floor and incline are frictionless. Its maximum (vertical) height on the incline is 5.00 m. What are (a) the spring constant and (b) the maximum speed? (c) If the angle of the incline is increased, what happens to the maximum (vertical) height?

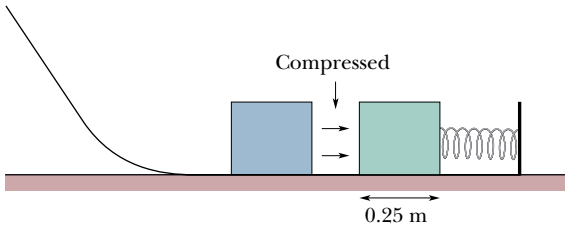

Figure 8-72 Problem 70.

71. In Fig. 8-73, a chain is held on a frictionless table with one-fourth of its length hanging over the edge. If the chain has length $L = 24$ cm and mass $m = 0.016$ kg, how much work is required to pull the hanging part back onto the table?

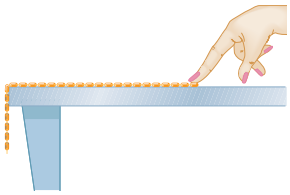

Figure 8-73 Problem 71.

72. In Fig. 8-74 a spring with $k = 170$ N/m is at the top of a frictionless incline of angle $\theta = 37.0°$. The lower end of the incline is distance $D = 1.00$ m from the end of the spring, which is at its relaxed length. A 2.00 kg canister is pushed against the spring until the spring is compressed 0.200 m and released from rest. (a) What is the speed of the canister at the instant the spring returns to its relaxed

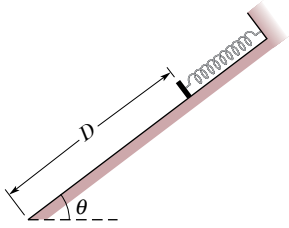

Figure 8-74 Problem 72.

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length (which is when the canister loses contact with the spring)? (b) What is the speed of the canister when it reaches the lower end of the incline?

73. In Fig. 8-70, a block of mass $m = 3.20$ kg slides from rest a distance d down a frictionless incline at angle $\theta = 30.0°$ where it runs into a spring of spring constant 431 N/m. When the block momentarily stops, it has compressed the spring by 21.0 cm. What are (a) distance d and (b) the distance between the point of the first block–spring contact and the point where the block's speed is greatest?

74. Two children are playing a game in which they try to hit a small box on the floor with a marble fired from a spring-loaded gun that is mounted on a table. The target box is horizontal distance $D = 2.20$ m from the edge of the table; see Fig. 8-75. Bobby compresses the spring 1.10 cm, but the center of the marble falls 26.3 cm short of the center of the box. How far should Rhoda compress the spring to score a direct hit? Assume that neither the spring nor the ball encounters friction in the gun.

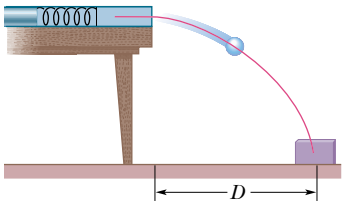

Figure 8-75 Problem 74.

75. A uniform chain hangs over the edge of a horizontal platform. A machine does 1.0 J of work in pulling the rest of the chain onto the platform. The chain has a mass of 2.0 kg and a length of 3.0 m. What length was initially hanging over the edge? On the Moon, the gravitational acceleration is 1/6 of 9.8 m/s^2.

76. Figure 8-76 shows a plot of potential energy U versus position x of a 0.200 kg particle that can travel only along an x axis under the influence of a conservative force. The graph has these values: $U_A = 9.00$ J, $U_C = 20.00$ J, and $U_D = 24.00$ J. The particle is released at the point where U forms a "potential hill" of "height" $U_B = 12.00$ J, with kinetic energy 4.00 J. What is the speed of the particle at (a) $x = 3.5$ m and (b) $x = 6.5$ m? What is the position of the turning point on (c) the right side and (d) the left side?

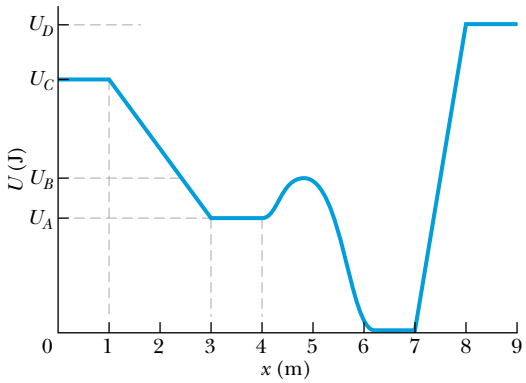

Figure 8-76 Problem 76.

77. Figure 8-77 shows a plot of potential energy U versus position x of a 0.90 kg particle that can travel only along an x axis. (Nonconservative forces are not involved.) Three values are $U_A = 15.0$ J, $U_B = 35.0$ J, and $U_C = 45.0$ J. The particle is released at $x = 4.5$ m with an initial speed of 7.0 m/s, headed in the negative x direction. (a) If the particle can reach $x = 1.0$ m, what is its speed there, and if it cannot, what is its turning point? What are the (b) magnitude and (c) direction of the force on the particle as it begins to move to the left of $x = 4.0$ m? Suppose, instead, the particle is headed in the positive x direction when it is released at $x = 4.5$ m at speed 7.0 m/s. (d) If the particle can reach $x = 7.0$ m, what is its speed there, and if it cannot, what is its turning point? What are the (e) magnitude and (f) direction of the force on the particle as it begins to move to the right of $x = 5.0$ m?

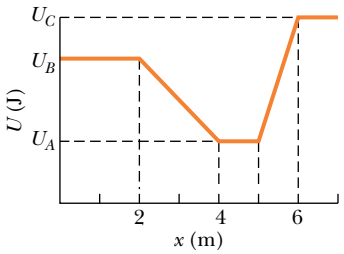

Figure 8-77 Problem 77.

78. A worker pushed a 23 kg block 8.4 m along a level floor at constant speed with a force directed 32° below the horizontal. If the coefficient of kinetic friction between block and floor was 0.20, what were (a) the work done by the worker's force and (b) the increase in thermal energy of the block–floor system?

79. In Fig. 8-78, a 3.5 kg block is accelerated from rest by a compressed spring of spring constant 640 N/m. The block leaves the spring at the spring's relaxed length and then travels over a horizontal floor with a coefficient of kinetic friction $\mu_k = 0.25$. The frictional force stops the block in distance $D = 7.8$ m. What are (a) the increase in the thermal energy of the block–floor system, (b) the maximum kinetic energy of the block, and (c) the original compression distance of the spring?

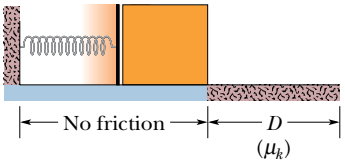

Figure 8-78 Problem 79.

80. A horizontal force of magnitude 41.0 N pushes a block of mass 4.00 kg across a floor where the coefficient of kinetic friction is 0.600. (a) How much work is done by that applied force on the block–floor system when the block slides through a displacement of 2.00 m across the floor? (b) During that displacement, the thermal energy of the block increases by 40.0 J. What is the increase in thermal

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energy of the floor? (c) What is the increase in the kinetic energy of the block?

81. An outfielder throws a baseball with an initial speed of 83.2 mi/h. Just before an infielder catches the ball at the same level, the ball's speed is 110 ft/s. In foot-pounds, by how much is the mechanical energy of the ball–Earth system reduced because of air drag? (The weight of a baseball is 9.0 oz.)

82. A small block of mass 1/4 kg has an initial kinetic energy of 500 J while moving over the frictionless surface shown in Fig. 8-79. (a) Find the kinetic energy of block at Q. (b) If we assume that the potential energy at P is equal to zero, what is the potential energy of the block at R? (c) Find the speed of the block at R. (d) Find the change in the potential energy as the block moves from Q to R.

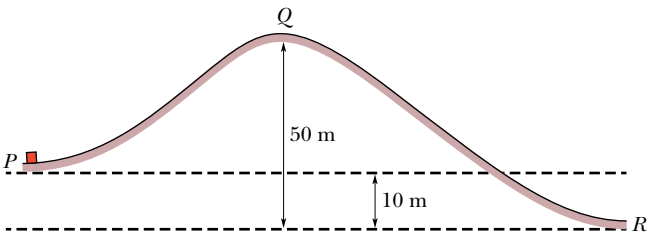

Figure 8-79 Problem 82.

83. In Fig. 8-80, a block slides down an incline. As it moves from point A to point B, which are 5.9 m apart, force $\vec{F}$ acts on the block, with magnitude 2.0 N and directed down the incline. The magnitude of the frictional force acting on the block is 10 N. If the kinetic energy of the block increases by 35 J between A and B, how much work is done on the block by the gravitational force as the block moves from A to B?

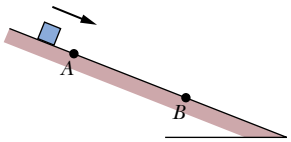

Figure 8-80 Problem 83.

84. In Fig. 8-81, a 50 kg child rides a 2.0 kg seat down a frictionless slope from a height of 7.0 m. Upon reaching the floor, the child and seat slide along it. There the coefficient of kinetic friction is 0.30. How far along the floor do they slide?

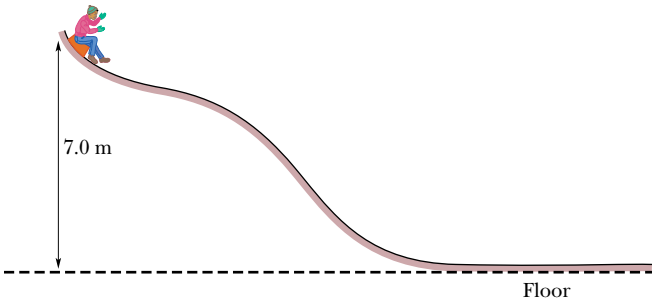

Figure 8-81 Problem 84.

85. A 60 kg skier leaves the end of a ski-jump ramp with a velocity of 27 m/s directed 25° above the horizontal. Suppose that as a result of air drag the skier returns to the ground with a speed of 22 m/s, landing 14 m vertically below the end of the ramp. From the launch to the return to the ground, by how much is the mechanical energy of the skier–Earth system reduced because of air drag?

86. In Fig. 8-82, a block of mass m slides down three slides, each from a height H. The coefficient of kinetic friction is the same value μ for the slides. In terms of m, H, μ, and g, what is the final kinetic energy of the block as it emerges from (a) slide a, (b) slide b, and (c) slide c?

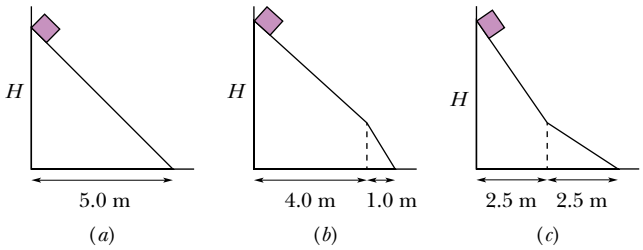

Figure 8-82 Problem 86.

87. A large fake cookie sliding on a horizontal surface is attached to one end of a horizontal spring with spring constant $k = 360$ N/m; the other end of the spring is fixed in place. The cookie has a kinetic energy of 20.0 J as it passes through the spring's equilibrium position. As the cookie slides, a frictional force of magnitude 10.0 N acts on it. (a) How far will the cookie slide from the equilibrium position before coming momentarily to rest? (b) What will be the kinetic energy of the cookie as it slides back through the equilibrium position?

88. A swimmer moves through the water at an average speed of 0.22 m/s. The average drag force is 110 N. What average power is required of the swimmer?

89. A child whose weight is 267 N slides down a 6.5 m playground slide that makes an angle of 20° with the horizontal. The coefficient of kinetic friction between slide and child is 0.10. (a) How much energy is transferred to thermal energy? (b) If she starts at the top with a speed of 0.457 m/s, what is her speed at the bottom?

90. The total mechanical energy of a 2.00 kg particle moving along an x axis is 5.00 J. The potential energy is given as $U(x) = (x^4 - 2.00x^2)$ J, with x in meters. Find the maximum velocity.

91. You push a 2.0 kg block against a horizontal spring, compressing the spring by 12 cm. Then you release the block, and the spring sends it sliding across a tabletop. It stops 75 cm from where you released it. The spring constant is 170 N/m. What is the block–table coefficient of kinetic friction?

92. A block of mass 6.0 kg is pushed up an incline to its top by a man and then allowed to slide down to the bottom. The length of incline is 10 m and its height is 5.0 m. The coefficient of friction between block and incline is 0.40. Calculate (a) the work done by the gravitational force over the complete round trip of the block, (b) the work done by the man during the upward journey, (c) the mechanical energy loss due to friction over the round trip, and (d) the speed of the block when it reaches the bottom.

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93. A cookie jar is moving up a 40° incline. At a point 45 cm from the bottom of the incline (measured along the incline), the jar has a speed of 1.4 m/s. The coefficient of kinetic friction between jar and incline is 0.15. (a) How much farther up the incline will the jar move? (b) How fast will it be going when it has slid back to the bottom of the incline? (c) Do the answers to parts (a) and (b) increase, decrease, or remain the same if we decrease the coefficient of kinetic friction (but do not change the given speed or location)?

94. A stone with a weight of 5.29 N is launched vertically from ground level with an initial speed of 20.0 m/s, and the air drag on it is 0.265 N throughout the flight. What are (a) the maximum height reached by the stone and (b) its speed just before it hits the ground?

95. A 4.0 kg bundle starts up a 30° incline with 150 J of kinetic energy. How far will it slide up the incline if the coefficient of kinetic friction between bundle and incline is 0.36?

96. A 10.0 kg block falls 30.0 m onto a vertical spring whose lower end is fixed to a platform. When the spring reaches its maximum compression of 0.200 m, it is locked in place. The block is then removed and the spring apparatus is transported to the Moon, where the gravitational acceleration is $g/6$. A 50.0 kg astronaut then sits on top of the spring and the spring is unlocked so that it propels the astronaut upward. How high above that initial point does the astronaut rise?

97. In Fig. 8-83, a block slides along a path that is without friction until the block reaches the section of length $L = 0.65$ m, which begins at height $h = 2.0$ m on a ramp of angle $\theta = 30°$. In that section, the coefficient of kinetic friction is 0.40. The block passes through point A with a speed of 8.0 m/s. If the block can reach point B (where the friction ends), what is its speed there, and if it cannot, what is its greatest height above A?

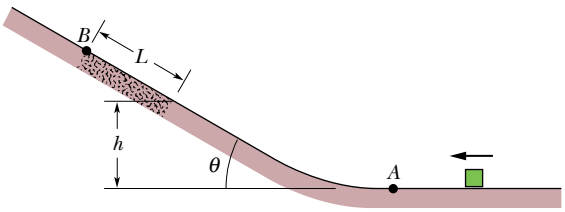

Figure 8-83 Problem 97.

98. The cable of the 1800 kg elevator cab in Fig. 8-84 snaps when the cab is at rest at the first floor, where the cab bottom is a distance $d = 3.7$ m above a spring of spring constant $k = 0.15$ MN/m. A safety device clamps the cab against guide rails so that a constant frictional force of 4.4 kN opposes the cab's motion. (a) Find the speed of the cab just before it hits the spring. (b) Find the maximum distance x that the spring is compressed (the frictional force still acts during this compression). (c) Find the distance that the cab will bounce back up the shaft. (d) Using conservation of energy, find the approximate total distance that the cab will move before coming to rest. (Assume that the frictional force on the cab is negligible when the cab is stationary.)

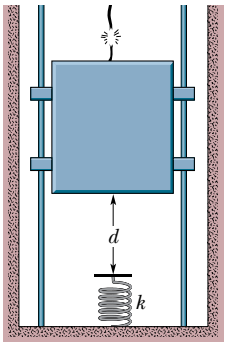

Figure 8-84 Problem 98.

99. In Fig. 8-85, a block is released from rest at height $d = 40$ cm and slides down a frictionless ramp and onto a first plateau, which has length d and where the coefficient of kinetic friction is 0.50. If the block is still moving, it then slides down a second frictionless ramp through height $d/2$ and onto a lower plateau, which has length $d/2$ and where the coefficient of kinetic friction is again 0.50. If the block is still moving, it then slides up a frictionless ramp until it (momentarily) stops. Where does the block stop? If its final stop is on a plateau, state which one and give the distance L from the left edge of that plateau. If the block reaches the ramp, give the height H above the lower plateau where it momentarily stops.

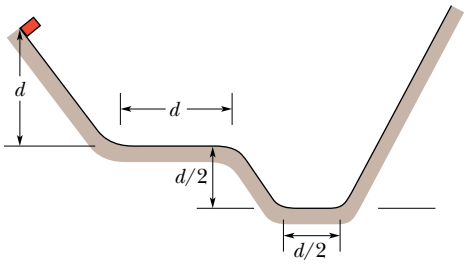

Figure 8-85 Problem 99.

100. A particle can slide along a track with elevated ends and a flat central part, as shown in Fig. 8-86. The flat part has length $L = 40$ cm. The curved portions of the track are frictionless, but for the flat part the coefficient of kinetic friction is $\mu_k = 0.20$. The particle is released from rest at point A, which is at height $h = L/2$. How far from the left edge of the flat part does the particle finally stop?

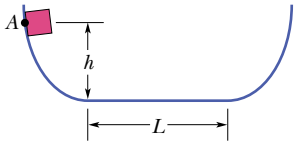

Figure 8-86 Problem 100.

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PRACTICE QUESTIONS

Single Correct Choice Type

1. Which one of the following statements about kinetic energy is true?
- **(a)** Kinetic energy can be measured in watts.
- **(b)** Kinetic energy is always equal to the potential energy.
- **(c)** Kinetic energy is always positive.
- **(d)** Kinetic energy is a quantitative measure of inertia.

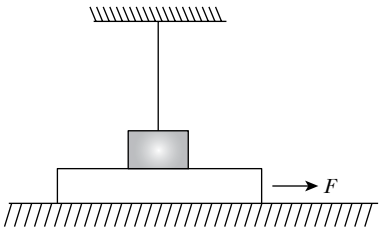
2. A fighter jet is launched from an aircraft carrier with the aid of its own engines and a steam-powered catapult. The thrust of its engines is 2.3×10^5 N. In being launched from rest position, it moves through a distance of 87 m and has a kinetic energy of 4.5×10^7 J at lift-off. What is the work done on the jet by the catapult?
- **(a)** 2.0×10^6 J **(b)** 2.5×10^7 J
- **(c)** 6.0×10^6 J **(d)** 6.5×10^7 J

3. A person pulls a sledge for a distance of 35.0 m along the snow with a rope directed 25.0° above the snow. The tension in the rope is 94.0 N. How much work is done on the sledge by the tension force?
- **(a)** 1390 J **(b)** 2740 J
- **(c)** 2980 J **(d)** 3290 J

4. A block of mass M is hanging over a smooth and light pulley through a light string. The other end of the string is pulled by a constant force F. The kinetic energy of the block increases by 20 J in 1 s. Now, choose the correct statement:
- **(a)** The tension in the string is Mg.
- **(b)** The tension in the string is F.
- **(c)** The work done by the tension on the block is 20 J in the above 1 s.
- **(d)** The work done by the force of gravity is –20 J in the above 1 s.

5. Use the work–energy theorem to find the force required to accelerate an electron ($m = 9.11 \times 10^{-31}$ kg) from rest to a speed of 1.50×10^7 m/s in a distance of 0.0125 m.
- **(a)** 8.20×10^{-15} N **(b)** 8.20×10^{-17} N
- **(c)** 5.47×10^{-22} N **(d)** 1.64×10^{-14} N

6. A block is placed on a plank which is placed on a horizontal plane shown in the given figure. A massless but elastic string deviates by an angle with vertical when a force F is applied to the plank to shift it to the right making the block side over it. If F_P and F_B are frictional forces between plank and plane and between block and plane, respectively, then choose the correct statement about the work done by applied force:



- **(a)** Work done against friction acting on the plank + Energy in the elastic string + Work done by friction acting on block.
- **(b)** Work done against frictions acting on the plank and the block + Elastic energy.
- **(c)** Elastic energy of the string.
- **(d)** Difference of work done against frictions acting on the plank and the block.

7. How much power is needed to lift a 75-kg student vertically upward at a constant speed of 0.33 m/s?
- **(a)** 12.5 W **(b)** 115 W
- **(c)** 243 **(d)** 230 W

8. Two blocks of masses m_1 and m_2 are connected by a spring having spring constant K. Initially, the spring is at its natural length. The coefficient of friction between the blocks and the surface is μ. What minimum constant force has to be applied in the horizontal direction to the block of mass m_1, in order to shift the other block?
- **(a)** $F = \mu\left(m_1 + \frac{m_2}{2}\right)g$ **(b)** $F = \mu\left(\frac{m_1}{2} + m_2\right)g$
- **(c)** $F = \mu\left(m_1^2 + \frac{m_2^2}{2}\right)g$ **(d)** $F = \mu^2\left(m_1^2 + \frac{m_2}{2}\right)g$

9. A pitcher throws a 0.140-kg baseball, and it approaches the bat at a speed of 40.0 m/s. The bat does $W_{nc} = 70.0$ J of work on the ball in hitting it. Ignoring air resistance, determine the speed of the ball after the ball leaves the bat and is 25.0 m above the point of impact.
- **(a)** 23.4 m/s **(b)** 48.5 m/s
- **(c)** 45.9 m/s **(d)** 51.0 m/s

10. Mike is cutting the grass using a lawn mower. He pushes the mower with a force of 45 N directed at an angle of 41° below the horizontal direction. Calculate the work that Mike does on the mower in pushing it 9.1 m across the yard.
- **(a)** 510 J **(b)** 410 J
- **(c)** 310 J **(d)** 360 J

11. A bead can slide on a smooth circular wire frame of radius R which is fixed in a vertical plane. The bead is displaced slightly from the highest point of the wire frame. The speed of the bead subsequently as a function of the angle θ made by the bead with the vertical line is
- **(a)** $\sqrt{2gR}$ **(b)** $\sqrt{2gR(1-\sin\theta)}$
- **(c)** $\sqrt{2gR(1-\sin\theta)}$ **(d)** $2\sqrt{gR}$

12. A 5.00-kg block of ice is sliding across a frozen pond at 2.00 m/s. A 7.60-N force is applied in the direction of motion. After the ice block slides 15.0 m, the force is removed. The work done by the applied force is
- **(a)** –114 J **(b)** –735 J
- **(c)** +19.7 J **(d)** +114 J

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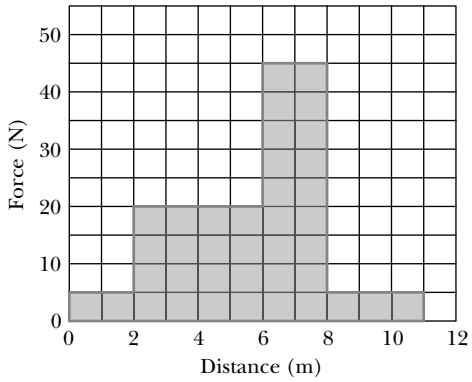
13. A 5.0×10^4 kg space probe is traveling at a speed of 11000 m/s through deep space. Auxiliary rockets are fired along the line of motion to reduce the probe's speed. The auxiliary rockets generate a force of 4.0×10^5 N over a distance of 2500 km. What is the final speed of the probe?

(a) 10000 m/s **(b)** 8000 m/s
(c) 9000 m/s **(d)** 7000 m/s

14. The graph shows the force component along the displacement as a function of the magnitude of the displacement. Determine the work done by the force during the interval from 2 m to 10 m.



(a) 140 J **(b)** 190 J
(c) 270 J **(d)** 450 J

15. An object of mass (m) is located on the horizontal plane at the origin O. The body acquires horizontal velocity V. The mean power developed by the frictional force during the whole time of motion is (μ = frictional coefficient)

(a) μmgV **(b)** $\frac{1}{2}\mu mgV$
(c) $\mu mg\frac{V}{4}$ **(d)** $\frac{3}{2}\mu mgV$

16. A force of magnitude 25 N directed at an angle of 37° above the horizontal moves a 10-kg crate along a horizontal surface at constant velocity. How much work is done by this force in moving the crate a distance of 15 m?

(a) 0 J **(b)** 40 J
(c) 300 J **(d)** 98 J

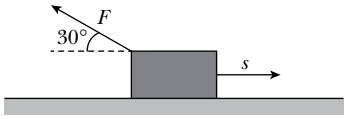
17. A 0.075-kg arrow is fired horizontally. The bowstring exerts an average force of 65 N on the arrow over a distance of 0.90 m. With what speed does the arrow leave the bow?

(a) 39 m/s **(b)** 28 m/s
(c) 82 m/s **(d)** 66 m/s

18. A 1,00,000 kg engine is moving up a slope of gradient 5° at a speed of 100 m/h. The coefficient of friction between the engine and rails is 0.1. If the engine has an efficiency of 4% for converting heat into work, find the amount of coal the engine has to burn up in one hour (burning of 1 kg of coal yields 50,000 J)

(a) 9.23×10^4 kg **(b)** 9.15×10^3 kg
(c) 8.41×10^4 kg **(d)** 8.65×10^3 kg

19. A constant force of 25 N is applied as shown to a block which undergoes a displacement of 7.5 m to the right along a frictionless surface while the force acts. What is the work done by the force?



(a) zero joules **(b)** –94 J
(c) –160 J **(d)** +160 J

20. The stopping distance for a vehicle, of mass M, moving with speed v along level road is (μ is the coefficient of friction between tyres and the road)

(a) $\frac{2v^2}{\mu g}$ **(b)** $\frac{v^2}{2\mu g}$
(c) $\frac{2v^2}{\mu^2 g}$ **(d)** $\frac{v^2}{2\mu^2 g}$

21. A 63-kg skier goes up a snow-covered hill that makes an angle of 25° with the horizontal. The initial speed of the skier is 6.6 m/s. After coasting a distance of 1.9 m up the slope, the speed of the skier is 4.4 m/s. Find the work done by the kinetic frictional force that acts on the skis.

(a) –760 J **(b)** +270 J
(c) +300 J **(d)** –270 J

22. A 10.0-g bullet traveling horizontally at 755 m/s strikes a stationary target and stops after penetrating 14.5 cm into the target. What is the average force of the target on the bullet?

(a) 1.97×10^4 N
(b) 6.26×10^3 N
(c) 2.07×10^5 N
(d) 3.13×10^4 N

23. A 3.00 kg model rocket is launched straight up. It reaches a maximum height of 1.00×10^2 m above where its engine cuts out, even though air resistance performs -8.00×10^2 J of work on the rocket. What would have been this height if there were no air resistance?

(a) 111 m **(b)** 135 m
(c) 159 m **(d)** 127 m

24. A car is traveling at 7.0 m/s when the driver applies the brakes. The car moves 1.5 m before it comes to a complete stop. If the car had been moving at 14 m/s, how far would it have continued to move after the brakes were applied? Assume the braking force is constant.

(a) 1.5 m **(b)** 4.5 m
(c) 7.5 m **(d)** 6.0 m

25. A man pulls a 5 kg block by 20 m along a horizontal surface at a constant speed with force directed above the horizontal. If the coefficient of kinetic friction is 0.2, how much work is done by the man in pulling block?

(a) 197.50 J **(b)** 219.76 J
(c) 162.32 J **(d)** 334.15 J

26. A bicyclist is traveling at a speed of 20.0 m/s as he approaches the bottom of a hill. He decides to coast up the

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hill and stops upon reaching the top. Neglecting friction, determine the vertical height of the hill.

(a) 28.5 m **(b)** 11.2 m
(c) 20.4 m **(d)** 40.8 m

27. A 55-kg box is being pushed a distance of 7.0 m across the floor by a force $\vec{P}$ whose magnitude is 150 N. The force $\vec{P}$ is parallel to the displacement of the box. The coefficient of kinetic friction is 0.25. Determine the work done on the box by each of the *four* forces that act on the box. Be sure to include the proper plus or minus sign for the work done by each force. (The four forces are the force P, the frictional force f, the normal force N, and the force due to gravity mg.)

	W_P	W_f	W_N	W_{mg}
(a)	1.0×10^3 J	–940 J	-3.8×10^3 J	3.8×10^3 J
(b)	1.0×10^3 J	–940 J	zero J	zero J
(c)	zero J	940 J	-3.8×10^3 J	3.8×10^3 J
(d)	-1.0×10^3 J	zero J	zero J	-3.8×10^3 J

28. What power is needed to lift a 49-kg person a vertical distance of 5.0 m in 20.0 s?

(a) 12.5 W **(b)** 60 W
(c) 25 W **(d)** 120 W

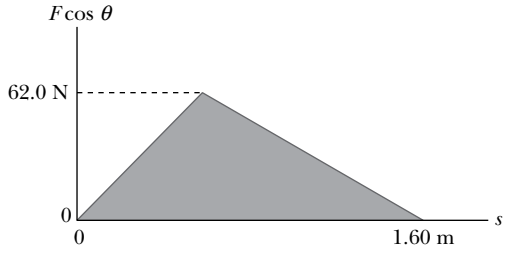
29. A motorcycle (mass of cycle plus rider = 2.50×10^2 kg) is traveling at a steady speed of 20.0 m/s. The force of air resistance acting on the cycle and rider is 2.00×10^2 N. Find the power necessary to sustain this speed if the road is sloped upward at 37.0° with respect to the horizontal.

(a) 5.75×10^3 W **(b)** 2.17×10^4 W
(c) 3.35×10^4 W **(d)** 4.90×10^4 W

30. The initial velocity of a 4.0-kg box is 11 m/s, due west. After the box slides 4.0 m horizontally, its speed is 1.5 m/s. Determine the magnitude and the direction of the non-conservative force acting on the box as it slides.

(a) 42 N, due west
(b) 31 N, due east
(c) 83 N, due west
(d) 59 N, due east

31. The graph shows how the force component $F \cos \theta$ along the displacement varies with the magnitude s of the displacement. Find the work done by the force. (*Hint:* Recall how the area of a triangle is related to the triangle's base and height.)



(a) 24.8 J **(b)** 55.1 J
(c) 99.2 J **(d)** 49.6 J

32. A small block of mass m is kept on a rough inclined surface of inclination θ fixed in an elevator. The elevator goes up with a uniform velocity v and the block does not slide on the wedge. The work done by the force of friction on the block in time t is

(a) zero
(b) $mgvt \cos^2\theta$
(c) $mgvt \sin^2\theta$
(d) $mgvt \sin^2\theta$

33. A 51-kg woman runs up a flight of stairs in 5.0 s. Her net upward displacement is 5.0 m. Approximately, what average power did the woman exert while she was running?

(a) 5.0 kW **(b)** 0.75 kW
(c) 0.25 kW **(d)** 0.50 kW

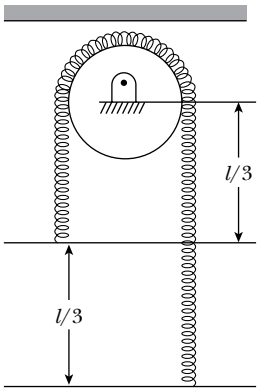
34. A particle of mass 100 g is thrown vertically upward with a speed of 5 m/s. The work done by the gravity during the time the particle goes up is

(a) 0.5 J **(b)** –0.5 J
(c) –1.25 J **(d)** 1.25 J

35. An escalator is 30.0 meters long and slants at 30.0° relative to the horizontal. If it moves at 1.00 m/s, at what rate does it do work in lifting a 50.0-kg woman from the bottom to the top of the escalator?

(a) 49.3 W **(b)** 245 W
(c) 98.0 W **(d)** 292 W

36. Two ends A and B of a smooth chain of mass m and length l are situated at distance $l/3$ as shown in the figure. If an external agent pulls A till it comes to same level of B, the work done by external agent is



(a) $\frac{mgl}{36}$ **(b)** $\frac{mgl}{15}$
(c) $\frac{mgl}{9}$ **(d)** none of these

37. A warehouse worker uses a forklift to lift a crate of pickles on a platform to a height 2.75 m above the floor. The combined mass of the platform and the crate is 207 kg. If the power expended by the forklift is 1440 W, how long does it take to lift the crate?

(a) 37.2 s **(b)** 3.87 s
(c) 1.86 s **(d)** 5.81 s

38. A uniform rope of linear mass density λ and length l is coiled on a smooth horizontal surface as shown in the figure. One end is pulled up with constant velocity v. Then,

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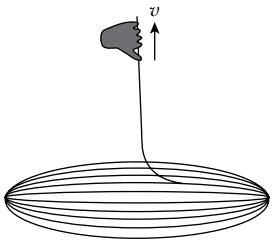
the average power applied by the external agent in pulling the entire rope just off the ground is



(a) $\frac{1}{2}\lambda l v^2 + \frac{\lambda l^2 g}{2}$ **(b)** $\lambda l g v$

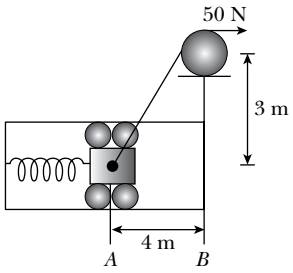
(c) $\frac{1}{2}\lambda v^3 + \frac{\lambda l g v}{2}$ **(d)** $\lambda g v + \frac{1}{2}\lambda v^3$

39. A dam is used to block the passage of a river and to generate electricity. Approximately 5.73×10^4 kg of water fall each second through a height of 19.6 m. If one half of the gravitational potential energy of the water were converted to electrical energy, how much power would be generated?

(a) 5.50×10^6 W **(b)** 1.10×10^7 W

(c) 2.70×10^9 W **(d)** 1.35×10^9 W

40. A 20 kg block attached to a spring of spring constant 5 N/m is released from rest at A. The spring at this instant is having an elongation of 1 m. The block is allowed to move in smooth horizontal slot with the help of a constant force of 50 N in the rope as shown in the figure. The velocity of the block as it reaches B is (assume the rope to be light)



(a) 4 m/s **(b)** 2 m/s

(c) 1 m/s **(d)** 3 m/s

41. A motorist driving a 1000-kg car wishes to increase her speed from 20 m/s to 30 m/s in 5 s. Determine the horsepower required to accomplish this increase. Neglect friction.

(a) 70 hp **(b)** 90 hp

(c) 30 hp **(d)** 80 hp

42. A skier slides horizontally along the snow for a distance of 21 m before coming to rest. The coefficient of kinetic friction between the skier and the snow is $\mu_k = 0.050$. Initially, how fast was the skier going?

(a) 6.4 m/s **(b)** 4.5 m/s

(c) 2.7 m/s **(d)** 5.4 m/s

43. An automobile approaches a barrier at a speed of 20 m/s along a level road. The driver locks the brakes at a distance of 50 m from the barrier. What minimum coefficient of kinetic friction is required to stop the automobile before it hits the barrier?

(a) 0.4 **(b)** 0.6

(c) 0.8 **(d)** 0.5

44. A 1900-kg car experiences a combined force of air resistance and friction that has the same magnitude whether the car goes up or down a hill at 27 m/s. Going up a hill, the car's engine needs to produce 47 hp more power to sustain the constant velocity than it does going down the same hill. At what angle is the hill inclined above the horizontal?

(a) 1.4° **(b)** 2.5°

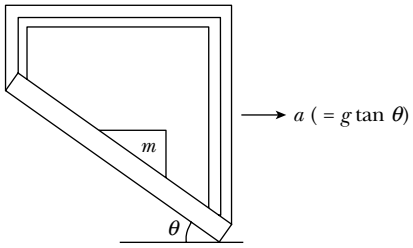
(c) 1.1° **(d)** 2.0°

45. A block m is kept stationary on the surface of an accelerating cage as shown in the given figure. At the given instant, study the following statements regarding the block:

(i) Normal reaction performs positive work on the block.

(ii) Frictional work done on the block is negative.

(iii) No net work is done by normal reaction and friction on the block.



Now mark the correct answer:

(a) only statement (i) is correct.

(b) only statement (ii) is correct.

(c) only statement (iii) is correct.

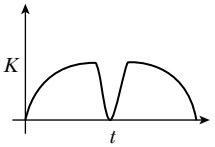
(d) all the statement correct.

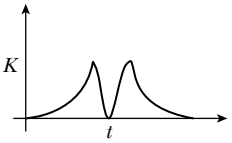
46. A racing car with a mass of 500.0 kg starts from rest and completes a quarter mile (402 m) race in a time of 5.0 s. The race car's final speed is 130 m/s. Neglecting friction, what average power was needed to produce this final speed?

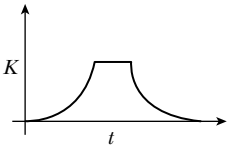
(a) 140 hp **(b)** 1100 hp

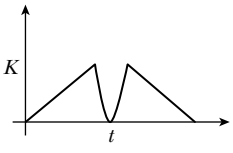
(c) 8.5×10^5 hp **(d)** 750 hp

47. A tennis ball is dropped on a horizontal smooth surface. It bounces back to its original position after hitting the surface. The force on the ball during the collision is proportional to the length of compression of the ball. Which one of the following sketches describes the variation of its kinetic energy K with time t most appropriately? The figures are only illustrative and not to the scale.

(a) 

(b) 

(c) 

(d) 

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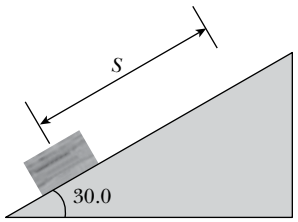
48. A physics student shoves a 0.50-kg block from the bottom of a frictionless 30.0° inclined plane. The student performs 4.0 J of work and the block slides a distance s along the incline before it stops. Determine the value of s.

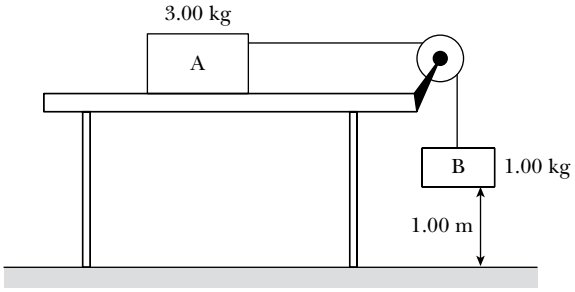


(a) 82 cm **(b)** 330 cm
(c) 16 cm **(d)** 160 cm

49. A 1.00×10^2 kg crate is being pushed across a horizontal floor by a force $\vec{P}$ that makes an angle of 30.0° below the horizontal. The coefficient of kinetic friction is 0.200. What should be the magnitude of $\vec{P}$, so that the net work done by it and the kinetic frictional force is zero?

(a) 256 N **(b)** 354 N
(c) 203 N **(d)** 287 N

50. Two boxes are connected to each other as shown. The system is released from rest and the 1.00 kg box falls through a distance of 1.00 m. The surface of the table is frictionless. What is the kinetic energy of box B just before it reaches the floor?



(a) 2.45 J **(b)** 29.4 J
(c) 4.90 J **(d)** 9.80 J

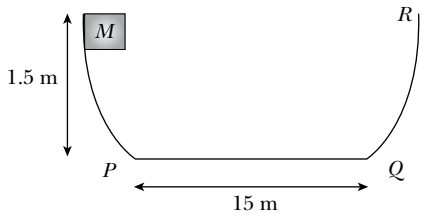
51. The potential energy of a 1-kg particle, which is free to move along the x-axis, is given by $V(x) = (x^4/4 - x^2/2)$ J. If the total mechanical energy of the particle is 2 J, then the maximum speed (in m/s) is

(a) 2 **(b)** $3\sqrt{2}$
(c) $\sqrt{2}$ **(d)** $1\sqrt{2}$

52. A 1500-kg elevator moves upward with constant speed through a vertical distance of 25 m. How much work was done by the tension in the elevator cable?

(a) 990 J **(b)** 8100 J
(c) 140,000 J **(d)** 370,000 J

53. A block of mass M slides along the sides of a bowl as shown in the figure. The walls of the bowl are frictionless and the base has coefficient of friction 0.2. If the block is released from the top of the side, which is 1.5 m high, where will the block come to rest? Given that the length of the base is 15 m.

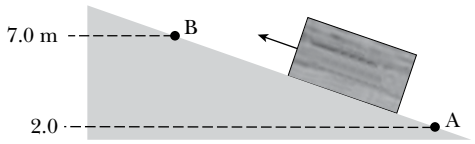


(a) 1 m from point P **(b)** Midpoint
(c) 2 m from point P **(d)** At point Q

54. Two balls of equal size are dropped from the same height from the roof of a building. One ball has twice the mass of the other. When the balls reach the ground, how do the kinetic energies of the two balls compare?

(a) The lighter one has one fourth as much kinetic energy as the other does.
(b) The lighter one has one half as much kinetic energy as the other does.
(c) The lighter one has the same kinetic energy as the other does.
(d) The lighter one has twice as much kinetic energy as the other does.

55. A 12-kg crate is pushed up an incline from point A to point B as shown in the figure. What is the change in the gravitational potential energy of the crate?



(a) +590 J **(b)** –590 J
(c) +1200 J **(d)** –1200 J

56. A woman stands on the edge of a cliff and throws a stone *vertically downward* with an initial speed of 10 m/s. The instant before the stone hits the ground below, it has 450 J of kinetic energy. If she were to throw the stone *horizontally outward* from the cliff with the same initial speed of 10 m/s, how much kinetic energy would it have just before it hits the ground?

(a) 50 J **(b)** 100 J
(c) 450 J **(d)** 800 J

57. A stone rolls off the roof of a School hall and falls vertically. Just before it reaches the ground, the stone's speed is 17 m/s. Neglect air resistance and determine the height of the School Hall.

(a) 42 m **(b)** 33 m
(c) 26 m **(d)** 15 m

58. A skier leaves the top of a slope with an initial speed of 5.0 m/s. Her speed at the bottom of the slope is 13 m/s. What is the height of the slope?

(a) 1.1 m **(b)** 4.6 m
(c) 6.4 m **(d)** 7.3 m

59. A 55.0-kg skateboarder starts out with a speed of 1.80 m/s. He does +80.0 J of work on himself by pushing with his feet against the ground. In addition, friction does –265 J of work on him. In both cases, the forces doing the work are

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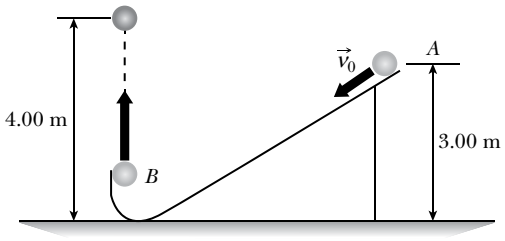
non-conservative. The final speed of the skateboarder is 6.00 m/s. Calculate the change $[\Delta(\text{P.E.}) = (\text{P.E.})_f - (\text{P.E.})_0]$ in the gravitational potential energy.

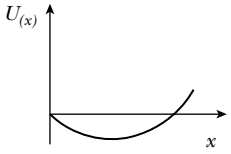
(a) –1246 J **(b)** –1086 J
(c) –821 J **(d)** –716 J

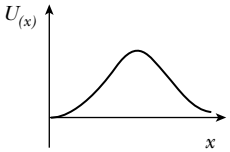
60. A particle, starting from point A in the figure, is projected down the curved runway. Upon leaving the runway at point B, the particle is traveling straight upward and reaches a height of 4.00 m above the floor before falling back down. Ignoring friction and air resistance, find the speed of the particle at point A.

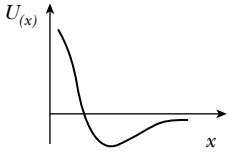


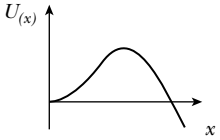
(a) 8.85 m/s **(b)** 7.67 m/s
(c) 3.13 m/s **(d)** 4.43 m/s

61. A particle which is constrained to move along the x-axis, is subjected to a force in the same direction which varies with the distance x of the particle from the origin. The functional form of the potential energy of the particle is (Here, k and a are positive constants.)

(a)


(b)


(c)


(d)


62. A bullet fired into a fixed target loses half of its velocity after penetrating 3 cm. How much further it will penetrate before coming to rest assuming that it faces constant resistance to motion?

(a) 3.0 cm **(b)** 2.0 cm
(c) 1.5 cm **(d)** 1.0 cm

63. A particle of mass m is moving in a horizontal circle of radius r, under a centripetal force equal to $(-k/r^2)$, where k is constant. The total energy of the particle is

(a) $-\frac{k}{r}$ **(b)** $-\frac{k}{2r}$

(c) $+\frac{k}{r}$ **(d)** $+\frac{k}{2r}$

64. Potential energy function describing the interaction between two atoms of a diatomic molecule is $U(x) = a/x^{12} - b/x^6$. In stable equilibrium, the distance between them would be

(a) $\left(\frac{32b}{a}\right)^{1/3}$ **(b)** $\left(\frac{2a}{b}\right)^{1/6}$

(c) $\left(\frac{3a}{b}\right)^{1/2}$ **(d)** $\left(\frac{3b}{a}\right)^{1/2}$

65. A block is dropped from a high tower and is falling freely under the influence of gravity. Which one of the following statements is true concerning this situation? Neglect air resistance.

(a) As the block falls, the net work done by all of the forces acting on the block is zero joules.
(b) The kinetic energy increases by equal amounts over equal distances.
(c) The kinetic energy of the block increases by equal amounts in equal times.
(d) The potential energy of the block decreases by equal amounts in equal times.

66. If the potential energy of a gas molecule is $U = (M/r^6) - (N/r^{12})$ where M and N are positive constants, the potential energy at equilibrium must be

(a) zero **(b)** $\frac{M^2}{4N}$

(c) $\frac{N^2}{4M}$ **(d)** $\frac{MN^2}{4}$

More than One Choice Correct Type

67. A boy pulls 5 kg block by 20 m along a horizontal surface at constant speed with a force directed 45° above the horizontal. If the coefficient of friction is 0.2, then

(a) the work done by the boy on the block 163.32 J.
(b) the normal force on the surface by the block is 40.8 N.
(c) the normal on the surface by the block is μmg.
(d) none of these.

68. Which of the following statement(s) is/are correct?

(a) Total work done by internal forces of a system on the system is sometimes zero.
(b) Total work done by internal forces of a system on the system is always zero.
(c) Total work done by internal forces acting between the particles of a rigid body is sometimes zero.
(d) Total work done by internal forces acting between the particles of a rigid body is always zero.

69. Choose the correct statement(s):

(a) Total work done by internal forces of a system on the system is always zero.
(b) Total work done by internal forces of a system on the system is sometimes zero.
(c) Total work done by internal forces acting between the particles of a rigid body is always zero.
(d) Total work done by internal forces acting between the particles of a rigid body is sometimes zero.

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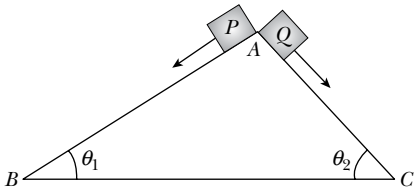
70. Choose the correct statement(s):
- **(a)** When power is constant, then variation of force with velocity is hyperbolic.
- **(b)** When constant power is applied on the stationary body in time t, then variation of velocity with time is parabolic.
- **(c)** When power is constant, then variation of velocity with time is linear.
- **(d)** None of these.

71. A block hangs freely from the end of a spring. A boy then slowly pushes the block upward so that the spring becomes strain free. The gain in gravitational potential energy of the block during this process is not equal to
- **(a)** the work done by the boy against the gravitational force acting on the block.
- **(b)** the loss of energy stored in the spring minus the work done by the tension in the spring.
- **(c)** the work done on the block by the boy minus the work done by the tension in the pring plus the loss of energy stored in the spring.
- **(d)** the work done on the block by the boy minus the work done by the tension in the spring.

72. Two inclined frictionless tracks of different inclinations meet at A from where two blocks P and Q of different masses are allowed to slide down from rest at the same time, one on each track, as shown in the figure. Then

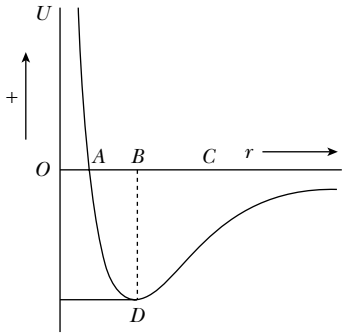


- **(a)** both blocks will reach the bottom at the same time.
- **(b)** block Q will reach the bottom earlier than block B.
- **(c)** both blocks will reach the bottom with the same speed.
- **(d)** block Q will reach the bottom with a higher speed than block P.

73. A particle is acted upon by a force of constant magnitude which is always perpendicular to the velocity of the particle. The motion of the particle takes place in a plane. It follows that
- **(a)** its velocity is constant.
- **(b)** its acceleration is constant.
- **(c)** its kinetic energy is constant.
- **(d)** it moves in circular path.

74. Which of the following statements are correct?
- **(a)** A pendulum bob suspended by a string of length l is pulled to one side so that it is at a height $l/4$ above the rest position. If the bob is now released from rest, its speed at the lowest point will be $gl/2$.
- **(b)** If the mass and velocity of a moving body are increased three times and two times, respectively then the kinetic energy is increased by a factor of 12.
- **(c)** A boat is being towed at a velocity of 20 m/s. If the tension in the tow-line is 6 kN, then the power supplied to the boat is 120 kW.
- **(d)** If the work done in increasing the extension of a spring from 0.4 m to 0.5 m is 18 J, then the spring constant is 400 N/m.

75. The potential energy curve for interaction between two molecules is shown in the figure. Which of the following statement(s) is/are true?



- **(a)** The molecules have maximum attraction for $r = OA$.
- **(b)** The molecules have maximum kinetic energy for $r = OB$.
- **(c)** The intermolecular force is zero for $r = OB$.
- **(d)** For the gaseous state, the depth BD of the potential well is much smaller than KT.

76. A particle is taken from point A to point B under the influence of a force field. Now it is taken back from B to A and it is observed that the work done in taking the particle from A to B is not equal to the work done in taking it from B to A. If W_{nc} and W_c is the work done by non-conservative and conservative forces present in the system, respectively, ΔU is the change in potential energy and ΔK is the change in kinetic energy, then
- **(a)** $W_{nc} - \Delta U = \Delta K$
- **(b)** $W_{nc} - \Delta U = -\Delta K$
- **(c)** $W_{nc} + W_c = \Delta K$
- **(d)** $W_c = -\Delta U$

77. A particle of mass 5 kg moving in the x-y plane has its potential energy given by $U = (-7x + 24y)$ J, where x and y are in meter. The particle is initially at origin and has a velocity $\vec{u} = (14.4\hat{i} + 4.2\hat{j})$ m/s.
- **(a)** The particle has a speed of 25 m/s at $t = 4$ s.
- **(b)** The particle has an acceleration of 5 m/s^2.
- **(c)** The acceleration of the particle is perpendicular to its initial velocity.
- **(d)** None of the above is correct.

78. The potential energy U, in joule of a particle of mass 1 kg, moving in the xy-plane, obeys the law $U = 3x + 4y$, where (x, y) are the coordinates of the particle in meter. If the particle is at rest at $(6, 4)$ at time $t = 0$, then
- **(a)** the particle has constant acceleration.
- **(b)** the work done by the external forces, from the position of rest of the particle and the instant of the particle crossing x axis is 25 J.
- **(c)** the speed of the particle when it crosses the y axis is 10 m/s.
- **(d)** the coordinates of the particle at time $t = 4$ s are $(-18, -28)$.

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Linked Comprehension

Paragraph for Questions 79 and 80: A block of mass m is released from rest at a height R above a horizontal surface. The acceleration due to gravity is g. The block slides along the inside of a frictionless circular hoop of radius R.

79. Which one of the following expressions gives the speed of the mass at the bottom of the hoop?

(a) zero m/s^2

(b) $v = \dfrac{mg}{2R}$

(c) $v^2 = 2gR$

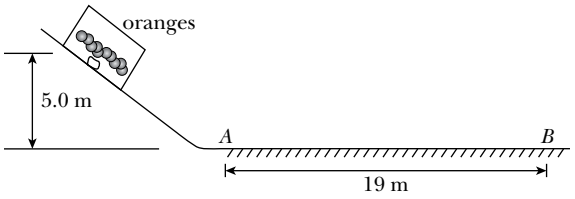
(d) $v = mgR$

80. What is the magnitude of the normal force exerted on the block by the hoop when the block has reached the bottom?

(a) mg **(b)** $3\,mg$

(c) $\dfrac{mg^2}{R}$ **(d)** $2\,mg$

Paragraph for Questions 81 and 82: A 9.0 kg box of oranges slides from rest down a frictionless incline from a height of 5.0 m. A constant frictional force, introduced at point A, brings the block to rest at point B, 19 m to the right of point A.



81. What is the speed of the block just before it reaches point A?

(a) 98 m/s **(b)** 9.9 m/s

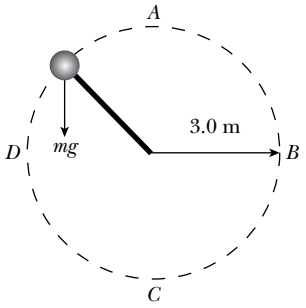
(c) 21 m/s **(d)** 5.7 m/s

82. What is the coefficient of kinetic friction, μ_k, of the surface from A to B?

(a) 0.11 **(b)** 0.33

(c) 0.26 **(d)** 0.47

Paragraph for Questions 83 and 84: A 0.50 kg ball on the end of a rope is moving in a vertical circle of radius 3.0 m near the surface of the Earth where the acceleration due to gravity, g, is 9.8 m/s^2. Point A is at the top of the circle; C is at the bottom. Points B and D are exactly halfway between A and C.



83. Which one of the following statements concerning the tension in the rope is true?

(a) The tension is smallest at A.

(b) The tension is the same at A and C.

(c) The tension is smallest at C.

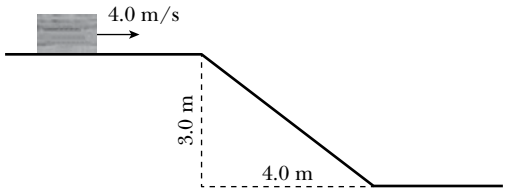
(d) The tension is smallest at both B and D.

84. The ball moves on the circle from A to C under the influence of gravity alone. If the kinetic energy of the ball is 35 J at A, what is its kinetic energy at C?

(a) 35 J **(b)** 64 J

(c) 29 J **(d)** 44 J

Paragraph for Questions 85—87: A 10.0 kg crate slides along a horizontal frictionless surface at a constant speed of 4.0 m/s. The crate then slides down a frictionless incline and across a second horizontal surface as shown in the figure.



85. While the crate slides along the upper surface, how much gravitational potential energy does it have compared to what it would have on the lower surface?

(a) 140 J **(b)** 490 J

(c) 80 J **(d)** 290 J

86. What is the kinetic energy of the crate as it slides on the lower surface?

(a) 290 J

(b) 370 J

(c) 570 J

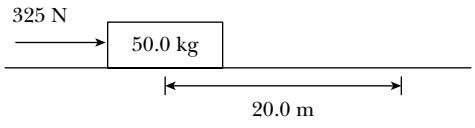
(d) 320 J

87. What minimum coefficient of kinetic friction is required to bring the crate to a stop over a distance of 5.0 m along the lower surface?

(a) 0.60 **(b)** 0.76

(c) 0.32 **(d)** 0.66

Paragraph for Questions 88—90: A 325 N force accelerates a 50.0 kg crate from rest along a horizontal frictionless surface for a distance of 20.0 m as shown in the figure.



88. What is the final speed of the crate?

(a) 11.4 m/s **(b)** 32.2 m/s

(c) 16.1 m/s **(d)** 131 m/s

89. How much work is done on the crate?

(a) 3.82×10^3 J

(b) 1.50×10^4 J

(c) 7.70×10^2 J

(d) 6.50×10^3 J

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90. What coefficient of friction would be required to keep the crate moving at constant speed under the action of the 325-N force?

(a) 0.508 **(b)** 0.747
(c) 0.321 **(d)** 0.663

Paragraph for Questions 91 and 92: A 2.0-kg projectile is fired with initial velocity components $v_{ox} = 30$ m/s and $v_{oy} = 40$ m/s from a point on the earth's surface. Neglect any effects due to air resistance.

91. What is the kinetic energy of the projectile when it reaches the highest point in its trajectory?

(a) zero joules **(c)** 1600 J
(b) 900 J **(d)** 2500 J

92. How much work was done in firing the projectile?

(a) 900 J **(c)** 2500 J
(b) 1600 J **(d)** 4900 J

Paragraph for Questions 93—96: Potential energy of a body is the energy possessed by the body by virtue of its position. Potential energy is mgh, where the symbols have their usual meaning. Kinetic energy of a body is the energy possessed by the body by virtue of its velocity and is given by $1/2\, mv^2$. Energy can neither be created nor be destroyed. However, energy can be changed from one form to the other, such that energy appearing in one form is equal to the energy disappearing in the other form.

93. Kinetic energy of the body 5 s after it starts falling is

(a) 1250 J **(b)** 2500 J
(c) 625 J **(d)** 25,000 J

94. The body will attain this kinetic energy when it falls freely from a height of

(a) 125 m **(b)** 250 m
(c) 1250 m **(d)** 2500 m

95. Velocity of the body on striking the ground will be

(a) 25 m/s **(b)** 12.5 m/s
(c) 50 m/s **(d)** 100 m/s

96. The ratio of potential energy to kinetic energy at a height of 62.5 m above the ground is

(a) 2 **(b)** 1
(c) 3 **(d)** 4

Matrix-Match

97. Match the statements in Column I with the statements in Column II:

Column I	Column II
(a) Force is equal to	**(p)** the work is path independent.
(b) For the conservative force	**(q)** the rate of change of linear momentum.
(c) Power is equal to	**(r)** the rate of work done.
(d) Area of power–time curve gives	**(s)** the product of force to the velocity.
	(t) the work done.
	(u) the negative of the potential energy gradient.

98. In the given table, Column I shows statements related to inertial and non-inertial frames. Match the related statements of Column I with that give in Column II.

Column I	Column II
(a) Observation depends on the choice of the frame of reference.	**(p)** Decrease or increase in potential energy.
(b) Observation does not depend on the choice of frame of reference.	**(q)** Potential energy.
(c) The law hold good in inertial as well as non-inertial frame of reference.	**(r)** Law of conservation of mechanical energy.
(d) The law does not hold in non-inertial frame of reference.	**(s)** Work–energy theorem.

Directions for Questions 99—102: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

99. In the given table, Column I shows the value of angle between the direction of force and direction of displacement, Column II shows the value of the cosine of the angle and Column III shows the direction of force and direction of displacement.

Column I	Column II	Column III
(I) $\theta = 0°$	**(i)** $\cos\theta = -1$	**(J)** Force and displacement are in the same direction.
(II) $0° < \theta < 90°$	**(ii)** $\cos\theta = 1$	**(K)** Force is perpendicular to displacement.
(III) $\theta = 180°$	**(iii)** $\cos\theta$ is positive.	**(L)** Force has its one component in the direction of displacement.
(IV) $\theta = 90°$	**(iv)** $\cos\theta = 0$	**(M)** Force and displacement are in opposite direction.

(1) When is the work done by constant force minimum?

(a) (I) (iv) (M) **(b)** (IV) (ii) (L)
(c) (II) (i) (K) **(d)** (III) (i) (M)

(2) When is the work done by constant force maximum?

(a) (I) (ii) (J)
(b) (IV) (iii) (J)
(c) (II) (iii) (L)
(d) (I) (i) (M)

(3) When is the work done by constant force zero?

(a) (II) (ii) (J)
(b) (IV) (iv) (K)
(c) (IV) (i) (L)
(d) (II) (iii) (M)

100. In the given table, Column I shows different values of masses of the body (car/person/football player), Column II shows the initial and final values of velocities of the body and Column III shows the time for which force is exerted on the body or work done by/on the body.

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Column I	Column II	Column III
(I) Mass = 1000 kg	**(i)** $v_1 = 88$ m/s, $v_2 = 100$ m/s	**(J)** $t = 10$ s
(II) Mass = 60 kg	**(ii)** $v_1 = 5$ m/s, $v_2 = 10$ m/s	**(K)** $t = 30$ s
(III) Mass = 120 kg	**(iii)** $v_1 = 10$ m/s, $v_2 = 20$ m/s	**(L)** $t = 2$ s
(IV) Mass = 500 kg	**(iv)** $v_1 = 5$ m/s, $v_2 = 7$ m/s	**(M)** $t = 1$ s

(1) What will be the mass, velocities and time of a car with power 3.8×10^4 W?
(a) (I) (iii) (L) **(b)** (III) (ii) (K)
(c) (I) (i) (K) **(d)** (I) (iii) (J)

(2) What will be the mass, velocities and time of a person with power 360 W?
(a) (II) (iv) (L) **(b)** (III) (ii) (J)
(c) (II) (ii) (K) **(d)** (III) (i) (M)

(3) What will be the mass, velocities and time of a football player with power 4500 W?
(a) (III) (i) (L) **(b)** (IV) (i) (J)
(c) (I) (iv) (J) **(d)** (III) (ii) (M)

101. In the given table, Column I shows different positions of the particle, Column II shows the potential energy of the particle and Column III shows examples related to potential energy.

Column I	Column II	Column III
(I) When a particle is slightly displaced from its position, it does not experience any force acting on it and it continues to be in equilibrium in the displaced position.	**(i)** Potential energy is zero.	**(J)** *Example:* A marble placed on horizontal table.
(II) When a particle is displaced slightly from its present position, a force acting on the particle brings the particle back to the initial position.	**(ii)** Potential energy is minimum.	**(K)** *Example:* A marble placed at the bottom of a hemispherical bowl.
(III) When a particle is displaced slightly from its present position, then a force acting on it does not bring it back to the initial position.	**(iii)** Potential energy is maximum.	**(L)** *Example:* A marble balanced on top of a hemispherical bowl.
(IV) When a particle is displaced slightly from its present position, then a force acting on it tries to displace the particle further away from the equilibrium position.	**(iv)** Potential energy is constant.	**(M)** A marble placed at the bottom of a horizontal table.

(1) Determine the characteristics of stable equilibrium.
(a) (I) (iv) (M) **(b)** (IV) (ii) (L)
(c) (III) (i) (K) **(d)** (II) (ii) (K)

(2) Determine the characteristics of unstable equilibrium.
(a) (I) (ii) (J) **(b)** (IV) (iii) (J)
(c) (IV) (iii) (L) **(d)** (I) (i) (M)

(3) Determine the characteristics of neutral equilibrium?
(a) (II) (ii) (J) **(b)** (I) (iv) (J)
(c) (IV) (i) (L) **(d)** (II) (iii) (M)

102. In the given table, Column I shows different values of load on a cart, Column II shows the different values of acceleration due to gravity on Earth and Column III shows the height up to which the load is lifted.

Column I	Column II	Column III
(I) Mass = 3 kg	**(i)** $g = 9.8$ m/s^2	**(J)** height = 0.24 m
(II) Mass = 30 kg	**(ii)** $g = 9.75$ m/s^2	**(K)** height = 10 m
(III) Mass = 0.229 kg	**(iii)** $g = 10$ m/s^2	**(L)** height = 80 m
(IV) Mass = 5 kg	**(iv)** $g = 19.8$ m/s^2	**(M)** height = 0.45 m

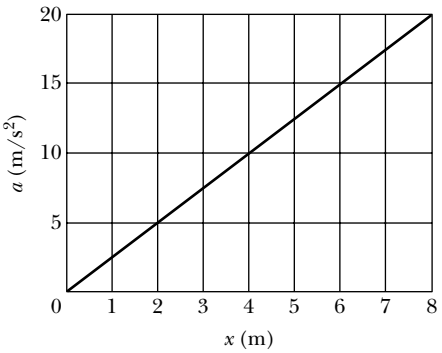
(1) What will be the mass and height of a loaded cart with potential energy 13.2 J?
(a) (I) (iii) (L) **(b)** (III) (ii) (K)
(c) (I) (i) (M) **(d)** (I) (iii) (J)

(2) What will be the mass and height of a loaded cart with potential energy 23,520 J?
(a) (II) (i) (L) **(b)** (III) (ii) (J)
(c) (II) (ii) (K) **(d)** (III) (i) (M)

(3) What will be the mass and height of a loaded cart with potential energy 22.5 J?
(a) (III) (i) (L) **(b)** (IV) (i) (J)
(c) (I) (iv) (J) **(d)** (III) (i) (K)

Integer Type

103. A 10-kg object moves along the x axis. Its acceleration as a function of its position is shown in the figure. What is the net work (in J) performed on the object as it moves from $x = 0$ to $x = 8.0$ m?



104. A car of mass 900 kg accelerates uniformly from rest to a speed of 60 km/h in a time of 2 s when traveling on a level road. If there is a constant resistance to motion of 20 N. Find the maximum power (in W) of the engine.

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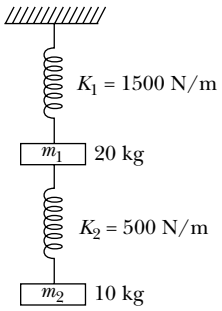
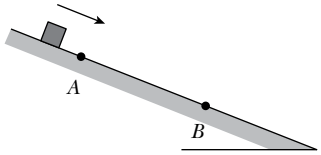
105. If the speed of a car increases 4 times, the stopping distance (in m) for this will increase by _____.

106. A swimmer moves through the water at a speed of 0.22 m/s. The drag force opposing this motion is 110 N. How much power (in W) is developed by the swimmer?

107. Find the potential energy (in J) stored in the springs, as shown in the following figure, in equilibrium.



108. A block slides down an incline as shown in the figure. As it moves from point A to point B, which are 5.0 m apart, force $\vec{F}$ acts on the block, with magnitude 2.0 N and directed down the incline. The magnitude of the frictional force acting on the block is 10 N. If the kinetic energy of the block increases by 35 J between A and B, how much work is done (in J) on the block by the gravitational force as the block moves from A to B?



109. A river descends 15 m through rapids. The speed of the water is 3.2 m/s upon entering the rapids and 13 m/s upon leaving. What percentage of the gravitational potential energy of the water–Earth system is transferred to kinetic energy during the descent? (*Hint:* Consider the descent of, say, 10 kg of water.)

ANSWER KEY

Checkpoints

1. $F_N.ut$; zero **2.** (a) Decrease; (b) Same; (c) Negative, zero **3.** (a) Positive; (b) Negative; (c) Zero

4. No (consider round trip on the small loop) **5.** 3, 1, 2 (see Eq. 8-47) **6.** (a) All tie; (b) All tie **7.** Zero

Problems

1. (a) 2.3×10^7 m/s; (b) 1.5×10^{-16} J **2.** 1.8×10^{13} J **3.** (a) 11 J; (b) –21 J

4. (a) $v_i = 2.4$ m/s; (b) $2v_i = 4.8$ m/s **5.** 7.1 J **6.** 0.96 J **7.** 6.6×10^3 J

8. 20 J **9.** 8.5 J **10.** (a) 73.2°; (b) 107° **11.** (a) 3.00 N; (b) 9.00 J

12. (a) 1.7×10^2 N; (b) 3.4×10^2 m; (c) -5.8×10^4 J; (d) 3.4×10^2 N; (e) 1.7×10^2 m; (f) -5.8×10^4 J **13.** 11.6 J

14. 3.7 m/s **15.** (a) 1.3×10^4 J; (b) -1.2×10^4 J; (c) 1.2×10^3 J; (d) 5.6 m/s

16. –25 J (the minus sign arises from the fact that the pull from the rope is anti-parallel to the direction of motion of the block). Thus, the kinetic energy would have been 25 J greater if the rope had not been attached (given the same displacement).

17. 30 N. **18.** (a) $W_F = -Fd = -\frac{3}{4}Mgd$; (b) $W_g = F_g d = Mgd$; (c) $W_{\text{net}} = W_F + W_g = \frac{1}{4}Mgd$; (d) $v = \sqrt{\frac{gd}{2}}$

19. (a) 1.11×10^4 J; (b) 1.00×10^4 J; (c) 8.93×10^4 J. **20.** 4.41 J **21.** (a) +3.03 J; (b) 1.42 m/s

22. (a) 2.59×10^4 J; (b) 2.45 N **23.** 19 J **24.** (a) 7.2 J; (b) 7.2 J; (c) 0 J; (d) –25 J

25. (a) 0.90 J; (b) 2.1 J; (c) 0 J **26.** (a) 12 N; (b) $k = 12$ N/m

27. (a) 7.3 m/s; (b) 5.5 m **28.** (a) 16 J; (b) 16 J; (c) 0 J; (d) –14 J

29. (a) 0.12 m; (b) 0.36 J; (c) –0.36 J; (d) 0.060 m; (e) 0.090 J

30. (a) Zero; (b) Zero **31.** 25 J

32. (a) 42 J; (b) 30 J; (c) 12 J; (d) $v = 6.5$ m/s, velocity vector pointing in the $+x$ direction; (e) $v = 5.5$ m/s, its velocity vector still points in the $+x$ direction; (f) $v = 3.5$ m/s, its velocity vector *still* points in the $+x$ direction.

33. (a) 2.3 J; (b) 2.6 J **34.** 2.2 J **35.** 0.21 J

36. 3.6×10^4 J (rounded off to two significant figures).

37. 46.18 J **38.** (a) 0.83 J; (b) 2.5 J; (c) 4.2 J; (d) 5.0 W

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39. (a) 8.8×10^2 J, just as it was in the first case; (b) 1.3×10^2 W; (c) 2.5×10^2 W

40. The net rate is zero **41.** 4.5×10^5 W **42.** (a) 1.0×10^2 J; (b) 8.4 W

43. No work is required by the motor.

44. (a) 28 W; (b) $v = 7.5$ m/s **45.** (a) 32.0 J; (b) 8.00 W; (c) 78.2°

46. $dT = \dfrac{T}{3P} dP$ **47.** 1 : 9

48. (a) 167 J; (b) –167 J; (c) 196 J; (d) 29 J; (e) 167 J; (f) –167 J; (g) 296 J; (h) 129 J.

49. (a) 1.86 ; (b) –1.86 J; (c) No work is done during this displacement; (d) –1.86 J; (e) 1.86 J; (f) $\Delta U = 0$ as it goes to the point at the same; (g) The change in the gravitational potential energy depends only on the initial and final positions of the ball, not on its speed anywhere. The change in the potential energy is the *same* since the initial and final positions are the same height.

50. (a) 4.31×10^{-3} J; (b) -4.31×10^{-3} J; (c) The potential energy when the flake is at the top is greater than when it is at the bottom by $|\Delta U|$. If $U = 0$ at the bottom, then $U = +4.31 \times 10^{-3}$ J at the top.(d) If $U = 0$ at the top, then $U = -4.31 \times 10^{-3}$ J at the bottom; (e) All the answers are proportional to the mass of the flake. If the mass is doubled, all answers are doubled.

51. (a) 0.13 J; (b) 0.094 J; (c) 0.16 J; (d) 0.031 J; (e) 0.063 J; (f) The new information ($v_i \neq 0$) is not involved in any of the preceding computations; the above results are unchanged.

52. (a) 13.1 J; (b) –13.1 J; (c) 13.1 J; (d) As the angle increases, we intuitively see that the height h increases (and, less obviously, from the mathematics, we see that cos θ decreases so that $1 - \cos\theta$ increases), so the answers to parts (a) and (c) increase, and the absolute value of the answer to part (b) also increases.

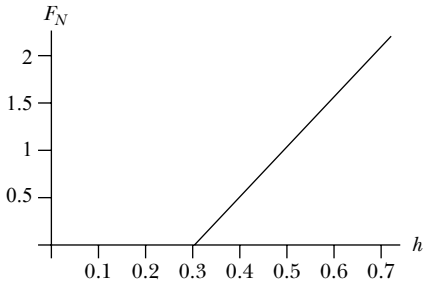
53. (a) 169 J; (b) –169 J; (c) The potential energy when it reaches the ground is less than the potential energy when it is fired by $|\Delta U|$, so $U = -169$ J when the snowball hits the ground.

54. (a) 21.9 m/s; (b) 21.9 m/s; (c) 21.9 m/s **55.** (a) 0.98 J; (b) –0.98 J; (c) 3.1 N/cm

56. (a) The ramp must be about 2.6×10^2 m long if friction is negligible; (b) The minimum length does not depend on the mass of the truck. Thus, the answer remains the same if the mass is reduced; (c) If the speed is decreased, then both h and L decrease.

57. (a) 8.1 J; (b) –8.1 J; (c) 0.99 m; (d) 0.26 m

58. (a) 2.5 N; (b) 0.31 N; (c) 0.30 m; (d)



59. (a) 2.29 m/s; (b) A different mass for the ball must not change the result.

60. The stone goes 1. 9 m above the spring's rest-length height.

61. (a) 4.5 m/s; (b) 74°; (c) 64 J

62. (a) 4.8 m; (b) new value for the mass will yield the same result as before.

63. (a) 4.85 m/s; (b) 2.42 m/s **64.** 0.11 m **65.** -3.2×10^{-2} J

66. (a) $U = 27 + 12x - 3x^2$; (b) $U = 39$ J; (c) $x = -1.6$ m; (d) $x = 5.6$ m.

67. (a) $x = 0.055$ m; (b) $v = 2.6$ m/s **68.** (a) $\mu_k = 0.55$; (b) 0.12 m

69. (a) $v = 0.86$ m/s; (b) $d = 0.24$ m; (c) $a = 6.3$ m/s^2; (d) The acceleration is up the incline.

70. (a) 1.6×10^3 N/m; (b) 9.9 m/s; (c) If the angle of the slope is increased, there would not be any change in the height acquired because the total energy is not dissipated in any form.

71. 0.0012 J. **72.** (a) $v_2 = 2.40$m s; (b) $v_3 = 4.19$ m/s

73. (a) $d = 0.396$ m; (b) $x = 0.0364$ m = 3.64 cm; this is long before the block finally stops (36.0 cm before it stops).

74. 1.25 cm **75.** ≈1.4 m **76.** (a) 8.37 m/s; (b) 12.6 m/s; (c) 7.67 m; (d) 1.73 m

77. (a) 2.1 m/s; (b) +10 N; (c) Since the magnitude $F_x > 0$, the force points in the $+x$ direction; (d) $x = 5.7$ m; (e) 30 N; (f) The force points in the $-x$ direction.

78. (a) $\approx 4.3 \times 10^2$ J; (b) 4.3×10^2 J **79.** (a) 67 J; (b) 67 J; (c) 0.46 m

80. (a) 82.0 J; (b) 7.0 J; (c) 35.0 J **81.** 25 ft·lb **82.** (a) 402 J; (b) –24.5 J; (c) ≈ 64.8 m/s; (d) ≈ 123 J

83. 82 J. **84.** ≈ 23 m **85.** 1.6×10^4 J

86. (a) $mg(H - 5.0\mu)$; (b) $mg(H - 5.0\mu)$; (c) $mg(H - 5.0\mu)$ **87.** (a) 0.307 m; (b) 13.9 J

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88. (a) 24 W
89. (a) 1.63 10 J; (b) 5.6 m/s
90. (a) $v_{max} = \pm\sqrt{6} \approx \pm 2.45 \text{m/s}$
91. $\mu_k = 0.083$
92. (a) 0 J; (b) 498 J; (c) 407 J; (d) 5.5 m/s
93. (a) 0.13 m; (b) 2.5 m/s; (c) In part (a) it is clear that d increases if μ_k decreases and in part (b), the factor itself increases in value.
94. (a) 19.4 m; (b) 19.0 m/s
95. 4.7 m
96. 36.0 m
97. 3.7 m/s
98. (a) 1800 kg; (b) 0.90 m (considering the positive root); (c) 2.8 m; (d) 15 m
99. The block (momentarily) stops on the inclined ramp at the right, at a height of $H = 0.75d = 0.75$ (40 cm) = 30 cm, measured from the lowest plateau.
100. $d = 20$ cm and this occurs on the particle's third pass through the flat region.

Practice Questions

Single Correct Choice Type

1. (c)
2. (b)
3. (c)
4. (b)
5. (a)
6. (a)
7. (c)
8. (a)
9. (c)
10. (c)
11. (c)
12. (d)
13. (c)
14. (b)
15. (b)
16. (c)
17. (a)
18. (b)
19. (b)
20. (b)
21. (d)
22. (a)
23. (d)
24. (d)
25. (c)
26. (c)
27. (b)
28. (d)
29. (c)
30. (d)
31. (d)
32. (c)
33. (d)
34. (c)
35. (b)
36. (a)
37. (b)
38. (c)
39. (a)
40. (b)
41. (a)
42. (b)
43. (a)
44. (d)
45. (a)
46. (b)
47. (b)
48. (d)
49. (a)
50. (a)
51. (b)
52. (d)
53. (b)
54. (b)
55. (a)
56. (c)
57. (d)
58. (d)
59. (b)
60. (d)
61. (d)
62. (d)
63. (b)
64. (b)
65. (b)
66. (b)

More than One Correct Choice Type

67. (a), (b)
68. (a), (d)
69. (b), (c)
70. (a), (b)
71. (a), (b), (c), (d)
72. (b), (c)
73. (c), (d)
74. (a), (b), (c), (d)
75. (b), (c), (d)
76. (a), (c), (d)
77. (a), (b), (c)
78. (a), (b), (c), (d)

Linked Comprehension

79. (c)
80. (b)
81. (b)
82. (c)
83. (a)
84. (b)
85. (d)
86. (b)
87. (b)
88. (c)
89. (d)
90. (d)
91. (b)
92. (c)
93. (a)
94. (a)
95. (c)
96. (b)

Matrix-Match

97. (a) → (q), (u); (b) → (p); (c) → (r), (s); (d) → (t)
98. (a) → (q); (b) → (p); (c) → (s); (d) → (r)
99. (1) → (d); (2) → (a); (3) → (b)
100. (1) → (c); 2 → (a); (3) → (d)
101. (1) → (d); (2) → (c); 3 → (b)
102. (1) → (c); (2) → (a); (3) → (d)

Integer Type

103. 800
104. 125
105. 16
106. 24
107. 40
108. 75
109. 54

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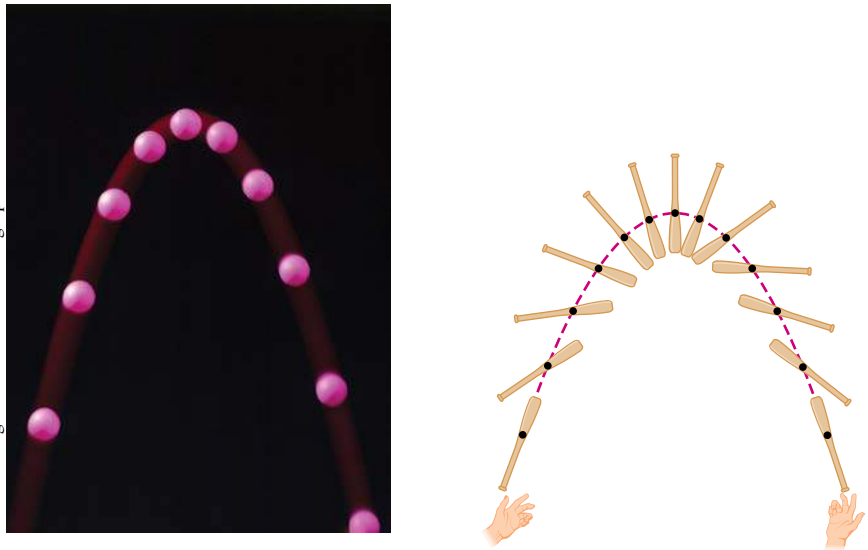
CHAPTER 9

Center of Mass

9.1 | WHAT IS PHYSICS?

Every mechanical engineer who is hired as a courtroom expert witness to reconstruct a traffic accident uses physics. Every dance trainer who coaches a ballerina on how to leap uses physics. Indeed, analyzing complicated motion of any sort requires simplification via an understanding of physics. In this chapter we discuss how the complicated motion of a system of objects, such as a car or a ballerina, can be simplified if we determine a special point of the system—the *center of mass* of that system.

Here is a quick example. If you toss a ball into the air without much spin on the ball (Fig. 9-1*a*), its motion is simple—it follows a parabolic path, as we discussed in Chapter 4, and the ball can be treated as a particle. If, instead, you flip a baseball bat into the air (Fig. 9-1*b*), its motion is more



Richard Megna/Fundamental Photographs

(*a*) (*b*)

Figure 9-1 (*a*) A ball tossed into the air follows a parabolic path. (*b*) The center of mass (black dot) of a baseball bat flipped into the air follows a parabolic path, but all other points of the bat follow more complicated curved paths.

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9.2 The Center of Mass
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9.4 Linear Momentum
9.5 The Linear Momentum of a System of Particles
9.6 Impulse and Momentum
9.7 Conservation of Linear Momentum
9.8 Collisions
9.9 Inelastic Collisions in One Dimension
9.10 Elastic Collisions in One Dimension
9.11 Collisions in Two Dimensions
9.12 *C*-Frame
9.13 Impulse Momentum Equation for Continuous Processes
9.14 Systems with Varying Mass: A Rocket
9.15 Some Derivations Pertaining to com of Objects

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complicated. Because every part of the bat moves differently, along paths of many different shapes, you cannot represent the bat as a particle. Instead, it is a system of particles each of which follows its own path through the air. However, the bat has one special point—the center of mass—that *does* move in a simple parabolic path. The other parts of the bat move around the center of mass. (To locate the center of mass, balance the bat on an outstretched finger; the point is above your finger, on the bat's central axis.)

You cannot make a career of flipping baseball bats into the air, but you can make a career of advising long-jumpers or dancers on how to leap properly into the air while either moving their arms and legs or rotating their torso. Your starting point would be to determine the person's center of mass because of its simple motion.

9.2 | THE CENTER OF MASS

Key Concept

- The center of mass of a system of n particles is defined to be the point whose coordinates are given by

$$x_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i, \qquad y_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i, \qquad z_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i,$$

or

$$\vec{r}_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i,$$

where M is the total mass of the system.

We define the **center of mass (com)** of a system of particles (such as a person) in order to predict the possible motion of the system.

The center of mass of a system of particles is the point that moves as though (1) all of the system's mass were concentrated there and (2) all external forces were applied there.

Here we discuss how to determine where the center of mass of a system of particles is located. We start with a system of only a few particles, and then we consider a system of a great many particles (a solid body, such as a baseball bat). Later in the chapter, we discuss how the center of mass of a system moves when external forces act on the system.

System of Particles

Two Particles. Figure 9-2*a* shows two particles of masses m_1 and m_2 separated by distance d. We have arbitrarily chosen the origin of an x axis to coincide with the particle of mass m_1. We *define* the position of the center of mass of this two-particle system to be

$$x_{com} = \frac{m_2}{m_1 + m_2} d. \tag{9-1}$$

Suppose, as an example, that $m_2 = 0$. Then there is only one particle, of mass m_1, and the center of mass must lie at the position of that particle; Eq. 9-1 dutifully reduces to $x_{com} = 0$. If $m_1 = 0$, there is again only one particle (of mass m_2), and we have, as we expect, $x_{com} = d$. If $m_1 = m_2$, the center of mass should be halfway between the two particles; Eq. 9-1 reduces to $x_{com} = 1/2\, d$, again as we expect. Finally, Eq. 9-1 tells us that if neither m_1 nor m_2 is zero, x_{com} can have only values that lie between zero and d; that is, the center of mass must lie somewhere between the two particles.

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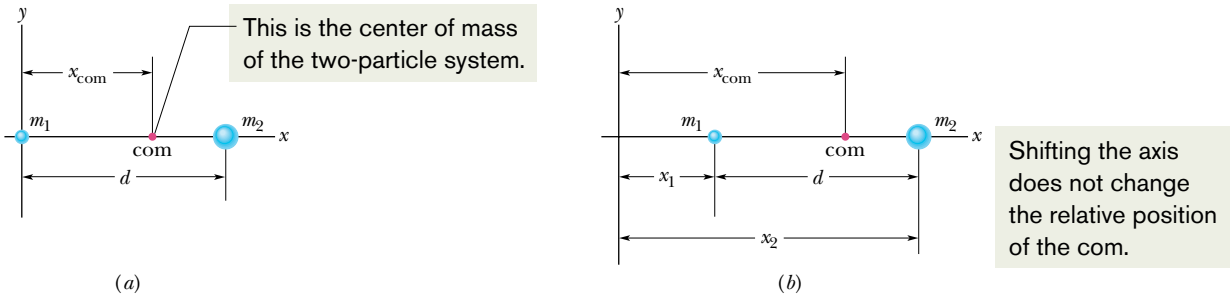

Figure 9-2 (*a*) Two particles of masses m_1 and m_2 are separated by distance d. The dot labeled com shows the position of the center of mass, calculated from Eq. 9-1. (*b*) The same as (*a*) except that the origin is located farther from the particles. The position of the center of mass is calculated from Eq. 9-2. The location of the center of mass with respect to the particles is the same in both cases.

We are not required to place the origin of the coordinate system on one of the particles. Figure 9-2*b* shows a more generalized situation, in which the coordinate system has been shifted leftward. The position of the center of mass is now defined as

$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}. \tag{9-2}$$

Note that if we put $x_1 = 0$, then x_2 becomes d and Eq. 9-2 reduces to Eq. 9-1, as it must. Note also that inspite of the shift of the coordinate system, the center of mass is still the same distance from each particle. The com is a property of the physical particles, not the coordinate system we happen to use.

We can rewrite Eq. 9-2 as

$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{M}, \tag{9-3}$$

in which M is the total mass of the system. (Here, $M = m_1 = m_2$.)

Many Particles. We can extend this equation to a more general situation in which n particles are strung out along the x axis. Then the total mass is $M = m_1 + m_2 + \cdots + m_n$, and the location of the center of mass is

$$x_{com} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots + m_n x_n}{M} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i. \tag{9-4}$$

The subscript i is an index that takes on all integer values from 1 to n.

Three Dimensions. If the particles are distributed in three dimensions, the center of mass must be identified by three coordinates. By extension of Eq. 9-4, they are

$$x_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i, \quad y_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i, \quad z_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i. \tag{9-5}$$

We can also define the center of mass with the language of vectors. First recall that the position of a particle at coordinates x_i, y_i, and z_i is given by a position vector (it points from the origin to the particle):

$$\vec{r}_i = x_i \hat{\mathrm{i}} + y_i \hat{\mathrm{j}} + z_i \hat{\mathrm{k}}. \tag{9-6}$$

Here the index identifies the particle, and $\hat{\mathrm{i}}$, $\hat{\mathrm{j}}$, and $\hat{\mathrm{k}}$ are unit vectors pointing, respectively, in the positive direction of the x, y, and z axes. Similarly, the position of the center of mass of a system of particles is given by a position vector:

$$\vec{r}_{com} = x_{com} \hat{\mathrm{i}} + y_{com} \hat{\mathrm{j}} + z_{com} \hat{\mathrm{k}}. \tag{9-7}$$

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If you are a fan of concise notation, the three scalar equations of Eq. 9-5 can now be replaced by a single vector equation,

$$\vec{r}_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i, \tag{9-8}$$

where again M is the total mass of the system. You can check that this equation is correct by substituting Eqs. 9-6 and 9-7 into it, and then separating out the x, y, and z components. The scalar relations of Eq. 9-5 result.

Solid Bodies

An ordinary object, such as a baseball bat, contains so many particles (atoms) that we can best treat it as a continuous distribution of matter. The "particles" then become differential mass elements dm, the sums of Eq. 9-5 become integrals, and the coordinates of the center of mass are defined as

$$x_{com} = \frac{1}{M}\int x\, dm, \quad y_{com} = \frac{1}{M}\int y\, dm, \quad z_{com} = \frac{1}{M}\int z\, dm, \tag{9-9}$$

where M is now the mass of the object. The integrals effectively allow us to use Eq. 9-5 for a huge number of particles, an effort that otherwise would take many years.

Evaluating these integrals for most common objects (such as a television set or a moose) would be difficult, so here we consider only *uniform* objects. Such objects have uniform *density*, or mass per unit volume; that is, the density ρ (Greek letter rho) is the same for any given element of an object as for the whole object. From Eq. 1-8, we can write

$$\rho = \frac{dm}{dV} = \frac{M}{V}, \tag{9-10}$$

where dV is the volume occupied by a mass element dm, and V is the total volume of the object. Substituting $dm = (M/V)\,dV$ from Eq. 9-10 into Eq. 9-9 gives

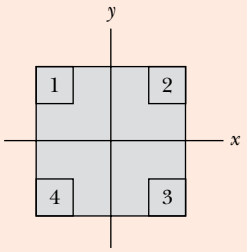
$$x_{com} = \frac{1}{V}\int x\, dV, \quad y_{com} = \frac{1}{V}\int y\, dV, \quad z_{com} = \frac{1}{V}\int z\, dV. \tag{9-11}$$

Symmetry as a Shortcut. You can bypass one or more of these integrals if an object has a point, a line, or a plane of symmetry. The center of mass of such an object then lies at that point, on that line, or in that plane. For example, the center of mass of a uniform sphere (which has a point of symmetry) is at the center of the sphere (which is the point of symmetry). The center of mass of a uniform cone (whose axis is a line of symmetry) lies on the axis of the cone. The center of mass of a banana (which has a plane of symmetry that splits it into two equal parts) lies somewhere in the plane of symmetry.

The center of mass of an object need not lie within the object. There is no dough at the com of a doughnut, and no iron at the com of a horseshoe.

CHECKPOINT 1

The figure shows a uniform square plate from which four identical squares at the corners will be removed. (a) Where is the center of mass of the plate originally? Where is it after the removal of (b) square 1; (c) squares 1 and 2; (d) squares 1 and 3; (e) squares 1, 2, and 3; (f) all four squares? Answer in terms of quadrants, axes, or points (without calculation, of course).



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SAMPLE PROBLEM 9.01

com of three particles

Three particles of masses $m_1 = 1.2$ kg, $m_2 = 2.5$ kg, and $m_3 = 3.4$ kg form an equilateral triangle of edge length $a = 140$ cm. Where is the center of mass of this system?

KEY IDEA

We are dealing with particles instead of an extended solid body, so we can use Eq. 9-5 to locate their center of mass. The particles are in the plane of the equilateral triangle, so we need only the first two equations.

Calculations: We can simplify the calculations by choosing the x and y axes so that one of the particles is located at the origin and the x axis coincides with one of the triangle's sides (Fig. 9-3). The three particles then have the following coordinates:

Particle	Mass (kg)	x (cm)	y (cm)
1	1.2	0	0
2	2.5	140	0
3	3.4	70	120

The total mass M of the system is 7.1 kg.

From Eq. 9-5, the coordinates of the center of mass are

$$x_{com} = \frac{1}{M}\sum_{i=1}^{3} m_i x_i = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{M}$$

$$= \frac{(1.2\text{ kg})(0) + (2.5\text{ kg})(140\text{ cm}) + (3.4\text{ kg})(70\text{ cm})}{7.1\text{ kg}}$$

$$= 83\text{ cm} \qquad \text{(Answer)}$$

and

$$y_{com} = \frac{1}{M}\sum_{i=1}^{3} m_i y_i = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{M}$$

$$= \frac{(1.2\text{ kg})(0) + (2.5\text{ kg})(0) + (3.4\text{ kg})(120\text{ cm})}{7.1\text{ kg}}$$

$$= 58\text{ cm} \qquad \text{(Answer)}$$

In Fig. 9-3, the center of mass is located by the position vector $\vec{r}_{com}$, which has components x_{com} and y_{com}. If we had chosen some other orientation of the coordinate system, these coordinates would be different but the location of the com relative to the particles would be the same.

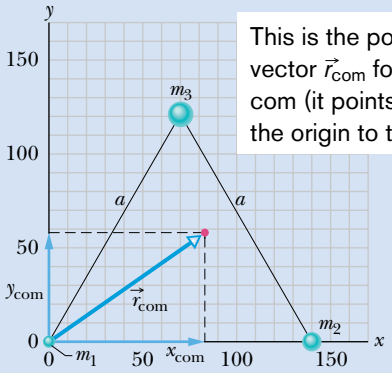

Figure 9-3 Three particles form an equilateral triangle of edge length a. The center of mass is located by the position vector $\vec{r}_{com}$.

SAMPLE PROBLEM 9.02

com of plate with missing piece

This sample problem has lots of words to read, but they will allow you to calculate a com using easy algebra instead of challenging integral calculus. Figure 9-4*a* shows a uniform metal plate P of radius $2R$ from which a disk of radius R has been stamped out (removed) in an assembly line. The disk is shown in Fig. 9-4*b*. Using the xy coordinate system shown, locate the center of mass com_P of the remaining plate.

KEY IDEAS

(1) Let us roughly locate the center of plate P by using symmetry. We note that the plate is symmetric about the x axis (we get the portion below that axis by rotating the upper portion about the axis). Thus, com_P must be on the x axis. The plate (with the disk removed) is not symmetric about the y axis. However, because there is

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somewhat more mass on the right of the y axis, com_P must be somewhat to the right of that axis. Thus, the location of com_P should be roughly as indicated in Fig. 9-4*a*.

(2) Plate P is an extended solid body, so in principle we can use Eqs. 9-11 to find the actual coordinates of the center of mass of plate P. Here we want the xy coordinates of the center of mass because the plate is thin and uniform. If it had any appreciable thickness, we would just say that the center of mass is midway across the thickness. Still, using Eqs. 9-11 would be challenging because we would need a function for the shape of the plate with its hole, and then we would need to integrate the function in two dimensions.

(3) Here is a much easier way: In working with centers of mass, we can assume that the mass of a uniform object (as we have here) is concentrated in a particle at the object's center of mass. Thus we can treat the object as a particle and avoid any two-dimensional integration.

Calculations: First, put the stamped-out disk (call it disk S) back into place (Fig. 9-4*c*) to form the original composite plate (call it plate C). Because of its circular symmetry, the center of mass com_S for disk S is at the center of S, at $x = -R$ (as shown). Similarly, the center of mass com_C for composite plate C is at the center of C, at the origin (as shown). We then have the following:

Plate	Center of Mass	Location of com	Mass
P	com_P	$x_P = ?$	m_P
S	com_S	$x_S = -R$	m_S
C	com_C	$x_C = 0$	$m_C = m_S + m_P$

Assume that mass m_S of disk S is concentrated in a particle at $x_S = -R$, and mass m_P is concentrated in a particle at x_P (Fig. 9-4*d*). Next we use Eq. 9-2 to find the center of mass x_{S+P} of the two-particle system:

$$x_{S+P} = \frac{m_S x_S + m_P x_P}{m_S + m_P}. \qquad (9\text{-}12)$$

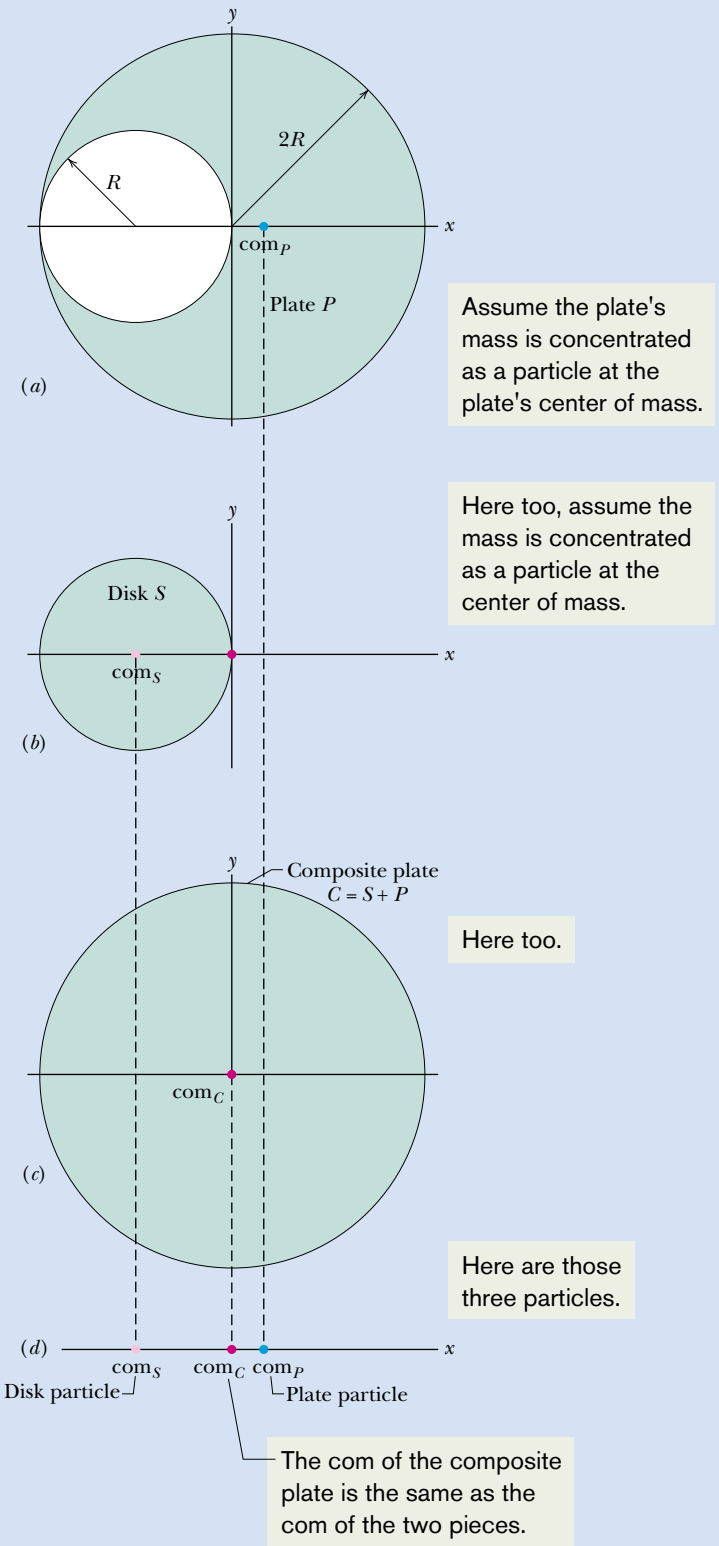

Figure 9-4 (*a*) Plate P is a metal plate of radius $2R$, with a circular hole of radius R. The center of mass of P is at point com_P. (*b*) Disk S. (*c*) Disk S has been put back into place to form a composite plate C. The center of mass com_S of disk S and the center of mass com_C of plate C are shown. (*d*) The center of mass com_{S+P} of the combination of S and P coincides with com_C, which is at $x = 0$.

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Next note that the combination of disk S and plate P is composite plate C. Thus, the position x_{S+P} of com_{S+P} must coincide with the position x_C of com_C, which is at the origin; so $x_{S+P} = x_C = 0$. Substituting this into Eq. 9-12, we get

$$x_P = -x_S \frac{m_S}{m_P}. \tag{9-13}$$

We can relate these masses to the face areas of S and P by noting that

$$\begin{aligned}\text{mass} &= \text{density} \times \text{volume} \\ &= \text{density} \times \text{thickness} \times \text{area}.\end{aligned}$$

Then

$$\frac{m_S}{m_P} = \frac{\text{density}_S}{\text{density}_P} \times \frac{\text{thickness}_S}{\text{thickness}_P} \times \frac{\text{area}_S}{\text{area}_P}.$$

Because the plate is uniform, the densities and thicknesses are equal; we are left with

$$\begin{aligned}\frac{m_S}{m_P} &= \frac{\text{area}_S}{\text{area}_P} = \frac{\text{area}_S}{\text{area}_C - \text{area}_S} \\ &= \frac{\pi R^2}{\pi(2R)^2 - \pi R^2} = \frac{1}{3}.\end{aligned}$$

Substituting this and $x_S = -R$ into Eq. 9-13, we have

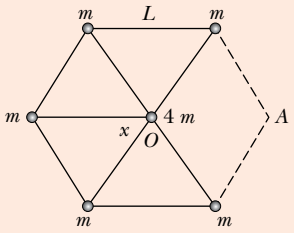
$$x_P = \frac{1}{3}R. \qquad \text{(Answer)}$$

PROBLEM-SOLVING TACTICS

Tactic 1: Center-of-Mass Problems Sample Problems 9.01 and 9.02 provide three strategies for simplifying center-of-mass problems. (1) Make full use of the symmetry of the object, be it about a point, a line, or a plane. (2) If the object can be divided into several parts, treat each of these parts as a particle, located at its own center of mass. (3) Choose your axes wisely: If your system is a group of particles, choose one of the particles as your origin. If your system is a body with a line of symmetry, let that line of symmetry be the x axis or y axis. The choice of origin can be completely arbitrary because the location of the center of mass is the same regardless of the origin from which it is measured.

CHECKPOINT 2

Find location of com of the arrangement of regular hexagon that has point masses at five vertices.



SAMPLE PROBLEM 9.03

com of several thin rods

Figure 9-5*a* shows two thin uniform rods, each of length 0.20 m and mass 0.020 kg. What are the x and y coordinates of the center of mass of the two-rod system?

KEY IDEA

A *uniform* object has a uniform (or even) distribution of mass. The com of a uniform rod must be at the

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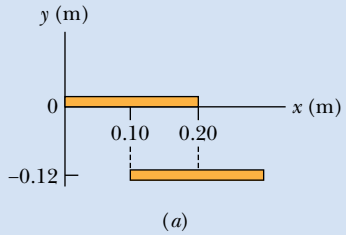

Figure 9-5 (a) Two thin uniform rods of different mass and length.

rod's center. We can pretend that the mass of the rod is concentrated there, and then we can treat the rod as being a particle. The com of the system of two rods is located at the com of the system of two particles.

Calculations: The com of the top rod is located at the (x, y) coordinates (0.10 m, 0), and the com of the bottom rod is located at coordinates (0.20 m, –0.12 m). The associated particles are shown in Fig. 9-5*b*. The x coordinate of the system's com is then given by Eq. 9-2:

$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$
$$= \frac{(0.020\ \text{kg})(0.10\ \text{m}) + (0.020\ \text{kg})(0.20\ \text{m})}{0.020\ \text{kg} + 0.020\ \text{kg}}$$
$$= 0.15\ \text{m}. \qquad \text{(Answer)}$$

Similarly, the y coordinate of the system's com is given by

$$y_{com} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$$
$$= \frac{(0.020\ \text{kg})(0) + (0.020\ \text{kg})(-0.12\ \text{m})}{0.020\ \text{kg} + 0.020\ \text{kg}}$$
$$= -0.05\ \text{m}. \qquad \text{(Answer)}$$

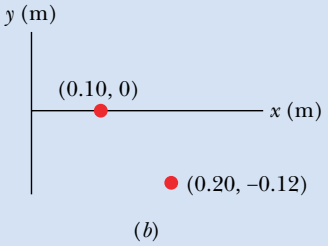

Figure 9-5 (b) The associated particles of two rods.

SAMPLE PROBLEM 9.04

com of uniform plate after a piece is removed

Figure 9-6 shows a uniform square plate with edge length 6.0 m; the plate is centered on the origin of an xy coordinate system. A rectangular portion measuring 1.0 m by 3.0 m is to be removed from the lower left. What is the x component of the remaining plate's com?

KEY IDEA

A uniform distribution of mass can be replaced with a particle of the same mass, located at the distribution's com. In an "easy" geometric shape, the com is at the obvious center of the shape. The com of a system of several particles can be calculated with Eq. 9-5 ($x_{com} = \sum m_1 x_1$) with M being the total mass).

Method 1: The com of the whole plate is located at the origin, at point W in Fig. 9-6. The com of the deleted piece is located at the center of the piece, at point D which has coordinates (–2.5 m, –1.5 m). We want the coordinates of the com of the remaining piece, at point R.

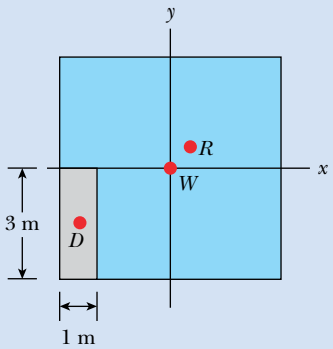

Figure 9-6 Uniform square plate and a rectangular portion removed from the lower left.

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The mass m of any part of the plate is proportional to its face area A. So, for the deleted piece and the whole plate, we have

$$\frac{m_D}{m_W} = \frac{A_D}{A_W} = \frac{3\,\text{m}^2}{36\,\text{m}^2}$$

or

$$m_D = \frac{3}{36} m_W .$$

If (3/36) of the whole-plate mass is deleted, then the remaining piece must have mass $(33/36)m_W$.

Let us go back to the whole plate, which has a com at coordinates $x_{\text{com},W} = 0$ and $y_{\text{com},W} = 0$ and mass m_W. It consists of a piece to be deleted and a remaining piece. The deleted piece can be represented by a particle at (–2.5 m, –1.5 m) and with a mass of $(3/36)m_W$. The remaining piece can be represented by a particle at unknown coordinates (x_R, y_R) and with a mass of $(33/36)m_W$. Equation 9-5 then gives us

$$x_{\text{com},W} = \frac{m_R x_R + m_W x_W}{m_R + m_W}$$

$$0 = \frac{\left(\frac{33}{36} m_W\right) x_R + \left(\frac{3}{36} m_W\right)(-2.4\ \text{m})}{\left(\frac{33}{36} m_W + \frac{3}{36} m_W\right)}$$

$$x_R = 0.23\ \text{m}. \qquad \text{(Answer)}$$

Method 2: Split the remaining plate into four rectangles. Replace each rectangle with a particle at the rectangle's center and with the rectangle's mass. That mass is proportional to the face area. So, first find the face areas:

Piece	com coordinates	Face area
Upper-left square	(–1.5 m, 1.5 m)	9 m²
Upper-right square	(1.5 m, 1.5 m)	9 m²
Lower-right square	(1.5 m, –1.5 m)	9 m²
Lower-left rectangle	(–1.0 m, –1.5 m)	6 m²

The total area of these rectangles is 33 m². Again, mass is proportional to face area. So, the masses of the pieces can be written in terms of the total mass M of the rectangles:

$$\frac{\text{Mass of piece}}{\text{Total mass}} = \frac{\text{Area of piece}}{\text{Total area}}$$

Here are the masses:

Piece	com coordinates	Face area	Mass
Upper-left square	(–1.5 m, 1.5 m)	9 m²	$(9/33)M$
Upper-right square	(1.5 m, 1.5 m)	9 m²	$(9/33)M$
Lower-right square	(1.5 m, –1.5 m)	9 m²	$(9/33)M$
Lower-left rectangle	(–1.0 m, –1.5 m)	6 m²	$(6/33)M$

Finally, we use Eq. 9-5 to find the x component x_R of the com of this system of four particles that represent the four rectangles.

$$x_R = \frac{\left(\frac{9}{33}M\right)(-1.5) + \left(\frac{9}{33}M\right)(1.5) + \left(\frac{9}{33}M\right)(1.5) + \left(\frac{6}{33}M\right)(-1.0)}{M}$$

$$= 0.2. \qquad \text{(Answer)}$$

SAMPLE PROBLEM 9.05

com of uniform objects

Figure 9-7*a* shows two uniform plates made of the same material and having the same thickness (not shown). Figure 9-7*b* shows a uniform, solid cylinder of radius $r = 0.030$ m and height $h = 0.20$ m. Locate the com of the two-plate system and the com of the cylinder.

KEY IDEA

The com of a uniform object is at the center of the object. If a system consists of several uniform objects, they each can be replaced with a particle located at the object's com, as if all the object's mass is concentrated

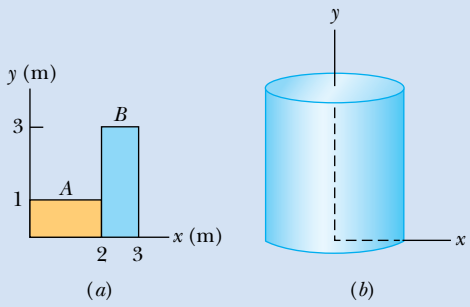

Figure 9-7 Two uniform plates made of the same material.

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there. The com of the system is then calculated in terms of these replacement particles.

Two-plate system: We replace each plate with a particle located at its center. Because the plates are uniform and have the same density and thickness, the mass of each plate is proportional to the face area.

Plate	com location	Area
A	(1.0 m, 0.50 m)	2.0 m^2
B	(2.5 m, 1.50 m)	3.0 m^2

Because the plates are uniform and have the same density and thickness, the mass of each plate is proportional to the face area. The total face area of 5.0 m^2 is proportional to the total mass M.

So, $$\frac{\text{Mass of plate}}{\text{Total mass}} = \frac{\text{Area of plate}}{\text{Total area}}$$

becomes $$\frac{\text{Mass of plate}}{M} = \frac{\text{Area of plate}}{5.0\ \text{m}^2}$$

and we have the following table:

Plate	com location	Area	Mass
A	(1.0 m, 0.50 m)	2.0 m^2	$(2.0/5.0)M$
B	(2.5 m, 1.5 m)	3.0 m^2	$(3.0/5.0)M$

We can now calculate the coordinates x_{com} and y_{com} of the system's com by using the first two parts of Eq. 9-5:

$$x_{com} = \frac{m_A x_A + m_B x_B}{M}$$
$$= \frac{\left(\frac{2.0}{5.0}M\right)(1.0) + \left(\frac{3.0}{5.0}M\right)(2.5)}{M}$$
$$= 1.9\ \text{m}. \qquad \text{(Answer)}$$

$$y_{com} = \frac{m_A y_A + m_B y_B}{M}$$
$$= \frac{\left(\frac{2.0}{5.0}M\right)(0.50) + \left(\frac{3.0}{5.0}M\right)(1.5)}{M}$$
$$= 1.1\ \text{m}. \qquad \text{(Answer)}$$

Cylinder: The com of the cylinder is at the center of the cylinder, at coordinates

$$x_{com} = 0 \quad \text{and} \quad y_{com} = 0.5\,h = 0.10\ \text{m}.$$

SAMPLE PROBLEM 9.06

com of uniform triangular plate

Find the center of mass of the uniform triangular plate that is, shown in Fig. 9-8.

KEY IDEA

Figure 9-8*a* shows the plate divided into thin slats, parallel to one side of the triangle. The key idea here is that, from symmetry, the center of mass of a thin, uniform slat is at its mid-point. Thus, the center of mass of the triangular plate must lie somewhere along the line that connects the midpoints of all slats.

That bisecting line also connects the upper vertex with the midpoint of the opposite side. The plate would balance if it were placed on a knife-edge coinciding with this line of symmetry.

Calculations: In Figs. 9-8*b* and 9-8*c*, we subdivide the plate into slats parallel to the other two sides of the triangle. Again, the center of mass must lie somewhere along each of the bisecting lines shown. Hence the center of mass of the plate must lie at the intersection of these three symmetry lines, as Fig. 9-8*d* shows. That intersection is the only point that the three lines have in common.

You can check this conclusion experimentally by taking advantage of the (correct) intuitive notion that an object suspended from one of its points will orient itself so that its center of mass lies vertically below that point. Suspend the triangle from each vertex in turn, and draw a line vertically downward from the suspension point, as in Fig. 9-8*e*. The center of mass of the triangle will be at the intersection of the three lines.

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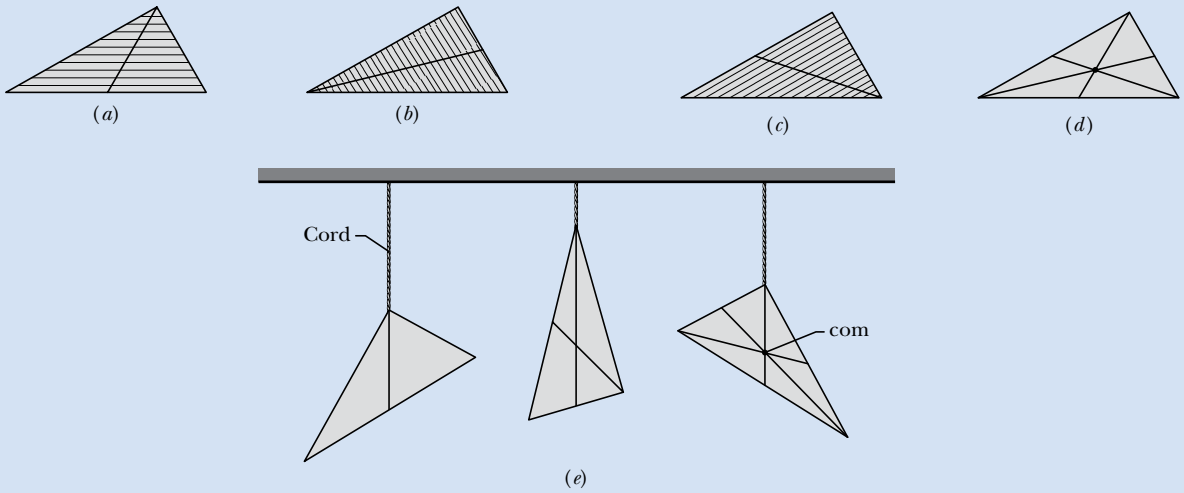

Figure 9-8 In (*a*), (*b*), and (*c*), the triangular plate is divided into thin slats, parallel to one side. The center of mass must lie along the bisecting line shown, (*d*) The dot, the only point common to all three lines, is the position of the center of mass, (*e*) Finding the center of mass by suspending the triangle from each vertex in turn.

SAMPLE PROBLEM 9.07

com of U-shaped object

The U-shaped object pictured in Fig. 9-9 has outer dimensions of 100 mm on each side, and each of its three sides is 20 mm wide. It was cut from a uniform sheet of plastic 6.0 mm thick. Locate the center of mass of this object.

KEY IDEA

Here we break the U-shaped object up into pieces, each having an easily located center of mass of that piece, and then use Eq. 9-5 to locate the center of mass of the whole object.

Calculations: As shown in Fig. 9-9 we can think of the U-shaped object as made up of two vertical bars, each 100 mm long 20 mm wide, joined together by one horizontal bar 60 mm long and 20 mm wide. Let's place the origin of our coordinate system at the lower-left rear corner of the U, with the x axis across its base and the y axis along its left edge. To locate the center of mass of each of the bars, we will use the key idea that the center of mass of a symmetric object of uniform density is located at its geometric center. This means that the center of mass of each bar is exactly halfway from either end, halfway from either side, and halfway between the top and bottom surfaces of the plastic sheet. Putting a dot at the center of each of the bars, A, B and C, we can now write down their locations in the coordinate system pictured there:

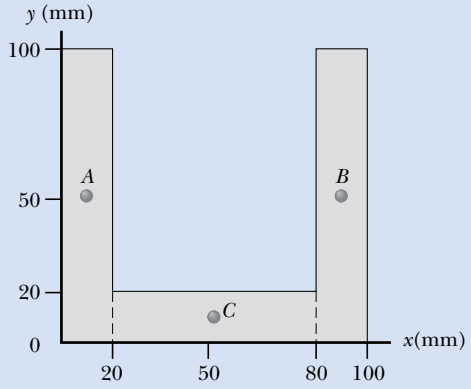

Figure 9-9 U-shaped object.

Object	Mass	X (mm)	Y (mm)	Z (mm)
Left bar	M_A	90	50	3
Right bar	M_B	90	50	3
Bottom bar	M_C	50	10	3

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To learn more about the relative masses of the three bars, we can use the key idea that the mass of each bar is proportional to its volume since the bars have a uniform common density. In this case, the relationship is even simpler: since the three bars are of the same width and thickness, each one's mass is directly proportional to its length, so

$$M_A = M_B = M \quad \text{and} \quad M_C = M(60\text{ mm})/(100\text{ mm}) = 0.6\,M$$
$$\text{and} \quad M_{\text{sys}} = M_A + M_B + M_C = M + M + 0.6\,M = 2.6\,M$$

Replacing each bar by a point mass at its center of mass, Eq. 9-5 gives us the location of the center of mass of the entire U-shaped object.

$$X_{\text{com}} = \frac{M_A X_A + M_B X_B + M_C X_C}{M_{\text{sys}}} = \frac{M(10\text{ mm}) + M(90\text{ mm}) + 0.6M(50\text{ mm})}{2.6M} = 50\text{ mm}$$

$$Y_{\text{com}} = \frac{M_A Y_A + M_B Y_B + M_C Y_C}{M_{\text{sys}}} = \frac{M(50\text{ mm}) + M(50\text{ mm}) + 0.6M(10\text{ mm})}{2.6M} = 41\text{ mm}$$

$$Z_{\text{com}} = \frac{M(3.0\text{ mm}) + M(3.0\text{ mm}) + 0.6M(3.0\text{ mm})}{2.6M} = \frac{(7.8\text{ mm})M}{2.6M} = 3\text{ mm}$$

So, $R_{\text{com}} = (50\text{ mm})\hat{\mathbf{i}} + (41\text{ mm})\hat{\mathbf{j}} + (3\text{ mm})\hat{\mathbf{k}}$ (Answer)

9.3 | NEWTON'S SECOND LAW FOR A SYSTEM OF PARTICLES

Key Concept

- The motion of the center of mass of any system of particles is governed by Newton's second law for a system of particles, which is

$$\vec{F}_{\text{net}} = M\vec{a}_{\text{com}}.$$

Here $\vec{F}_{\text{net}}$ is the net force of all the *external* forces acting on the system, M is the total mass of the system, and $\vec{a}_{\text{com}}$ is the acceleration of the system's center of mass.

Now that we know how to locate the center of mass of a system of particles, we discuss how external forces can move a center of mass. Let us start with a simple system of two billiard balls.

If you roll a cue ball at a second billiard ball that is at rest, you expect that the two-ball system will continue to have some forward motion after impact. You would be surprised, for example, if both balls came back toward you or if both moved to the right or to the left. You already have an intuitive sense that *something* continues to move forward.

What continues to move forward, its steady motion completely unaffected by the collision, is the center of mass of the two-ball system. If you focus on this point—which is always halfway between these bodies because they have identical masses—you can easily convince yourself by trial at a billiard table that this is so. No matter whether the collision is glancing, head-on, or somewhere in between, the center of mass continues to move forward, as if the collision had never occurred. Let us look into this center-of-mass motion in more detail.

Motion of a System's com. To do so, we replace the pair of billiard balls with a system of n particles of (possibly) different masses. We are interested not in the individual motions of these particles but *only* in the motion of the center of mass of the system. Although the center of mass is just a point, it moves like a particle whose mass is equal to the total mass of the system; we can assign a position, a velocity, and an acceleration to it. We state

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(and shall prove next) that the vector equation that governs the motion of the center of mass of such a system of particles is

$$\vec{F}_{\text{net}} = M\vec{a}_{\text{com}} \quad \text{(system of particles)}. \tag{9-14}$$

This equation is Newton's second law for the motion of the center of mass of a system of particles. Note that its form is the same as the form of the equation ($\vec{F}_{\text{net}} = m\vec{a}$) for the motion of a single particle. However, the three quantities that appear in Eq. 9-14 must be evaluated with some care:

1. $\vec{F}_{\text{net}}$ is the net force of *all external forces* that act on the system. Forces on one part of the system from another part of the system (*internal forces*) are not included in Eq. 9-14.
2. M is the *total mass* of the system. We assume that no mass enters or leaves the system as it moves, so that M remains constant. The system is said to be **closed**.
3. $\vec{a}_{\text{com}}$ is the acceleration of the *center of mass* of the system. Equation 9-14 gives no information about the acceleration of any other point of the system.

Equation 9-14 is equivalent to three equations involving the components of $\vec{F}_{\text{net}}$ and $\vec{a}_{\text{com}}$ along the three coordinate axes. These equations are

$$F_{\text{net},x} = Ma_{\text{com},x} \quad F_{\text{net},y} = Ma_{\text{com},y} \quad F_{\text{net},z} = Ma_{\text{com},z}. \tag{9-15}$$

Billiard Balls. Now we can go back and examine the behavior of the billiard balls. Once the cue ball has begun to roll, no net external force acts on the (twoball) system. Thus, because $\vec{F}_{\text{net}} = 0$, Eq. 9-14 tells us that $\vec{a}_{\text{com}} = 0$ also. Because acceleration is the rate of change of velocity, we conclude that the velocity of the center of mass of the system of two balls does not change. When the two balls collide, the forces that come into play are *internal* forces, on one ball from the other. Such forces do not contribute to the net force $\vec{F}_{\text{net}}$, which remains zero. Thus, the center of mass of the system, which was moving forward before the collision, must continue to move forward after the collision, with the same speed and in the same direction.

Solid Body. Equation 9-14 applies not only to a system of particles but also to a solid body, such as the bat of Fig. 9-1*b*. In that case, M in Eq. 9-14 is the mass of the bat and $\vec{F}_{\text{net}}$ is the gravitational force on the bat. Equation 9-14 then tells us that $\vec{a}_{\text{com}} = \vec{g}$. In other words, the center of mass of the bat moves as if the bat were a single particle of mass M, with force $\vec{F}_g$ acting on it.

Exploding Bodies. Figure 9-10 shows another interesting case. Suppose that at a fireworks display, a rocket is launched on a parabolic path. At a certain point, it explodes into fragments. If the explosion had not occurred, the rocket would have continued along the trajectory shown in the figure. The forces of the explosion are *internal* to the system (at first the system is just the rocket, and later it is its fragments); that is, they are forces on parts of the system from other parts. If we ignore air drag, the net *external* force $\vec{F}_{\text{net}}$ acting on the system is the gravitational force on the system, regardless of whether the rocket explodes. Thus, from Eq. 9-14, the acceleration $\vec{a}_{\text{com}}$ of the center of mass of the fragments (while they are in flight) remains equal to $\vec{g}$. This means that the center of mass of the fragments follows the same parabolic trajectory that the rocket would have followed had it not exploded.

The internal forces of the explosion cannot change the path of the com.

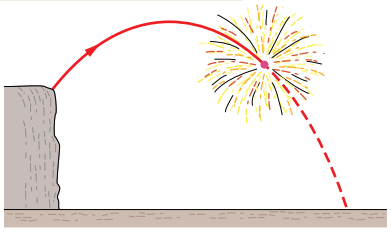

Figure 9-10 A fireworks rocket explodes in flight. In the absence of air drag, the center of mass of the fragments would continue to follow the original parabolic path, until fragments began to hit the ground.

Ballet Leap. When a ballet dancer leaps across the stage in a grand jeté, she raises her arms and stretches her legs out horizontally as soon as her feet leave the stage (Fig. 9-11). These actions shift her center of mass upward through her body. Although the shifting center of mass faithfully follows a parabolic path across the stage, its movement relative to the body decreases the height that is attained by her head and torso, relative to that of a normal jump. The result is that the head and torso follow a nearly horizontal path, giving an illusion that the dancer is floating.

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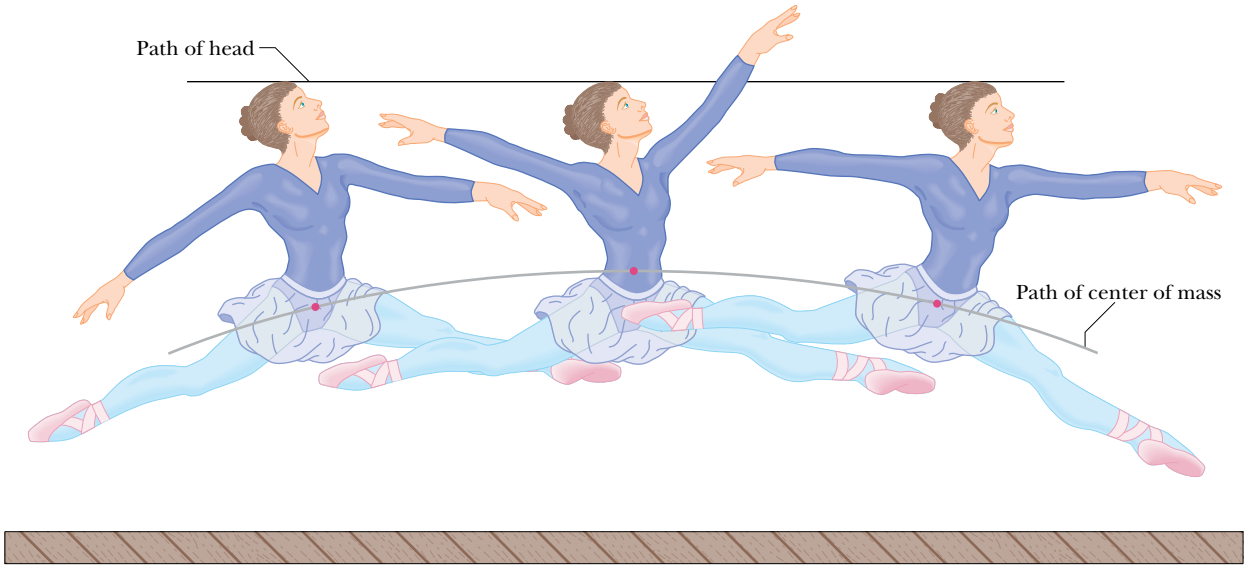

Figure 9-11 A grand jeté. (Based on *The Physics of Dance*, by Kenneth Laws, Schirmer Books, 1984.)

Proof of Eq. 9-14

Now let us prove this important equation. From Eq. 9-8 we have, for a system of n particles,

$$M\vec{r}_{com} = m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots + m_n\vec{r}_n, \tag{9-16}$$

in which M is the system's total mass and $\vec{r}_{com}$ is the vector locating the position of the system's center of mass.

Differentiating Eq. 9-16 with respect to time gives

$$M\vec{v}_{com} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots + m_n\vec{v}_n. \tag{9-17}$$

Here $\vec{v}_i (= d\vec{r}_i/dt)$ is the velocity of the ith particle, and $\vec{v}_{com} (= d\vec{r}_{com}/dt)$ is the velocity of the center of mass.

Differentiating Eq. 9-17 with respect to time leads to

$$M\vec{a}_{com} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \cdots + m_n\vec{a}_n. \tag{9-18}$$

Here $\vec{a}_i (= d\vec{v}_i/dt)$ is the acceleration of the ith particle, and $\vec{a}_{com} (= d\vec{v}_{com}/dt)$ is the acceleration of the center of mass. Although the center of mass is just a geometrical point, it has a position, a velocity, and an acceleration, as if it were a particle.

From Newton's second law, $m_i\vec{a}_i$ is equal to the resultant force $\vec{F}_i$ that acts on the ith particle. Thus, we can rewrite Eq. 9-18 as

$$M\vec{a}_{com} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots + \vec{F}_n. \tag{9-19}$$

Among the forces that contribute to the right side of Eq. 9-19 will be forces that the particles of the system exert on each other (internal forces) and forces exerted on the particles from outside the system (external forces). By Newton's third law, the internal forces form third-law force pairs and cancel out in the sum that appears on the right side of Eq. 9-19. What remains is the vector sum of all the *external* forces that act on the system. Equation 9-19 then reduces to Eq. 9-14, the relation that we set out to prove.

CHECKPOINT 3

Two skaters on frictionless ice hold opposite ends of a pole of negligible mass. An axis runs along it, with the origin at the center of mass of the two-skater system. One skater, Fred, weighs twice as much as the other skater, Ethel. Where do the skaters meet if (a) Fred pulls hand over hand along the pole so as to draw himself to Ethel, (b) Ethel pulls hand over hand to draw herself to Fred, and (c) both skaters pull hand over hand?

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SAMPLE PROBLEM 9.08

Motion of the com of three particles

If the particles in a system all move together, the com moves with them—no trouble there. But what happens when they move in different directions with different accelerations? Here is an example.

The three particles in Fig. 9-12*a* are initially at rest. Each experiences an *external* force due to bodies outside the three-particle system. The directions are indicated, and the magnitudes are $F_1 = 6.0\,\text{N}$, $F_2 = 12\,\text{N}$, and $F_3 = 14\,\text{N}$. What is the acceleration of the center of mass of the system, and in what direction does it move?

KEY IDEAS

The position of the center of mass is marked by a dot in the figure. We can treat the center of mass as if it were a real particle, with a mass equal to the system's total mass $M = 16$ kg. We can also treat the three external forces as if they act at the center of mass (Fig. 9-12*b*).

Calculations: We can now apply Newton's second law ($\vec{F}_{\text{net}} = m\vec{a}$) to the center of mass, writing

$$\vec{F}_{\text{net}} = M\vec{a}_{\text{com}} \tag{9-20}$$

or

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = M\vec{a}_{\text{com}}$$

so

$$\vec{a}_{\text{com}} = \frac{\vec{F}_1 + \vec{F}_2 + \vec{F}_3}{M}. \tag{9-21}$$

Equation 9-20 tells us that the acceleration $\vec{a}_{\text{com}}$ of the center of mass is in the same direction as the net external force $\vec{F}_{\text{net}}$ on the system (Fig. 9-12*b*). Because the particles are initially at rest, the center of mass must also be at rest. As the center of mass then begins to accelerate, it must move off in the common direction of $\vec{a}_{\text{com}}$ and $\vec{F}_{\text{net}}$.

We can evaluate the right side of Eq. 9-21 directly on a vector-capable calculator, or we can rewrite Eq. 9-21 in component form, find the components of $\vec{a}_{\text{com}}$, and then find $\vec{a}_{\text{com}}$. Along the x axis, we have

$$a_{\text{com},x} = \frac{F_{1x} + F_{2x} + F_{3x}}{M}$$
$$= \frac{-6.0\,\text{N} + (12\,\text{N})\cos 45° + 14\,\text{N}}{16\,\text{kg}} = 1.03\,\text{m/s}^2.$$

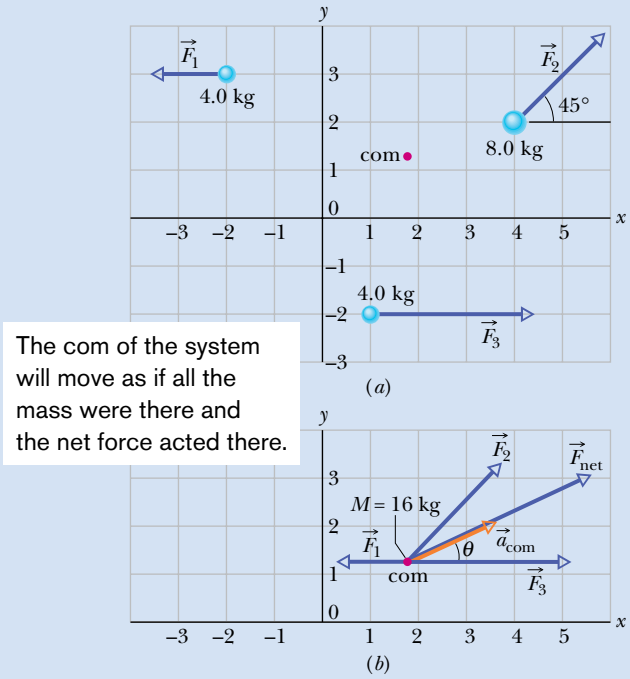

Figure 9-12 (*a*) Three particles, initially at rest in the positions shown, are acted on by the external forces shown. The center of mass (com) of the system is marked. (*b*) The forces are now transferred to the center of mass of the system, which behaves like a particle with a mass M equal to the total mass of the system. The net external force $\vec{F}_{\text{net}}$ and the acceleration $\vec{a}_{\text{com}}$ of the center of mass are shown.

Along the y axis, we have

$$a_{\text{com},y} = \frac{F_{1y} + F_{2y} + F_{3y}}{M}$$
$$= \frac{0 + (12\,\text{N})\sin 45° + 0}{16\,\text{kg}} = 0.530\,\text{m/s}^2.$$

From these components, we find that $\vec{a}_{\text{com}}$ has the magnitude

$$a_{\text{com}} = \sqrt{(a_{\text{com},x})^2 + (a_{\text{com},y})^2}$$
$$= 1.16\,\text{m/s}^2 \approx 1.2\,\text{m/s}^2 \qquad \text{(Answer)}$$

and the angle (from the positive direction of the x axis)

$$\theta = \tan^{-1}\frac{a_{\text{com},y}}{a_{\text{com},x}} = 27°. \qquad \text{(Answer)}$$

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SAMPLE PROBLEM 9.09

Displacement of com of two particle system

At time $t = 0$, forces begin to act on particles A and B, which are initially at rest.

Particle	Mass	Force acting on the particle
A	2.0 kg	$2\hat{i}$
B	4.0 kg	$4\hat{j}$

What is the displacement of the com of the two-particle system at $t = 6.0$ s?

KEY IDEA

We can treat the com of the system as being a single particle with the total mass M of the system and experiencing the net force $\vec{F}_{net}$ of the forces applied to the individual particles. Because the forces are constant, the acceleration $\vec{a}_{com}$ of the com is also and thus the constant-acceleration equations of Chapter 2 can be applied.

Vector calculations: We can recast the constant-acceleration equations into vector notation to find the displacement vector $\vec{r}_{com}$ of the com. In particular, Eq. 2-15 becomes

$$\vec{r}_{com} = \vec{v}_{0,com}\, t + \frac{1}{2}\vec{a}_{com}t^2. \qquad (9\text{-}22)$$

To find $\vec{a}_{com}$, we apply Newton's second law ($\vec{F}_{net} = m\vec{a}_{com}$) to the system's com, as if it were a real particle with mass $M = 6.0$ kg.

$$2.0\hat{i} + 4.0\hat{j} = 6.0\vec{a}_{com}$$

$$\vec{a}_{com} = \left(\frac{2.0}{6.0}\hat{i} + \frac{4.0}{6.0}\hat{j}\right) \text{m/s}^2.$$

Substituting this into Eq. 9-22, along with $t = 6.0$ s and the fact that the particles have initial velocities of zero, we find

$$\vec{r}_{com} = 0t + \frac{1}{2}\left(\frac{2.0}{6.0}\hat{i} + \frac{4.0}{6.0}\hat{j}\right)(6)^2$$

$$= (6.0\ m)\hat{i} + (12.0\ m)\hat{j} \qquad \text{(Answer)}$$

Component calculations: We can also do component calculations. The acceleration of the com has an x component

$$a_x = \frac{F_x}{m} = \frac{2.0}{6.0}\ \text{m/s}^2.$$

Thus, Eq. 2-15 gives

$$r_{com,x} = v_{0,x}t + \frac{1}{2}a_x t^2$$

$$= 0 + \left(\frac{1}{2}\right)\left(\frac{2.0}{6.0}\right)(6)^2 = 6.0\ \text{m}.$$

Similarly, $$a_y = \frac{F_y}{m} = \frac{4.0}{6.0}\ \text{m/s}^2.$$

and we also have

$$r_{com,y} = v_{0,y}t + \frac{1}{2}a_y t^2$$

$$= 0 + \left(\frac{1}{2}\right)\left(\frac{4.0}{6.0}\right)(6)^2 = 12.0\ \text{m}.$$

Putting these components into unit-vector notation gives us (again)

$$\vec{r}_{com} = (6.0\ \text{m})\hat{i} + (12.0\ \text{m})\hat{j}. \qquad \text{(Answer)}$$

SAMPLE PROBLEM 9.10

velocity and acceleration of com of two particle system

Particles A and B move along an x-axis. Particle A has mass $m_A = 4.0$ kg and its position is given by $x_A = 3.0t$ in meters and seconds. Particle B has mass $m_B = 2.0$ kg and its position is given by $x_A = 3.0t + 6.0t^2$. What are the velocity v_{com} and acceleration a_{com} of the com of the two-particle system at $t = 3.0$ s?

KEY IDEA

Velocity is the time derivative of the position function. Acceleration is the time derivative of the velocity function.

Velocity: The position function of the system's com is given by Eq. 9-2:

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$$x_{com} = \frac{m_A x_A + m_B x_B}{m_A + m_B}$$
$$= \frac{(4.0)(3.0)t + (2.0)(3.0 + 6.0t^2)}{4.0 + 2.0}$$
$$= 2.0t + 1.0t + 2.0t^2.$$

Taking a derivative with respect to time, we find

$$v_{com} = \frac{d}{dt} x_{com}$$
$$= 2.0 + 4.0t. \qquad (9\text{-}23)$$

So, at $t = 3.0$ s, the velocity of the com is

$$v_{com} = 14.0 \text{ m/s} \qquad \text{(Answer)}$$

Acceleration: Taking the time derivative of Eq. 9-23, we obtain

$$a_{com} = 4.0 \text{ m/s}^2.$$

Because the result does not depend on t, the acceleration is constant. If we wanted, we could now apply the constant-acceleration equations of Chapter 2 as if the com is a real particle.

SAMPLE PROBLEM 9.11

System com stationary while objects move

Figure 9-13 shows the position of an armadillo at one end of a uniform slab on a frictionless floor. The armadillo has mass $m_1 = 10$ kg; the slab has mass $m_2 = 20$ kg. Relative to an x axis on the floor, the armadillo moves 1.0 m in the positive direction. How far does it move relative to the slab?

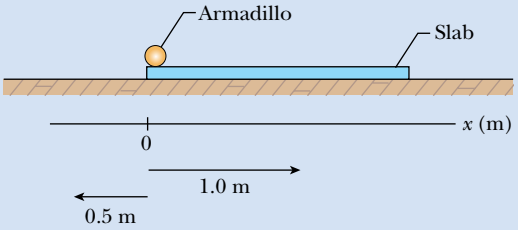

Figure 9-13 Position of an armadillo at one end of a uniform slab.

KEY IDEA

(1) The forces between the armadillo and the slab are internal to the armadillo-slab system and thus cannot move the system's center of mass. (2) The armadillo and the slab can each be replaced with a particle at their individual center of mass, as if all their mass is concentrated in the particles.

Calculations: In the figure, the armadillo particle is represented by a dot at some coordinate x_1. The slab particle is at the center of the slab at some coordinate x_1. Initially, the system com is given by Eq. 9-2:

$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

When the armadillo walks 1.0 m rightward to a new coordinate x_1', the slab's center moves to a new coordinate x_2'. So,

$$x_{com} = \frac{m_1 x_1' + m_2 x_2'}{m_1 + m_2}.$$

Because the system's com cannot move, we have

$$\frac{m_1 x_1' + m_2 x_2'}{m_1 + m_2} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2},$$

which can be manipulated to be

$$m_1(x_1' - x_1) = -m_2(x_2' - x_2).$$

Inserting $m_1 = 10$ kg, $m_2 = 20$ kg, and $x_1' - x_1 = 1.0$ m, we find

$$x_2' - x_2 = -0.50 \text{ m}.$$

Thus, the slab moves 0.50 m leftward in the figure while the armadillo moves 1.0 rightward. The distance the armadillo moves over the slab is therefore

$$1.0 \text{ m} + 0.50 \text{ m} = 1.50 \text{ m}. \qquad \text{(Answer)}$$

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SAMPLE PROBLEM 9.12

Motion of come of system of man on a raft

A man of mass m_1 is located on a raft of mass m_2 which is floating on the surface of a lake. The man travels a distance l on the raft and then stops. The resistance of the water is negligible. Find the corresponding displacement of the raft and displacement of the man relative to the ground.

KEY IDEA

If you have ever walked on a stationary raft or boat, you can imagine that as you go forward the raft goes backward.

Although the problem is given only in terms of the displacements, it cannot be solved by kinematics equations as function of the man's displacement is unknown. In this case, the resultant of all external forces acting on the "man-raft" system is equal to zero; the force of interaction between man and raft is an internal force.

Calculations: The velocity of center of mass of the system will remain zero even when man walks on the raft,

$$m_1\vec{v}_1 + m_2\vec{v}_2 = 0$$

where v_1 and v_2 are the velocities of the man and the raft with respect to the shore.

This means that the position of the center of mass of the given system does not change in the process of motion of the man (and the raft), that is,

$$m_1\vec{r}_1 + m_2\vec{r}_2 = \text{constant}$$

where r_1 and r_2 are the position vectors describing the positions of the center of mass of the man and the raft relative to a certain point on the shore. From this equality, we find the relationship between the change of the vectors r_1 and r_2 as

$$m_1\Delta\vec{r}_1 + m_2\Delta\vec{r}_2 = 0$$

The vectors Δr_1 and Δr_2 represent the displacements of the man and the raft with respect to the shore. Also, the displacement of the man relative to the shore may be represented in the form $\Delta r_1 = \Delta r_2 + \Delta r_1'$, where $\Delta r_1'$ is the displacement of the man relative to the raft, l.

We can find the displacement of the raft using all above relations and rearranging them as

$$\Delta\vec{r}_2 = \frac{m_1}{m_1 + m_2}\Delta r_1'$$

so,

$$\Delta\vec{r}_2 = -\frac{m_1}{m_1 + m_2}l$$

Learn: It can be seen from this that the displacement Δr_2 of the raft does not depend on the nature of the man's motion, that is, on the function $\Delta r_1'(t)$. Thus, he may have walked in any pattern at any speed, the displacement of the raft will be same if he stops at the other end. If we make $m_2 \gg m_1$, then as we expect, answer should be zero because the man will not be able to move the infinitely heavy raft.

SAMPLE PROBLEM 9.13

com of displacement of system of mass sliding down a wedge

A block of mass m slides down the wedge of mass M as shown in Fig. 9-14*a*. Find distance moved by wedge when the block reaches ground. Assume all surfaces are frictionless.

KEY IDEA

(1) It can be imagined with the help of Fig. 9-14*b* that when block slides down on the wedge, the wedge will start moving towards left. The horizontal component of the normal contact force due to the block will push the wedge toward left.

(2) You may want to proceed directly by calculating acceleration and then displacements. In fact if you have time go ahead do it; may be when you have gone through that painful and tedious solution you will be able to appreciate concept of momentum better.

Calculations: Since no external force will act in horizontal direction, so momentum of the system consisting of the block and wedge remains conserved. Since initially both the bodies are at rest, the velocity of center of mass

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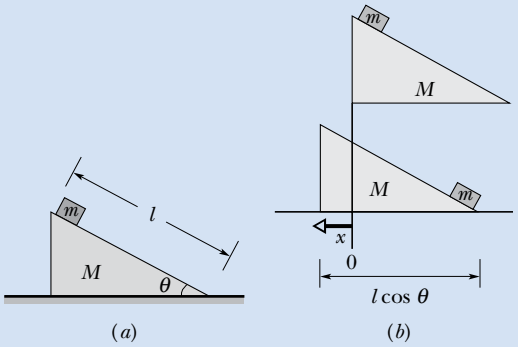

Figure 9-14 (*a*) A block kept on a smooth wedge. (*b*) Movement of block on the wedge and simultaneously the wedge also shifts.

initially is zero and its x component will remain zero. Hence, displacement of the center of mass along x will remain zero. Thus, similar to the previous example, we can say that $M\Delta\vec{x}_1 + m\Delta\vec{x}_2 = 0$.

$\Delta\vec{x}_1$ is the x component of the displacement of the wedge with respect to ground. Considering direction along right to be positive, $\Delta\vec{x}_1 = -x$. $\Delta\vec{x}_2$ is the x component of the displacement of the block with respect to ground, given by $\Delta\vec{x}_2 = (l\cos\theta - x)$.

Putting the values of Δx_1 and Δx_2, we get

$$-Mx + m(l\cos\theta - x) = 0$$

$$ml\cos\theta = x(M+m)$$

$$x = \frac{ml\cos\theta}{(M+m)}$$

Learn: If we make $\theta = 0$, we get same result as in the previous example because the wedge becomes raft. If we make $M \cong m$, then as we expect, answer should be zero because the small block will not be able to move the infinitely heavy wedge.

SAMPLE PROBLEM 9.14

com of a system of missile being fired from a submarine

A missile fired into the air suddenly explodes into several fragments as shown in Fig. 9-15.

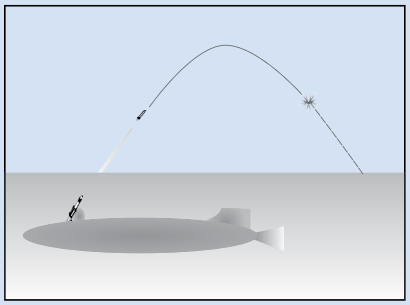

Figure 9-15 A missile being fired from a submarine.

(a) What can be said about the motion of the center of mass of the system after the explosion?

KEY IDEA

Neglecting air resistance, the only external force on the missile is the gravitational force. Therefore, if the missile did not explode, it would continue to move along the parabolic path indicated by the dashed line in the figure and would have landed at a distance R from its launch point.

Calculation: Because the forces caused by the explosion are internal, they will not affect the motion of the center of mass of the system (the pieces). Therefore, even after the explosion, the center of mass of the fragments follows the same parabolic path that the projectile would have followed if no explosion had occurred.

(b) Suppose the explosion splits it into two pieces of mass ratio 1:2. Lighter piece lands at a distance $3R$ from the launch point. Where does the heavier piece land?

Calculation: As discussed in part (a), the center of mass of the two-piece system lands at a distance R from the launch point. One of the pieces lands at a further distance $2R$ from the landing point (or a distance $3R$ from the launch point) to the right in the figure. Applying the equation for locating the center of mass of the two piece system, $Mx_{com} = (M/3)x_1 + (2M/3)x_2$ considering the launch point as origin.

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9.4 | LINEAR MOMENTUM

Key Concept

- For a single particle, we define a quantity $\vec{p}$ called its linear momentum as

$$\vec{p} = m\vec{v},$$

which is a vector quantity that has the same direction as the particle's velocity. We can write Newton's second law in terms of this momentum:

$$\vec{F}_{net} = \frac{d\vec{p}}{dt}.$$

Here we discuss only a single particle instead of a system of particles, in order to define two important quantities. Then we shall extend those definitions to systems of many particles.

The first definition concerns a familiar word—*momentum*—that has several meanings in everyday language but only a single precise meaning in physics and engineering. The **linear momentum** of a particle is a vector quantity $\vec{p}$ that is defined as

$$\vec{p} = m\vec{v} \quad \text{(linear momentum of a particle)}, \tag{9-24}$$

in which m is the mass of the particle and $\vec{v}$ is its velocity. (The adjective *linear* is often dropped, but it serves to distinguish $\vec{p}$ from *angular* momentum, which is introduced in Chapter 11 and which is associated with rotation.) Since m is always a positive scalar quantity, Eq. 9-24 tells us that $\vec{p}$ and $\vec{v}$ have the same direction. From Eq. 9-24, the SI unit for momentum is the kilogram-meter per second (kg · m/s).

Force and Momentum. Newton expressed his second law of motion in terms of momentum:

> The time rate of change of the momentum of a particle is equal to the net force acting on the particle and is in the direction of that force.

In equation form this becomes

$$\vec{F}_{net} = \frac{d\vec{p}}{dt}. \tag{9-25}$$

In words, Eq. 9-25 says that the net external force $\vec{F}_{net}$ on a particle changes the particle's linear momentum $\vec{p}$. Conversely, the linear momentum can be changed only by a net external force. If there is no net external force, $\vec{p}$ *cannot* change. As we shall see in Section 9.7, this last fact can be an extremely powerful tool in solving problems.

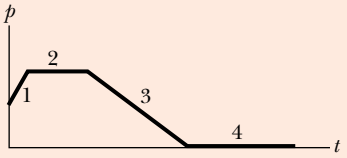
Manipulating Eq. 9-25 by substituting for $\vec{p}$ from Eq. 9-24 gives, for constant mass m,

$$\vec{F}_{net} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} = m\vec{a}.$$

Thus, the relations $\vec{F}_{net} = d\vec{p}/dt$ and $\vec{F}_{net} = m\vec{a}$ are equivalent expressions of Newton's second law of motion for a particle.

CHECKPOINT 4

The figure gives the magnitude p of the linear momentum versus time t for a particle moving along an axis. A force directed along the axis acts on the particle. (a) Rank the four regions indicated according to the magnitude of the force, greatest first. (b) In which region is the particle slowing?



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9.5 | THE LINEAR MOMENTUM OF A SYSTEM OF PARTICLES

Key Concept

- For a system of particles, the relations for linear momentum and force become

$$\vec{P} = M\vec{v}_{com} \quad \text{and} \quad \vec{F}_{net} = \frac{d\vec{P}}{dt}.$$

Let's extend the definition of linear momentum to a system of particles. Consider a system of n particles, each with its own mass, velocity, and linear momentum. The particles may interact with each other, and external forces may act on them. The system as a whole has a total linear momentum $\vec{P}$, which is defined to be the vector sum of the individual particles' linear momenta. Thus,

$$\begin{aligned}\vec{P} &= \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots + \vec{p}_n \\ &= m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots + m_n\vec{v}_n.\end{aligned} \tag{9-26}$$

If we compare this equation with Eq. 9-17, we see that

$$\vec{P} = M\vec{v}_{com} \quad \text{(linear momentum, system of particles)}, \tag{9-27}$$

which is another way to define the linear momentum of a system of particles:

The linear momentum of a system of particles is equal to the product of the total mass M of the system and the velocity of the center of mass.

Force and Momentum. If we take the time derivative of Eq. 9-27 (the velocity can change but not the mass), we find

$$\frac{d\vec{P}}{dt} = M\frac{d\vec{v}_{com}}{dt} = M\vec{a}_{com}. \tag{9-28}$$

Comparing Eqs. 9-14 and 9-28 allows us to write Newton's second law for a system of particles in the equivalent form

$$\vec{F}_{net} = \frac{d\vec{P}}{dt} \quad \text{(system of particles)}, \tag{9-29}$$

where $\vec{F}_{net}$ is the net external force acting on the system. This equation is the generalization of the single-particle equation $\vec{F}_{net} = d\vec{P}/dt$ to a system of many particles. In words, the equation says that the net external force $\vec{F}_{net}$ on a system of particles changes the linear momentum $\vec{P}$ of the system. Conversely, the linear momentum can be changed only by a net external force. If there is no net external force, $\vec{P}$ *cannot* change. Again, this fact gives us an extremely powerful tool for solving problems.

SAMPLE PROBLEM 9.15

Obtaining momentum vector from time-dependent position vector

A 0.200 kg particle's position is given by

$$\vec{r} = 50t\hat{i} - 20t^2\hat{j}$$

in meters and seconds. What is the particle's momentum $\vec{p}$?

KEY IDEA

Momentum is the product of mass and velocity as given by Eq. 9-22 ($\vec{p} = m\vec{v}$).

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Calculations: We obtain the velocity from the position vector by taking a derivative with respect to time:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(50t\hat{\mathrm{i}} - 20t^2\hat{\mathrm{j}}) = 50\hat{\mathrm{i}} - 40t\hat{\mathrm{j}}$$

in meters per second. The momentum is then

$$\vec{p} = m\vec{v} = (0.200\ \mathrm{kg})(50\hat{\mathrm{i}} - 40t\hat{\mathrm{j}}) = (10\hat{\mathrm{i}} - 8.0t\hat{\mathrm{j}})\ \mathrm{kg \cdot m/s} \qquad \text{(Answer)}$$

SAMPLE PROBLEM 9.16

Boat moving at constant speed, change of momentum

A boat is moving at constant speed through the water. The passengers on the boat want to arrive at the shore early. They decide to try to speed up the boat by gathering at the front and running together toward the back of the boat.

(1) While they are running toward the back, is the speed of the boat (*a*) higher than it was before, (*b*) unchanged, (*c*) lower than it was before, or (*d*) impossible to determine? (2) The passengers stop running when they reach the back of the boat. After they have all stopped running, is the speed of the boat (*a*) higher than it was before they started running, (*b*) unchanged from what it was before they started running, (*c*) lower than it was before they started running, or (*d*) impossible to determine?

Calculation 1: This effect is the same one as the man walking on the raft that we just discussed. The boat—passengers system is isolated. If all the passengers start running one way, the speed of the boat increases the other way. So, correct answer is (a).

Calculation 2: Once the passengers stop running, the momentum of the system is the same as it was before they started running; you cannot change the momentum of an isolated system by means of internal forces. So, correct answer is (b).

Reasoning: One may think that the passengers could run to the back of the boat repeatedly to take advantage of the speed increase. However, it must be remembered that they will slow the ship down every time they return to the front of the boat.

CHECKPOINT 5

In Sample Problem 9.16, what happens to the work done by the passengers?

Internal Forces and Momentum of System

If we pull a rod from one end (Fig. 9-16*a*) then at any point on the rod at a distance x from the left end, there will be a tension (internal force) driving the remaining body (Fig. 9-16*b*).

$$T = \frac{F}{m} \times \left[\left(\frac{m}{l}\right)x\right].$$

From Sample Problem 9.16, we can understand that momentum of a system depends only on external forces and internal forces cannot affect net momentum of system. But internal forces can still contribute in changing energy of system

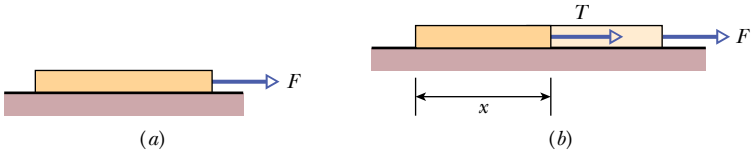

Figure 9-16 (*a*) Rod being pulled by a force. (*b*) Free-body diagram of left part of the rod.

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and momentum of parts of system. For example, in chin—up exercise on horizontal bar, force is required to generate momentum in vertical direction but it is not doing work. Increase in potential energy during rising up is due to work done by internal force of body. But internal forces alone cannot change momentum of the body.

SAMPLE PROBLEM 9.17

Man lying on a surface of negligible friction shoves stone

A 91 kg man lying on a surface of negligible friction shoves a 68 g stone away from himself, giving it a speed of 4.0 m/s. What speed does the man acquire as a result?

KEY IDEA

This problem deals with momentum conservation. Since no external forces with horizontal components act on the man-stone system and the vertical forces sum to zero, the total momentum of the system is conserved.

Calculations: Since the man and the stone are initially at rest, the total momentum is zero both before and after the stone is kicked. Let m_s be the mass of the stone and v_s be its velocity after it is kicked. Also, let m_m be the mass of the man and v_m be his velocity after he kicks the stone. Then, by momentum conservation,

$$m_s v_s + m_m v_m = 0 \quad \Rightarrow \quad v_m = \frac{m_s}{m_m} v_s.$$

We take the axis to be positive in the direction of motion of the stone. Then

$$v_m = -\frac{m_s}{m_m} v_s = -\frac{0.068 \text{ kg}}{91 \text{ kg}} (4.0 \text{ m/s})$$
$$= -3.0 \times 10^{-3} \text{ m/s}$$

or

$$|v_m| = 3.0 \times 10^{3} \text{ m/s}.$$

Reasoning: The negative sign in v_m indicates that the man moves in the direction opposite to the motion of the stone. Note that his speed is much smaller (by a factor of m_s/m_m) compared to the speed of the stone.

9.6 | IMPULSE AND MOMENTUM

Key Concepts

- Applying Newton's second law in momentum form to a particle-like body involved in a collision leads to the impulse–linear momentum theorem:

$$\vec{p}_f - \vec{p}_i = \Delta\vec{p} = \vec{J},$$

where $\vec{p}_f - \vec{p}_i = \Delta\vec{p}$ is the change in the body's linear momentum, and $\vec{J}$ is the impulse due to the force $\vec{F}(t)$ exerted on the body by the other body in the collision:

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)\, dt.$$

- If F_{avg} is the average magnitude of $\vec{F}(t)$ during the collision and Δt is the duration of the collision, then for one-dimensional motion

$$J = F_{avg}\, \Delta t.$$

- When a steady stream of bodies, each with mass m and speed v, collides with a body whose position is fixed, the average force on the fixed body is

$$F_{avg} = -\frac{n}{\Delta t} \Delta p = -\frac{n}{\Delta t} m\, \Delta v,$$

where $n/\Delta t$ is the rate at which the bodies collide with the fixed body, and Δv is the change in velocity of each colliding body. This average force can also be written as

$$F_{avg} = -\frac{\Delta m}{\Delta t} \Delta v,$$

where $\Delta m/\Delta t$ is the rate at which mass collides with the fixed body. The change in velocity is $\Delta v = -v$ if the bodies stop upon impact and $\Delta v = -2v$ if they bounce directly backward with no change in their speed.

The momentum $\vec{p}$ of any particle-like body cannot change unless a net external force changes it. For example, we could push on the body to change its momentum. More dramatically, we could arrange for the body to collide with a baseball bat. In such a *collision* (or *crash*), the external force on the body is brief, has large magnitude, and suddenly

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The collision of a ball with a bat collapses part of the ball.

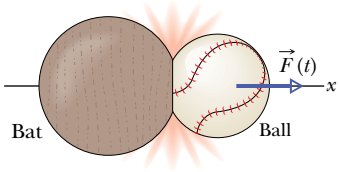

Figure 9-17 Force $\vec{F}(t)$ acts on a ball as the ball and a bat collide.

changes the body's momentum. Collisions occur commonly in our world, but before we get to them, we need to consider a simple collision in which a moving particle-like body (a *projectile*) collides with some other body (a *target*).

Single Collision

Let the projectile be a ball and the target be a bat. The collision is brief, and the ball experiences a force that is great enough to slow, stop, or even reverse its motion. Figure 9-17 depicts the collision at one instant. The ball experiences a force $\vec{F}(t)$ that varies during the collision and changes the linear momentum $\vec{p}$ of the ball. That change is related to the force by Newton's second law written in the form $\vec{F} = d\vec{p}/dt$. By rearranging this second-law expression, we see that, in time interval dt, the change in the ball's momentum is

$$d\vec{p} = \vec{F}(t)\,dt. \tag{9-30}$$

We can find the net change in the ball's momentum due to the collision if we integrate both sides of Eq. 9-30 from a time t_i just before the collision to a time t_f just after the collision:

$$\int_{t_i}^{t_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F}(t)\,dt. \tag{9-31}$$

The left side of this equation gives us the change in momentum: $\vec{p}_f - \vec{p}_i = \Delta\vec{p}$. The right side, which is a measure of both the magnitude and the duration of the collision force, is called the **impulse** $\vec{J}$ of the collision:

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)\,dt \quad \text{(impulse defined)}. \tag{9-32}$$

Thus, the change in an object's momentum is equal to the impulse on the object:

$$\Delta\vec{p} = \vec{J} \quad \text{(linear momentum–impulse theorem)}. \tag{9-33}$$

This expression can also be written in the vector form

$$\vec{p}_f - \vec{p}_i = \vec{J} \tag{9-34}$$

and in such component forms as

$$\Delta p_x = J_x \tag{9-35}$$

and

$$p_{fx} - p_{ix} = \int_{t_i}^{t_f} F_x\,dt. \tag{9-36}$$

Integrating the Force. If we have a function for $\vec{F}(t)$, we can evaluate $\vec{J}$ (and thus the change in momentum) by integrating the function. If we have a plot of $\vec{F}$ versus time t, we can evaluate $\vec{J}$ by finding the area between the curve and the t axis, such as in Fig. 9-18*a*. In many situations we do not know how the force varies with time but we do know the average magnitude F_{avg} of the force and the duration $\Delta t\,(=t_f - t_i)$ of the collision. Then we can write the magnitude of the impulse as

$$J = F_{avg}\,\Delta t. \tag{9-37}$$

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The average force is plotted versus time as in Fig. 9-18*b*. The area under that curve is equal to the area under the curve for the actual force $F(t)$ in Fig. 9-18*a* because both areas are equal to impulse magnitude J.

Instead of the ball, we could have focused on the bat in Fig. 9-17. At any instant, Newton's third law tells us that the force on the bat has the same magnitude but the opposite direction as the force on the ball. From Eq. 9-32, this means that the impulse on the bat has the same magnitude but the opposite direction as the impulse on the ball.

CHECKPOINT 6

A paratrooper whose chute fails to open lands in snow; he is hurt slightly. Had he landed on bare ground, the stopping time would have been 10 times shorter and the collision lethal. Does the presence of the snow increase, decrease, or leave unchanged the values of (a) the paratrooper's change in momentum, (b) the impulse stopping the paratrooper, and (c) the force stopping the paratrooper?

Series of Collisions

Now let's consider the force on a body when it undergoes a series of identical, repeated collisions. For example, as a prank, we might adjust one of those machines that fire tennis balls to fire them at a rapid rate directly at a wall. Each collision would produce a force on the wall, but that is not the force we are seeking. We want the average force F_{avg} on the wall during the bombardment—that is, the average force during a large number of collisions.

In Fig. 9-19, a steady stream of projectile bodies, with identical mass m and linear momenta $m\vec{v}$, moves along an x axis and collides with a target body that is fixed in place. Let n be the number of projectiles that collide in a time interval Δt. Because the motion is along only the x axis, we can use the components of the momenta along that axis. Thus, each projectile has initial momentum mv and undergoes a change Δp in linear momentum because of the collision. The total change in linear momentum for n projectiles during interval Δt is $n\ \Delta p$. The resulting impulse $\vec{J}$ on the target during Δt is along the x axis and has the same magnitude of $n\ \Delta p$ but is in the opposite direction. We can write this relation in component form as

$$J = -n\ \Delta p, \tag{9-38}$$

where the minus sign indicates that J and Δp have opposite directions.

Average Force. By rearranging Eq. 9-37 and substituting Eq. 9-38, we find the average force F_{avg} acting on the target during the collisions:

$$F_{avg} = \frac{J}{\Delta t} = -\frac{n}{\Delta t}\Delta p = -\frac{n}{\Delta t} m\Delta v. \tag{9-39}$$

This equation gives us F_{avg} in terms of $n/\Delta t$, the rate at which the projectiles collide with the target, and Δv, the change in the velocity of those projectiles.

Velocity Change. If the projectiles stop upon impact, then in Eq. 9-39 we can substitute, for Δv,

$$\Delta v = v_f - v_i = 0 - v = -v, \tag{9-40}$$

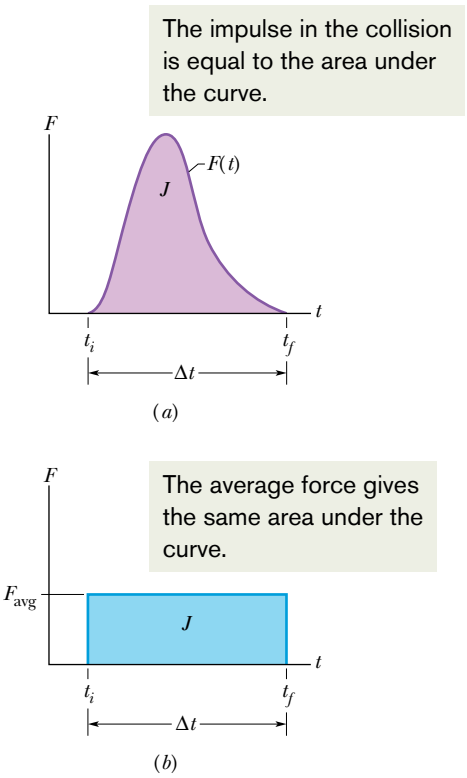

Figure 9-18 (*a*) The curve shows the magnitude of the time-varying force $F(t)$ that acts on the ball in the collision of Fig. 9-13. The area under the curve is equal to the magnitude of the impulse $\vec{J}$ on the ball in the collision. (*b*) The height of the rectangle represents the average force F_{avg} acting on the ball over the time interval Δt. The area within the rectangle is equal to the area under the curve in (*a*) and thus is also equal to the magnitude of the impulse $\vec{J}$ in the collision.

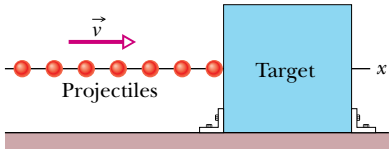

Figure 9-19 A steady stream of projectiles, with identical linear momenta, collides with a target, which is fixed in place. The average force F_{avg} on the target is to the right and has a magnitude that depends on the rate at which the projectiles collide with the target or, equivalently, the rate at which mass collides with the target.

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where $v_i (=v)$ and $v_f (=0)$ are the velocities before and after the collision, respectively. If, instead, the projectiles bounce (rebound) directly backward from the target with no change in speed, then $v_f = -v$ and we can substitute

$$\Delta v = v_f - v_i = -v - v = -2v. \tag{9-41}$$

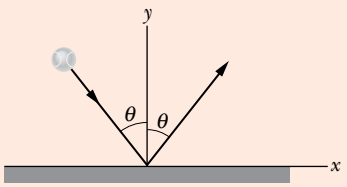
In time interval Δt, an amount of mass $\Delta m = nm$ collides with the target. With this result, we can rewrite Eq. 9-39 as

$$F_{avg} = -\frac{\Delta m}{\Delta t}\Delta v. \tag{9-42}$$

This equation gives the average force F_{avg} in terms of $\Delta m/\Delta t$, the rate at which mass collides with the target. Here again we can substitute for Δv from Eq. 9-40 or 9-41 depending on what the projectiles do.

CHECKPOINT 7

The figure shows an overhead view of a ball bouncing from a vertical wall without any change in its speed. Consider the change $\Delta\vec{p}$ in the ball's linear momentum. (a) Is Δp_x positive, negative, or zero? (b) Is Δp_y positive, negative, or zero? (c) What is the direction of $\Delta\vec{p}$?



SAMPLE PROBLEM 9.18

Two-dimensional impulse, race car–wall collision

Figure 9-20*a* is an overhead view of the path taken by a race car driver as his car collides with the racetrack wall. Just before the collision, he is traveling at speed $v_i = 70$ m/s along a straight line at 30° from the wall. Just after the collision, he is traveling at speed $v_f = 50$ m/s along a straight line at 10° from the wall. His mass m is 80 kg.

(a) What is the impulse $\vec{J}$ on the driver due to the collision?

KEY IDEAS

We can treat the driver as a particle-like body and thus apply the physics of this module. However, we cannot calculate $\vec{J}$ directly from Eq. 9-32 because we do not know anything about the force $\vec{F}(t)$ on the driver during the collision. That is, we do not have a function of $\vec{F}(t)$ or a plot for it and thus cannot integrate to find $\vec{J}$. However, we *can* find $\vec{J}$ from the change in the driver's linear momentum $\vec{p}$ via Eq. 9-34 ($\vec{J} = \vec{p}_f - \vec{p}_i$).

Calculations: Figure 9-20*b* shows the driver's momentum $\vec{p}_i$ before the collision (at angle 30° from the positive x direction) and his momentum $\vec{p}_f$ after the collision (at angle −10°). From Eqs. 9-34 and 9-24 ($\vec{p} = m\vec{v}$), we can write

$$\vec{J} = \vec{p}_f - \vec{p}_i = m\vec{v}_f - m\vec{v}_i = m(\vec{v}_f - \vec{v}_i). \tag{9-43}$$

We could evaluate the right side of this equation directly on a vector-capable calculator because we know m is 80 kg, $\vec{v}_f$ is 50 m/s at −10°, and $\vec{v}_i$ is 70 m/s at 30°. Instead, here we evaluate Eq. 9-43 in component form.

x component: Along the x axis we have

$$\begin{aligned} J_x &= m(v_{fx} - v_{ix}) \\ &= (80\text{ kg})[(50\text{ m/s})\cos(-10°) - (70\text{ m/s})\cos 30°] \\ &= -910\text{ kg·m/s} \end{aligned}$$

y component: Along the y axis,

$$\begin{aligned} J_y &= m(v_{fy} - v_{iy}) \\ &= (80\text{ kg})[(50\text{ m/s})\sin(-10°) - (70\text{ m/s})\sin 30°] \\ &= -3495\text{ kg·m/s} \approx -3500\text{ kg·m/s}. \end{aligned}$$

Impulse: The impulse is then

$$\vec{J} = (-910\hat{i} - 3500\hat{j})\text{ kg·m/s}, \qquad \text{(Answer)}$$

which means the impulse magnitude is

$$J = \sqrt{J_x^2 + J_y^2} = 3616\text{ kg·m/s} \approx 3600\text{ kg·m/s}.$$

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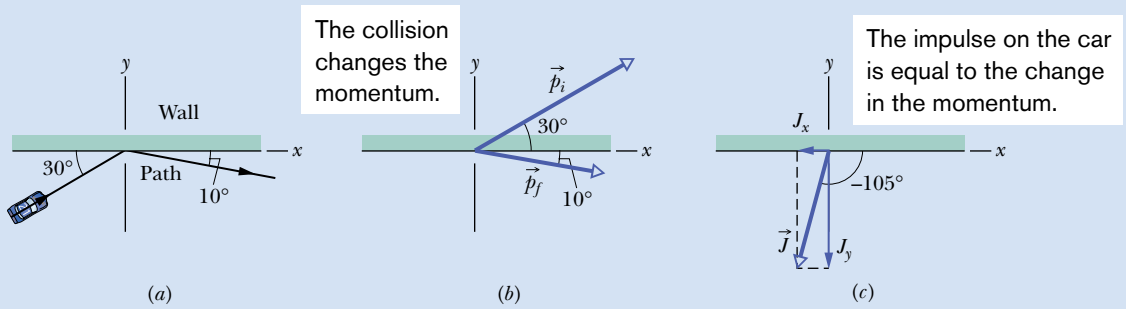

Figure 9-20 (*a*) Overhead view of the path taken by a race car and its driver as the car slams into the racetrack wall. (*b*) The initial momentum $\vec{p}_i$ and final momentum $\vec{p}_f$ of the driver. (*c*) The impulse $\vec{J}$ on the driver during the collision.

The angle of $\vec{J}$ is given by

$$\theta = \tan^{-1}\frac{J_y}{J_x}, \qquad \text{(Answer)}$$

which a calculator evaluates as 75.4°. Recall that the physically correct result of an inverse tangent might be the displayed answer plus 180°. We can tell which is correct here by drawing the components of $\vec{J}$ (Fig. 9-20*c*). We find that θ is actually 75.4° + 180° = 255.4°, which we can write as

$$\theta = -105°. \qquad \text{(Answer)}$$

(b) The collision lasts for 14 ms. What is the magnitude of the average force on the driver during the collision?

KEY IDEA

From Eq. 9-37 ($J = F_{avg}\,\Delta t$), the magnitude F_{avg} of the average force is the ratio of the impulse magnitude J to the duration Δt of the collision.

Calculations: We have

$$F_{avg} = \frac{J}{\Delta t} = \frac{3616\ \text{kg}\cdot\text{m/s}}{0.014\ \text{s}}$$
$$= 2.583\times10^5\ \text{N} \simeq 2.6\times10^5\ \text{N}. \qquad \text{(Answer)}$$

Using $F = ma$ with $m = 80$ kg, you can show that the magnitude of the driver's average acceleration during the collision is about 3.22×10^3 m/s^2 = 329*g*, which is fatal.

Surviving: Mechanical engineers attempt to reduce the chances of a fatality by designing and building racetrack walls with more "give," so that a collision lasts longer. For example, if the collision here lasted 10 times longer and the other data remained the same, the magnitudes of the average force and average acceleration would be 10 times less and probably survivable.

SAMPLE PROBLEM 9.19

Moving car, impulse, average force

A 1400 kg car moving at 5.3 m/s is initially traveling north along the positive direction of a *y* axis. After completing a 90° right-hand turn in 4.6 s, the inattentive operator drives into a tree, which stops the car in 350 ms. In unit-vector notation, what is the impulse on the car (a) due to the turn and (b) due to the collision? What is the magnitude of the average force that acts on the car (c) during the turn and (d) during the collision? (e) What is the direction of the average force during the turn?

KEY IDEA

The momentum of the car changes as it turns and collides with a tree. Let the initial and final momenta of the car be $\vec{p}_i = m\vec{v}_i$ $m\vec{v}_i$ and $\vec{p}_f = m\vec{v}_f$, respectively. The impulse on it equals the change in its momentum:

$$\vec{J} = \Delta\vec{p} = \vec{p}_f - \vec{p}_i = m(\vec{v}_f - \vec{v}_i).$$

The average force over the duration Δt is given by $F_{avg} = \vec{J}/\Delta t$.

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Calculations: (a) The initial momentum of the car is

$$\vec{p}_i = m\vec{v}_i = (1400\text{ kg})(5.3\text{ m/s})\hat{j}.$$

and the final momentum after making the turn is $\vec{p}_f = (7400\text{ kg}\cdot\text{m/s})\hat{j}$ (note that the magnitude remains the same, only the direction is changed). Thus, the impulse is

$$\vec{J} = \vec{p}_f - \vec{p}_i = (7.4\times10^3\text{ N}\cdot\text{s})(\hat{i}-\hat{j}).$$

(b) The initial momentum of the car after the turn is $p_i' = (7400\text{ kg}\cdot\text{m/s})\hat{j}$ and the final momentum after colliding with a tree is $\vec{p}_f' = 0$. The impulse acting on it is

$$\vec{J}' = \vec{p}_f - \vec{p}_i = (-7.4\times10^3\text{ N}\cdot\text{s})\hat{i}.$$

(c) The average force on the car during the turn is

$$\vec{F}_{avg} = \frac{\Delta\vec{p}}{\Delta t} = \frac{\vec{J}}{\Delta t} = \frac{(7400\text{ kg}\cdot\text{m/s})(\hat{i}-\hat{j})}{4.6\text{ s}} = (1600\text{ N})(\hat{i}-\hat{j})$$

and its magnitude is

$$F_{avg} = (1600\text{ N})\sqrt{2} = 2.3\times10^3\text{ N}.$$

(d) The average force during the collision with the tree is

$$\vec{F}_{avg}' = \frac{\Delta\vec{p}}{\Delta t} = \frac{\vec{J}'}{\Delta t} = \frac{(-7400\text{ kg}\cdot\text{m/s})\hat{i}}{350\times10^{-3}\text{ s}} = (-2.1\times10^4\text{ N})\hat{i}$$

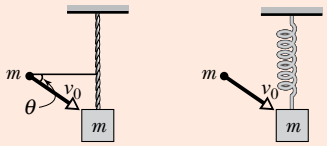
and its magnitude is $\vec{F}_{avg}' = 2.1\times10^4$ N.

(e) As shown in (c), the average force during the turn, in unit vector notation, is $\vec{F}_{avg} = (1600\text{ N})(\hat{i}-\hat{j})$. The force is 45° below the positive x axis.

Reasoning: During the turn, the average force $\vec{F}_{avg}$ is in the same direction as $\vec{J}$, or $\Delta\vec{p}$. Its x and y components have equal magnitudes. The x component is positive and the y component is negative, so the force is 45° below the positive x axis.

CHECKPOINT 8

A block of mass m is hanging from an inextensible string and another block of mass m is hanging from a spring as shown. When a particle of mass m is fired towards the block with velocity v_0, it immediately sticks to the larger block. Find the velocity of combined mass immediately after this collision in both the cases?



SAMPLE PROBLEM 9.20

Force on the wall due to avalanche

An engineer has been hired to design wall to protect a hillside home from a possible snow avalanche. The engineer makes these assumptions: Snow in an avalanche has a density ρ of 500 kg/m^3, would hit the wall perpendicularly at a speed of 20 m/s and then stop, and would hit it at the rate of 1000 m^3/s. For these assumptions, what would be the magnitude of the average force on the wall from the snow?

KEY IDEAS

The snow hits the wall as a steady stream of bodies, as in Fig. 9-19.

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Thus, we can use Eq. 9-42 to find the magnitude of the average force on the wall. However, because the snow hits in a continuous flow, we take a differential limit of $\Delta m/\Delta t$ and write

$$F_{avg} = -\frac{dm}{dt}\Delta v \tag{9-44}$$

Calculations: Because the snow stops on hitting the wall, we use Eq. 9-40 ($\Delta v = -v$) for the change Δv in the velocity of the snow. We are given the volume rate $dV/dt = 1000\ m^3/s$ at which snow collides with the wall. We can find the mass rate dm/dt from

$$\frac{\text{mass}}{\text{second}} = \frac{\text{mass}}{\text{volume}} \times \frac{\text{volume}}{\text{second}}$$

or,

$$\frac{dm}{dt} = \rho\frac{dV}{dt}$$

Substituting $-v$ for Δv and $\rho\, dV/dt$ for dm/dt in Eq. 9-44, we find

$$F_{avg} = \rho\frac{dV}{dt} = (500\ kg/m^3)(20\ m/s)(1000\ m^3/s)$$
$$= 10\times10^7 N \qquad \text{(Answer)}$$

The answer suggests why avalanches typically flatten homes and other structures in their paths.

9.7 | CONSERVATION OF LINEAR MOMENTUM

Key Concepts

- If a system is closed and isolated so that no net *external* force acts on it, then the linear momentum $\vec{P}$ must be constant even if there are internal changes:

  $\vec{P} = \text{constant}$ (closed, isolated system).

- This conservation of linear momentum can also be written in terms of the system's initial momentum and its momentum at some later instant:

  $\vec{P}_i = \vec{P}_f$ (closed, isolated system).

Suppose that the net external force $\vec{F}_{net}$ (and thus the net impulse $\vec{J}$) acting on a system of particles is zero (the system is isolated) and that no particles leave or enter the system (the system is closed). Putting $\vec{F}_{net} = 0$ in Eq. 9-29 then yields $d\vec{P}/dt = 0$, which means that

$$\vec{P} = \text{constant} \quad \text{(closed, isolated system).} \tag{9-45}$$

In words,

If no net external force acts on a system of particles, the total linear momentum $\vec{P}$ of the system cannot change.

This result is called the **law of conservation of linear momentum** and is an extremely powerful tool in solving problems. For solving problems we usually write the law as

$$\vec{P}_i = \vec{P}_f \quad \text{(closed, isolated system).} \tag{9-46}$$

In words, this equation says that, for a closed, isolated system,

$$\begin{pmatrix}\text{total linear momentum}\\ \text{at some initial time } t_i\end{pmatrix} = \begin{pmatrix}\text{total linear momentum}\\ \text{at some later time } t_f\end{pmatrix}.$$

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Caution: Momentum should not be confused with energy. In the Sample Problems of this section, momentum is conserved but energy is definitely not.

Equations 9-45 and 9-46 are vector equations and, as such, each is equivalent to three equations corresponding to the conservation of linear momentum in three mutually perpendicular directions as in, say, an xyz coordinate system. Depending on the forces acting on a system, linear momentum might be conserved in one or two directions but not in all directions. However,

If the component of the net *external* force on a closed system is zero along an axis, then the component of the linear momentum of the system along that axis cannot change.

While solving a problem, how can you know if linear momentum can be conserved along, say, an x axis? Check the force components along that axis. If the net of any such components is zero, then the conservation applies. As an example, suppose that you toss a grapefruit across a room. During its flight, the only external force acting on the grapefruit (which we take as the system) is the gravitational force $\vec{F}_g$, which is directed vertically downward. Thus, the vertical component of the linear momentum of the grapefruit changes, but since no horizontal external force acts on the grapefruit, the horizontal component of the linear momentum cannot change.

Note that we focus on the external forces acting on a closed system. Although internal forces can change the linear momentum of portions of the system, they cannot change the total linear momentum of the entire system. For example, there are plenty of forces acting between the organs of your body, but they do not propel you across the room (thankfully).

The Sample Problems in this section involve explosions that are either one-dimensional (meaning that the motions before and after the explosion are along a single axis) or two-dimensional (meaning that they are in a plane containing two axes). In the following sections we consider collisions.

CHECKPOINT 9

An initially stationary device lying on a frictionless floor explodes into two pieces, which then slide across the floor, one of them in the positive x direction. (a) What is the sum of the momenta of the two pieces after the explosion? (b) Can the second piece move at an angle to the x axis? (c) What is the direction of the momentum of the second piece?

SAMPLE PROBLEM 9.21

Determination of velocity from conservation of momentum after a collision

(a) Consider a block of mass M kept on a rough surface being hit by a particle of mass m moving with speed v_0 (Fig. 9-21a). Find the velocity of combined mass immediately after the particle sticks with the block.

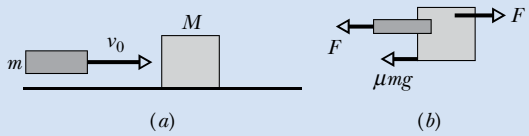

Figure 9-21 (a) A block of mass M hit by a particle of mass m. (b) Free-body diagram.

Calculation: Impulse of μmg (frictional force) is negligible thus momentum is conserved just after collision.

According to conservation of momentum, initial momentum = final momentum.

Initially block of mass M is stationary, so its initial velocity is zero while velocity of block of mass m is v_0. After collision, both the blocks are stuck and move with a common final velocity v. Therefore,

$$mv_0 + 0 = (m+M)v$$

$$\Rightarrow v = \frac{mv_0}{M+m}$$

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(b) Consider a block of mass M connected to a spring kept on smooth surface being hit by a particle of mass m moving with speed v_0 (Fig. 9-22). Find the velocity of combined mass immediately after the particle sticks with the block

Calculation: Spring force will be very small and impulse by it is negligible, thus momentum is conserved just after collision. Therefore,

$$v = \frac{mv_0}{M+m}$$

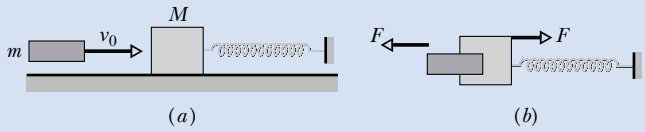

Figure 9-22 (*a*) A block of mass M connected to a spring and kept on a smooth surface being hit by a particle of mass m. (*b*) Free-body diagram.

SAMPLE PROBLEM 9.22

One-dimensional explosion, relative velocity, space hauler

One-dimensional explosion: Figure 9-23*a* shows a space hauler and cargo module, of total mass M, traveling along an x axis in deep space. They have an initial velocity $\vec{v}_i$ of magnitude 2100 km/h relative to the Sun. With a small explosion, the hauler ejects the cargo module, of mass $0.20M$ (Fig. 9-23*b*). The hauler then travels 500 km/h faster than the module along the x axis; that is, the relative speed v_{rel} between the hauler and the module is 500 km/h. What then is the velocity $\vec{v}_{HS}$ of the hauler relative to the Sun?

The explosive separation can change the momentum of the parts but not the momentum of the system.

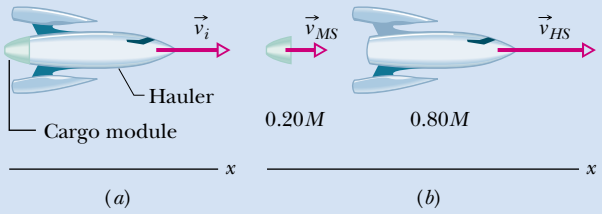

Figure 9-23 (*a*) A space hauler, with a cargo module, moving at initial velocity $\vec{v}_i$. (*b*) The hauler has ejected the cargo module. Now the velocities relative to the Sun are $\vec{v}_{MS}$ for the module and $\vec{v}_{HS}$ for the hauler.

KEY IDEA

Because the hauler–module system is closed and isolated, its total linear momentum is conserved; that is,

$$\vec{P}_i = \vec{P}_f, \tag{9-47}$$

where the subscripts i and f refer to values before and after the ejection, respectively. (We need to be careful here: Although the momentum of the *system* does not change, the momenta of the hauler and module certainly do.)

Calculations: Because the motion is along a single axis, we can write momenta and velocities in terms of their x components, using a sign to indicate direction. Before the ejection, we have

$$P_i = Mv_i. \tag{9-48}$$

Let v_{MS} be the velocity of the ejected module relative to the Sun. The total linear momentum of the system after the ejection is then

$$P_f = (0.20M)v_{MS} + (0.80M)v_{HS}, \tag{9-49}$$

where the first term on the right is the linear momentum of the module and the second term is that of the hauler.

We can relate the v_{MS} to the known velocities with

$$\begin{pmatrix}\text{velocity of}\\ \text{hauler relative}\\ \text{to Sun}\end{pmatrix} = \begin{pmatrix}\text{velocity of}\\ \text{hauler relative}\\ \text{to module}\end{pmatrix} + \begin{pmatrix}\text{velocity of}\\ \text{module relative}\\ \text{to Sun}\end{pmatrix}.$$

In symbols, this gives us

$$v_{HS} = v_{rel} + v_{MS} \tag{9-50}$$

or

$$v_{MS} = v_{HS} - v_{rel}.$$

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Substituting this expression for v_{MS} into Eq. 9-49, and then substituting Eqs. 9-48 and 9-49 into Eq. 9-47, we find

$$Mv_i = 0.20M(v_{HS} - v_{rel}) + 0.80Mv_{HS},$$

which gives us

$$v_{HS} = v_i + 0.20v_{rel}$$

or

$$v_{HS} = 2100 \text{ km/h} + (0.20)(500 \text{ km/h})$$
$$= 2200 \text{ km/h}. \quad \text{(Answer)}$$

SAMPLE PROBLEM 9.23

Two-dimensional explosion, momentum, coconut

Two-dimensional explosion: A firecracker placed inside a coconut of mass M, initially at rest on a frictionless floor, blows the coconut into three pieces that slide across the floor. An overhead view is shown in Fig. 9-24*a*. Piece C, with mass $0.30M$, has final speed $v_{fc} = 5.0$ m/s.

(a) What is the speed of piece B, with mass $0.20M$?

KEY IDEA

First we need to see whether linear momentum is conserved. We note that (1) the coconut and its pieces form a closed system, (2) the explosion forces are internal to that system, and (3) no net external force acts on the system. Therefore, the linear momentum of the system is conserved. (We need to be careful here: Although the momentum of the system does not change, the momenta of the pieces certainly do.)

Calculations: To get started, we superimpose an xy coordinate system as shown in Fig. 9-24*b*, with the negative direction of the x axis coinciding with the direction of $\vec{v}_{fA}$.The x axis is at 80° with the direction of $\vec{v}_{fC}$ and 50° with the direction of $\vec{v}_{fB}$.

Linear momentum is conserved separately along each axis. Let's use the y axis and write

$$P_{iy} = P_{fy}, \quad (9\text{-}51)$$

where subscript i refers to the initial value (before the explosion), and subscript y refers to the y component of $\vec{P}_i$ or $\vec{P}_f$.

The component P_{iy} of the initial linear momentum is zero, because the coconut is initially at rest. To get an expression for P_{fy}, we find the y component of the final linear momentum of each piece, using the y-component version of Eq. 9-24 ($p_y = mv_y$):

$$p_{fA,y} = 0,$$
$$p_{fB,y} = -0.20Mv_{fB,y} = -0.20Mv_{fB} \sin 50°,$$
$$p_{fC,y} = 0.30Mv_{fC,y} = 0.30Mv_{fC} \sin 80°.$$

The explosive separation can change the momentum of the parts but not the momentum of the system.

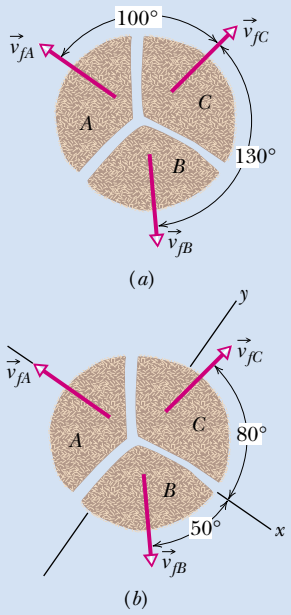

Figure 9-24 Three pieces of an exploded coconut move off in three directions along a frictionless floor. (*a*) An overhead view of the event. (*b*) The same with a two-dimensional axis system imposed.

(Note that $p_{fA,y} = 0$ because of our nice choice of axes.) Equation 9-51 can now be written as

$$P_{iy} = P_{fy} = p_{fA,y} + p_{fB,y} + p_{fC,y}.$$

Then, with $v_{fC} = 5.0$ m/s, we have

$$0 = 0 - 0.20Mv_{fB} \sin 50° + (0.30M)(5.0 \text{ m/s}) \sin 80°,$$

from which we find

$$v_{fB} = 9.64 \text{ m/s} \approx 9.6 \text{ m/s}. \quad \text{(Answer)}$$

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(b) What is the speed of piece A?

Calculations: Linear momentum is also conserved along the x axis because there is no net external force acting on the coconut and pieces along that axis. Thus we have

$$P_{ix} = P_{fx}, \tag{9-52}$$

where $P_{ix} = 0$ because the coconut is initially at rest. To get P_{fx}, we find the x components of the final momenta, using the fact that piece A must have a mass of $0.50M$ ($=M - 0.20M - 0.30M$):

$$p_{fA,x} = -0.50Mv_{fA},$$
$$p_{fB,x} = 0.20Mv_{fB,x} = 0.20Mv_{fB} \sin 50°,$$
$$p_{fC,x} = 0.30Mv_{fC,x} = 0.30Mv_{fC} \cos 80°.$$

Equation 9-52 for the conservation of momentum along the x axis can now be written as

$$P_{ix} = P_{fx} = p_{fA,x} + p_{fB,x} + p_{fC,x}.$$

Then, with $v_{fC} = 5.0$ m/s and $v_{fB} = 9.64$ m/s, we have

$$0 = -0.50Mv_{fA} + 0.20M(9.64 \text{ m/s}) \cos 50° + 0.30M(5.0 \text{ m/s}) \cos 80°,$$

from which we find

$$v_{fA} = 3.0 \text{ m/s}. \qquad \text{(Answer)}$$

PROBLEM-SOLVING TACTICS

Tactic 2: Conservation of Linear Momentum For problems involving the conservation of linear momentum, first make sure that you have chosen a closed, isolated system. "Closed" means that no matter (no particles) passes through the system boundary in any direction. "Isolated" means that the net external force acting on the system is zero. If it is not isolated, then remember that each component of linear momentum is conserved separately if the corresponding component of the net external force is zero. So, you might conserve one component and not another. Next, select two appropriate states of the system (which you may choose to call the initial state and the final state) and write expressions for the linear momentum of the system in each of these two states. In writing these expressions, make sure that you know what inertial reference frame you are using, and make sure also that you include the entire system, not missing any part of it and not including objects that do not belong to your system.

Finally, set your expressions for $\vec{P}_i$ and $\vec{P}_f$ equal to each other and solve for what is requested.

SAMPLE PROBLEM 9.24

Conversation of momentum in block-spring system

Figure 9-25 shows two blocks that are connected by an ideal spring and are free to slide along an x axis on a frictionless horizontal surface. Block 1 has mass m_1 and block 2 has mass m_2. The blocks are pulled in opposite directions (stretching the spring) and then released from rest.

(a) What is the ratio v_1/v_2 of the velocity of block 1 to the velocity of block 2 as the separation between the blocks decreases?

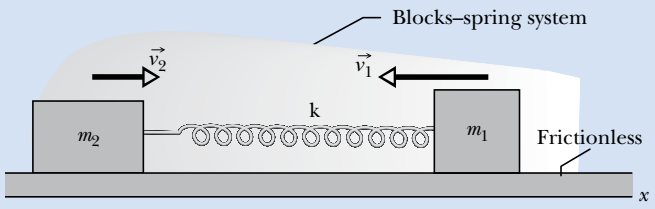

Figure 9-25 Two blocks, which are on a frictionless surface and connected by a spring, are pulled away from each other and then released from rest. The blocks-spring system is indicated.

KEY IDEAS

(1) The blocks are to be treated as particles as there is no requirement to find motion as function of time we need not attempt the problem from force analyses.

(2) We can use the law of conservation of linear momentum. If so, maybe we can relate the velocities of the blocks by relating their momenta. To make this check, we consider the system consisting of the two blocks and

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the spring. Then the spring force that accelerates the blocks is internal to the system and so cannot change the total linear momentum P of the system; thus, we need not consider that force. What *can* change P is an external force. The only *external* forces acting on the system are vertical, so the horizontal component of P cannot change as the blocks move.

Calculations: We will apply the law of conservation of momentum in the form $P_i = P_f$, for x components of the momenta. We let i correspond to the release of the blocks from rest. Each block has a momentum of zero just then, so $P_i = 0$. We also let f correspond to any instant as the blocks approach each other. Then the total momentum is the sum of momenta m_1v_1 for block 1 and m_2v_2 for block 2. For our system, we can now write the conservation law as

$$0 = m_1v_1 + m_2v_2 \tag{9-53}$$

which gives us

$$\frac{v_1}{v_2} = -\frac{m_2}{m_1} \tag{9-54}$$

The minus sign tells us that the two velocities always have opposite signs. Equation 9-53 holds at every instant after release, no matter what the actual speeds of the blocks are.

(b) What is the ratio K_1/K_2 of the kinetic energies of the blocks as their separation decreases?

Calculations: The key idea here is that we can find the ratio of the kinetic energies by using the definition of kinetic energy, that is,

$$\frac{K_1}{K_2} = -\frac{\frac{1}{2}m_1v_1^2}{\frac{1}{2}m_2v_2^2} = \frac{m_1}{m_2}\left(\frac{v_1}{v_2}\right)^2$$

$$\frac{K_1}{K_2} = \frac{m_2}{m_1} \tag{9-55}$$

Learn: As the blocks move toward each other and the stretch of the spring decreases, energy is transferred from the elastic potential energy of the spring to the kinetic energies K_1 and K_2 of the blocks. Although K_1 and K_2 then increase, Eq. 9-53 tells us that their ratio does not change but is preset by the ratio of the masses. After the spring reaches its rest length and begins to be compressed by the blocks, the energy transfer is reversed, but Eq. 9-55 still holds.

Equations 9-53 through 9-55 apply to other situations in which two bodies attract (or repel) each other. For example, they apply to a stone falling toward Earth. The stone corresponds to, say, block 1 in Fig. 9-25, and Earth to block 2. The gravitational force between the stone and Earth corresponds to the mutual force between the blocks that is, provided by the spring in the figure. Our reference frame is the frame in which the center of mass of the stone-Earth system is stationary. *In this frame,* Eq. 9-55 tells us that the magnitudes of the linear momenta of the stone and Earth remain equal to each other throughout the fall. Equations 9-52 and 9-53 tell us that, because $m_2 \gg m_1$, the stone has much greater speed and kinetic energy than Earth during the fall, again *in the given frame.*

SAMPLE PROBLEM 9.25

Determining velocity using conservation of energy in a wedge-block system

A block of mass m slides down the wedge of mass M as shown in Fig. 9-26*a*.

Find speed of the wedge when block of mass m is about to reach the ground, assuming that all surfaces are frictionless. Mass of the block and of wedge is 1 kg each. Length of incline is $l = 0.51$ m and it is at an angle 37° to the horizontal.

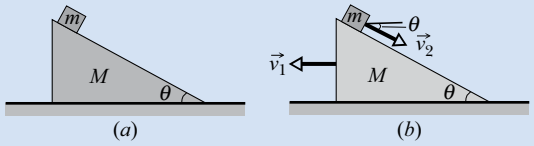

Figure 9-26 (*a*) A block kept on a smooth wedge. (*b*) Block comes down and the wedge moves to the right with velocity v_1.

KEY IDEA

(1) We go back and review Sample Problem 9.13. There we had to just find displacement here we have to find velocity.

(2) As discussed already, solving the problem by calculating acceleration and then displacements and velocities is a tedious solution. We will attempt this by conserving momentum and energy; here energy can be conserved as no dissipative force is present in the system.

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Calculations: Let us say that the block has velocity v_2' with respect to the wedge and the wedge has velocity v_1 with respect to ground. Considering direction of motion along right horizontal to be positive value of x component of velocity for block and wedge can be given as

$$\vec{v}_{2x} = v_2' \cos\theta - v_1 \qquad \text{(of block)}$$

and $$\vec{v}_{1x} = -v_1 \qquad \text{(of wedge)}$$

Similarly, considering direction of motion along vertically downward positive y axis, we can find y component of velocity for block and wedge that is,

$$\vec{v}_{2y} = v_2' \sin\theta \qquad \text{(of block)}$$

and $$\vec{v}_{1y} = 0 \qquad \text{(of wedge)}$$

Thus, net velocity of block is

$$v_2^2 = (v_2' \cos\theta - v_1)^2 + (v_2' \sin\theta)^2$$

Since no external force will act in horizontal direction, so momentum of the system consisting of the block and wedge remains conserved in horizontal direction. Initially both the bodies are at rest the velocity of center of mass initially is zero and its x component will remain zero.

$$-Mv_1 + m(v_2' \cos\theta - v_1) = 0 \qquad (9\text{-}56)$$

Also, from conservation of energy

$$mgl\sin\theta = \frac{1}{2}Mv_1^2 + \frac{1}{2}m[(v_2' \cos\theta - v_1)^2 + (v_2' \sin\theta)^2]$$

From Eq. 9-56, we get $v_2' \cos\theta - v_1 = Mv_1/m$

and $$v_2' = \left(\frac{M+m}{m\cos\theta}\right)v_1$$

Using above results, we get

$$mgl\sin\theta = \frac{1}{2}M v_1^2 + \frac{1}{2}m\left(\frac{M^2v_1^2}{m^2} + \frac{(M+m)^2\sin^2\theta v_1^2}{m^2\cos^2\theta}\right)$$

Solving, we get $v_1 = 1.2$ m/s. (Answer)

Learn: If we make $M \gg m$, then as we expect, answer should be zero because the small block will not be able to move the infinitely heavy wedge.

The magnitude of normal due to ground F_N. It must be less than $(M+m)g$ as the center of mass of the system must be accelerating downwards as the block has a component of acceleration vertically downward.

SAMPLE PROBLEM 9.26

Impulsive forces: Relative means after the event

A soldier is belted on a trolley which is traveling at the speed v_0 on a smooth horizontal surface; he is carrying a shoulder fired rocket launcher (Fig. 9-27*a*). The combined mass of soldier, rocket launcher and trolley is M. The launcher has a rocket of mass m (which makes the total mass as $M + m$). The muzzle velocity of rocket is V. What will be the velocities of the soldier.

KEY IDEA

(1) We know that the recoil of firing will apply a impulse on soldier toward left. Thus, we should expect him to be moving slowly or moving toward left. (2) Firing will involve large impulsive forces of explosion which cannot be analyzed directly. Taking soldier and rocket as one system, then momentum along horizontal direction is conserved.

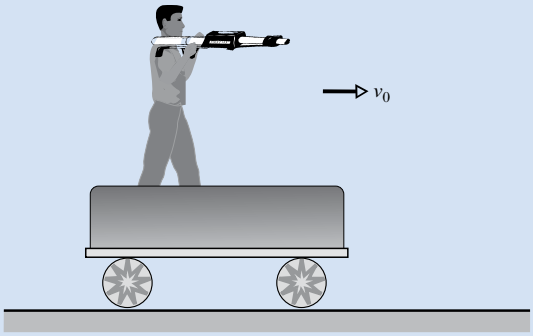

Figure 9-27 (*a*) Soldier belted on a trolley travelling at speed u with rocket launcher on his shoulder.

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Calculations: Let us say v_2 and v_1 are the velocities of the rocket and soldier, respectively, just after firing (Fig. 9-27*b*). Muzzle velocity means the velocity of the projectile with respect to the launcher. But as soon as the velocity of rocket increases from v_0 to v_2, the velocity of the man would also be changed to v_1 as the force of explosion acts simultaneously on both the launcher and the rocket according to Newton's third law. So, the equation of relative velocity is written in terms of velocities just after firing.

Using Eq. 9-48 and rearranging, we get

$$v_2 - v_1 = V$$

Taking man and rocket as our system, momentum along horizontal direction is conserved.

So, $$(m+M)v_0 = Mv_1 + Mv_2$$

Solving, we get

$$v_2 = v_0 + \frac{Mv}{M+m} \quad \text{and} \quad v_1 = v_0 - \left(\frac{mv}{M+m}\right) \quad \text{(Answer)}$$

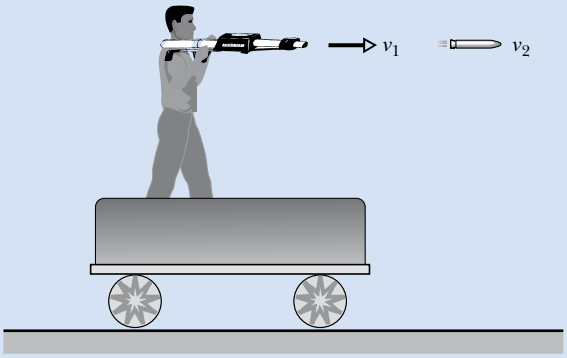

Figure 9-27 (*b*) Motion of soldier and rocket after rocket is fired.

Learn: As expected, the velocity of the soldier has decreased or if $mV/(M+m) > v_0$ he would be going toward left. Also, if we assume M to be very large in comparison to m, then there will be no change in the velocity of the soldier. Also, the rocket will have a higher velocity as all the energy of explosion will be going only to the rocket.

9.8 | COLLISIONS

Key Concepts

- Collision is an event in which strong force acts between two or more bodies for a short time.
- If the kinetic energy of the system is *conserved* (it is the same before and after the collision). Such a collision is called an **elastic collision**.
- If the kinetic energy of the system is *not conserved* (it is always transferred from kinetic energy to other forms of energy). Such a collision is called an **inelastic collision**.
- According to Newton's law of collision

$$\frac{v_{2f} - v_{1f}}{v_{2i} - v_{1i}} = -e$$

where v_{1f} and v_{2f} are the components of velocities of colliding points of the bodies 1 and 2, along the line of contact, after collision (with sign); v_{1i} and v_{2i} are the components of velocities of colliding points of the bodies, along the line of contact, before collision (with sign) and e is the coefficient of restitution.

- In general, e has a value between zero and one, but the physical meaning of these two limits is
  - For $e = 1$: Perfectly elastic collision.
  - For $0 < e < 1$: Inelastic collision.
  - For $e = 0$: Perfectly inelastic collision (bodies will move together).

In Section 9.6, we considered the collision of two particle-like bodies but focused on only one of the bodies at a time. For the next several sections we switch our focus to the system itself, with the assumption that the system is closed and isolated. In Section 9.7, we discussed a rule about such a system: The total linear momentum $\vec{P}$ of the system cannot change because there is no net external force to change it. This is a very powerful rule because it can allow us to determine the results of a collision *without* knowing the details of the collision (such as how much damage is done).

We shall also be interested in the total kinetic energy of a system of two colliding bodies. If that total happens to be unchanged by the collision, then the kinetic energy of the system is *conserved* (it is the same before and after the collision). Such a collision is called an **elastic collision**. In everyday collisions of common bodies, such as two cars or a ball and a bat, some energy is always transferred from kinetic energy to other forms of energy, such as thermal energy or energy of sound. Thus, the kinetic energy of the system is *not* conserved. Such a collision is called an **inelastic collision**.

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Collision is an event in which strong force acts between two or more bodies for a short time. Some important points:

1. In a collision, we consider a situation before and after the collision. The duration of collision is negligible as compared to the time for which event is observed.
2. In a collision, effect of external non-impulsive forces (e.g., gravity, spring) can be neglected because they are very small compared to interaction forces and time of collision is very small.
3. When two bodies collide, they exert force on each other through point of contact, which is perpendicular to the plane of contact. The direction of force of interaction is line of collision.
4. In case of collision, if the external impulsive forces are not acting in a direction of collision, the total momentum of system in that direction remains conserved:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

5. After collision only the components of velocity along the line of collision change, the perpendicular components of velocity remain unaffected.
6. According to initial velocities and line of collision, collision can be of two types: head-on collision and oblique collision.

Head-on collision occurs when the direction of motion of the two colliding particles is along a line passing through the center of mass of the particles. This line is called the **line of collision** (Fig. 9-28*a*). When the motion of one or both of the particles is at an angle with the line of collision (Figs. 9-28*b*, 9-28*c* and 9-28*d*), the collision is said to be **oblique collision**.

Consider the two smooth particles 1 and 2 undergoing a head-on collision.

1. Let's say the particles have the initial momentum as shown in Fig. 9-28*a*. Collision will occur if $v_{1i} > v_{2i}$.
2. During the collision, the bodies must be thought of as deformable or non-rigid. The bodies will undergo a period of deformation during which they exert an equal but opposite deformation impulse $\int P\,dt$ on each other. Thus, their velocities will vary continuously such that velocity of particle 1 will decrease and velocity of particle 2 will increase.
3. When both particles will have a common velocity v, the deformation of both particles will be maximum.
4. After achieving common velocity, a period of restitution occurs in which the bodies try to return to their original shapes. The equal and opposite restitution impulse $\int R\,dt$ pushes the particles apart from one another. The deformation impulse is always greater than that of restitution, that is, $\int P\,dt > R\,dt$.
5. Once the restitution impulse stops acting, the particles will have a final velocity and hence momentum and kinetic energy.

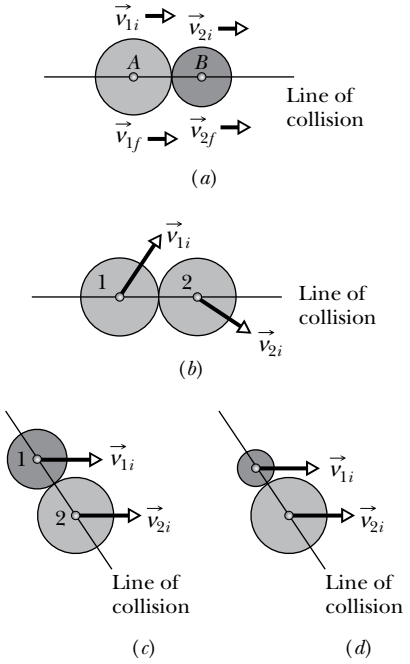

Figure 9-28 Head-on and oblique collision.

Let us determine their final velocities v_{1f} and v_{2f}. Only the internal impulses of deformation and restitution occur and no external impulse is acting.

Applying the principle of impulse and momentum to each particle, during the deformation phase for particles 1 and 2, we have

$$-\int P\,dt = m_1 v_{1i} - m_1 v_{1f} \quad \text{and} \quad \int P\,dt = m_2 v_{2i} - m_2 v_{2f}$$

For the restitution phase, we have

$$-\int R\,dt = m_1 v_{1f} - m_1 v_{1i} \quad \text{and} \quad \int R\,dt = m_2 v_{2f} - m_2 v_{2i}$$

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The ratio of the restitution impulse to the deformation impulse is called the coefficient of restitution, e. From the above equations, this value for particles 1 and 2 is

$$e = \frac{\int R\,dt}{\int P\,dt} = \frac{v - v_{1f}}{v_{1i} - v} \quad \text{and} \quad e = \frac{\int R\,dt}{\int P\,dt} = \frac{v_{2f} - v}{v - v_{2i}}$$

If the unknown v is eliminated from the above two equations, the coefficient of restitution can be expressed in terms of the particles' initial and final velocities as

$$e = \frac{v_{2f} - v_{1f}}{v_{1f} - v_{2i}} \tag{9-57}$$

It is important to choose a positive direction and then use it consistently while writing all quantities with sign. We have defined the positive direction to the right, if a negative value comes as a result of the solution, it indicates motion is to the left.

When applying the equation, the direction of an unknown velocity can be assumed. If the solution yields a negative magnitude, the velocity acts in the opposite direction.

Newton's Law of Collisions

According to Newton's law of collision

$$\frac{v_{2f} - v_{1f}}{v_{2i} - v_{1i}} = -e \tag{9-58}$$

where v_{1f} and v_{2f} are the components of velocities of colliding points of the bodies 1 and 2, along the line of contact, after collision (with sign); v_{1i} and v_{2i} are the components of velocities of colliding points of the bodies, along the line of contact, before collision (with sign) and e is the coefficient of restitution.

It can also be expressed as

$$v_{2f} - v_{1f} = e(v_{2i} - v_{1i})$$

Relative velocity of receding after impact = $e \times$ (Relative velocity of approach before impact)

This law is valid even when momentum of the system is not conserved.

An example of sign convention followed when bodies move towards each other before collision and in the same direction after collision is illustrated in Fig. 9-29.

$$-e = \frac{v_2 - v_1}{(-3) - (+4)} \quad \text{or} \quad v_2 - v_1 = 7e$$

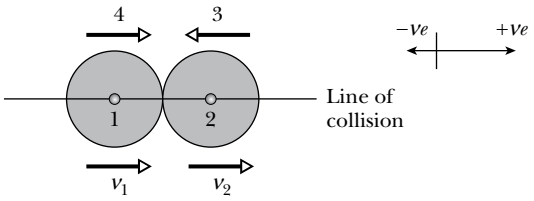

Figure 9-29 Bodies move towards each other before collision and in the same direction after collision.

The collision between bodies moving in same direction before and after collision is shown in Fig. 9-30.

In general, e has a value between zero and one, and one should be aware of the physical meaning of these two limits.

- **For $e = 1$:** Perfectly elastic collision.
- **For $0 < e < 1$:** Inelastic collision.
- **For $e = 0$:** Perfectly inelastic collision (bodies will move together).

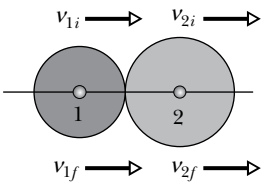

Figure 9-30 Bodies move in same direction before and after collision.

Perfectly Elastic Collision ($e = 1$): If the collision between the two bodies is perfectly elastic, the deformation impulse $\int P\,dt$ is

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equal and opposite to the restitution impulse $\int R\,dt$. Although in reality, the condition $e = 1$ for an elastic collision can never be achieved.

Perfectly inelastic collision ($e = 0$): The collision is said to be perfectly inelastic when $e = 0$. In this case of colliding bodies, there is no tendency to regain original shape. The event of collision is over when they have maximum deformation. Thus restitution impulse $\int R\,dt = 0$, so that after collision both the bodies stick together and move with a common velocity. In this case, the loss of kinetic energy is maximum as energy is lost during deformation.

From the above discussion it should be clear that the principle of work and energy is not useful for the analysis of this problem. Using the particle's velocities before and after collision, the energy lost during collision can be calculated. This energy loss $\sum U_{1-2} = \sum U_2 - \sum U_1$, occurs because some part of the initial kinetic energy of the bodies is converted into thermal energy as well as creating sound and deformation of the material when the collision occurs. If the collision is perfectly elastic, no energy is lost in the collision; whereas if the collision is perfectly inelastic, the energy lost during collision is a maximum.

However, in some situations,we can *approximate* a collision of common bodies as elastic. Suppose that you drop a Superball onto a hard floor. If the collision between the ball and floor (or Earth) were elastic, the ball would lose no kinetic energy because of the collision and would rebound to its original height. However, the actual rebound height is somewhat short, showing that at least some kinetic energy is lost in the collision and thus that the collision is somewhat inelastic. Still, we might choose to neglect that small loss of kinetic energy to approximate the collision as elastic.

The inelastic collision of two bodies always involves a loss in the kinetic energy of the system. The greatest loss occurs if the bodies stick together, in which case the collision is called a **completely inelastic collision** or **perfectly inelastic collision**. The collision of a baseball and a bat is inelastic. However, the collision of a wet putty ball and a bat is completely inelastic because the putty sticks to the bat.

9.9 | INELASTIC COLLISIONS IN ONE DIMENSION

Key Concepts

- In an inelastic collision of two bodies, the kinetic energy of the two-body system is not conserved. If the system is closed and isolated, the total linear momentum of the system *must* be conserved, which we can write in vector form as

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f},$$

  where subscripts i and f refer to values just before and just after the collision, respectively.
- If the motion of the bodies is along a single axis, the collision is one-dimensional and we can write the equation in terms of velocity components along that axis:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}.$$

- If the bodies stick together, the collision is a completely inelastic collision and the bodies have the same final velocity V (because they *are* stuck together).
- The center of mass of a closed, isolated system of two colliding bodies is not affected by a collision. In particular, the velocity $\vec{v}_{com}$ of the center of mass cannot be changed by the collision.

One-Dimensional Inelastic Collision

Figure 9-31 shows two bodies just before and just after they have a onedimensional collision. The velocities before the collision (subscript i) and after the collision (subscript f) are indicated. The two bodies form our system, which is closed and isolated. We can write the law of conservation of linear momentum for this two-body system as

$$\begin{pmatrix}\text{total momentum } \vec{P}_i \\ \text{before the collision}\end{pmatrix} = \begin{pmatrix}\text{total momentum } \vec{P}_f \\ \text{after the collision}\end{pmatrix},$$

which we can symbolize as

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f} \quad \text{(conservation of linear momentum)}. \tag{9-59}$$

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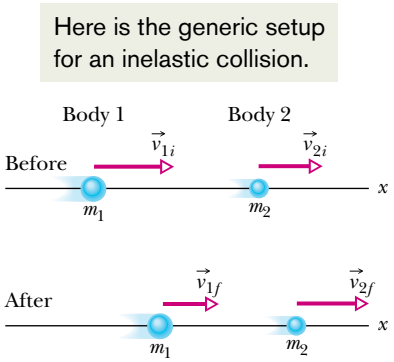

Figure 9-31 Bodies 1 and 2 move along an x axis, before and after they have an inelastic collision.

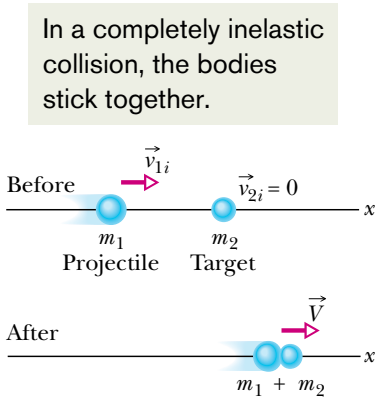

Figure 9-32 A completely inelastic collision between two bodies. Before the collision, the body with mass m_2 is at rest and the body with mass m_1 moves directly toward it. After the collision, the stuck-together bodies move with the same velocity $\vec{V}$.

Because the motion is one-dimensional, we can drop the overhead arrows for vectors and use only components along the axis, indicating direction with a sign. Thus, from $p = mv$, we can rewrite Eq. 9-59 as

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}. \tag{9-60}$$

If we know values for, say, the masses, the initial velocities, and one of the final velocities, we can find the other final velocity with Eq. 9-60.

One-Dimensional Completely Inelastic Collision

Figure 9-32 shows two bodies before and after they have a completely inelastic collision (meaning they stick together). The body with mass m_2 happens to be initially at rest ($v_{2i} = 0$). We can refer to that body as the *target* and to the incoming body as the *projectile*. After the collision, the stuck-together bodies move with velocity V. For this situation, we can rewrite Eq. 9-60 as

$$m_1 v_{1i} = (m_1 + m_2)V \tag{9-61}$$

or

$$V = \frac{m_1}{m_1 + m_2} v_{1i}. \tag{9-62}$$

If we know values for, say, the masses and the initial velocity v_{1i} of the projectile, we can find the final velocity V with Eq. 9-62. Note that V must be less than v_{1i} because the mass ratio $m_1/(m_1 + m_2)$ must be less than unity.

Velocity of the Center of Mass

In a closed, isolated system, the velocity $\vec{v}_{com}$ of the center of mass of the system cannot be changed by a collision because, with the system isolated, there is no net external force to change it. To get an expression for $\vec{v}_{com}$, let us return to the two-body system and one-dimensional collision of Fig. 9-31. From Eq. 9-27 ($\vec{P} = M\vec{v}_{com}$), we can relate $\vec{v}_{com}$ to the total linear momentum $\vec{P}$ of that two-body system by writing

$$\vec{P} = M\vec{v}_{com} = (m_1 + m_2)\vec{v}_{com}. \tag{9-63}$$

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The total linear momentum $\vec{P}$ is conserved during the collision; so it is given by either side of Eq. 9-59. Let us use the left side to write

$$\vec{P} = \vec{p}_{1i} + \vec{p}_{2i}. \tag{9-64}$$

Substituting this expression for $\vec{P}$ in Eq. 9-63 and solving for $\vec{v}_{com}$ give us

$$\vec{v}_{com} = \frac{\vec{P}}{m_1 + m_2} = \frac{\vec{p}_{1i} + \vec{p}_{2i}}{m_1 + m_2}. \tag{9-65}$$

The right side of this equation is a constant, and $\vec{v}_{com}$ has that same constant value before and after the collision.

For example, Fig. 9-33 shows, in a series of freeze-frames, the motion of the center of mass for the completely inelastic collision of Fig. 9-32. Body 2 is the target, and its initial linear momentum in Eq. 9-65 is $\vec{p}_{2i} = m_2\vec{v}_{2i} = 0$. Body 1 is the projectile, and its initial linear momentum in Eq. 9-65 is $\vec{p}_{1i} = m_1\vec{v}_{1i}$. Note that as the series of freeze-frames progresses to and then beyond the collision, the center of mass moves at a constant velocity to the right. After the collision, the common final speed V of the bodies is equal to $\vec{v}_{com}$ because then the center of mass travels with the stuck-together bodies.

CHECKPOINT 10

Body 1 and body 2 are in a completely inelastic one-dimensional collision. What is their final momentum if their initial momenta are, respectively, (a) 10 kg·m/s and 0; (b) 10 kg·m/s and 4 kg·m/s; (c) 10 kg·m/s and –4 kg·m/s?

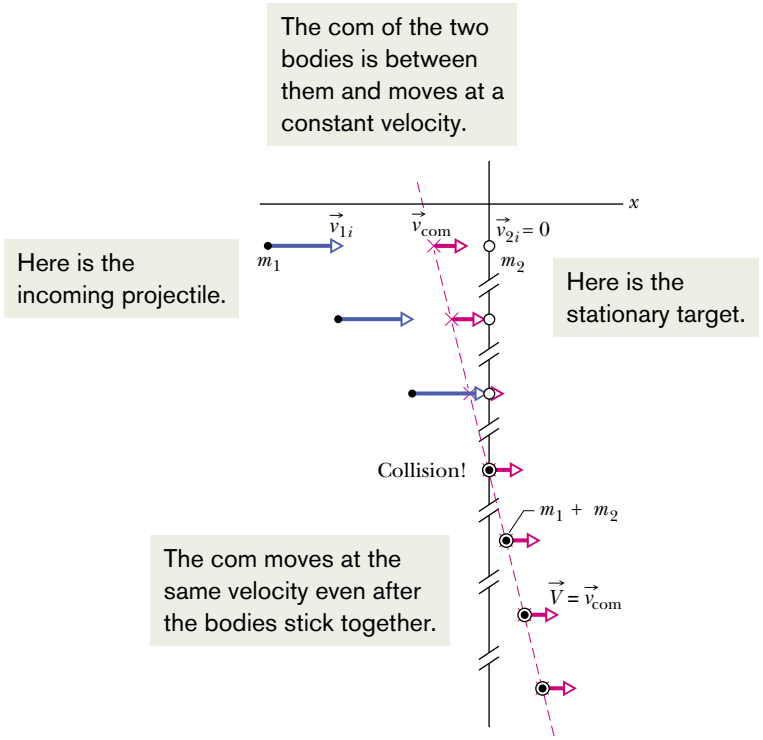
The com of the two bodies is between them and moves at a constant velocity.
Here is the incoming projectile.
Here is the stationary target.
The com moves at the same velocity even after the bodies stick together.
x
$\vec{v}_{1i}$
$\vec{v}_{com}$
$\vec{v}_{2i} = 0$
m_1
m_2
Collision!
$m_1 + m_2$
$\vec{V} = \vec{v}_{com}$

Figure 9-33 Some freeze-frames of the two-body system in Fig. 9-32, which undergoes a completely inelastic collision. The system's center of mass is shown in each freeze-frame. The velocity $\vec{v}_{com}$ of the center of mass is unaffected by the collision. Because the bodies stick together after the collision, their common velocity $\vec{V}$ must be equal to $\vec{v}_{com}$.

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SAMPLE PROBLEM 9.27

Conservation of momentum, ballistic pendulum

Here is an example of a common technique in physics. We have a demonstration that cannot be worked out as a whole (we don't have a workable equation for it). So, we break it up into steps that can be worked separately (we have equations for them).

The *ballistic pendulum* was used to measure the speeds of bullets before electronic timing devices were developed. The version shown in Fig. 9-34 consists of a large block of wood of mass $M = 5.4$ kg, hanging from two long cords. A bullet of mass $m = 9.5$ g is fired into the block, coming quickly to rest. The *block + bullet* then swing upward, their center of mass rising a vertical distance $h = 6.3$ cm before the pendulum comes momentarily to rest at the end of its arc. What is the speed of the bullet just prior to the collision?

KEY IDEAS

We can see that the bullet's speed v must determine the rise height h. However, we cannot use the conservation of mechanical energy to relate these two quantities because surely energy is transferred from mechanical energy to other forms (such as thermal energy and energy to break apart the wood) as the bullet penetrates the block. Nevertheless, we can split this complicated motion into two steps that we can separately analyze: (1) the bullet–block collision and (2) the bullet–block rise, during which mechanical energy is conserved.

Reasoning step 1: Because the collision within the bullet–block system is so brief, we can make two important assumptions: (1) During the collision, the gravitational force on the block and the force on the block from the cords are still balanced. Thus, during the collision, the net external impulse on the bullet–block system is zero. Therefore, the system is isolated and its total linear momentum is conserved:

$$\begin{pmatrix}\text{total momentum}\\ \text{before the collision}\end{pmatrix} = \begin{pmatrix}\text{total momentum}\\ \text{after the collision}\end{pmatrix}. \quad (9\text{-}66)$$

(2) The collision is one-dimensional in the sense that the direction of the bullet and block *just after* the *collision* is in the bullet's original direction of motion.

Because the collision is one-dimensional, the block is initially at rest, and the bullet sticks in the block,we use Eq. 9-62 to express the conservation of linear momentum. Replacing the symbols there with the corresponding symbols here, we have

$$V = \frac{m}{m+M}v. \quad (9\text{-}67)$$

Reasoning step 2: As the bullet and block now swing up together, the mechanical energy of the bullet–block–Earth system is conserved:

$$\begin{pmatrix}\text{mechanical energy}\\ \text{at bottom}\end{pmatrix} = \begin{pmatrix}\text{mechanical energy}\\ \text{at top}\end{pmatrix}. \quad (9\text{-}68)$$

(This mechanical energy is not changed by the force of the cords on the block, because that force is always directed perpendicular to the block's direction of travel.) Let's take the block's initial level as our reference level of zero gravitational potential energy. Then conservation of mechanical energy means that the system's kinetic energy at the start of the swing must equal its gravitational potential energy at the highest point of the swing. Because the speed of the bullet and block at the start of the swing is the speed V immediately after the collision, we may write this conservation as

$$\frac{1}{2}(m+M)V^2 = (m+M)gh. \quad (9\text{-}69)$$

Combining steps: Substituting for V from Eq. 9-62 leads to

$$v = \frac{m+M}{m}\sqrt{2gh} \quad (9\text{-}70)$$

$$= \left(\frac{0.0095\text{ kg} + 5.4\text{ kg}}{0.0095\text{ kg}}\right)\sqrt{(2)(9.8\text{ m/s}^2)(0.063\text{ m})}$$

$$= 630\text{ m/s}. \quad \text{(Answer)}$$

The ballistic pendulum is a kind of "transformer," exchanging the high speed of a light object (the bullet) for the low—and thus more easily measurable—speed of a massive-object (the block).

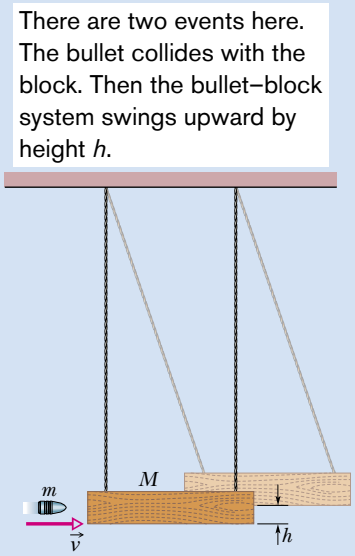

Figure 9-34 A ballistic pendulum, used to measure the speeds of bullets.

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9.10 | ELASTIC COLLISIONS IN ONE DIMENSION

Key Concept

- An elastic collision is a special type of collision in which the kinetic energy of a system of colliding bodies is conserved. If the system is closed and isolated, its linear momentum is also conserved. For a one-dimensional collision in which body 2 is a target and body 1 is an incoming projectile, conservation of kinetic energy and linear momentum yield the following expressions for the velocities immediately after the collision:

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}$$

and

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}.$$

As we discussed in Section 9.8, everyday collisions are inelastic but we can approximate some of them as being elastic; that is, we can approximate that the total kinetic energy of the colliding bodies is conserved and is not transferred to other forms of energy:

$$\begin{pmatrix}\text{total kinetic energy}\\ \text{before the collision}\end{pmatrix} = \begin{pmatrix}\text{total kinetic energy}\\ \text{after the collision}\end{pmatrix}. \tag{9-71}$$

This means:

In an elastic collision, the kinetic energy of each colliding body may change, but the total kinetic energy of the system does not change.

For example, the collision of a cue ball with an object ball in a game of pool can be approximated as being an elastic collision. If the collision is head-on (the cue ball heads directly toward the object ball), the kinetic energy of the cue ball can be transferred almost entirely to the object ball. (Still, the collision transfers some of the energy to the sound you hear.)

Stationary Target

Figure 9-35 shows two bodies before and after they have a one-dimensional collision, like a head-on collision between pool balls. A projectile body of mass m_1 and initial velocity v_{1i} moves toward a target body of mass m_2 that is initially at rest ($v_{2i} = 0$). Let's assume that this two-body system is closed and isolated. Then the net linear momentum of the system is conserved, and from Eq. 9-60 we can write that conservation as

$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f} \quad \text{(linear momentum)}. \tag{9-72}$$

If the collision is also elastic, then the total kinetic energy is conserved and we can write that conservation as

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \quad \text{(kinetic energy)}. \tag{9-73}$$

In each of these equations, the subscript i identifies the initial velocities and the subscript f the final velocities of the bodies. If we know the masses of the bodies and if we also know v_{1i}, the initial velocity of body 1, the only unknown quantities are v_{1f} and v_{2f}, the final velocities of the two bodies. With two equations at our disposal, we should be able to find these two unknowns.

To do so, we rewrite Eq. 9-72 as

$$m_1(v_{1i} - v_{1f}) = m_2 v_{2f} \tag{9-74}$$

Here is the generic setup for an elastic collision with a stationary target.

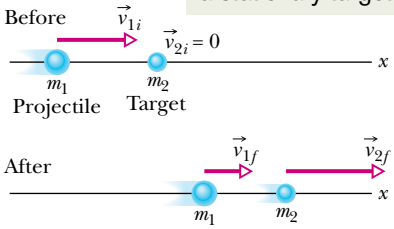

Figure 9-35 Body 1 moves along an x axis before having an elastic collision with body 2, which is initially at rest. Both bodies move along that axis after the collision.

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and Eq. 9-73 as*

$$m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2 v_{2f}^2. \tag{9-75}$$

After dividing Eq. 9-75 by Eq. 9-74 and doing some more algebra, we obtain

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \tag{9-76}$$

and

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}. \tag{9-77}$$

Note that v_{2f} is always positive (the initially stationary target body with mass m_2 always moves forward). From Eq. 9-76 we see that v_{1f} may be of either sign (the projectile body with mass m_1 moves forward if $m_1 > m_2$ but rebounds if $m_1 < m_2$).

Let us look at a few special situations.

1. ***Equal masses*** If $m_1 = m_2$, Eqs. 9-76 and 9-77 reduce to

$$v_{1f} = 0 \quad \text{and} \quad v_{2f} = v_{1i},$$

which we might call a pool player's result. It predicts that after a head-on collision of bodies with equal masses, body 1 (initially moving) stops dead in its tracks and body 2 (initially at rest) takes off with the initial speed of body 1. In head-on collisions, bodies of equal mass simply exchange velocities. This is true even if body 2 is not initially at rest.

2. ***A massive target*** In Fig. 9-35, a massive target means that $m_2 \gg m_1$. For example, we might fire a golf ball at a stationary cannonball. Equations 9-76 and 9-77 then reduce to

$$v_{1f} \approx -v_{1i} \quad \text{and} \quad v_{2f} \approx \left(\frac{2m_1}{m_2}\right) v_{1i}. \tag{9-78}$$

This tells us that body 1 (the golf ball) simply bounces back along its incoming path, its speed essentially unchanged. Initially stationary body 2 (the cannonball) moves forward at a low speed, because the quantity in parentheses in Eq. 9-78 is much less than unity. All this is what we should expect.

3. ***A massive projectile*** This is the opposite case; that is, $m_1 \gg m_2$. This time, we fire a cannonball at a stationary golf ball. Equations 9-76 and 9-77 reduce to

$$v_{1f} \approx v_{1i} \quad \text{and} \quad v_{2f} \approx 2v_{1i}. \tag{9-79}$$

Equation 9-79 tells us that body 1 (the cannonball) simply keeps on going, scarcely slowed by the collision. Body 2 (the golf ball) charges ahead at twice the speed of the cannonball. Why twice the speed? Recall the collision described by Eq. 9-78, in which the velocity of the incident light body (the golf ball) changed from $+v$ to $-v$, a velocity *change* of $2v$. The same *change* in velocity (but now from zero to $2v$) occurs in this example also.

Moving Target

Now that we have examined the elastic collision of a projectile and a stationary target, let us examine the situation in which both bodies are moving before they undergo an elastic collision.

For the situation of Fig. 9-36, the conservation of linear momentum is written as

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}, \tag{9-80}$$

* In this step, we use the identity $a^2 - b^2 = (a - b)(a + b)$. It reduces the amount of algebra needed to solve the simultaneous equations Eqs. 9-69 and 9-70.

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and the conservation of kinetic energy is written as

$$\frac{1}{2}m_1v_{1i}^2+\frac{1}{2}m_2v_{2i}^2=\frac{1}{2}m_1v_{1f}^2+\frac{1}{2}m_2v_{2f}^2. \tag{9-81}$$

To solve these simultaneous equations for v_{1f} and v_{2f}, we first rewrite Eq. 9-80 as

$$m_1(v_{1i}-v_{1f})=-m_2(v_{2i}-v_{2f}), \tag{9-82}$$

and Eq. 9-81 as

$$m_1(v_{1i}-v_{1f})(v_{1i}+v_{1f})=-m_2(v_{2i}-v_{2f})(v_{2i}+v_{2f}). \tag{9-83}$$

Here is the generic setup for an elastic collision with a moving target.

Figure 9-36 Two bodies headed for a onedimensional elastic collision.

After dividing Eq. 9-83 by Eq. 9-82 and doing some more algebra, we obtain

$$v_{1f}=\frac{m_1-m_2}{m_1+m_2}v_{1i}+\frac{2m_2}{m_1+m_2}v_{2i} \tag{9-84}$$

and

$$v_{2f}=\frac{2m_1}{m_1+m_2}v_{1i}+\frac{m_2-m_1}{m_1+m_2}v_{2i}. \tag{9-85}$$

Note that the assignment of subscripts 1 and 2 to the bodies is arbitrary. If we exchange those subscripts in Fig. 9-36 and in Eqs. 9-84 and 9-85, we end up with the same set of equations. Note also that if we set $v_{2i}=0$, body 2 becomes a stationary target as in Fig. 9-35, and Eqs. 9-84 and 9-85 reduce to Eqs. 9-76 and 9-77, respectively.

CHECKPOINT 11

What is the final linear momentum of the target in Fig. 9-35 if the initial linear momentum of the projectile is 6 kg·m/s and the final linear momentum of the projectile is (a) 2 kg·m/s and (b) –2 kg·m/s? (c) What is the final kinetic energy of the target if the initial and final kinetic energies of the projectile are, respectively, 5 J and 2 J?

SAMPLE PROBLEM 9.28

Chain reaction of elastic collisions

In Fig. 9-37*a*, block 1 approaches a line of two stationary blocks with a velocity of v_{1i} = 10 m/s. It collides with block 2, which then collides with block 3, which has mass m_3 = 6.0 kg. After the second collision, block 2 is again stationary and block 3 has velocity v_{3f} = 5.0 m/s (Fig. 9-37*b*). Assume that the collisions are elastic. What are the masses of blocks 1 and 2? What is the final velocity v_{1f} of block 1?

KEY IDEAS

Because we assume that the collisions are elastic, we are to conserve mechanical energy (thus energy losses to sound, heating, and oscillations of the blocks are negligible). Because no external horizontal force acts on the blocks, we are to conserve linear momentum along the *x* axis. For these two reasons, we can apply Eqs. 9-76 and 9-77 to each of the collisions.

Calculations: If we start with the first collision, we have too many unknowns to make any progress: we do not

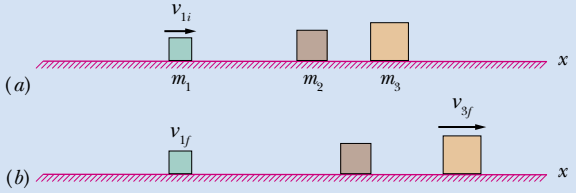

Figure 9-37 Block 1 collides with stationary block 2, which then collides with stationary block 3.

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know the masses or the final velocities of the blocks. So, let's start with the second collision in which block 2 stops because of its collision with block 3. Applying Eq. 9-76 to this collision, with changes in notation, we have

$$v_{2f} = \frac{m_2 - m_3}{m_2 + m_3} v_{2i},$$

where v_{2i} is the velocity of block 2 just before the collision and v_{2f} is the velocity just afterward. Substituting $v_{2f} = 0$ (block 2 stops) and then $m_3 = 6.0$ kg gives us

$$m_2 = m_3 = 6.00 \text{ kg}. \qquad \text{(Answer)}$$

With similar notation changes, we can rewrite Eq. 9-77 for the second collision as

$$v_{3f} = \frac{2m_2}{m_2 + m_3} v_{2i},$$

where v_{3f} is the final velocity of block 3. Substituting $m_2 = m_3$ and the given $v_{3f} = 5.0$ m/s, we find

$$v_{2i} = v_{3f} = 5.0 \text{ m/s}.$$

Next, let's reconsider the first collision, but we have to be careful with the notation for block 2: its velocity v_{2f} just after the first collision is the same as its velocity v_{2f} (=5.0 m/s) just before the second collision. Applying Eq. 9-77 to the first collision and using the given $v_{1f} = 10$ m/s, we have

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i},$$

$$5.0 \text{ m/s} = \frac{2m_1}{m_1 + m_2} (10 \text{ m/s}),$$

which leads to

$$m_1 = \frac{1}{3} m_2 = \frac{1}{3}(6.0 \text{ kg}) = 2.0 \text{ kg}. \qquad \text{(Answer)}$$

Finally, applying Eq. 9-76 to the first collision with this result and the given v_{1i}, we write

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i},$$

$$= \frac{\frac{1}{3}m_2 - m_2}{\frac{1}{3}m_2 + m_2} (10 \text{ m/s}) = -5.0 \text{ m/s}. \qquad \text{(Answer)}$$

9.11 | COLLISIONS IN TWO DIMENSIONS

Key Concepts

- If two bodies collide and their motion is not along a single axis (the collision is not head-on), the collision is two-dimensional. If the two-body system is closed and isolated, the law of conservation of momentum applies to the collision and can be written as

$$\vec{P}_{1i} + \vec{P}_{2i} = \vec{P}_{1f} + \vec{P}_{2f}.$$

In component form, the law gives two equations that describe the collision (one equation for each of the two dimensions). If the collision is also elastic (a special case), the conservation of kinetic energy during the collision gives a third equation:

$$K_{1i} + K_{2i} = K_{1f} + K_{2f}.$$

When two bodies collide, the impulse between them determines the directions in which they then travel. In particular, when the collision is not head-on, the bodies do not end up traveling along their initial axis. For such two-dimensional collisions in a closed, isolated system, the total linear momentum must still be conserved:

$$\vec{P}_{1i} + \vec{P}_{2i} = \vec{P}_{1f} + \vec{P}_{2f}. \qquad (9\text{-}86)$$

If the collision is also elastic (a special case), then the total kinetic energy is also conserved:

$$K_{1i} + K_{2i} = K_{1f} + K_{2f}. \qquad (9\text{-}87)$$

Equation 9-86 is often more useful for analyzing a two-dimensional collision if we write it in terms of components on an xy coordinate system. For example, Fig. 9-38 shows a *glancing collision* (it is not head-on) between a projectile body and a target body initially at rest. The impulses between the bodies have sent the bodies off at angles θ_1 and θ_2

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A glancing collision that conserves both momentum and kinetic energy.

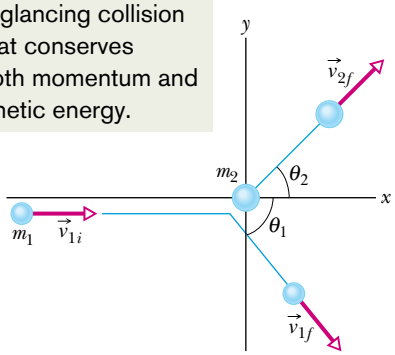

Figure 9-38 An elastic collision between two bodies in which the collision is not headon. The body with mass m_2 (the target) is initially at rest.

to the x axis, along which the projectile initially traveled. In this situation we would rewrite Eq. 9-86 for components along the x axis as

$$m_1 v_{1i} = m_1 v_{1f} \cos \theta_1 + m_2 v_{2f} \cos \theta_2, \tag{9-88}$$

and along the y axis as

$$0 = -m_1 v_{1f} \sin \theta_1 + m_2 v_{2f} \sin \theta_2. \tag{9-89}$$

We can also write Eq. 9-87 (for the special case of an elastic collision) in terms of speeds:

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \quad \text{(kinetic energy)}. \tag{9-90}$$

Equations 9-88 to 9-90 contain seven variables: two masses, m_1 and m_2; three speeds, v_{1i}, v_{1f}, and v_{2f}; and two angles, θ_1 and θ_2. If we know any four of these quantities, we can solve the three equations for the remaining three quantities.

Oblique Collision

When an oblique collision occurs between two smooth bodies, the bodies have velocities of unknown directions and unknown magnitudes after collision. Provided the initial velocities are known, four unknowns present in the problem can be calculated. As shown in Fig. 9-39*a*, these unknowns may be represented either as v_{Af}, v_{Bf}, θ_2, and f_2, or as the x and y components of the final velocities.

If the y axis is taken as the plane of contact and the x axis along the line of collision, the impulsive forces of deformation and restitution act only in the x direction (Fig. 9-39*b*). Resolving the velocity or momentum vectors into components along the x and y axes (Fig. 9-39*b*), it is possible to write four independent scalar equations in order to determine v_{Axf}, v_{Ayf}, v_{Bxf}, and v_{Byf}.

1. Momentum of the system is conserved along x direction as no external impulse is acting on the system.
2. The coefficient of restitution

$$e = \frac{v_{Bxf} - v_{Axf}}{v_{Axi} - v_{Bxi}} \tag{9-91}$$

relates the relative-velocity components of the particles along the line of collision (x axis).

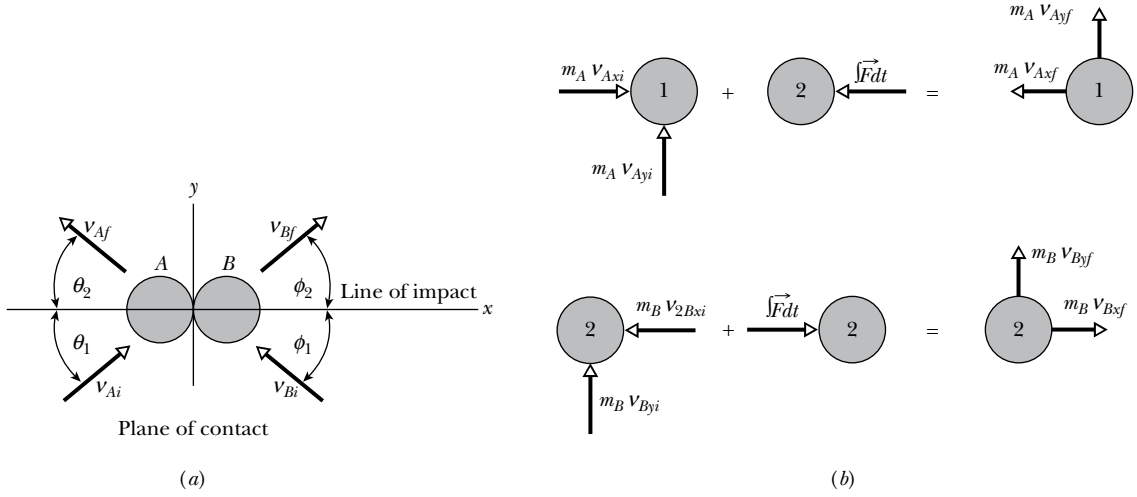

Figure 9-39 Oblique collision (*a*) Bodies just before collision. (*b*) Bodies during and just after collision.

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3. Momentum of particle A is conserved along the y axis, perpendicular to the line of collision, since no impulse acts on particle A in this direction.
4. Momentum of particle B is conserved along the y axis, perpendicular to the line of collision, since no impulse acts on particle B in this direction.

CHECKPOINT 12

In Fig. 9-29, suppose that the projectile has an initial momentum of 6 kg·m/s, a final x component of momentum of 4 kg·m/s, and a final y component of momentum of –3 kg·m/s. For the target, what then are (a) the final x component of momentum and (b) the final y component of momentum?

SAMPLE PROBLEM 9.29

Moving car, impulse, average force

A particle of mass 2.0 m is projected at an angle of 45° with the horizontal with a speed of $20\sqrt{2}$ m/s. After 1.0 s, an explosion occurs and the particle is broken into two equal pieces. One piece is momentarily at rest before it falls. Find the maximum height attained by the other piece.

KEY IDEA

In a projectile motion (Fig. 9-40), since no acceleration exists along x direction, the x-component of the velocity remains constant at all instants.

Calculations: After 1 s, the velocity components are

$$v_x = u\cos 45° = 20\sqrt{2}\times\frac{1}{\sqrt{2}} = 20 \text{ m/s}$$

$$v_y = u\sin 45° - gt = \left(20\sqrt{2}\times\frac{1}{\sqrt{2}}\right) - 9.9(1) = 10.2 \text{ m/s}$$

Therefore, $\vec{v} = v_x + v_y = 20\hat{i} + 10.2\hat{j}$. Applying conservation of momentum (at $t = 1$ s) for before explosion, we get

$$\vec{p}_i = 2m\vec{v} = 2m(20\hat{i} + 10.2\hat{j})$$

For after explosion, we have

$$\vec{p}_f = m(0) + m\vec{v}'$$

$$\vec{p}_i = p'_f$$

$$2m(20\hat{i} + 10.2\hat{j}) = m(0) + m\vec{v}'$$

$$\vec{v}' = 40\hat{i} + 20.4\hat{j}$$

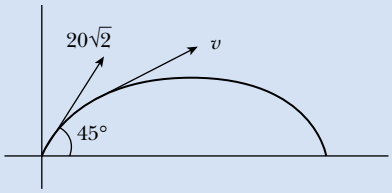

Figure 9-40 Projectile motion if a particle at an angle of 45°.

As measured from the point of explosion, the maximum height attained is

$$\frac{v_y^2}{2g} = \frac{(20.4)^2}{2\times 9.8} = 21.2 \text{ m}.$$

The height of the point of explosion as measured from the ground is

$$h = u_y(1) - \frac{1}{2}g(1)^2 = u\sin 45°(1) - \frac{1}{2}g(1)^2$$

$$= 20\sqrt{2}\times\frac{1}{\sqrt{2}}(1) - \frac{1}{2}\times 9.8\times 1$$

$$= 15.1 \text{ m}.$$

Hence, the maximum height attained (as measured from the ground) is

$$(15.1 + 21.2) = 36 \text{ m}. \qquad \text{(Answer)}$$

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9.12 | C-FRAME

Key Concepts

- The frame of reference attached to the center of mass of a system is called the **center of mass frame of reference** or ***C*-frame of reference**.

$$\vec{P} = \sum \vec{P}_i = 0 \quad \text{(in the } C\text{-frame of reference)}$$

- The kinetic energy of system in ground frame related to kinetic energy in *C*-frame is

$$K_{sys} = K_{sys/c} + \frac{p_c^2}{2M}$$

- The total kinetic energy of the two particles in the *C*-frame is

$$K_{sys/c} = \frac{\mu v_{rel}^2}{2} + \frac{m v_c^2}{2\mu} \quad (\text{where } m = m_1 + m_2)$$

We can attach a frame of reference to the center of mass of a system, this is called the center of mass or ***C*-frame of reference** (Fig. 9-41). Relative to this frame, the center of mass is at rest ($v_{com} = 0$) and according to equation $P = M\vec{v}_{com}$, the total momentum of a system of particles in the *C*-frame of reference is always zero.

$$\vec{P} = \sum \vec{P}_i = 0 \quad \text{(in the } C\text{-frame of reference)} \tag{9-92}$$

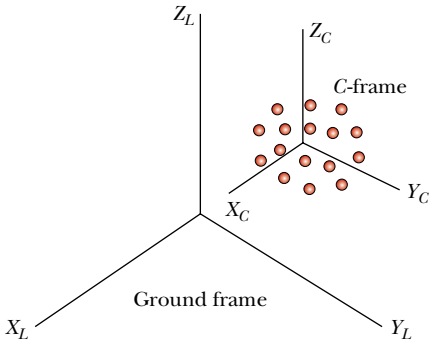

Figure 9-41 *C*-frame of reference.

The *C*-frame is important because many situations can be more simply analyzed in this frame. It is clear that the *C*-frame moves with a velocity v_{com} relative to the ground frame. When no external forces act on a system, the *C*-frame becomes an inertial frame.

Let us find relation between kinetic energy of a system from ground frame and *C*-frame. We have a system consisting of many particles; let us say that the speed of the *i*th particle is v_i. Then kinetic energy of system, *K*, in ground frame will be summation of individual kinetic energies.

$$K_{sys} = \sum\left(\frac{1}{2} m_i v_i^2\right)$$

Now,

$$v_1 = v_{i/c} + v_c$$

where v_i is velocity of the *i*th particle in ground frame, $v_{i/c}$ is velocity of the *i*th particle in reference to frame attached to the center of mass and v_c is velocity of center of mass in ground frame.

$$K_{sys} = \frac{1}{2}\sum m_i(\vec{v}_{i/c} + \vec{v}_c)$$

$$K_{sys} = \frac{1}{2}\sum m_i \vec{v}_{i/c}^2 + \frac{1}{2}\sum m_i \vec{v}_c^2 + \frac{1}{2}\times 2\left(\sum m_i \cdot \vec{v}_{i/c} \cdot \vec{v}_c\right)$$

$$K_{sys} = \sum \frac{1}{2} m_i \vec{v}_{i/c}^2 + \frac{1}{2}\left(\sum m_i\right)\vec{v}_c^2 + \left(\sum m_i \cdot \vec{v}_{i/c}\right)\vec{v}_c$$

we can take v_c out of summation in second and third term as it is constant. Now, third term becomes zero, as

$$\sum m_i \vec{v}_{i/c} = M\vec{v}_{c/c} \tag{9-93}$$

where $v_{c/c}$ is velocity of center of mass in frame of com, which is zero. Also, it represents momentum of system in *C*-frame which is zero.

$$\left(\frac{1}{2}\sum m_i \vec{v}_{i/c}^2\right) = K_{sys/c}$$

Thus, we get

$$K_{sys} = K_{sys/c} + \frac{1}{2} m v_c^2 \tag{9-94}$$

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where $K_{sys/c}$ means kinetic energy of system in C-frame. This important conclusion will be useful again in rotational dynamics; we can do little manipulation to write the equation as

$$K_{sys} = K_{sys/c} + \frac{p_c^2}{2M} \tag{9-95}$$

A System of Two Particles

Suppose the masses of the particles are equal to m_1 and m_2 and their velocities in the K reference frame be $\vec{v}_1$ and $\vec{v}_2$, respectively. Let us find the expressions defining their moment and the total kinetic energy in the C-frame. The momentum of the first particle in the C-frame is

$$\vec{P}_{1/c} = m_1\vec{v}_{1/c} = m_1(\vec{v}_1 - \vec{v}_c)$$

where v_c is the velocity of the center of mass of the system in the ground frame. Substituting in this formula expression

$$\vec{v}_c = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$$

$$\vec{P}_{1/c} = \mu(\vec{v}_1 - \vec{v}_2) \tag{9-96}$$

where μ is the reduced mass of the system, given by

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Similarly, the momentum of the second particle in the C-frame is

$$\vec{P}_{2/c} = \mu(\vec{v}_2 - \vec{v}_1) \tag{9-97}$$

Thus, the momenta of the two particles in the C-frame are equal in magnitude and opposite in direction; the modulus of the momentum of each particle is

$$\vec{P}_{1/c} = \mu v_{rel} \tag{9-98}$$

where $v_{rel} = |\vec{v}_1 - \vec{v}_2|$ is the velocity of one particle relative to another. Finally, let us consider kinetic energy. The total kinetic energy of the two particles in the C-frame is

$$K_{sys/c} = K_1 + K_2 = \frac{\vec{P}^2}{2m_1} + \frac{\vec{P}^2}{2m_2}$$

We know that

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad \text{or} \quad \frac{1}{m_1} + \frac{1}{m_2} = \frac{1}{\mu}$$

Then

$$K_{sys/c} = \frac{\vec{P}^2}{2\mu} = \frac{\mu v_{rel}^2}{2}$$

as

$$K_{sys} = K_{sys/c} + K_c \tag{9-99}$$

we get

$$K_{sys/c} = \frac{\mu v_{rel}^2}{2} + \frac{m v_c^2}{2\mu} \quad (\text{where } m = m_1 + m_2) \tag{9-100}$$

SAMPLE PROBLEM 9.30

Velocity of two particles relative to center of mass using C-frame reference

The velocities of two particles of masses m_1 and m_2 relative to an observer in an inertial frame are v_1 and v_2. Determine the velocity of the center of mass relative to the observer and the velocity of each particle relative to the center of mass.

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Calculations: From definition

$$v_{com} = \frac{dr_{com}}{dt} = \frac{1}{M}\sum_i m_i \frac{dr_i}{dt} = \frac{\sum_i m_i v_i}{M}$$

The velocity of the center of mass relative to the observer is

$$v_{com} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

The velocity of each particle relative to the center of mass (Fig. 9-42) using the relative motion equations for velocities is

$$v_1' = v_1 - v_{com} = v_1 - \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

$$= \frac{m_2(v_1 - v_2)}{m_1 + m_2} = \frac{m_2 v_{12}}{m_1 + m_2}$$

$$v_2' = v_2 - v_{com} = \frac{m_1(v_2 + v_1)}{m_1 + m_2} = -\frac{m_1 v_{12}}{m_1 + m_2}$$

where $v_{12} = v_1 - v_2$ is the relative velocity of the two particles.

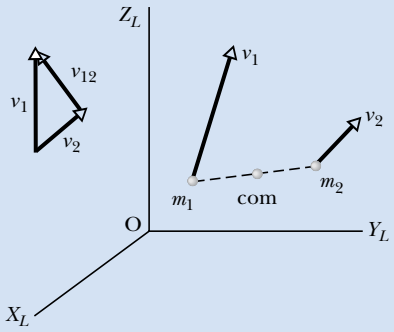

Figure 9-42 Velocity of two particles relative to center of mass.

Thus, in the *C*-frame, the two particles appear to be moving in opposite directions with velocities inversely proportional to their masses.

Also relative to the center of mass, the two particles move with equal but opposite momentum since

$$p_1' = m_1 v_1' = \frac{m_1 m_2 v_{12}}{(m_1 + m_2)} = p_2'$$

Note: The expressions for two particle problems are much simpler when they are related to the *C*-frame of reference.

SAMPLE PROBLEM 9.31

Kinetic energy in a perfectly inelastic collision

Two particles of mass m_1, m_2 moving with initial velocity u_1 and u_2 collide head-on (Fig. 9-43). Find minimum kinetic energy that system has during collision. Thus, prove that maximum kinetic energy is lost in perfectly inelastic collision

Calculations: In *C*-frame, initial kinetic energy of system is

$$\frac{1}{2}\mu(u_2 - u_1)^2$$

where $$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

During collision, at the instant of maximum deformation, we get minimum kinetic energy in *C*-frame as particles attain same velocity, thus relative velocity becomes zero.

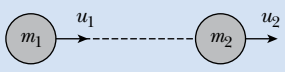

Figure 9-43 Particles moving with velocities u_1 and u_2 in the same direction.

When an isolated system has minimum kinetic energy in *C*-frame, it will also have minimum kinetic energy in ground frame, as velocity of center of mass is constant. Thus, minimum kinetic energy during collision is

$$\frac{1}{2}(m_1 + m_2)v_c^2$$

where $$v_c = \frac{(m_1 u_1 + m_2 u_2)}{m_1 + m_2}$$

Learn: In a perfectly inelastic collision, since both the particles move together, the relative velocity becomes zero. Thus, final kinetic energy is, $1/2\, m_s v_c^2 (m_s = m_1 + m_2)$ as velocity of center of mass is constant. This is the minimum possible kinetic energy that a system will have because in all other cases there will be one more term adding in the kinetic energy of system because of particles having relative velocity.

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9.13 | IMPULSE MOMENTUM EQUATION FOR CONTINUOUS PROCESSES

Momentum and the Flow of Mass

Analyzing a system having mass flowing in or out may become really confusing if we try to apply Newton's laws directly. A rocket is the best example of such a system.

These problems can be handled easily using $F = dP/dt$, provided that we keep clearly in mind exactly what is included in the system. When this equation is applied in the integral form,

$$\int_{t_a}^{t_b} F dt = P(t_b) - P(t_a) \tag{9-101}$$

it is essential to deal with the same set of particles throughout the time interval t_a to t_b; that is, all the particles which are originally in the system are taken in account. Thus, the mass of the system does not change during the time of interest.

9.14 | SYSTEMS WITH VARYING MASS: A ROCKET

Key Concepts

- In the absence of external forces a rocket accelerates at an instantaneous rate given by

  $$Rv_{\text{rel}} = Ma \quad \text{(first rocket equation)},$$

  in which M is the rocket's instantaneous mass (including unexpended fuel), R is the fuel consumption rate, and v_{rel} is the fuel's exhaust speed relative to the rocket. The term Rv_{rel} is the thrust of the rocket engine.
- For a rocket with constant R and v_{rel}, whose speed changes from v_i to v_f when its mass changes from M_i to M_f,

  $$v_f - v_i = v_{\text{rel}} \ln \frac{M_i}{M_f} \quad \text{(second rocket equation)}.$$

So far, we have assumed that the total mass of the system remains constant. Sometimes, as in a rocket, it does not. Most of the mass of a rocket on its launching pad is fuel, all of which will eventually be burned and ejected from the nozzle of the rocket engine. We handle the variation of the mass of the rocket as the rocket accelerates by applying Newton's second law, not to the rocket alone but to the rocket and its ejected combustion products taken together. The mass of *this* system does *not* change as the rocket accelerates.

The ejection of mass from the rocket's rear increases the rocket's speed.

Finding the Acceleration

Assume that we are at rest relative to an inertial reference frame, watching a rocket accelerate through deep space with no gravitational or atmospheric drag forces acting on it. For this one-dimensional motion, let M be the mass of the rocket and v its velocity at an arbitrary time t (see Fig. 9-44*a*).

Figure 9-44*b* shows how things stand a time interval dt later. The rocket now has velocity $v + dv$ and mass $M + dM$, where the change in mass dM is a *negative quantity*. The exhaust products released by the rocket during interval dt have mass $-dM$ and velocity U relative to our inertial reference frame.

Conserve Momentum. Our system consists of the rocket and the exhaust products released during interval dt. The system is closed and isolated, so the linear momentum of the system must be conserved during dt; that is,

$$P_i = P_f, \tag{9-102}$$

where the subscripts i and f indicate the values at the beginning and end of time interval dt. We can rewrite Eq. 9-102 as

$$Mv = -dM\,U + (M + dM)(v + dv), \tag{9-103}$$

where the first term on the right is the linear momentum of the exhaust products released during interval dt and the second term is the linear momentum of the rocket at the end of interval dt.

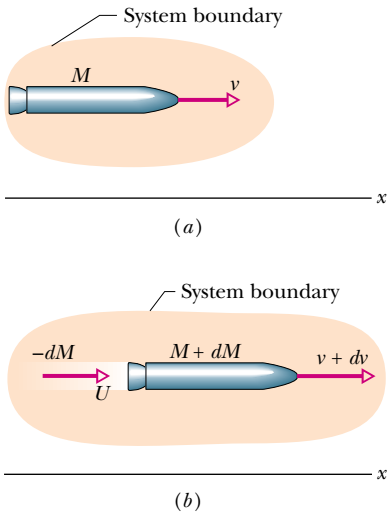

Figure 9-44 (*a*) An accelerating rocket of mass M at time t, as seen from an inertial reference frame. (*b*) The same but at time $t + dt$. The exhaust products released during interval dt are shown.

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Use Relative Speed. We can simplify Eq. 9-103 by using the relative speed v_{rel} between the rocket and the exhaust products, which is related to the velocities relative to the frame with

$$\begin{pmatrix}\text{velocity of rocket}\\ \text{relative to frame}\end{pmatrix}=\begin{pmatrix}\text{velocity of rocket}\\ \text{relative to products}\end{pmatrix}+\begin{pmatrix}\text{velocity of products}\\ \text{relative to frame}\end{pmatrix}.$$

In symbols, this means

$$(v + dv) = v_{rel} + U,$$

or

$$U = v + dv - v_{rel}. \tag{9-104}$$

Substituting this result for U into Eq. 9-103 yields, with a little algebra,

$$-dM\, v_{rel} = M\, dv. \tag{9-105}$$

Dividing each side by dt gives us

$$-\frac{dM}{dt}v_{rel} = M\frac{dv}{dt}. \tag{9-106}$$

We replace dM/dt (the rate at which the rocket loses mass) by $-R$, where R is the (positive) mass rate of fuel consumption, and we recognize that dv/dt is the acceleration of the rocket. With these changes, Eq. 9-106 becomes

$$Rv_{rel} = Ma \quad \text{(first rocket equation)}. \tag{9-107}$$

Equation 9-107 holds for the values at any given instant.

Note the left side of Eq. 9-107 has the dimensions of force ($kg/s \cdot m/s = kg \cdot m/s^2 = N$) and depends only on design characteristics of the rocket engine—namely, the rate R at which it consumes fuel mass and the speed v_{rel} with which that mass is ejected relative to the rocket. We call this term Rv_{rel} the **thrust** of the rocket engine and represent it with T. Newton's second law emerges if we write Eq. 9-107 as $T = Ma$, in which a is the acceleration of the rocket at the time that its mass is M.

Finding the Velocity

How will the velocity of a rocket change as it consumes its fuel? From Eq. 9-105, we have

$$dv = -v_{rel}\frac{dM}{M}.$$

Integrating leads to

$$\int_{v_i}^{v_f} dv = -v_{rel}\int_{M_i}^{M_f}\frac{dM}{M},$$

in which M_i is the initial mass of the rocket and M_f its final mass. Evaluating the integrals then gives

$$v_f - v_i = v_{rel}\ln\frac{M_i}{M_f} \quad \text{(second rocket equation)} \tag{9-108}$$

for the increase in the speed of the rocket during the change in mass from M_i to M_f. (The symbol "ln" in Eq. 9-108 means the *natural logarithm*.) We see here the advantage of multistage rockets, in which M_f is reduced by discarding successive stages when their fuel is depleted. An ideal rocket would reach its destination with only its payload remaining.

SAMPLE PROBLEM 9.32

Rocket engine, thrust, acceleration

In all previous examples in this chapter, the mass of a system is constant (fixed as a certain number). Here is an example of a system (a rocket) that is losing mass. A rocket whose initial mass M_i is 850 kg consumes fuel at the rate $R = 2.3$ kg/s. The speed v_{rel} of the exhaust gases relative to the rocket engine is 2800 m/s. What thrust does the rocket engine provide?

KEY IDEA

Thrust T is equal to the product of the fuel consumption rate R and the relative speed v_{rel} at which exhaust gases are expelled, as given by Eq. 9-107.

Calculations: Here we find

$$T = Rv_{rel} = (2.3\text{ kg/s})(2800\text{ m/s})$$
$$= 6440\text{ N} \approx 6400\text{ N}. \quad \text{(Answer)}$$

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(b) What is the initial acceleration of the rocket?

KEY IDEA

We can relate the thrust T of a rocket to the magnitude a of the resulting acceleration with $T = Ma$, where M is the rocket's mass. However, M decreases and a increases as fuel is consumed. Because we want the initial value of a here, we must use the intial value M_i of the mass.

Calculation: We find

$$a = \frac{T}{M_i} = \frac{6440\ \text{N}}{850\ \text{kg}} = 7.6\ \text{m/s}^2. \qquad \text{(Answer)}$$

To be launched from Earth's surface, a rocket must have an initial acceleration greater than $g = 9.8\ \text{m/s}^2$. That is, it must be greater than the gravitational acceleration at the surface. Put another way, the thrust T of the rocket engine must exceed the initial gravitational force on the rocket, which here has the magnitude $M_i g$, which gives us

$$(850\ \text{kg})(9.8\ \text{m/s}^2) = 8330\ \text{N}.$$

Because the acceleration or thrust requirement is not met (here $T = 6400$ N), our rocket could not be launched from Earth's surface by itself; it would require another, more powerful, rocket.

SAMPLE PROBLEM 9.33

Freight car with varying mass

Consider the following two situations for freight car filled with sand and moving with uniform velocity V_0.

(a) Sand falls from a stationary hopper onto a freight car which is moving with uniform velocity V_0. The sand falls at the rate m (Fig. 9-45). How much force is needed to keep the freight car moving at the speed V_0?

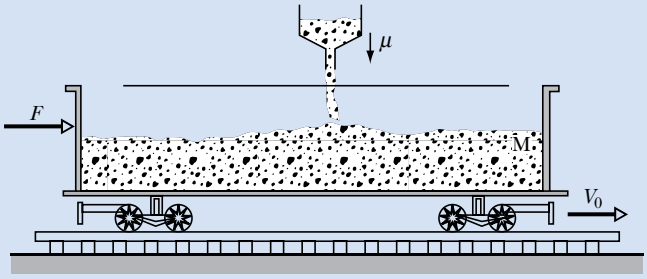

Figure 9-45 Freight car with a stationary hopper, moving with uniform velocity V_0.

KEY IDEA

(1) Although the velocity of the cart is constant but it may not remain so when the sand falls in it. The falling sand will apply force of friction on the cart; to counter this force we have to apply equal and opposite force on the cart.

(2) The required force on the cart cannot be calculated using $F = ma$ as the force on the cart due to sand is unknown. This problem will be solved by taking falling sand and cart as one system.

Calculations: At time t, the total mass of the freight car including the mass of sand present inside it is M.

$$M = M_0 + \mu t$$

where M_0 is the initial mass.

Let us consider the freight car, the sand inside it and small amount of sand which is going to fall in time Δt onto the car as one system. This small amount of the sand is in midair, falling freely and does not have any horizontal component of velocity. Thus, momentum of the system in horizontal direction at time t is:

$$P_i = (M_0 + \mu t)\, V_0 + \mu \Delta t(0)$$

Once the small amount of the sand joins the remaining sand in the car, it will have same velocity as the car due to impulsive friction. Thus, momentum of the system in horizontal direction at time $(t + \Delta t)$ is

$$P_f = (M + \mu \Delta t)\, V_0$$

Using

$$F = \frac{dp}{dt} = \frac{P_f - p_i}{\Delta t}$$

or

$$F\Delta t = P_f - P_i$$

$$F\Delta t = \mu \Delta t V_0$$

The required force is

$$F = \mu V_0$$

In Case of: What would happen if the upper surface of the freight car was slippery?

Calculation: Here the required force is $F = 0$ because sand particles will not move and fall down as there is no friction, hence no force. In this case, the initial speed of the sand is zero.

(b) Now consider a related case. The same freight car is leaking sand at the rate m (Fig. 9-46); what force is needed to keep the freight car moving uniformly with speed V_0?

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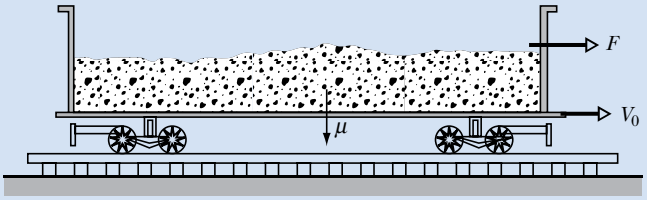

Figure 9-46 Freight car moving with constant velocity V_0 and sand leaking at the rate m.

KEY IDEA

The required force on the cart cannot be calculated using $F = ma$ as the force on the cart due to sand is not known. This problem will be solved by taking falling sand and cart as one system.

Calculations: At time t, the total mass of the freight car including the mass of sand present inside it is M.

$$M = M_0 - \mu t$$

Let us consider the freight car and the sand inside it as one system. Thus, momentum of the system in horizontal direction at time t is $P_i = (M_0 - \mu t)\, V_0$. Now consider small amount of the sand leaving the cart in time Δt, having mass $\mu \Delta t$. However, the velocity of sand after leaving the freight car is same as that of its initial velocity, and its momentum does not change.

Thus, momentum of the system in horizontal direction at time $(t + \Delta t)$ is:

$$P_f = \{M_0 - \mu(t + \Delta t)\} V_0 + \mu \Delta t\, V_0 = (M_0 - \mu t) V_0$$

Using $$F = \frac{dP}{dt} = \frac{\Delta P}{\Delta t}$$

or $$= \frac{P_f - P_i}{\Delta t}$$

$$F \Delta t = P_f - P_i$$

$$F \Delta t = 0$$

Since $dP/dt = 0$, no force is required. The sand does change its momentum when it hits the ground, and there is a resulting force on the ground, but that does not affect the motion of the freight car.

The required force is zero.

Learn: The falling sand need not apply any force on the cart while leaving it, so the required force on the cart is zero. Although the mass is varying but no force is required.

9.15 | SOME DERIVATIONS PERTAINING TO COM OF OBJECTS

Key Concepts

- Position coordinates of com of thin hemispherical shell of mass M and radius R, assuming uniform mass distribution is

$$x_{com} = 0 \quad \text{and} \quad y_{com} = \frac{R}{2}$$

- The com of a uniform hollow cone of height H and semi-vertex angle α is

$$y_{com} = \frac{2H}{3} \quad \text{(from the tip of cone)}$$

- The com of a uniform solid cone of height h and semi-angle α is

$$y_{com} = \frac{3h}{4}$$

- The com of a hemispherical object of uniform density has radius R

$$x_{com} = \frac{3R}{8},\ y_{com} = z_{com} = 0$$

Position of com of Thin Hemispherical Shell

Consider a thin hemispherical shell of mass M and radius R as shown in Fig. 9-47. Assuming the mass distribution to be uniform, consider a small circular strip of thickness ds. This thickness of ring subtends angle $d\theta$ at the center of the hemisphere as shown in Fig. 9-47. Radius of ring element is $R \cos \theta$.

Mass of the element is

$$dm = \text{Mass per unit area} \times \text{Area of circular strip}$$

$$= \frac{M}{2\pi R^2} \times (2\pi R \cos\theta) R d\theta$$

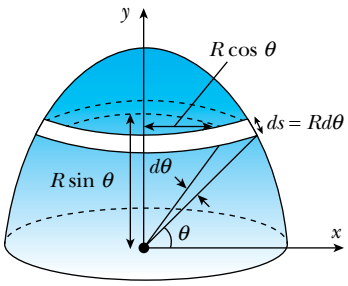

Figure 9-47 Thin hemispherical shell.

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Then $x_{com} = 0$ from symmetry of diagram and

$$y_{com} = \frac{1}{M}\int_0^{\pi/2} dm\, R\sin\theta$$

Substituting the value of dm, we get

$$y_{com} = \frac{1}{M}\int_0^{\pi/2} \frac{M}{2\pi R^2} \times (2\pi R\cos\theta R d\theta) R\sin\theta$$

$$= R\int_0^{\pi/2} \sin\theta\cos\theta d\theta = \frac{R}{2}$$

The com of a Uniform Hollow Cone

Consider a uniform hallow cone of height H and semi-vertex angle α as shown in Fig. 9-48. Mass per unit area:

$$\sigma = \frac{M}{\pi RL}$$

Consider a small elemental ring of thickness dx and radius r, then mass of the elemental ring,

$$dm = \alpha \times (2\pi r\, dx);\ y = x\cos\theta;\ dy = dx\cos\theta;\ r = y\tan\theta$$

$$y_{com} = \frac{\int_0^H y\left(\frac{M}{\pi RL}\right) \times 2\pi r dx}{M} = \frac{2H}{3} \quad \text{(from the tip of cone)}$$

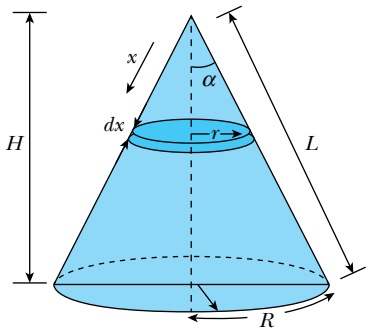

Figure 9-48 Uniform hollow cone.

The com of a Uniform Solid Cone

Consider a uniform solid cone of height 'h' and semi-angle α, as shown in Fig. 9-49. We place the apex of the cone at the origin. It is clear that the center of mass will lie along the y-axis. We divide the cone into disk of radius r and thickness dy. The volume of such a disk is

$$dV = \pi r^2 dy = \pi (y\ \tan\ \alpha)^2\, dy$$

The mass of the elemental disk is $dm = \rho\ dV$. First, let us determine the total mass of the cone M, where y goes from 0 to h, that is, the total height of cone.

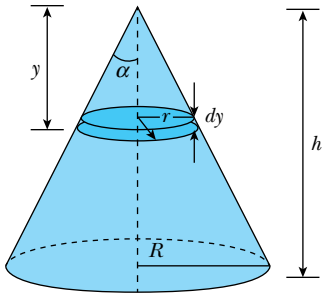

Figure 9-49 Uniform solid cone.

$$M = \int dm = \pi\rho\tan^2\alpha\int_0^h y^2 dy$$

$$= \pi\rho\tan^2\alpha\frac{h^3}{3} \tag{9-109}$$

The position of the com is given by

$$y_{com} = \frac{1}{M}\int_0^h y\, dm = \frac{1}{M}\pi\rho\tan^2\alpha\int_0^h y^3 dy$$

$$= \frac{1}{M}\pi\rho\tan^2\alpha\frac{h^4}{4} \tag{9-110}$$

From Eq. 9-109 and Eq. 9-110, we have

$$y_{com} = \frac{3h}{4}$$

This explains element of solid cone as disk and compares it with next problems element as ring.

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The com of a Hemispherical Object

Consider a hemispherical object with uniform distribution of mass as shown in Fig. 9-50. Let us prove by integration that the com of a hemispherical object of uniform density has radius R and center of mass lies along its axis of symmetry at a distance $3R/8$ from the center of the plane surface. As shown in Fig. 9-50, we choose a coordinate system with the origin at the center of the flat face and we let the yz plane be the plane of the face.

We now imagine slicing the hemisphere into disks parallel to the yz plane. A disk of thickness dx, located at a distance x from the plane face, has a radius $\sqrt{R^2 - x^2}$. Therefore, the mass of the disk $dm = \pi\rho(R^2 - x^2)dx$, where r is the mass density of the hemisphere. The center of mass of the hemisphere has an x coordinate x_{com} given by

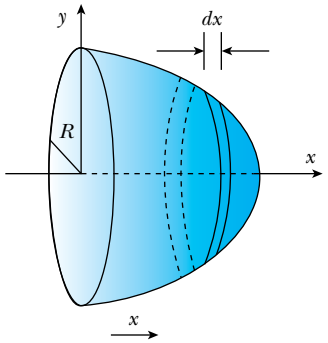

Figure 9-50 Hemispherical object of uniform density.

$$x_{com} = \frac{\int_0^R x\,dm}{\int_0^R dm} = \frac{\int_0^R \pi\rho x(\sqrt{R^2 - x^2})\,dx}{\int_0^R \pi\rho(\sqrt{R^2 - x^2})\,dx} = \frac{\left(\frac{x^2R^2}{2} - \frac{x^4}{4}\right)\Big|_0^R}{\left(xR^2 - \frac{x^3}{4}\right)\Big|_0^R} = \frac{R^4/4}{2R^3/3} = \frac{3R}{8}$$

$$y_{com} = z_{com} = 0$$

REVIEW AND SUMMARY

Center of Mass The **center of mass** of a system of n particles is defined to be the point whose coordinates are given by

$$x_{com} = \frac{1}{M}\sum_{i-1}^{n} m_i x_i, \quad y_{com} = \frac{1}{M}\sum_{i-1}^{n} m_i y_i, \quad z_{com} = \frac{1}{M}\sum_{i-1}^{n} m_i z_i, \qquad (9\text{-}5)$$

or

$$\vec{r}_{com} = \frac{1}{M}\sum_{i-1}^{n} m_i \vec{r}_i, \qquad (9\text{-}8)$$

where M is the total mass of the system.

Newton's Second Law for a System of Particles The motion of the center of mass of any system of particles is governed by **Newton's second law for a system of particles**, which is

$$\vec{F}_{net} = M\vec{a}_{com}. \qquad (9\text{-}14)$$

Here $\vec{F}_{net}$ is the net force of all the *external* forces acting on the system, M is the total mass of the system, and $\vec{a}_{com}$ is the acceleration of the system's center of mass.

Linear Momentum and Newton's Second Law For a single particle, we define a quantity $\vec{p}$ called its **linear momentum** as

$$\vec{p} = m\vec{v}, \qquad (9\text{-}24)$$

and can write Newton's second law in terms of this momentum:

$$\vec{F}_{net} = \frac{d\vec{p}}{dt}. \qquad (9\text{-}25)$$

Linear Momentum of a System of Particles For a system of particles, the relations for linear momentum and force become

$$\vec{P} = M\vec{v}_{com} \quad \text{and} \quad \vec{F}_{net} = \frac{d\vec{p}}{dt}. \qquad (9\text{-}27, 9\text{-}29)$$

Impulse and Momentum Applying Newton's second law in momentum form to a particle-like body involved in a collision leads to the **impulse–linear momentum theorem**:

$$\vec{p}_f - \vec{p}_i = \Delta\vec{p} = \vec{J}, \qquad (9\text{-}33, 9\text{-}34)$$

where $\vec{p}_f - \vec{p}_i = \Delta\vec{p}$ is the change in the body's linear momentum, and $\vec{J}$ is the **impulse** due to the force $\vec{F}(t)$ exerted on the body by the other body in the collision:

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)\,dt. \qquad (9\text{-}32)$$

If F_{avg} is the average magnitude of $\vec{F}(t)$ during the collision and Δt is the duration of the collision, then for one-dimensional motion

$$J = F_{avg}\,\Delta t. \qquad (9\text{-}33)$$

When a steady stream of bodies, each with mass m and speed v, collides with a body whose position is fixed, the average force on the fixed body is

$$F_{avg} = -\frac{n}{\Delta t}\Delta p = -\frac{n}{\Delta t} m\,\Delta v, \qquad (9\text{-}39)$$

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where $n/\Delta t$ is the rate at which the bodies collide with the fixed body, and Δv is the change in velocity of each colliding body. This average force can also be written as

$$F_{avg} = -\frac{\Delta m}{\Delta t}\Delta v, \tag{9-42}$$

where $\Delta m/\Delta t$ is the rate at which mass collides with the fixed body. In Eqs. 9-39 and 9-42, $\Delta v = -v$ if the bodies stop upon impact and $\Delta v = -2v$ if they bounce directly backward with no change in their speed.

Conservation of Linear Momentum If a system is isolated so that no net *external* force acts on it, the linear momentum $\vec{P}$ of the system remains constant:

$$\vec{P} = \text{constant} \quad \text{(closed, isolated system)}. \tag{9-45}$$

This can also be written as

$$\vec{P}_i = \vec{P}_f \quad \text{(closed, isolated system)}, \tag{9-46}$$

where the subscripts refer to the values of $\vec{P}$ at some initial time and at a later time. Equations 9-44 and 9-45 are equivalent statements of the **law of conservation of linear momentum**.

Collisions Collision is an event in which strong force acts between two or more bodies for a short time.

If the kinetic energy of the system is *conserved* (it is the same before and after the collision). Such a collision is called an **elastic collision**.

If the kinetic energy of the system is *not conserved* (it is always transferred from kinetic energy to other forms of energy). Such a collision is called an **inelastic collision**.

According to Newton's law of collision

$$\frac{v_{2f} - v_{1f}}{v_{2i} - v_{1i}} = -e \tag{9-58}$$

where v_{1f} and v_{2f} are the components of velocities of colliding points of the bodies 1 and 2, along the line of contact, after collision (with sign); v_{1i} and v_{2i} are the components of velocities of colliding points of the bodies, along the line of contact, before collision (with sign) and e is the coefficient of restitution.

In general, e has a value between zero and one, but the physical meaning of these two limits is

- For $e = 1$: Perfectly elastic collision.
- For $0 < e < 1$: Inelastic collision.
- For $e = 0$: Perfectly inelastic collision (bodies will move together).

Inelastic Collision in One Dimension In an *inelastic collision* of two bodies, the kinetic energy of the two-body system is not conserved (it is not a constant). If the system is closed and isolated, the total linear momentum of the system *must* be conserved (it *is* a constant), which we can write in vector form as

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}, \tag{9-59}$$

where subscripts i and f refer to values just before and just after the collision, respectively.

If the motion of the bodies is along a single axis, the collision is one-dimensional and we can write Eq. 9-59 in terms of velocity components along that axis:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}. \tag{9-60}$$

If the bodies stick together, the collision is a *completely inelastic collision* and the bodies have the same final velocity V (because they *are* stuck together).

Velocity of the Center of Mass The center of mass of a closed, isolated system of two colliding bodies is not affected by a collision. In particular, the velocity $\vec{v}_{com}$ of the center of mass cannot be changed by the collision.

Elastic Collisions in One Dimension An *elastic collision* is a special type of collision in which the kinetic energy of a system of colliding bodies is conserved. If the system is closed and isolated, its linear momentum is also conserved. For a onedimensional collision in which body 2 is a target and body 1 is an incoming projectile, conservation of kinetic energy and linear momentum yield the following expressions for the velocities immediately after the collision:

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \tag{9-76}$$

and

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}. \tag{9-77}$$

Collisions in Two Dimensions If two bodies collide and their motion is not along a single axis (the collision is not head-on), the collision is two-dimensional. If the two-body system is closed and isolated, the law of conservation of momentum applies to the collision and can be written as

$$\vec{P}_{1i} + \vec{P}_{2i} = \vec{P}_{1f} + \vec{P}_{2f}. \tag{9-86}$$

In component form, the law gives two equations that describe the collision (one equation for each of the two dimensions). If the collision is also elastic (a special case), the conservation of kinetic energy during the collision gives a third equation:

$$K_{1i} + K_{2i} = K_{1f} + K_{2f}. \tag{9-87}$$

***C*-frame** The frame of reference attached to the center of mass of a system is called the **center of mass frame of reference** or ***C*-frame of reference**.

$$\vec{P} = \sum \vec{P}_i = 0 \quad \text{(in the } C\text{-frame of reference)} \tag{9-92}$$

The kinetic energy of system in ground frame related to kinetic energy in C-frame is

$$K_{sys} = K_{sys/c} + \frac{p_c^2}{2M} \tag{9-95}$$

The total kinetic energy of the two particles in the C-frame is

$$K_{sys/c} = \frac{\mu v_{rel}^2}{2} + \frac{m v_c^2}{2\mu} \quad (\text{where } m = m_1 + m_2) \tag{9-100}$$

Variable-Mass Systems: A Rocket In the absence of external forces a rocket accelerates at an instantaneous rate given by

$$Rv_{rel} = Ma \quad \text{(first rocket equation)}, \tag{9-107}$$

in which M is the rocket's instantaneous mass (including unexpended fuel), R is the fuel consumption rate, and v_{rel} is

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the fuel's exhaust speed relative to the rocket. The term Rv_{rel} is the **thrust** of the rocket engine. For a rocket with constant R and v_{rel}, whose speed changes from v_i to v_f when its mass changes from M_i to M_f,

$$v_f - v_i = v_{rel} \ln \frac{M_i}{M_f} \quad \text{(second rocket equation).} \qquad (9\text{-}108)$$

Some Derivations Pertaining to com of Objects Position coordinates of com of thin hemispherical shell of mass M and radius R, assuming uniform mass distribution

$$x_{com} = 0 \quad \text{and} \quad y_{com} = \frac{R}{2}$$

The com of a uniform hollow cone of height H and semi-vertex angle α

$$y_{com} = \frac{2H}{3} \quad \text{(from the tip of cone)}$$

The com of a uniform solid cone of height h and semi-angle α

$$y_{com} = \frac{3h}{4}$$

The com of a hemispherical object of uniform density has radius R

$$x_{com} = \frac{3R}{8}, \; y_{com} = z_{com} = 0$$

PROBLEMS

1. Three particles of mass 1.0 kg, 2.0 kg, and 3.0 kg are placed at the vertices A, B, and C, respectively, of an equilateral triangle ABC of edge 1.0 m (Fig. 9-51). Find the distance of their center of mass from A.

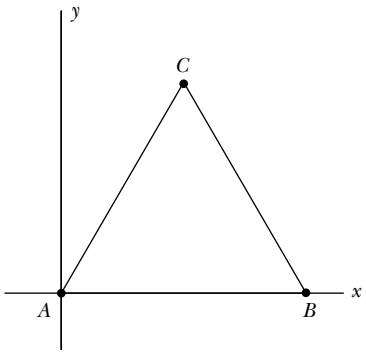

Figure 9-51 Problem 1.

2. Figure 9-52 shows a three-particle system, with masses $m_1 = 2.0$ kg, $m_2 = 4.0$ kg, and $m_3 = 8.0$ kg. The scales on the axes are set by $x_s = 2.0$ m and $y_s = 2.0$ m. What are (a) the x coordinate and (b) the y coordinate of the system's center of mass? (c) If m_3 is gradually increased, does the center of mass of the system shift toward or away from that particle, or does it remain stationary?

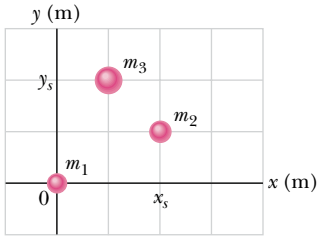

Figure 9-52 Problem 2.

3. In Fig. 9-53, three uniform thin rods, each of length $L = 24$ cm, form an inverted U. The vertical rods each have a mass of 14 g; the horizontal rod has a mass of 42 g. What are (a) the x coordinate and (b) the y coordinate of the system's center of mass?

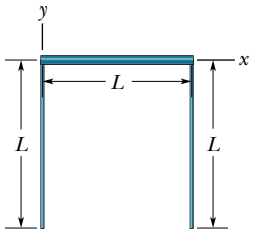

Figure 9-53 Problem 3.

4. Figure 9-54 shows a cubical box that has been constructed from uniform metal plate of negligible thickness. The box is open at the top and has edge length $L = 50$ cm. Find (a) the x coordinate, (b) the y coordinate, and (c) the z coordinate of the center of mass of the box.

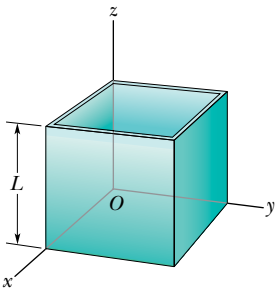

Figure 9-54 Problem 4.

5. A uniform soda can of mass 0.140 kg is 12.0 cm tall and filled with 0.354 kg of soda (Fig. 9-55). Then small holes are drilled in the top and bottom (with negligible loss of metal) to drain the soda. What is the height h of the com of the can and contents (a) initially and (b) after the can loses all the soda? (c) What happens to h as the soda drains out? (d) If x is the height of the remaining soda at any given instant, find x when the com reaches its lowest point.

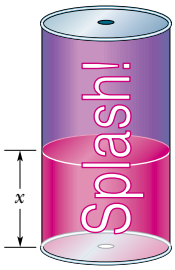

Figure 9-55 Problem 5.

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6. In the arrangement shown in Fig. 9-56, m_A = 2.0 kg and m_B = 1.0 kg. The pulley is massless; the string is massless and long. The system is released at t = 0 s. Find (a) the acceleration of the center of mass of the blocks, (b) the displacement of the center of mass at t = 2.0 s, and (c) the speed of the center of mass when m_A strikes the floor.

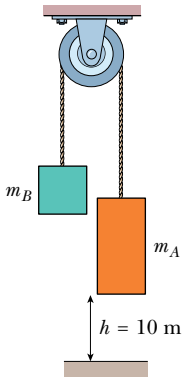

Figure 9-56 Problem 6.

7. A 1000 kg automobile is at rest at a traffic signal. At the instant the light turns green, the automobile starts to move with a constant acceleration of 3.0 m/s². At the same instant a 2000 kg truck, traveling at a constant speed of 8.0 m/s, overtakes and passes the automobile. (a) How far is the com of the automobile–truck system from the traffic light at t = 5.0 s? (b) What is the speed of the com then?

8. A big olive (m = 0.50 kg) lies at the origin of an xy coordinate system, and a big Brazil nut (M = 1.5 kg) lies at the point (1.0, 2.0) m. At t = 0, a force $\vec{F}_o = (2.0\hat{i} + 3.0\hat{j})$ N begins to act on the olive, and a force $\vec{F}_n = (-3.0\hat{i} - 2.0\hat{j})$ N begins to act on the nut. In unit-vector notation, what is the displacement of the center of mass of the olive–nut system at t = 4.0 s, with respect to its position at t = 0?

9. Two skaters, one with mass 75 kg and the other with mass 40 kg, stand on an ice rink holding a pole of length 10 m and negligible mass. Starting from the ends of the pole, the skaters pull themselves along the pole until they meet. How far does the 40 kg skater move?

10. A shell is shot with an initial velocity $\vec{v}_0$ of 20 m/s, at an angle of θ_0 = 60° with the horizontal. At the top of the trajectory, the shell explodes into two fragments of equal mass (Fig. 9-57). One fragment, whose speed immediately after the explosion is zero, falls vertically. How far from the gun does the other fragment land, assuming that the terrain is level and that air drag is negligible?

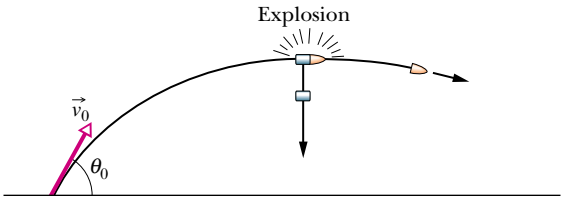

Figure 9-57 Problem 10.

11. In Fig. 9-58, two particles are launched from the origin of the coordinate system at time t = 0. Particle 1 of mass m_1 = 5.00 g is shot directly along the x axis on a frictionless floor, with constant speed 10.0 m/s. Particle 2 of mass m_2 = 3.00 g is shot with a velocity of magnitude 20.0 m/s, at an upward angle such that it always stays directly above particle 1. (a) What is the maximum height H_{max} reached by the com of the two-particle system? In unit-vector notation, what are the (b) velocity and (c) acceleration of the com when the com reaches H_{max}?

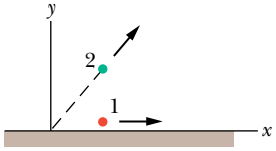

Figure 9-58 Problem 11.

12. Richard, of mass 80 kg, and Camelia, who is lighter, are in a 30 kg canoe on a lake. When the canoe is at rest in the placid water, they exchange seats, which are 3.0 m apart and symmetrically located with respect to the canoe's center. If the canoe moves 45 cm horizontally relative to a pier post, what is Camelia's mass?

13. In Fig. 9-59*a*, a 4.5 kg dog stands on an 18 kg flatboat at distance D = 6.1 m from the shore. It walks 2.4 m along the boat toward shore and then stops. Assuming no friction between the boat and the water, find how far the dog is then from the shore. (*Hint:* See Fig. 9-59*b*.)

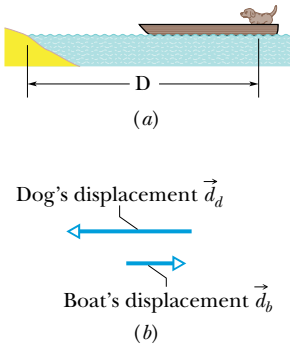

Figure 9-59 Problem 13.

14. A 0.70 kg ball moving horizontally at 6.0 m/s strikes a vertical wall and rebounds with speed 3.5 m/s. What is the magnitude of the change in its linear momentum?

15. A 100 kg motorbike moves along AB at 10.0 km/h, and after some time, the motorbike turns to travel along BC at the same speed as shown in Fig. 9-60. Find (a) the change in its kinetic energy and the (b) magnitude and (c) direction (relative to +x) of the change in its momentum.

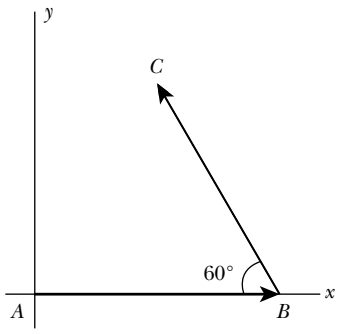

Figure 9-60 Problem 15.

16. A ball of mass 50 g moving with a speed of 2.0 m/s strikes a wall at an angle of incidence 45° and is reflected from the wall at the same angle and with the same speed. See the overhead view in Fig. 9-61. Calculate (a) the magnitude of the change $\Delta\vec{p}$ in the momentum of the ball, (b) the change in the magnitude of the momentum $\vec{p}$ of the ball, and (c) the change in the magnitude of the momentum of the wall.

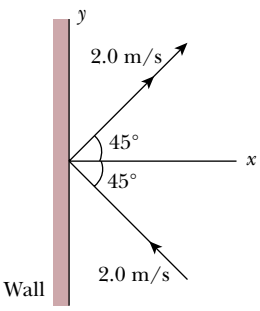

Figure 9-61 Problem 16.

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17. Figure 9-62 gives an overhead view of the path taken by a 0.150 kg cue ball as it bounces from a rail of a pool table. The ball's initial speed is 2.00 m/s, and the angle θ_1 is 30.0°. The bounce reverses the y component of the ball's velocity but does not alter the x component. What are (a) angle θ_2 and (b) the change in the ball's linear momentum in unit-vector notation? (The fact that the ball rolls is irrelevant to the problem.)

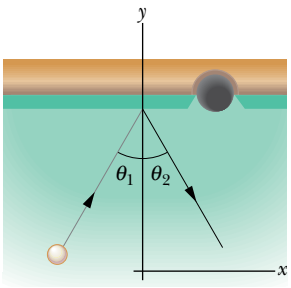

Figure 9-62 Problem 17.

18. Until his seventies, Henri LaMothe (Fig. 9-63) excited audiences by belly-flopping from a height of 12 m into 30 cm of water. Assuming that he stops just as he reaches the bottom of the water and estimating his mass, find the magnitude of the impulse on him from the water.

George Long/Getty Images, Inc.

Figure 9-63 Problem 18. Belly-flopping into 30 cm of water.

19. In February 1955, a paratrooper fell 370 m from an airplane without being able to open his chute but happened to land in snow, suffering only minor injuries. Assume that his speed at impact was 56 m/s (terminal speed), that his mass (including gear) was 85 kg, and that the magnitude of the force on him from the snow was at the survivable limit of 1.2×10^5 N. What are (a) the minimum depth of snow that would have stopped him safely and (b) the magnitude of the impulse on him from the snow?

20. A 5.00 g bullet moving at 100 m/s strikes a log. Assume that the bullet undergoes a uniform deceleration and stops after penetrating 6.00 cm. Find (a) the time taken by the bullet to stop, (b) the impulse on the log, and (c) the magnitude of the average force experienced by the log.

21. In a common but dangerous prank, a chair is pulled away as a person is moving downward to sit on it, causing the victim to land hard on the floor. Suppose the victim falls by 0.50 m, the mass that moves downward is 75 kg, and the collision on the floor lasts 0.088 s. What are the magnitudes of the (a) impulse and (b) average force acting on the victim from the floor during the collision?

22. A 3.00 kg block slides on a frictionless horizontal surface, first moving to the left at 50.0 m/s. It collides with a spring whose other end is fixed to a wall, compresses the spring, and is brought to rest momentarily. Then it continues to be accelerated toward the right by the force of the compressed spring. The block acquires a final speed of 40.0 m/s. It is in contact with the spring for 0.020 s. Find (a) the magnitude and (b) the direction of the impulse of the spring force on the block. (c) What is the magnitude of the spring's average force on the block?

23. In tae-kwon-do, a hand is slammed down onto a target at a speed of 13 m/s and comes to a stop during the 5.5 ms collision. Assume that during the impact the hand is independent of the arm and has a mass of 0.70 kg. What are the magnitudes of the (a) impulse and (b) average force on the hand from the target?

24. A ball of mass 1.00 kg is attached to a loose string fixed to a ceiling. The ball is released from rest and falls 2.00 m, where the string suddenly stops it. Find the impulse on it from the string.

25. *Jumping up before the elevator hits.* After the cable snaps and the safety system fails, an elevator cab free-falls from a height of 36 m. During the collision at the bottom of the elevator shaft, a 90 kg passenger is stopped in 5.0 ms. (Assume that neither the passenger nor the cab rebounds.) What are the magnitudes of the (a) impulse and (b) average force on the passenger during the collision? If the passenger were to jump upward with a speed of 7.0 m/s relative to the cab floor just before the cab hits the bottom of the shaft, what are the magnitudes of the (c) impulse and (d) average force (assuming the same stopping time)?

26. A 2.5 kg toy car can move along an x axis; Fig. 9-64 gives F_x of the force acting on the car, which begins at rest at time $t = 0$. The scale on the F_x axis is set by $F_{xs} = 5.0$ N. In unit-vector notation, what is $\vec{p}$ at (a) $t = 4.0$ s and (b) $t = 7.0$ s, and (c) what is $\vec{v}$ at $t = 9.0$ s?

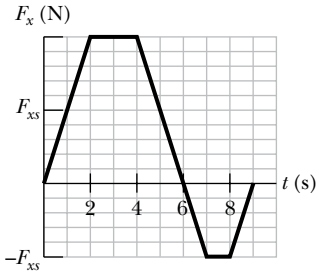

Figure 9-64 Problem 26.

27. Figure 9-65 shows a 0.300 kg baseball just before and just after it collides with a bat. Just before, the ball has velocity

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$\vec{v}_1$ of magnitude 12.0 m/s and angle $\theta_1 = 35.0°$. Just after, it is traveling directly upward with velocity $\vec{v}_2$ of magnitude 10.0 m/s. The duration of the collision is 2.00 ms. What are the (a) magnitude and (b) direction (relative to the positive direction of the x axis) of the impulse on the ball from the bat? What are the (c) magnitude and (d) direction of the average force on the ball from the bat?

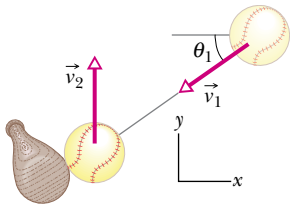

Figure 9-65 Problem 27.

28. Figure 9-66 shows an approximate plot of force magnitude F versus time t during the collision of a 58 g Superball with a wall. The initial velocity of the ball is 34 m/s perpendicular to the wall; the ball rebounds directly back with approximately the same speed, also perpendicular to the wall. What is F_{max}, the maximum magnitude of the force on the ball from the wall during the collision?

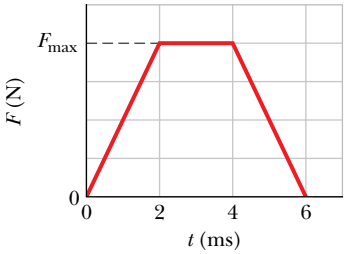

Figure 9-66 Problem 28.

29. A 0.25 kg puck is initially stationary on an ice surface with negligible friction. At time $t = 0$, a horizontal force begins to move the puck. The force is given by $\vec{F} = (12.0 - 3.00t^2)\hat{i}$, with $\vec{F}$ in newtons and t in seconds, and it acts until its magnitude is zero. (a) What is the magnitude of the impulse on the puck from the force between $t = 0.750$ s and $t = 1.25$ s? (b) What is the change in momentum of the puck between $t = 0$ and the instant at which $F = 0$?

30. A particle of unknown mass is acted upon by a force $\vec{F} = (100e^{-2t}\hat{i})$ N. If at $t = 0.00$ s the particle is at rest, for the time interval $t = 0.00$ s to $t = 2.00$ s find (a) the impulse on the particle and (b) the average force on the particle.

31. In the overhead view of Fig. 9-67, a 300 g ball with a speed v of 6.0 m/s strikes a wall at an angle θ of 30° and then rebounds with the same speed and angle. It is in contact with the wall for 10 ms. In unitvector notation, what are (a) the impulse on the ball from the wall and (b) the average force on the wall from the ball?

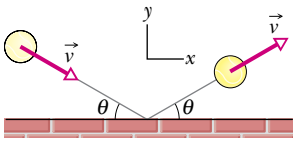

Figure 9-67 Problem 31.

32. A man of mass $m_1 = 80$ kg is standing on a platform of mass $m_2 = 20$ kg that lies on a frictionless horizontal surface. The man starts moving on the platform with a velocity $v_r = 10$ m/s relative to the platform. Find the recoil speed of the platform.

33. A space vehicle is traveling at 4800 km/h relative to Earth when the exhausted rocket motor (mass $4m$) is disengaged and sent backward. The relative speed between motor and command module (mass m) is then 82 km/h. What is the speed of the command module relative to Earth just after the separation?

34. A 15.0 kg package is moving at a speed of 10.0 m/s vertically upward along a y axis when it explodes into three fragments: a 2.00 kg fragment is shot upward with an initial speed of 20.0 m/s and a 3.00 kg fragment is shot in the positive direction of a horizontal x axis with an initial speed of 5.00 m/s. Find (a) the speed of the third fragment right after the explosion and (b) the total kinetic energy provided by the explosion.

35. In Fig. 9-68, a stationary block explodes into two pieces L and R that slide across a frictionless floor and then into regions with friction, where they stop. Piece L, with a mass of 2.0 kg, encounters a coefficient of kinetic friction $\mu_L = 0.35$ and slides to a stop in distance $d_L = 0.15$ m. Piece R encounters a coefficient of kinetic friction $\mu_R = 0.50$ and slides to a stop in distance $d_R = 0.30$ m. What was the mass of the block?

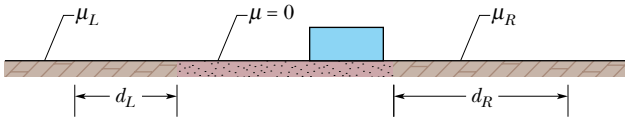

Figure 9-68 Problem 35.

36. A 4.0 kg mess kit sliding on a frictionless surface explodes into two 2.0 kg parts: 3.0 m/s, due north, and 6.0 m/s, 30° north of east. What is the original speed of the mess kit?

37. Particle A and particle B are held together with a compressed spring between them. When they are released, the spring pushes them apart, and they then fly off in opposite directions, free of the spring. The mass of A is 2.00 times the mass of B, and the energy stored in the spring was 80 J. Assume that the spring has negligible mass and that all its stored energy is transferred to the particles. Once that transfer is complete, what are the kinetic energies of (a) particle A and (b) particle B?

38. A bullet of mass 10 g strikes a ballistic pendulum of mass 2.0 kg. The center of mass of the pendulum rises a vertical distance of 12 cm. Assuming that the bullet remains embedded in the pendulum, calculate the bullet's initial speed.

39. A 5.20 g bullet moving at 700 m/s strikes a 700 g wooden block at rest on a frictionless surface. The bullet emerges, traveling in the same direction with its speed reduced to 450 m/s. (a) What is the resulting speed of the block? (b) What is the speed of the bullet–block center of mass?

40. In Fig. 9-69a, a 3.50 g bullet is fired horizontally at two blocks at rest on a frictionless table. The bullet passes through block 1 (mass 1.20 kg) and embeds itself in block 2 (mass 1.80 kg). The blocks end up with speeds

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$v_1 = 0.630$ m/s and $v_2 = 1.40$ m/s (Fig. 9-69*b*). Neglecting the material removed from block 1 by the bullet, find the speed of the bullet as it (a) leaves and (b) enters block 1.

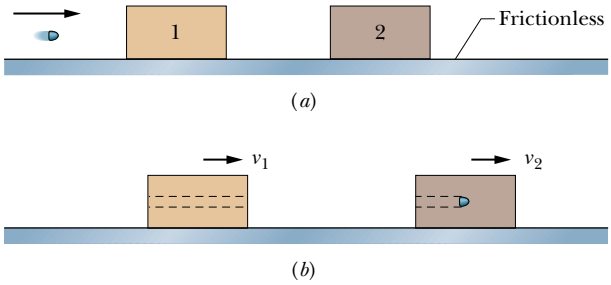

Figure 9-69 Problem 40.

41. In Fig. 9-70, a 10 g bullet moving directly upward at 1000 m/s strikes and passes through the center of mass of a 5.0 kg block initially at rest. The bullet emerges from the block moving directly upward at 300 m/s. To what maximum height does the block then rise above its initial position?

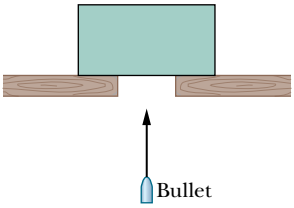

Figure 9-70 Problem 41.

42. A completely inelastic collision occurs between two balls of wet putty that move directly toward each other along a vertical axis. Just before the collision, one ball, of mass 3.0 kg, is moving upward at 20 m/s and the other ball, of mass 2.0 kg, is moving downward at 10 m/s. How high do the combined two balls of putty rise above the collision point? (Neglect air drag.)

43. Block 1 of mass 3.0 kg is sliding across a floor with speed $v_1 = 2.0$ m/s when it makes a head-on, one-dimensional, elastic collision with initially stationary block 2 of mass 2.0 kg. The coefficient of kinetic friction between the blocks and the floor is $\mu_k = 0.30$. Find the speeds of (a) block 1 and (b) block 2 just after the collision. Also find (c) their final separation after friction has stopped them and (d) the energy lost to thermal energy because of the friction.

44. In the "before" part of Fig. 9-71, car A (mass 1100 kg) is stopped at a traffic light when it is rear-ended by car B (mass 1400 kg). Both cars then slide with locked wheels until the frictional force from the slick road (with a low μ_k of 0.10) stops them, at distances $d_A = 8.2$ m and $d_B = 6.1$ m. What are the speeds of (a) car A and (b) car B at the start of the sliding, just after the collision? (c) Assuming that linear momentum is conserved during the collision, find the speed of car B just before the collision. (d) Explain why this assumption may be invalid.

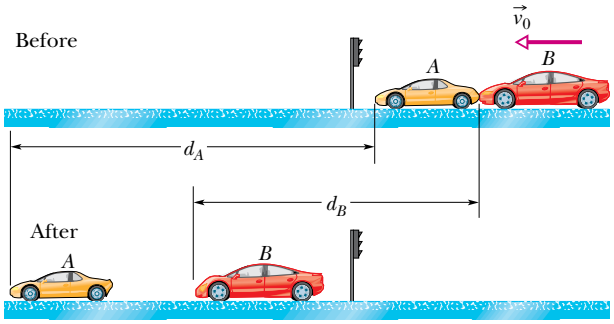

Figure 9-71 Problem 44.

45. In Fig. 9-72, a ball of mass $m = 60$ g is shot with speed $v_i = 22$ m/s into the barrel of a spring gun of mass $M = 240$ g initially at rest on a frictionless surface. The ball sticks in the barrel at the point of maximum compression of the spring. Assume that the increase in thermal energy due to friction between the ball and the barrel is negligible. (a) What is the speed of the spring gun after the ball stops in the barrel? (b) What fraction of the initial kinetic energy of the ball is stored in the spring?

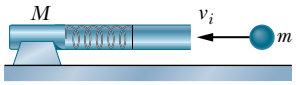

Figure 9-72 Problem 45.

46. In Fig. 9-73, block 2 (mass 1.0 kg) is at rest on a frictionless surface and touching the end of an unstretched spring of spring constant 230 N/m. The other end of the spring is fixed to a wall. Block 1 (mass 2.0 kg), traveling at speed $v_1 = 4.0$ m/s, collides with block 2, and the two blocks stick together. When the blocks momentarily stop, by what distance is the spring compressed?

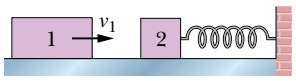

Figure 9-73 Problem 46.

47. In Fig. 9-74, block 1 (mass 2.0 kg) is moving rightward at 10 m/s and block 2 (mass 5.0 kg) is moving rightward at 3.0 m/s. The surface is frictionless, and a spring with a spring constant of 1120 N/m is fixed to block 2. When the blocks collide, the compression of the spring is maximum at the instant the blocks have the same velocity. Find the maximum compression.

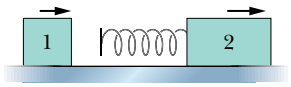

Figure 9-74 Problem 47.

48. In Fig. 9-75, block A (mass 1.6 kg) slides into block B (mass 2.4 kg), along a frictionless surface. The directions of three velocities before (i) and after (f) the collision are indicated; the corresponding speeds are $v_{Ai} = 5.5$ m/s, $v_{Bi} = 2.5$ m/s, and $v_{Bf} = 4.9$ m/s. What are the (a) speed and (b) direction (left or right) of velocity $\vec{v}_{Af}$? (c) Is the collision elastic?

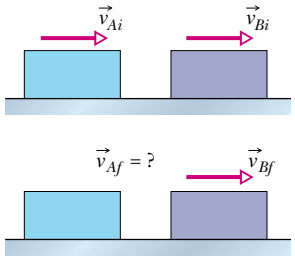

Figure 9-75 Problem 48.

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49. Two titanium spheres approach each other head-on with the same speed and collide elastically. After the collision, one of the spheres, whose mass is 250 g, remains at rest. (a) What is the mass of the other sphere? (b) What is the speed of the two sphere center of mass if the initial speed of each sphere is 2.00 m/s?

50. Block 1 of mass m_1 slides along a frictionless floor and into a one-dimensional elastic collision with stationary block 2 of mass $m_2 = 3m_1$. Prior to the collision, the center of mass of the two block system had a speed of 3.00 m/s. Afterward, what are the speeds of (a) the center of mass and (b) block 2?

51. A steel ball of mass 0.600 kg is fastened to a cord that is 70.0 cm long and fixed at the far end. The ball is then released when the cord is horizontal (Fig. 9-76). At the bottom of its path, the ball strikes a 2.80 kg steel block initially at rest on a frictionless surface. The collision is elastic. Find (a) the speed of the ball and (b) the speed of the block, both just after the collision.

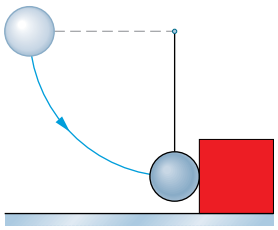

Figure 9-76 Problem 51.

52. Particle 1 with mass m and velocity v and particle 2 with mass $2m$ and velocity $-2v$ are moving toward each other along an x axis when they undergo a one-dimensional elastic collision. After the collision, what are the velocities of (a) particle 1 and (b) particle 2? What is the velocity of the center of mass of the two-particle system (c) before and (d) after the collision?

53. Block 1, with mass m_1 and speed 3.0 m/s, slides along an x axis on a frictionless floor and then undergoes a one-dimensional elastic collision with stationary block 2, with mass $m_2 = 0.40m_1$. The two blocks then slide into a region where the coefficient of kinetic friction is 0.50; there they stop. How far into that region do (a) block 1 and (b) block 2 slide?

54. In Fig. 9-77, particle 1 of mass m_1 = 0.30 kg slides rightward along an x axis on a frictionless floor with a speed of 2.0 m/s. When it reaches $x = 0$, it undergoes a one-dimensional elastic collision with stationary particle 2 of mass m_2 = 0.40 kg. When particle 2 then reaches a wall at x_w = 70 cm, it bounces from the wall with no loss of speed. At what position on the x axis does particle 2 then collide with particle 1?

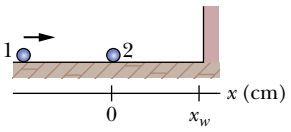

Figure 9-77 Problem 54.

55. In Fig. 9-78, block 1 of mass m_1 slides from rest along a frictionless ramp from height $h = 3.00$ m and then collides with stationary block 2, which has mass $m_2 = 2.00m_1$. After the collision, block 2 slides into a region where the coefficient of kinetic friction μ_k is 0.450 and comes to a stop in distance d within that region. What is the value of distance d if the collision is (a) elastic and (b) completely inelastic?

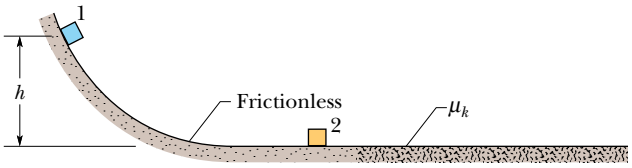

Figure 9-78 Problem 55.

56. A small ball of mass m is aligned above a larger ball of mass M = 0.63 kg (with a slight separation, as with the baseball and basketball of Fig. 9-79*a*), and the two are dropped simultaneously from a height of h = 1.8 m. (Assume the radius of each ball is negligible relative to h.) (a) If the larger ball rebounds elastically from the floor and then the small ball rebounds elastically from the larger ball, what value of m results in the larger ball stopping when it collides with the small ball? (b) What height does the small ball then reach (Fig. 9-79*b*)?

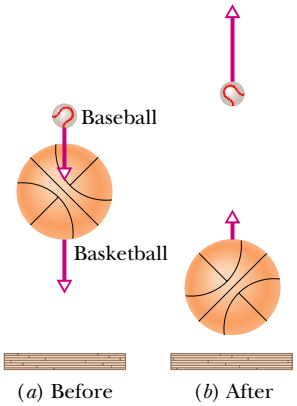

Figure 9-79 Problem 56.

57. In Fig. 9-80, puck 1 of mass $m_1 = 0.25$ kg is sent sliding across a frictionless lab bench, to undergo a one-dimensional elastic collision with stationary puck 2. Puck 2 then slides off the bench and lands a distance d from the base of the bench. Puck 1 rebounds from the collision and slides off the opposite edge of the bench, landing a distance $2d$ from the base of the bench. What is the mass of puck 2? (*Hint:* Be careful with signs.)

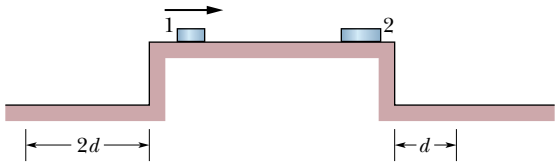

Figure 9-80 Problem 57.

58. In the two-dimensional collision in Fig. 9-38, the projectile particle has mass $m_1 = m$, initial speed $v_{1i} = 3v_0$, and final speed $v_{1f} = \sqrt{5v_0}$. The initially stationary target particle has mass $m_1 = 2m$ and final speed $v_{2f} = v_2$. The projectile is scattered at an angle given by $\tan \theta_1 = 2.0$. (a) Find angle θ_2. (b) Find v_2 in terms of v_0. (c) Is the collision elastic?

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59. After a completely inelastic collision, two objects of the same mass and same initial speed move away together at half their initial speed. Find the angle between the initial velocities of the objects.

60. A force $\vec{F}$ acts on two particles of masses m and $4.0m$ moving at the same speed but at right angles to each other, as shown in Fig. 9-81. The force acts on both the particles for a time T. Consequently, the particle of mass m moves with a velocity $4v$ in its original direction. (a) Find the new velocity v' of the other particle. (b) Also find the change in the kinetic energy of the system.

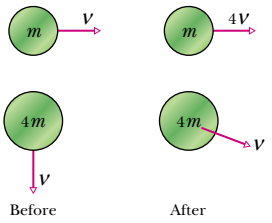

Figure 9-81 Problem 60.

61. In Fig. 9-82, a bob of mass $10m$ is suspended from an inextensible string with negligible mass. When the bob is in equilibrium (at rest), two particles each of mass m strike it simultaneously with the speeds indicated. The particles stick to the bob. Find (a) the magnitude of the net impulse on the string due to the collision, (b) the velocity of the system just after the collision, and (c) the mechanical energy lost in the collision.

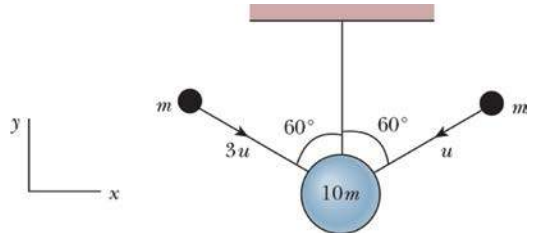

Figure 9-82 Problem 61.

62. A 6090 kg space probe moving nose-first toward Jupiter at 120 m/s relative to the Sun fires its rocket engine, ejecting 70.0 kg of exhaust at a speed of 253 m/s relative to the space probe. What is the final velocity of the probe?

63. Consider a rocket that is in deep space and at rest relative to an inertial reference frame. The rocket's engine is to be fired for a certain interval. What must be the rocket's *mass ratio* (ratio of initial to final mass) over that interval if the rocket's original speed relative to the inertial frame is to be equal to (a) the exhaust speed (speed of the exhaust products relative to the rocket) and (b) 2.0 times the exhaust speed?

64. A rocket that is set for a vertical launch has a mass of 50.0 kg and contains 450 kg of fuel. The rocket can have a maximum exhaust velocity of 2.00 km/s. What should be the minimum rate of fuel consumption (a) to just lift it off the launching pad and (b) to give it an acceleration of 20.0 m/s²? (c) If the consumption rate is set at 10.0 kg/s, what is the rocket speed at the moment when the fuel is fully consumed?

PRACTICE QUESTIONS

Single Correct Choice Type

1. A ball is dropped from a height h on a stationary floor and rebounds several times until it stops. If the coefficient of restitution is e, then the total distance covered by the ball before it stops, would be

(a) $\left(\frac{1+e^2}{1-e^2}\right)h$ **(b)** $\left(\frac{1-e^2}{1+e^2}\right)h$

(c) $\left(\frac{1+e}{1-e^2}\right)h$ **(d)** $\left(\frac{1+e^2}{1}\right)h$

2. Which one of the following statements concerning momentum is true?

(a) Momentum is a force.
(b) Momentum is a scalar quantity.
(c) The SI unit of momentum is kg·m²/s.
(d) Momentum and impulse are measured in the same units.

3. A 62.0-kg person, standing on a diving board, dives straight down into the water. Just before striking the water, her speed is 5.50 m/s. At a time of 1.65 s after she enters the water, her speed is reduced to 1.10 m/s. What is the net average force (magnitude and direction) that acts on her when she is in the water?

(a) 206 N, downward
(b) 248 N, upward
(c) 248 N, downward
(d) 165 N, upward

4. A ball hits a floor and rebounds after an inelastic collision. In this case,

(a) the momentum of the ball just after the collision is same as that just before the collision.
(b) the mechanical energy of the ball remains the same during the collision.
(c) the total momentum of the ball and the Earth is conserved.
(d) the total energy of the ball and the Earth remains the same.

5. A 1.0-kg ball has a velocity of 12 m/s downward just before it strikes the ground and bounces up with a velocity of 12 m/s upward. What is the change in momentum of the ball?

(a) zero kg·m/s
(b) 12 kg·m/s, downward
(c) 12 kg·m/s, upward
(d) 24 kg·m/s, upward

6. A 0.1-kg steel ball is dropped straight down onto a hard, horizontal floor and bounces straight up. The ball's speed

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just before *and just after* impact with the floor is 10 m/s. Determine the magnitude of the impulse delivered to the floor by the steel ball.

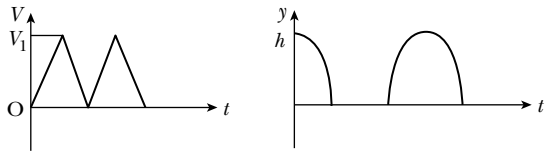
(a) zero N·s **(b)** 1 N·s
(c) 2 N·s **(d)** 10 N·s

7. A stick is thrown in the air and lands at some distance from the thrower. The center of mass of the stick will move along a parabolic path

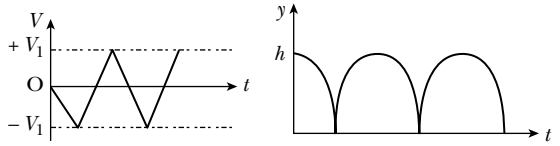
(a) in all cases.
(b) only if the stick is uniform.
(c) only if the stick does not have any rotational motion.
(d) only if the center of mass of the stick lies at some point on it and not outside it.

8. Consider a rubber ball freely falling from a height $h = 4.9$ m onto a horizontal elastic plate. Assume that the duration of collision is negligible and the collision with the plate is totally elastic. Then, the velocity as a function of time and the height as function of time will be

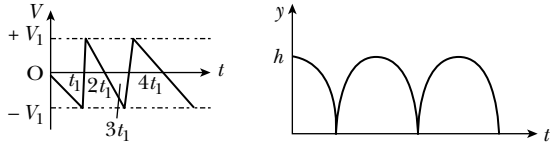
(a)



(b)



(c)



(d) None of these

9. A machine gun fires 50-g bullets at the rate of 4 bullets per second. The bullets leave the gun at a speed of 1000 m/s. What is the average recoil force experienced by the machine gun?

(a) 10 N **(b)** 20 N
(c) 100 N **(d)** 200 N

10. A particle of mass 1.0 g moving with a velocity $\vec{v}_1 = (3\hat{i} - 2\hat{j})$ m/s experiences a perfectly inelastic collision with another particle of mass 2.0 g and velocity $\vec{v}_2 = (4\hat{i} - 6\hat{j})$ m/s. Find the magnitude of the velocity vector $\vec{v}$ of the coalesced particles.

(a) 2.3 m/s **(b)** 5.93 m/s
(c) 3.2 m/s **(d)** 6.4 m/s

11. A 0.065-kg tennis ball moving to the right with a speed of 15 m/s is struck by a tennis racket, causing it to move to the left with a speed of 15 m/s. If the ball remains in contact with the racquet for 0.020 s, what is the magnitude of the average force experienced by the ball?

(a) 0 N **(b)** 98 N
(c) 160 N **(d)** 240 N

12. A space probe is traveling in outer space with a momentum that has a magnitude of 7.5×10^7 kg·m/s. A retrorocket is fired to slow down the probe. It applies a force to the probe that has a magnitude of 2.0×10^6 N and a direction opposite to the probe's motion. It fires for a period of 12 s. Determine the momentum of the probe after the retrorocket ceases to fire.

(a) $+5.1 \times 10^7$ kg·m/s **(b)** $+2.4 \times 10^7$ kg·m/s
(c) $+7.5 \times 10^7$ kg·m/s **(d)** $+9.9 \times 10^7$ kg·m/s

13. While a car is stopped at a traffic light in a storm, raindrops strike the roof of the car. The area of the roof is 5.0 m^2. Each raindrop has a mass of 3.7×10^{-4} kg and speed of 2.5 m/s before impact and is at rest after the impact. If, on average at a given time, 150 raindrops strike each square meter, what is the impulse of the rain striking the car?

(a) 0.69 N·s **(b)** 0.046 N·s
(c) 0.14 N·s **(d)** 11 N·s

14. A car weighing 2×10^3 kg and moving at 20 m/s along a main road collide with a lorry of mass 8×10^3 kg which emerges at 5 m/s from a crossroad at right angles to the main road. If the two vehicles lock, what will be their velocity after the collision?

(a) $4/\sqrt{2}$ m/s, 45° with cross-road.
(b) $4/\sqrt{2}$ m/s, 60° with cross-road.
(c) $4/\sqrt{2}$ m/s, 60° with main road.
(d) $4/\sqrt{2}$ m/s, 45° with main road.

15. A football player kicks a 0.41-kg football initially at rest; and the ball flies through the air. If the kicker's foot was in contact with the ball for 0.051 s and the ball's initial speed after the collision is 21 m/s, what was the magnitude of the average force on the football?

(a) 9.7 N **(b)** 46 N
(c) 81 N **(d)** 210 N

16. A man stands at one end of the open truck which can run on frictionless horizontal rails. Initially, the man and the truck are at rest. Man now walks to the other end and stops. Then, which of the following is true?

(a) The truck moves opposite to direction of motion of man even after the man ceases to walk.
(b) The center of mass of the man and truck remains at the same point throughout the man's walk.
(c) The kinetic energy of man and truck are exactly equal throughout the man's walk.
(d) The truck does not move at all during the man's walk.

17. A stationary bomb explodes in space breaking into a number of small fragments. At the location of the explosion, the net force due to gravity is 0 N. Which one of the following statements concerning this event is true?

(a) Kinetic energy is conserved in this process.
(b) The fragments must have equal kinetic energies.
(c) The sum of the kinetic energies of the fragments must be zero.
(d) The velocity of any one fragment must be equal to the velocity of any other fragment.

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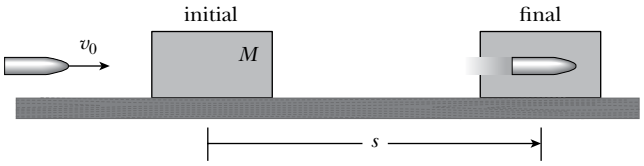
18. The law of conservation of momentum applies to a system of colliding objects only if
- **(a)** there is no change in kinetic energy of the system.
- **(b)** the coefficient of restitution is one.
- **(c)** the coefficient of restitution is zero.
- **(d)** the net external impulse is zero.

19. An object of mass $3m$, initially at rest, explodes breaking into two fragments of mass m and $2m$, respectively. Which one of the following statements concerning the fragments after the explosion is true?
- **(a)** They will fly off at right angles.
- **(b)** They will fly off in the same direction.
- **(c)** The smaller fragment will have twice the speed of the larger fragment.
- **(d)** The larger fragment will have twice the speed of the smaller fragment.

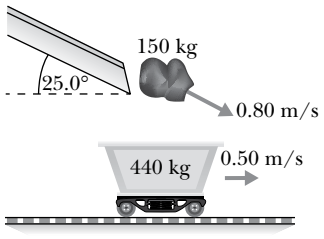
20. A 55-kg swimmer is standing on a stationary 210-kg boating raft. The swimmer then runs off the raft horizontally with a velocity of +4.6 m/s relative to the shore. Find the recoil velocity that the raft would have if there were no friction and resistance due to the water.
- **(a)** +1.2 m/s
- **(b)** −1.2 m/s
- **(c)** +0.60 m/s
- **(d)** −2.4 m/s

21. A bullet of mass m is fired at speed v_0 into a wooden block of mass M. The bullet instantaneously comes to rest in the block. The block with the embedded bullet slides along a horizontal surface with a coefficient of kinetic friction μ.



Which one of the following expressions determines how far the block slides before it comes to rest (the magnitude of displacement s is as shown in the figure)?
- **(a)** $s = \dfrac{mv_0^2}{M\mu g}$
- **(b)** $s = \dfrac{m}{m+M}\left(\dfrac{v_0^2}{\mu g}\right)$
- **(c)** $s = \left(\dfrac{m}{m+M}\right)^2 \dfrac{v_0^2}{2\mu g}$
- **(d)** $s = \left(\dfrac{m}{m+M}\right)^2 \sqrt{\dfrac{v_0^2}{2\mu g}}$

22. A mine car (mass = 440 kg) rolls at a speed of 0.50 m/s on a horizontal track, as the drawing shows. A 150-kg chunk of coal has a speed of 0.80 m/s when it leaves the chute. Determine the velocity of the car/coal system after the coal has come to rest in the car.



- **(a)** 0.56 m/s, to the left
- **(b)** 0.56 m/s, to the right
- **(c)** 0.28 m/s, to the left
- **(d)** 0.28 m/s, to the right

23. While in Earth's orbit, an 80-kg astronaut carrying a 20-kg tool kit is initially drifting toward a stationary (to her) space shuttle at a speed of 2 m/s. If she throws the tool kit toward the shuttle with a speed of 6 m/s as seen from the shuttle, her final speed is
- **(a)** 1 m/s toward the shuttle.
- **(b)** 4 m/s toward the shuttle.
- **(c)** 1 m/s away from the shuttle.
- **(d)** 6 m/s away from the shuttle.

24. A person stands in a stationary canoe and throws a 5.00-kg stone with a velocity of 8.00 m/s at an angle of 30.0° above the horizontal. The person and canoe have a combined mass of 105 kg. Ignoring air resistance and effects of the water, find the horizontal recoil velocity (magnitude and direction) of the canoe. Assume the horizontal component of the velocity of the stone is in the $+x$ direction.
- **(a)** 0.119 m/s, $+x$ direction
- **(b)** 0.119 m/s, $-x$ direction
- **(c)** 0.330 m/s, $+x$ direction
- **(d)** 0.330 m/s, $-x$ direction

25. On an interplanetary mission, a 58.5-kg astronaut is floating toward the front of her ship at 0.15 m/s, relative to the ship. She wishes to stop moving, relative to the ship. She decides to throw away the 2.50-kg book she's carrying. What should the speed and direction of the book be to achieve her goal?
- **(a)** 0.15 m/s, toward the front of the ship.
- **(b)** 3.5 m/s, toward the back of the ship.
- **(c)** 3.7 m/s, toward the front of the ship.
- **(d)** 0.30 m/s, toward the back of the ship.

26. Two objects of equal mass traveling toward each other with equal speeds undergo a head on collision. Which one of the following statements concerning their velocities after the collision is necessarily true?
- **(a)** They will exchange velocities.
- **(b)** Their velocities will be zero.
- **(c)** Their velocities will be reduced.
- **(d)** Their velocities may be zero.

27. A car (mass = 1100 kg) is traveling at 32 m/s when it collides head-on with a sport utility vehicle (mass = 2500 kg) traveling in the opposite direction. In the collision, the two vehicles come to a halt. At what speed was the sport utility vehicle traveling?
- **(a)** 11 m/s
- **(b)** 14 m/s
- **(c)** 16 m/s
- **(d)** 18 m/s

28. Momentum is conserved in a two-body collision only if
- **(a)** both bodies come to rest.
- **(b)** the collision is perfectly elastic.
- **(c)** the kinetic energy of the system is conserved.
- **(d)** the net external force acting on the two-body system is zero.

29. A completely inelastic collision occurs between two balls of wet putty that move directly toward each other along a vertical axis. Just before the collision, one ball, of mass 3.0 kg, is moving upward at 20 m/s and the other ball, of

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mass 2.0 kg, is moving downward at 12 m/s. How high do the combined two balls of putty rise above the collision point? (Neglect air drag.)

(a) 2.3 m **(b)** 2.0 m
(c) 2.6 m **(d)** 2.9 m

30. A tennis ball has a velocity of 12 m/s downward just before it strikes the ground and bounces up with a velocity of 12 m/s upward. Which statement is true concerning this situation?

(a) The momentum of the ball and the momentum of the earth both change.
(b) Neither the momentum of the ball nor the momentum of the earth changes.
(c) The momentum of the ball is changed; the momentum of the earth is not changed.
(d) The momentum of the ball is unchanged; the momentum of the earth is changed.

31. A railway flat car has an artillery gun installed on it. The combined system has a mass M and moves with a velocity u. The barrel of the gun makes an angle α with the horizontal. A shell of mass m leaves the barrel at a speed v relative to the barrel in the direction of flat car's motion. The speed of the flat car so that it may stop after the firing is

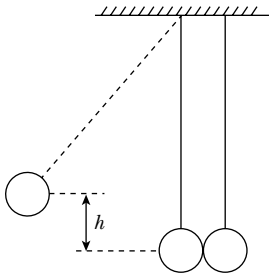
(a) $\dfrac{mv}{M+m}$ **(b)** $\left(\dfrac{Mv}{M+m}\right)\cos\alpha$

(c) $\left(\dfrac{mv}{M+m}\right)\cos\alpha$ **(d)** $(M+m)v\cos\alpha$

32. A 3.0-kg cart moving to the right with a speed of 1.0 m/s has a head-on collision with a 5.0-kg cart that is initially moving to the left with a speed of 2.0 m/s. After the collision, the 3.0-kg cart is moving to the left with a speed of 1.0 m/s. What is the final velocity of the 5.0-kg cart?

(a) zero m/s
(b) 0.80 m/s to the right
(c) 0.80 m/s to the left
(d) 2.0 m/s to the right

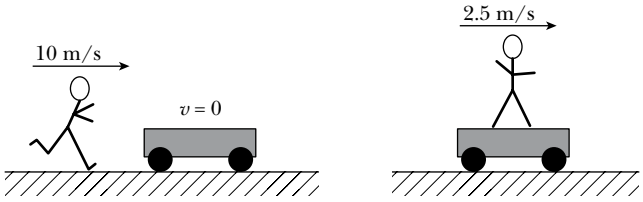
33. In the following figure, pendulum bob on the left side is pulled aside to a height h from its initial position. After it is released, it collides with the right pendulum bob at rest, which is of same mass. After the collision, the two bobs stick together and rise to a height.



(a) $\dfrac{h}{2}$ **(b)** $\dfrac{h}{4}$

(c) $\dfrac{3h}{3}$ **(d)** $\dfrac{3h}{4}$

34. A 50.0-kg boy runs at a speed of 10.0 m/s and jumps onto a cart as shown in the figure. The cart is initially at rest. If the speed of the cart with the boy on it is 2.50 m/s, what is the mass of the cart?



(a) 150 kg **(b)** 175 kg
(c) 210 kg **(d)** 260 kg

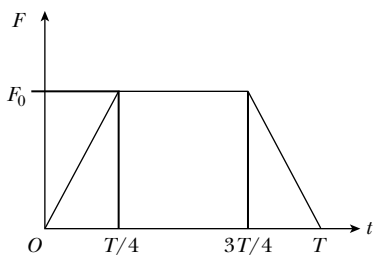
35. A student ($m = 63$ kg) falls freely from rest and strikes the ground. During the collision with the ground, he comes to rest in a time of 0.010 s. The average force exerted on him by the ground is +18,000 N, where the upward direction is taken to be the positive direction. From what height did the student fall? Assume that the only force acting on him during the collision is that due to the ground.

(a) 0.15 m **(b)** 0.21 m
(c) 0.30 m **(d)** 0.42 m

36. A 0.015-kg marble moving to the right at 0.40 m/s has a head-on, elastic collision with a 0.045-kg marble sitting at rest on a smooth, level surface. Which of the following are the correct magnitudes and directions of the velocities of the two marbles after the collision?

0.015-kg marble	0.045-kg marble
(a) 0.15 m/s, left	0.25 m/s, right
(b) 0.10 m/s, right	0.30 m/s, right
(c) zero m/s	0.25 m/s, right
(d) 0.20 m/s, left	0.20 m/s, right

37. A particle of mass m moving with a velocity v makes an elastic one-dimensional collision with a stationary particle of mass m establishing a contact with it for extremely small time T. Their force of contact increases from zero to F_0 linearly in time $T/4$, remains constant for a further time $T/2$ and decreases linearly from F_0 to zero in further time $T/4$ as shown in figure The magnitude possessed by F_0 is



(a) $\dfrac{mv}{T}$ **(b)** $\dfrac{2mv}{T}$

(c) $\dfrac{4mv}{3T}$ **(d)** $\dfrac{3mv}{4T}$

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38. A 2.5-kg ball and a 5.0-kg ball have an elastic collision. Before the collision, the 2.5-kg ball was at rest and the other ball had a speed of 3.5 m/s. What is the kinetic energy of the 2.5-kg ball after the collision?

(a) 1.7 J **(b)** 3.4 J
(c) 8.1 J **(d)** 27 J

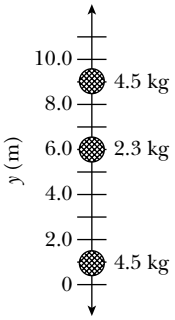
39. A 35-kg girl is standing near and to the left of a 43-kg boy on the frictionless surface of a frozen pond. The boy throws a 0.75-kg ice ball to the girl with a horizontal speed of 6.2 m/s. What are the velocities of the boy and the girl immediately after the girl catches the ice ball?

	Girl	Boy
(a)	0.81 m/s, left	0.67 m/s, right
(b)	0.17 m/s, left	0.14 m/s, left
(c)	0.18 m/s, right	0.13 m/s, left
(d)	0.13 m/s, left	0.11 m/s, right

40. A particle moving with a velocity of $(4\hat{i} - \hat{j})$ m/s strikes a fixed smooth wall and finally moves with a velocity of $(3\hat{i} + 2\hat{j})$ m/s. The coefficient of restitution between the wall and the particle in the collision will be

(a) $\frac{7}{3}$ **(b)** $\frac{3}{7}$
(c) $\sqrt{\frac{13}{17}}$ **(d)** $\sqrt{\frac{17}{13}}$

41. The figure shows two 4.5-kg balls located on the y axis at 1.0 and 9.0 m, respectively; a third ball with a mass 2.3 kg is located at 6.0 m. What is the location of the center of mass of this system?



(a) 4.8 m **(b)** 5.2 m
(c) 5.6 m **(d)** 6.0 m

42. During hockey practice, two pucks are sliding across the ice in the same direction. At one instant, a 0.18-kg puck is moving at 16 m/s while the other puck has a mass of 0.14 kg and a speed of 3.8 m/s. What is the velocity of the center of mass of the two pucks?

(a) 5.0 m/s **(b)** 9.0 m/s
(c) 7.0 m/s **(d)** 13 m/s

43. A dump truck is being filled with sand. The sand falls straight downward from rest from a height of 2.00 m above the truck bed, and the mass of sand that hits the truck per second is 55.0 kg/s. The truck is parked on the platform of a weight scale. By how much does the scale reading exceed the weight of the truck and sand?

(a) 626 N **(b)** 539 N
(c) 214 N **(d)** 344 N

44. A wagon is coasting at a speed v_A along a straight and level road. When ten percent of the wagon's mass is thrown off the wagon, parallel to the ground and in the forward direction, the wagon is brought to a halt. If the direction in which this mass is thrown is exactly reversed, but the speed of this mass relative to the wagon remains the same, the wagon accelerates to a new speed v_B. Calculate the ratio v_B/v_A.

(a) 0.49 **(b)** 1.5
(c) 0.58 **(d)** 2.0

45. A projectile (mass = 0.20 kg) is fired at and embeds itself in a target (mass = 2.50 kg). The target (with the projectile in it) flies off after being struck. What percentage of the projectile's incident kinetic energy does the target (with the projectile in it) carry off after being struck?

(a) 2.1% **(b)** 5.0%
(c) 3.8% **(d)** 7.4%

46. A ball is dropped from rest at the top of a 6.10-m-tall building, falls straight downward, collides inelastically with the ground, and bounces back. The ball loses 10.0% of its kinetic energy every time it collides with the ground. How many bounces can the ball make and still reach a windowsill that is 2.44 m above the ground?

(a) 2 **(b)** 3
(c) 4 **(d)** 8

47. A 0.10-kg cart traveling in the positive x direction at 10.0 m/s collides with a 0.30-kg cart at rest.

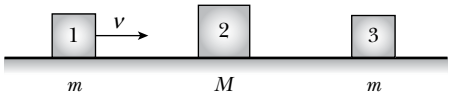
The collision is elastic. What is the velocity of the 0.10-kg cart after the collision?

(a) +2.5 m/s **(b)** +5 m/s
(c) −2.5 m/s **(d)** −5 m/s

48. The head of a hammer (m = 1.5 kg) moving at 4.5 m/s strikes a nail and bounces back with the same speed after an elastic collision lasting 0.075 s. What is the magnitude of the average force the hammer exerts on the nail?

(a) 6.8 N **(b)** 90 N
(c) 60 N **(d)** 180 N

49. Three blocks are placed on a smooth horizontal surface and along a straight line. Block-1 and Block-3 have mass m each, and Block-2 has mass $M > m$. Block-2 and Block-3 are initially stationary, while Block-1 is initially moving towards Block-2 with speed v as shown. Assuming all collisions are head-on and perfectly elastic, for what value of M/m will Block-1 and Block-3 have the same final speed?



(a) $5+\sqrt{2}$ **(b)** $5-\sqrt{2}$
(c) $2+\sqrt{5}$ **(d)** $3+\sqrt{5}$

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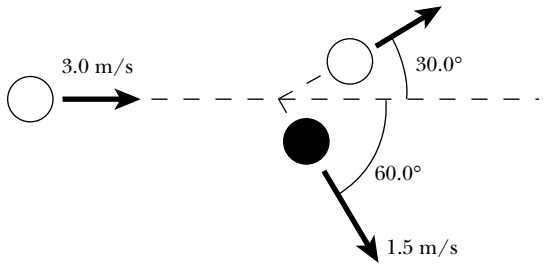
50. In the game of billiards, all the balls have approximately the same mass, about 0.17 kg. In the figure, the cue ball strikes another ball such that it follows the path shown. The other ball has a speed of 1.5 m/s immediately after the collision. What is the speed of the cue ball after the collision?



(a) 1.5 m/s **(b)** 1.8 m/s
(c) 2.6 m/s **(d)** 4.3 m/s

More than One Correct Choice Type

51. A point mass of 1 kg collides elastically with a stationary point mass of 5 kg. After their collision, the 1 kg mass reverses its direction and moves with a speed of 2 m/s. Which of the following statements are correct for the system of these two masses?
(a) Total momentum of the system is 3 kg m/s.
(b) Momentum of 5 kg mass after collision is 4 kg m/s.
(c) Kinetic energy of the center of mass is 0.75 J.
(d) Total kinetic energy of the system is 4 J.

52. A particle strikes a horizontal smooth floor with a velocity u making an angle θ with the floor and rebounds with velocity v making an angle ϕ with the floor. The coefficient of restitution between the particle and the floor is e.
(a) The impulse delivered by the floor to the body is $mu\,(1+e)\sin\theta$.
(b) $\tan\phi = e\tan\theta$.
(c) $v = u\sqrt{1-(1-e^2)\sin^2\theta}$.
(d) The ratio of the final kinetic energy to the initial kinetic energy is $\cos^2\theta + e^2\sin^2\theta$.

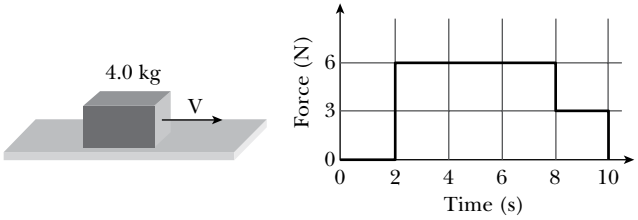
53. A non-zero external force acts on a system of particles. The velocity and the acceleration of the center of mass are found to be v_0 and a_0 at an instant t. It is possible that
(a) $v_0 = 0, a_0 = 0$. **(b)** $v_0 = 0, a_0 \neq 0$.
(c) $v_0 \neq 0, a_0 = 0$. **(d)** $v_0 \neq 0, a_0 \neq 0$.

54. Two balls having linear momentum, $\vec{p}_1 = p\hat{i}$ and $\vec{p}_2 = -p\hat{i}$, undergo a collision in free space. There is no external force acting on the balls. Let $\vec{p}_1'$ and $\vec{p}_2'$ be their final momentum. Choose the correct options that do not allow for any non-zero value of $p, a_1, a_2, b_1, b_2, c_1$, and c_2.
(a) $\vec{p}_1' = a_1\hat{i} + b_1\hat{j} + c_1\hat{k};\ \vec{p}_2' = a_2\hat{i} + b_2\hat{j}$
(b) $\vec{p}_1' = c_1\hat{k};\ \vec{p}_2' = c_2\hat{k}$
(c) $\vec{p}_1' = a_1\hat{i} + b_1\hat{j} + c_1\hat{k};\ \vec{p}_2' = a_2\hat{i} + b_2\hat{j} - c_1\hat{k}$
(d) $\vec{p}_1' = a_1\hat{i} + b_1\hat{j};\ \vec{p}_2' = a_2\hat{i} + b_2\hat{j}$

55. Choose the correct statement(s) of the following:
(a) Force acting on a particle for equal time intervals can produce the same change in momentum but different change in kinetic energy.
(b) Force acting on a particle for equal displacements can produce same change in kinetic energy but different change in momentum.
(c) Force acting on a particle for equal time intervals can produce different change in momentum but same change in kinetic energy.
(d) Force acting on a particle for equal displacements can produce different change in kinetic energy but same change in momentum.

Linked Comprehension

Paragraph for Questions 56 and 57: A 4.0-kg block slides along a frictionless surface with a constant speed of 5.0 m/s as shown. Two seconds after it begins sliding, a horizontal, time-dependent force is applied to the mass. The force is removed 8 s later. The graph shows how the force on the block varies with time.



56. What is the magnitude of the total impulse of the force acting on the block?
(a) 20 N s **(b)** 42 N s
(c) 48 N s **(d)** 54 N s

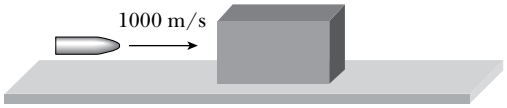
57. What is the approximate speed of the block at $t = 11$ s?
(a) 5.0 m/s **(b)** 16 m/s
(c) 25 m/s **(d)** 65 m/s

Paragraph for Questions 58 and 59: A comet fragment of mass 1.96×10^{13} kg is moving at 6.50×10^4 m/s when it crashes into Callisto, a moon of Jupiter. The mass of Callisto is 1.08×10^{23} kg. The collision is completely inelastic.

58. Assuming for this calculation that Callisto's initial momentum is zero kg m/s, what is the recoil speed of Callisto immediately after the collision?
(a) 3.34×10^{-18} m/s **(b)** 3.58×10^{-12} m/s
(c) 1.27×10^{-14} m/s **(d)** 1.18×10^{-5} m/s

59. How much kinetic energy was released in the collision?
(a) 8.28×10^{22} J **(b)** 3.51×10^{27} J
(c) 7.02×10^{27} J **(d)** 4.14×10^{22} J

Paragraph for Questions 60—62: A 2.0-kg pistol fires a 1.0-g bullet with a muzzle speed of 1000 m/s. The bullet then strikes a 10-kg wooden block resting on a horizontal frictionless surface. The block and the embedded bullet then slide across the surface.



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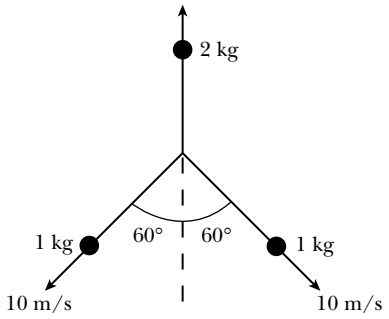
60. What is the kinetic energy of the bullet as it travels toward the block?
(a) 100 J **(b)** 500 J
(c) 1000 J **(d)** 5000 J

61. The explosive charge in the pistol acts for 0.001 s. What is the average force exerted on the bullet while it is being fired?
(a) 0.001 N **(b)** 100 N
(c) 1.0 N **(d)** 1000 N

62. What is the speed of the "bullet + block" system immediately after the bullet is embedded in the block?
(a) 0.1 m/s **(b)** 1000 m/s
(c) 10 m/s **(d)** 10 000 m/s

Paragraph for Questions 63 and 64: A stationary 4-kg shell explodes into three pieces. Two of the fragments have a mass of 1 kg each and move along the paths shown with a speed of 10 m/s. The third fragment moves upward as shown.



63. What is the speed of the third fragment?
(a) zero m/s **(b)** 10 m/s
(c) 1 m/s **(d)** 5 m/s

64. What is the speed of the center of mass of this system after the explosion?
(a) zero m/s **(b)** 1 m/s
(c) 3 m/s **(d)** 5 m/s

Matrix-Match

65. Match the statement in Column I with the statements in Column II. One or more than one choice from Column II can match with a statement from Column I.

Column I	Column II
(a) In a perfectly elastic collision,	**(p)** some kinetic energy is lost.
(b) In a perfectly inelastic collision,	**(q)** $0 < e < 1$.
(c) In a partially elastic collision,	**(r)** $e = 1$.
(d) In a partially inelastic collision,	**(s)** $e = 0$.

Directions for Questions 66 and 67: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

66.

$$e \text{ (coefficient of restitution)} = \frac{\text{Relative velocity after collision}}{\text{Relative velocity before collision}}$$

Column I	Column II	Column III
(I) $e = 1$	**(i)** $v_2 - v_1 < u_1 - u_2$	**(J)** The value of e shall depend upon the extent of loss of kinetic energy during collision.
(II) $e = 0$	**(ii)** $v_2 - v_1 > u_1 - u_2$	**(K)** Relative velocities of two bodies before and after collision are same.
(III) $e < 1$	**(iii)** $v_2 - v_1 = u_1 - u_2$	**(L)** Signifies that the two bodies shall move together with same velocity.
(IV) $e > 1$	**(iv)** $v_2 - v_1 = 0$, or $v_2 = v_1$	**(M)** Signifies that the two bodies shall not move together with same velocity.

(1) What are the characteristics when collision is perfectly elastic?
(a) (I) (ii) (J) **(b)** (I) (iii) (K)
(c) (II) (i) (M) **(d)** (II) (iii) (L)

(2) What are the characteristics when collision is perfectly inelastic?
(a) (I) (iii) (K) **(b)** (IV) (iii) (L)
(c) (II) (iii) (L) **(d)** (III) (i) (J)

(3) What are the characteristics when collision is inelastic?
(a) (III) (i) (K)
(b) (I) (i) (L)
(c) (II) (iv) (L)
(d) (II) (iii) (M)

67. Consider a man sitting on the cart that is moving with initial and final velocities.

Column I	Column II	Column III
(I) Mass of the man = 75.2 kg	**(i)** Mass of the cart = 38.6 kg	**(J)** Initial velocity = 2.33 m/s Final velocity = 0 m/s
(II) Mass of the man = 52 kg	**(ii)** Mass of the cart = 22 kg	**(K)** Initial velocity = 2.3 m/s Final velocity = 1 m/s
(III) Mass of the man = 62 kg	**(iii)** Mass of the cart = 23 kg	**(L)** Initial velocity = 2 m/s Final velocity = 2 m/s
(IV) Mass of the man = 55 kg	**(iv)** Mass of the cart = 36 kg	**(M)** Initial velocity = 2.5 m/s Final velocity = 1.1 m/s

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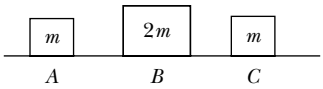
(1) Which has 4.54 m/s as the resulting change in speed of the cart?
- (a) (I) (iii) (L)
- (b) (IV) (i) (M)
- (c) (II) (iv) (K)
- (d) (I) (i) (J)

(2) Which has 3.347 m/s as the resulting change in speed of the cart?
- (a) (III) (ii) (L)
- (b) (IV) (iii) (M)
- (c) (II) (iii) (K)
- (d) (I) (i) (M)

(3) Which has 2.238 m/s as the resulting change in speed of the cart?
- (a) (III) (iv) (K)
- (b) (I) (i) (K)
- (c) (III) (iii) (L)
- (d) (I) (ii) (L)

Integer Type

68. Three objects A, B and C are kept in a straight line on a frictionless horizontal surface as shown in the figure. These have masses m, $2m$ and m, respectively. The object A moves towards B with a speed 9 m/s and makes an elastic collision with it. Thereafter, B makes completely inelastic collision with C. All motions occur on the same straight line. Then, the final speed (in m/s) of the object C is _____.



69. A block of mass $m_1 = 150$ kg is at rest on a very long frictionless table, one end of which is terminated in a wall. Another block of mass m_2 is placed between the first block and the wall, and set in motion towards m_1 with constant speed u_2. Assuming that all collisions are completely elastic, find the value of m_2 (in kg) for which both blocks move with the same velocity after m_2 has collided once with m_1 and once with the wall. (The wall has effectively infinite mass.)

70. A particle of mass 1 kg is projected upwards with velocity 60 m/s. Another particle of mass 2 kg is just dropped from a certain height. Is, after 2 s, when neither of the particles have collided with ground, the value of the velocity of com (in m/s) is _____.

71. A system of two objects has a total momentum of (18 kg m/s) $\hat{i}$ and its center of mass has the velocity of (3 m/s) $\hat{i}$. One of the objects has the mass 4 kg and velocity (1.5 m/s) $\hat{i}$. The mass of the other object (in kg) is _____.

ANSWER KEY

Checkpoints

1. (a) Origin; (b) Fourth quadrant; (c) On y axis below origin; (d) Origin; (e) Third quadrant; (f) origin.
2. At a distance $L/9$ left of origin.
3. (a)–(c) At the center of mass, still at the origin (their forces are internal to the system an cannot move the center of mass).
4. (Consider slopes and Eq. 9-25) (a) 1, 3 and then 2 and 4 tie (zero force); (b) 3.
5. It is equal to the negative work done by the friction between the ship and feet of the passengers.
6. (a) Unchanged; (b) Unchanged (see Eq. 9-34); (c) Changed (see Eq. 9-37).
7. (a) Zero; (b) Positive (initial py down y and final py up y); (c) Positive direction of y.
8. (a) $v_0 \cot\theta$; $v_0/2$.
9. (No net external force; P (with arrow on top) conserved.) (a) Zero; (b) No; (c) $-x$.
10. (a) 10 kg·m/s; (b) 14 kg·m/s; (c) 6 kg·m/s.
11. (a) 4 kg·m/s; (b) 8 kg·m/s; (c) 3 J.
12. 2 kg·m/s (conserve momentum along); (b) 3 kg·m/s (conserve momentum along y).

Problems

1. 0.73 m
2. (a) 1.1; (b) 1.4 m; (c) As the mass of m_3, the topmost particle, is increased, the center of mass shifts toward that particle. As we approach the limit where m_3 is infinitely more massive than the others, the center of mass becomes infinitesimally close to the position of m_3.
3. (a) 12 cm; (b) –4.8 cm
4. (a) 25 cm; (b) 25 cm; (c) 20 cm

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5. (a) 6.0 cm; (b) 6.0 cm; (c) 6.0 cm; (d) 4.2 cm
6. (a) $-1.1\ m/s^2$; (b) –2.2 m; (c) 2.7 m/s
7. (a) 39 m; (b) 10 m/s
8. $(-4.0\ m)\,\hat{i} + (4.0\ m)\,\hat{j}$
9. 6.5 m
10. 35.3 m
11. (a) 5.74 m; (b) $(10.0\ m/s)\,\hat{i}$; (c) $(-3.68\ m/s^2)\,\hat{j}$
12. 55 kg
13. 4.2 m from the shore.
14. $6.7\ kg \cdot m/s$
15. (a) 0; (b) 1732.05 kg·m/s; (c) Along the direction with the angle $\theta = 150°$.
16. (a) $(0.14\ kg \cdot m/s)\,\hat{i}$; (b) There is no change in the magnitude of the momentum of the ball; (c) Since the wall remains motionless, its initial and final momenta are zero, and hence there is no change in the magnitude of its momentum.
17. (a) $\theta_2 = 30.0°$; (b) $(-0.520\ kg \cdot m/s)\,\hat{j}$
18. 1.0×10^3 to $1.2 \times 10^3\ kg \cdot m/s$
19. (a) 1.1 m; (b) $4.8 \times 10^3\ kg \cdot m/s$
20. (a) 1.20×10^{-3} s; (b) 0.500 N·s; (c) 417 N
21. (a) 2.3×10^2 N·s; (b) $\approx 2.7 \times 10^3$ N
22. (a) 270 N·s; (b) Rightward; (c) 1.35×10^4 N
23. (a) 9.1 N·s; (b) $\approx 1.7 \times 10^3$ N
24. 6.26 N·s
25. (a) $\approx 2.39 \times 10^3$ N·s; (b) $\approx 4.78 \times 10^5$ N; (c) $\approx 1.76 \times 10^3$ N·s; (d) 3.52×10^5 N
26. (a) $(30\ kg \cdot m/s)\,\hat{i}$; (b) $(38\ kg \cdot m/s)\,\hat{i}$; (c) $(12\ m/s)\,\hat{i}$
27. (a) 5.86 N·s; (b) 59.8° measured counterclockwise from the $+x$ axis; (c) $\approx 2.93 \times 10^3$ N; (d) 59.8° measured counterclockwise from the $+x$ axis.
28. 9.9×10^2 N
29. (a) 4.47 N·s; (b) $(49.1\ N \cdot s)\,\hat{i}$; (c) $(24.5\ N)\,\hat{i}$
30. (a) $(49.1\ N \cdot s)\,\hat{i}$; (b) $(24.5\ N)\,\hat{i}$
31. (a) $(1.8\ N \cdot s)\,\hat{j}$; (b) $(-180\ N)\,\hat{j}$
32. 8.0 m/s
33. 4.9×10^3 km/h
34. (a) 11.1 m/s; (b) 304 J
35. 3.18 kg or about 3.2 kg.
36. 3.9 m/s
37. (a) 27 J; (b) 53 J
38. 3.1×10^2 m/s
39. (a) 1.86 m/s; (b) 5.16 m/s
40. (a) 721 m/s; (b) 937 m/s
41. 0.100 m
42. 3.3 m
43. (a) 0.40 m/s; (b) 2.4 m/s; (c) ≈ 0.95 m; (d) 6.0 J
44. (a) 4.0 m/s; (b) 3.5 m/s; (c) 6.6 m/s
45. (a) 4.4 m/s; (b) 0.80
46. 0.30 m
47. 0.25 m
48. (a) 1.9 m/s; (b) The block continues going to the right after the collision; (c) Since $K_i = K_f$, the collision is found to be elastic.
49. (a) 83.3 g; (b) 1.00 m/s
50. (a) 3.00 m/s; (b) 6.00 m/s
51. (a) 2.39 m/s; (b) 1.30 m/s
52. (a) $(-3v)\,\hat{i}$; (b) Particle 2 of mass $2m$ is brought to rest; (c) $-v\hat{i}$; (d) $-v\hat{i}$
53. (a) $d_1 = 0.1698\ m \approx 17$ cm; (b) $d_2 = 1.878\ m \approx 1.9$ m
54. $x = -28$ cm
55. (a) 2.96 m; (b) 0.741 m
56. (a) 0.21 kg; (b) 7.2 m
57. 1.3 kg
58. (a) $\theta_2 = 45°$; (b) $v_2 = \sqrt{2}v_0$; (c) The collision is elastic.
59. 120°
60. (a) $1.25v$; (b) $8.6\ mv^2$
61. (a) $J = \Delta p = 2.6mu$; (b) $0.14u$; (c) $4.9mu^2$
62. 123 m/s
63. (a) $e^1 \approx 2.7$; (b) $e^2 \approx 7.4$
64. (a) 2.45 kg/s; (b) 7.45 kg/s; (c) ≈4.16 km/s

Practice Questions

Single Correct Choice Type

1. (a)	2. (d)	3. (d)	4. (c)	5. (d)
6. (c)	7. (a)	8. (c)	9. (d)	10. (b)
11. (b)	12. (a)	13. (a)	14. (d)	15. (d)
16. (b)	17. (d)	18. (d)	19. (c)	20. (b)
21. (c)	22. (b)	23. (a)	24. (d)	25. (c)
26. (d)	27. (b)	28. (d)	29. (c)	30. (a)
31. (c)	32. (c)	33. (b)	34. (a)	35. (d)
36. (d)	37. (c)	38. (d)	39. (d)	40. (b)
41. (b)	42. (d)	43. (d)	44. (d)	45. (d)
46. (d)	47. (d)	48. (d)	49. (c)	50. (c)

More than One Correct Choice Type

51. (a), (c)	52. (a), (b), (c), (d)	53. (b), (d)	54. (a), (b)	55. (a), (b)

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Linked Comprehension

56. (b) **57.** (b) **58.** (d) **59.** (d) **60.** (b)
61. (d) **62.** (a) **63.** (d) **64.** (a)

Matrix-Match

65. (a) → (r); (b) → (p), (s); (c) → (p), (q); (d) → (p), (q)
66. (1) → (b); (2) → (d); (3) → (c)
67. (1) → (d); (2) → (b); (3) → (a)

Integer Type

68. 4 **69.** 50 **70.** 20 **71.** 2

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CHAPTER 10

Rigid Body Dynamics – I

10.1 | WHAT IS PHYSICS?

As we have discussed, one focus of physics is motion. A motion is said to be **translation motion** if any straight line inside the body shifts parallel to itself during the motion. **Rotation about a fixed axis**: So far, we have examined only the motion of **translation**, in which an object moves along a straight or curved line, as in Fig. 10-1*a*. We now turn to the motion of **rotation**, in which an object turns about an axis, as in Fig. 10-1*b*.

You see rotation in nearly every machine, you use it every time you open a beverage can with a pull tab, and you pay to experience it every time you go to an amusement park. Rotation is the key to many fun activities, such as hitting a long drive in golf (the ball needs to rotate in order for the air to

Mike Segar/Reuters/Landov LLC

Elsa/Getty Images, Inc.

(*a*) (*b*)

Figure 10-1 Figure skater Sasha Cohen in motion of (*a*) pure translation in a fixed direction and (*b*) pure rotation about a vertical axis.

Contents

10.1 What is Physics?
10.2 Rotational Variables
10.3 Are Angular Quantities Vectors?
10.4 Rotation with Constant Angular Acceleration
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10.7 Calculating the Rotational Inertia
10.8 Torque
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10.10 Vector Product and Torque
10.11 The Center of Gravity
10.12 The Rigid Object Under a Net Torque
10.13 Work and Rotational Kinetic Energy
10.14 Angular Momentum
10.15 Newton's Second Law in Angular Form
10.16 The Angular Momentum of a System of Particles
10.17 The Angular Momentum of a Rigid Body Rotating about a Fixed Axis
10.18 Conservation of Angular Momentum

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keep it aloft longer) and throwing a curveball in baseball (the ball needs to rotate in order for the air to push it left or right). Rotation is also the key to more serious matters, such as metal failure in aging airplanes.

In this motion, the particles forming the rigid body move in parallel planes along circles centered on the same fixed axis called the axis of rotation. If this axis, intersects the rigid body, the particles located on the axis have zero velocity and zero acceleration.

We begin our discussion of rotation by defining the variables for the motion, just as we did for translation in Chapter 2. As we shall see, the variables for rotation are analogous to those for one-dimensional motion and, as in Chapter 2, an important special situation is where the acceleration (here the rotational acceleration) is constant. We shall also see that Newton's second law can be written for rotational motion, but we must use a new quantity called *torque* instead of just force. Work and the work–kinetic energy theorem can also be applied to rotational motion, but we must use a new quantity called *rotational inertia* instead of just mass. In short, much of what we have discussed so far can be applied to rotational motion with, perhaps, a few changes.

Caution: In spite of this repetition of physics ideas, many students find this and the next chapter very challenging. Instructors have a variety of reasons as to why, but two reasons stand out: (1) There are a lot of symbols (with Greek letters) to sort out. (2) Although you are very familiar with linear motion (you can get across the room and down the road just fine), you are probably very unfamiliar with rotation (and that is one reason why you are willing to pay so much for amusement park rides). If a homework problem looks like a foreign language to you, see if translating it into the one-dimensional linear motion of Chapter 2 helps. For example, if you are to find, say, an *angular* distance, temporarily delete the word *angular* and see if you can work the problem with the Chapter 2 notation and ideas.

10.2 | ROTATIONAL VARIABLES

Key Concepts

- To describe the rotation of a rigid body about a fixed axis, called the rotation axis, we assume a reference line is fixed in the body, perpendicular to that axis and rotating with the body. We measure the angular position θ of this line relative to a fixed direction. When θ is measured in radians,

$$\theta = \frac{s}{r} \quad \text{(radian measure)},$$

  where s is the arc length of a circular path of radius r and angle θ.
- Radian measure is related to angle measure in revolutions and degrees by

$$1 \text{ rev} = 360^\circ = 2\pi \text{ rad}.$$

- A body that rotates about a rotation axis, changing its angular position from θ_1 to θ_2, undergoes an angular displacement

$$\Delta\theta = \theta_2 - \theta_1,$$

  where $\Delta\theta$ is positive for counterclockwise rotation and negative for clockwise rotation.
- If a body rotates through an angular displacement $\Delta\theta$ in a time interval Δt, its average angular velocity ω_{avg} (refer to Section 7.1) is

$$\omega_{avg} = \frac{\Delta\theta}{\Delta t}.$$

  The (instantaneous) angular velocity ω of the body is

$$\omega = \frac{d\theta}{dt}.$$

  Both ω_{avg} and ω are vectors, with directions given by a right-hand rule. They are positive for counterclockwise rotation and negative for clockwise rotation. The magnitude of the body's angular velocity is the angular speed.
- If the angular velocity of a body changes from ω_1 to ω_2 in a time interval $\Delta t = t_2 - t_1$ (refer to Section 7.1) the average angular acceleration α_{avg} of the body is

$$\alpha_{avg} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}.$$

  The (instantaneous) angular acceleration α of the body is

$$\alpha = \frac{d\omega}{dt}.$$

  Both α_{avg} and α are vectors.

We wish to examine the rotation of a rigid body about a fixed axis. A **rigid body** is a body that can rotate with all its parts locked together and without any change in its shape. A **fixed axis** means that the rotation occurs about an axis that does not move. Thus, we shall not examine an object like the Sun, because the parts of the Sun (a ball of gas)

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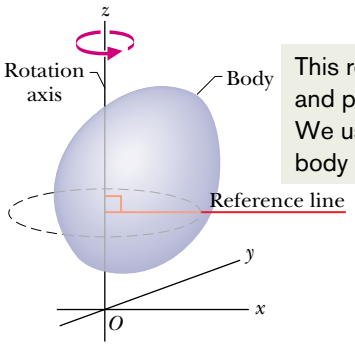


This reference line is part of the body and perpendicular to the rotation axis. We use it to measure the rotation of the body relative to a fixed direction.

Figure 10-2 A rigid body of arbitrary shape in pure rotation about the z axis of a coordinate system. The position of the *reference line* with respect to the rigid body is arbitrary, but it is perpendicular to the rotation axis. It is fixed in the body and rotates with the body.

are not locked together. We also shall not examine an object like a bowling ball rolling along a lane, because the ball rotates about a moving axis (the ball's motion is a mixture of rotation and translation).

Figure 10-2 shows a rigid body of arbitrary shape in rotation about a fixed axis, called the **axis of rotation** or the **rotation axis**. In pure rotation (*angular motion*), every point of the body moves in a circle whose center lies on the axis of rotation, and every point moves through the same angle during a particular time interval. In pure translation (*linear motion*), every point of the body moves in a straight line, and every point moves through the same *linear distance* during a particular time interval.

We deal now—one at a time—with the angular equivalents of the linear quantities position, displacement, velocity, and acceleration.

Angular Position

Figure 10-2 shows a *reference line*, fixed in the body, perpendicular to the rotation axis and rotating with the body. The **angular position** of this line is the angle of the line relative to a fixed direction, which we take as the **zero angular position**. In Fig. 10-3, the angular position θ is measured relative to the positive direction of the x axis. From geometry, we know that θ is given by

$$\theta = \frac{s}{r} \quad \text{(radian measure)}. \tag{10-1}$$

Here s is the length of a circular arc that extends from the x axis (the zero angular position) to the reference line, and r is the radius of the circle.

An angle defined in this way is measured in **radians** (rad) rather than in revolutions (rev) or degrees. The radian, being the ratio of two lengths, is a pure number and thus has no dimension. Because the circumference of a circle of radius r is $2\pi r$, there are 2π radians in a complete circle:

$$1 \text{ rev} = 360° = \frac{2\pi r}{r} = 2\pi \text{ rad}, \tag{10-2}$$

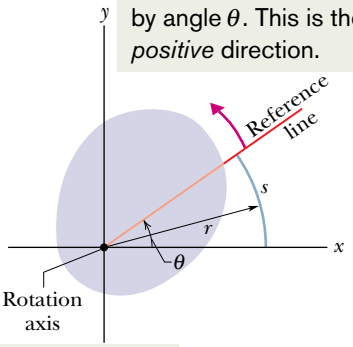
and thus

$$1 \text{ rad} = 57.3° = 0.159 \text{ rev}. \tag{10-3}$$

We do *not* reset θ to zero with each complete rotation of the reference line about the rotation axis. If the reference line completes two revolutions from the zero angular position, then the angular position θ of the line is $\theta = 4\pi$ rad.

For pure translation along an x axis, we can know all there is to know about a moving body if we know $x(t)$, its position as a function of time. Similarly, for pure rotation, we can know all there is to know about a rotating body if we know $\theta(t)$, the angular position of the body's reference line as a function of time.

The body has rotated *counterclockwise* by angle θ. This is the *positive* direction.



This dot means that the rotation axis is out toward you.

Figure 10-3 The rotating rigid body of Fig. 10-2 in cross section, viewed from above. The plane of the cross section is perpendicular to the rotation axis, which now extends out of the page, toward you. In this position of the body, the reference line makes an angle θ with the x axis.

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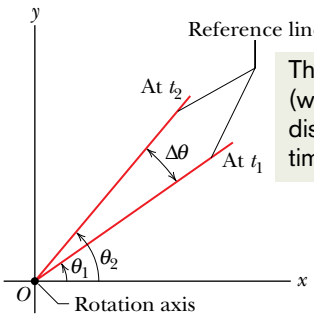
Angular Displacement

If the body of Fig. 10-3 rotates about the rotation axis as in Fig. 10-4, changing the angular position of the reference line from θ_1 to θ_2, the body undergoes an **angular displacement** $\Delta\theta$ given by

$$\Delta\theta = \theta_2 - \theta_1. \tag{10-4}$$

This definition of angular displacement holds not only for the rigid body as a whole but also for *every particle within that body*.

Clocks Are Negative. If a body is in translational motion along an x axis, its displacement Δx is either positive or negative, depending on whether the body is moving in the positive or negative direction of the axis. Similarly, the angular displacement $\Delta\theta$ of a rotating body is either positive or negative, according to the following rule:



This change in the angle of the reference line (which is part of the body) is equal to the angular displacement of the body itself during this time interval.

Figure 10-4 The reference line of the rigid body of Figs. 10-2 and 10-3 is at angular position θ_1 at time t_1 and at angular position θ_2 at a later time t_2.The quantity $\Delta\theta\,(=\theta_2-\theta_1)$ is the angular displacement that occurs during the interval $\Delta t\,(=t_2-t_1)$. The body itself is not shown.

An angular displacement in the counterclockwise direction is positive, and one in the clockwise direction is negative.

The phrase "*clocks are negative*" can help you remember this rule (they certainly are negative when their alarms sound off early in the morning).

CHECKPOINT 1

A disk can rotate about its central axis like a merry-go-round. Which of the following pairs of values for its initial and final angular positions, respectively, give a negative angular displacement: (a) –3 rad, +5 rad, (b) –3 rad, –7 rad, (c) 7 rad, –3 rad?

Let us recall **angular velocity** and **angular acceleration** that we learnt from Chapter 7 (Eqs. 7-2 to 7-5); we discuss them again in the following:

Angular Velocity

Suppose that our rotating body is at angular position θ_1 at time t_1 and at angular position θ_2 at time t_2 as in Fig. 10-4. We define the **average angular velocity** of the body in the time interval Δt from t_1 to t_2 to be

$$\omega_{avg} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}, \tag{10-5}$$

where $\Delta\theta$ is the angular displacement during Δt (ω is the lowercase omega).

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The **(instantaneous) angular velocity** ω, with which we shall be most concerned, is the limit of the ratio in Eq. 10-5 as Δt approaches zero. Thus,

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt}. \tag{10-6}$$

If we know $\theta(t)$, we can find the angular velocity ω by differentiation.

Equations 10-5 and 10-6 hold not only for the rotating rigid body as a whole but also for *every particle of that body* because the particles are all locked together. The unit of angular velocity is commonly the radian per second (rad/s) or the revolution per second (rev/s). Another measure of angular velocity was used during at least the first three decades of rock: Music was produced by vinyl (phonograph) records that were played on turntables at "33 1/3 rpm" or "45 rpm," meaning at 33 1/3 rev/min or 45 rev/min.

If a particle moves in translation along an x axis, its linear velocity v is either positive or negative, depending on its direction along the axis. Similarly, the angular velocity ω of a rotating rigid body is either positive or negative, depending on whether the body is rotating counterclockwise (positive) or clockwise (negative). ("Clocks are negative" still works.) The magnitude of an angular velocity is called the **angular speed**, which is also represented with ω.

Angular Acceleration

If the angular velocity of a rotating body is not constant, then the body has an angular acceleration. Let ω_2 and ω_1 be its angular velocities at times t_2 and t_1, respectively. The **average angular acceleration** of the rotating body in the interval from t_1 to t_2 is defined as

$$\alpha_{avg} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta \omega}{\Delta t}, \tag{10-7}$$

in which $\Delta\omega$ is the change in the angular velocity that occurs during the time interval Δt. The **(instantaneous) angular acceleration** α, with which we shall be most concerned, is the limit of this quantity as Δt approaches zero. Thus,

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt}. \tag{10-8}$$

As the name suggests, this is the angular acceleration of the body at a given instant. Equations 10-7 and 10-8 also hold for *every particle of that body.* The unit of angular acceleration is commonly the radian per second-squared (rad/s^2) or the revolution per second-squared (rev/s^2).

SAMPLE PROBLEM 10.01

Angular velocity derived from angular position

The disk in Fig. 10-5*a* is rotating about its central axis like a merry-go-round. The angular position $\theta(t)$ of a reference line on the disk is given by

$$\theta = -1.00 - 0.600t + 0.250t^2, \tag{10-9}$$

with t in seconds, θ in radians, and the zero angular position as indicated in the figure. (If you like, you can translate all this into Chapter 2 notation by momentarily dropping the word "angular" from "angular position" and replacing the symbol θ with the symbol x. What you then have is an equation that gives the position as a function of time, for the one dimensional motion of Chapter 2.)

(a) Graph the angular position of the disk versus time from $t = -3.0$ s to $t = 5.4$ s. Sketch the disk and its angular position reference line at $t = -2.0$ s, 0 s, and 4.0 s, and when the curve crosses the t axis.

KEY IDEA

The angular position of the disk is the angular position $\theta(t)$ of its reference line, which is given by Eq. 10-9 as a function of time t. So we graph Eq. 10-9; the result is shown in Fig. 10-5*b*.

Calculations: To sketch the disk and its reference line at a particular time, we need to determine θ for that time. To do so, we substitute the time into Eq. 10-9. For $t = -2.0$ s, we get

$$\theta = -1.00 - (0.600)(-2.0) + (0.250)(-2.0)^2$$

$$= 1.2 \text{ rad} = 1.2 \text{ rad} \frac{360^\circ}{2\pi \text{ rad}} = 69^\circ.$$

This means that at $t = -2.0$ s the reference line on the disk is rotated counterclockwise from the zero position by angle 1.2 rad = 69° (counterclockwise because θ

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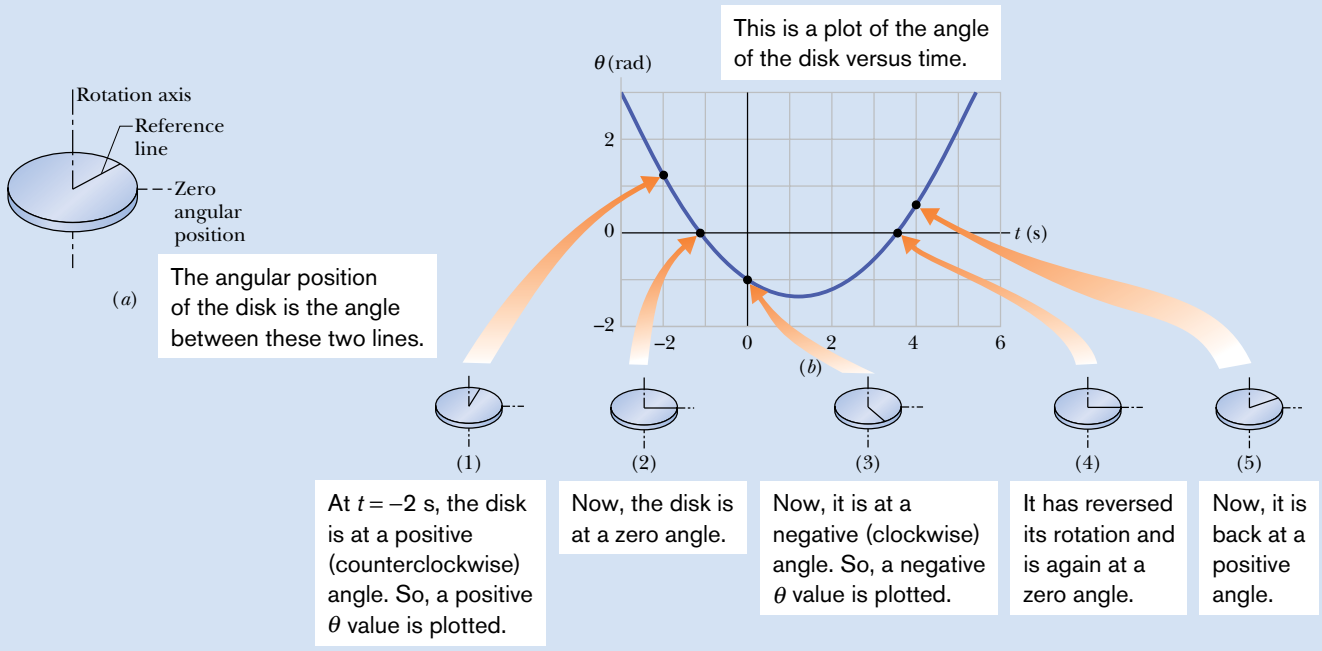
Rotation axis
Reference line
Zero angular position
(a)
The angular position of the disk is the angle between these two lines.
This is a plot of the angle of the disk versus time.
θ (rad)
2
0
−2
−2
0
2
4
6
t (s)
(b)
(1)
(2)
(3)
(4)
(5)
At t = −2 s, the disk is at a positive (counterclockwise) angle. So, a positive θ value is plotted.
Now, the disk is at a zero angle.
Now, it is at a negative (clockwise) angle. So, a negative θ value is plotted.
It has reversed its rotation and is again at a zero angle.
Now, it is back at a positive angle.

Figure 10-5 (*a*) A rotating disk. (*b*) A plot of the disk's angular position $\theta(t)$. Five sketches indicate the angular position of the reference line on the disk for five points on the curve. (*c*) A plot of the disk's angular velocity $\omega(t)$. Positive values of ω correspond to counterclockwise rotation, and negative values to clockwise rotation.

is positive). Sketch 1 in Fig. 10-5*b* shows this position of the reference line.

Similarly, for $t = 0$, we find $\theta = -1.00$ rad $= -57°$, which means that the reference line is rotated clockwise from the zero angular position by 1.0 rad, or 57°, as shown in sketch 3. For $t = 4.0$ s, we find $\theta = 0.60$ rad $= 34°$ (sketch 5). Drawing sketches for when the curve crosses the t axis is easy, because then $\theta = 0$ and the reference line is momentarily aligned with the zero angular position (sketches 2 and 4).

(b) At what time t_{min} does $\theta(t)$ reach the minimum value shown in Fig. 10-5*b*? What is that minimum value?

KEY IDEA

To find the extreme value (here the minimum) of a function, we take the first derivative of the function and set the result to zero.

Calculations: The first derivative of $\theta(t)$ is

$$\frac{d\theta}{dt} = -0.600 + 0.500t. \qquad (10\text{-}10)$$

Setting this to zero and solving for t give us the time at which $\theta(t)$ is minimum:

$$t_{min} = 1.20 \text{ s}. \qquad \text{(Answer)}$$

To get the minimum value of θ, we next substitute t_{min} into Eq. 10-9, finding

$$\theta = -1.36 \text{ rad} \approx -77.9°. \qquad \text{(Answer)}$$

This *minimum* of $\theta(t)$ (the bottom of the curve in Fig. 10-5*b*) corresponds to the *maximum clockwise* rotation of the disk from the zero angular position, somewhat more than is shown in sketch 3.

(c) Graph the angular velocity ω of the disk versus time from $t = -3.0$ s to $t = 6.0$ s. Sketch the disk and indicate the direction of turning and the sign of ω at $t = -2.0$ s, 4.0 s, and t_{min}.

KEY IDEA

From Eq. 10-6, the angular velocity ω is equal to $d\theta/dt$ as given in Eq. 10-10. So, we have

$$\omega = -0.600 + 0.500t. \qquad (10\text{-}11)$$

The graph of this function $\omega(t)$ is shown in Fig. 10-5*c*. Because the function is linear, the plot is a straight line. The slope is 0.500 rad/s^2 and the intercept with the vertical axis (not shown) is –0.600 rad/s.

Calculations: To sketch the disk at $t = -2.0$ s, we substitute that value into Eq. 10-11, obtaining

$$\omega = -1.6 \text{ rad/s}. \qquad \text{(Answer)}$$

The minus sign here tells us that at $t = -2.0$ s, the disk is turning clockwise (as indicated by the left-hand sketch in Fig. 10-5*c*).

Substituting $t = 4.0$ s into Eq. 10-11 gives us

$$\omega = 1.4 \text{ rad/s}. \qquad \text{(Answer)}$$

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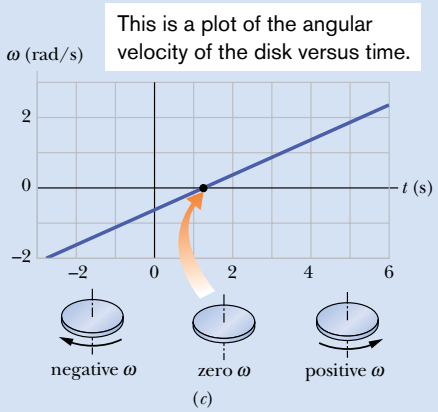

Figure 10-5 Continued.

The implied plus sign tells us that now the disk is turning counterclockwise (the right-hand sketch in Fig. 10-5*c*).

For t_{min}, we already know that $d\theta/dt = 0$. So, we must also have $\omega = 0$. That is, the disk momentarily stops when the reference line reaches the minimum value of θ in Fig. 10-5*b*, as suggested by the center sketch in Fig. 10-5*c*. On the graph of ω versus t in Fig. 10-5*c*, this momentary stop is the zero point where the plot changes from the negative clockwise motion to the positive counterclockwise motion.

(d) Use the results in parts (a) through (c) to describe the motion of the disk from $t = -3.0$ s to $t = 6.0$ s.

Description: When we first observe the disk at $t = -3.0$ s, it has a positive angular position and is turning clockwise but slowing. It stops at angular position $\theta = -1.36$ rad and then begins to turn counterclockwise, with its angular position eventually becoming positive again.

SAMPLE PROBLEM 10.02

Angular velocity derived from angular acceleration

A child's top is spun with angular acceleration

$$\alpha = 5t^3 - 4t,$$

with t in seconds and α in radians per second-squared. At $t = 0$, the top has angular velocity 5 rad/s, and a reference line on it is at angular position $\theta = 2$ rad.

(a) Obtain an expression for the angular velocity $\omega(t)$ of the top. That is, find an expression that explicitly indicates how the angular velocity depends on time. (We can tell that there *is* such a dependence because the top is undergoing an angular acceleration, which means that its angular velocity *is* changing.)

KEY IDEA

By definition, $\alpha(t)$ is the derivative of $\omega(t)$ with respect to time. Thus, we can find $\omega(t)$ by integrating $\alpha(t)$ with respect to time.

Calculations: Equation 10-8 tells us

$$d\omega = \alpha\, dt$$

so

$$\int d\omega = \int \alpha\, dt.$$

From this we find

$$\omega = \int (5t^3 - 4t)\, dt = \frac{5}{4}t^4 - \frac{4}{2}t^2 + C.$$

To evaluate the constant of integration C, we note that $\omega = 5$ rad/s at $t = 0$. Substituting these values in our expression for ω yields

$$5 \text{ rad/s} = 0 - 0 + C,$$

so $C = 5$ rad/s. Then

$$\omega = \frac{5}{4}t^4 - 2t^2 + 5. \qquad \text{(Answer)}$$

(b) Obtain an expression for the angular position $\theta(t)$ of the top.

KEY IDEA

By definition, $\omega(t)$ is the derivative of $\theta(t)$ with respect to time. Therefore, we can find $\theta(t)$ by integrating $\omega(t)$ with respect to time.

Calculations: Since Eq. 10-6 tells us that

$$d\theta = \omega\, dt,$$

we can write

$$\theta = \int \omega\, dt = \int \left(\frac{5}{4}t^4 - 2t^2 + 5\right) dt$$

$$= \frac{1}{4}t^5 - \frac{2}{3}t^3 + 5t + C'$$

$$= \frac{1}{4}t^5 - \frac{2}{3}t^3 + 5t + 2, \qquad \text{(Answer)}$$

where C'' has been evaluated by noting that $\theta = 2$ rad at $t = 0$.

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10.3 | ARE ANGULAR QUANTITIES VECTORS?

Key Concepts

- The angular velocity of the rotating body can be represented by the vector $\vec{\omega}$, lying along the axis and pointing down. We establish the direction of the angular velocity vector as downward by using a right-hand rule.
- Angular displacement (unless they are very small) *cannot* be treated as vector. We can certainly assign both magnitude and direction but it does not follow the rules of vector addition to qualify as vector quantity.

We can describe the position, velocity, and acceleration of a single particle by means of vectors. If the particle is confined to a straight line, however, we do not really need vector notation. Such a particle has only two directions available to it, and we can indicate these directions with plus and minus signs.

In the same way, a rigid body rotating about a fixed axis can rotate only clockwise or counterclockwise as seen along the axis, and again we can select between the two directions by means of plus and minus signs. The question arises: "Can we treat the angular displacement, velocity, and acceleration of a rotating body as vectors?" The answer is a qualified "yes" (see the caution below, in connection with angular displacements).

Angular Velocities. Consider the angular velocity. Figure 10-6*a* shows a vinyl record rotating on a turntable. The record has a constant angular speed ω (= 33 1/3 rev/min) in the clockwise direction. We can represent its angular velocity as a vector $\vec{\omega}$ pointing along the axis of rotation, as in Fig. 10-6*b*. Here's how: We choose the length of this vector according to some convenient scale, for example, with 1 cm corresponding to 10 rev/min. Then we establish a direction for the vector $\vec{\omega}$ by using a **right-hand rule**, as Fig. 10-6*c* shows: Curl your right hand about the rotating record, your fingers pointing *in the direction of rotation.* Your extended thumb will then point in the direction of the angular velocity vector. If the record were to rotate in the opposite sense, the right-hand rule would tell you that the angular velocity vector then points in the opposite direction.

It is not easy to get used to representing angular quantities as vectors. We instinctively expect that something should be moving *along* the direction of a vector. That is not the case here. Instead, something (the rigid body) is rotating *around* the direction of the vector. In the world of pure rotation, a vector defines an axis of rotation, not a direction in which something moves. Nonetheless, the vector also defines the motion. Furthermore, it obeys all the rules for vector manipulation discussed in Chapter 3. The angular acceleration $\vec{\alpha}$ is another vector, and it too obeys those rules.

In this chapter, we consider only rotations that are about a fixed axis. For such situations, we need not consider vectors—we can represent angular velocity with ω and angular acceleration with α, and we can indicate direction

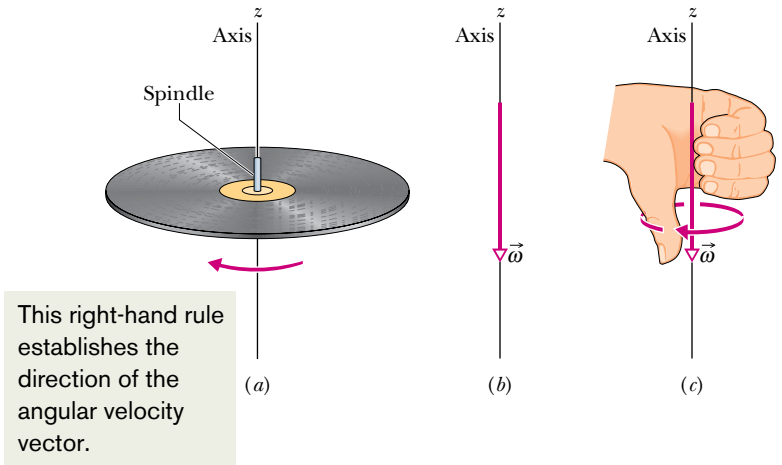

Figure 10-6 (*a*) A record rotating about a vertical axis that coincides with the axis of the spindle. (*b*) The angular velocity of the rotating record can be represented by the vector $\vec{\omega}$, lying along the axis and pointing down, as shown. (*c*) We establish the direction of the angular velocity vector as downward by using a right-hand rule. When the fingers of the right hand curl around the record and point the way it is moving, the extended thumb points in the direction of $\vec{\omega}$.

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with an implied plus sign for counterclockwise or an explicit minus sign for clockwise.

Angular Displacements. Now for the caution: Angular *displacements* (unless they are very small) *cannot* be treated as vectors. Why not? We can certainly give them both magnitude and direction, as we did for the angular velocity vector in Fig. 10-6. However, to be represented as a vector, a quantity must *also* obey the rules of vector addition, one of which says that if you add two vectors, the order in which you add them does not matter. Angular displacements fail this test.

Figure 10-7 gives an example. An initially horizontal book is given two 90° angular displacements, first in the order of Fig. 10-7*a* and then in the order of Fig. 10-7*b*. Although the two angular displacements are identical, their order is not, and the book ends up with different orientations. Here's another example. Hold your right arm downward, palm toward your thigh. Keeping your wrist rigid, (1) lift the arm forward until it is horizontal, (2) move it horizontally until it points toward the right, and (3) then bring it down to your side. Your palm faces forward. If you start over, but reverse the steps, which way does your palm end up facing? From either example, we must conclude that the addition of two angular displacements depends on their order and they cannot be vectors.

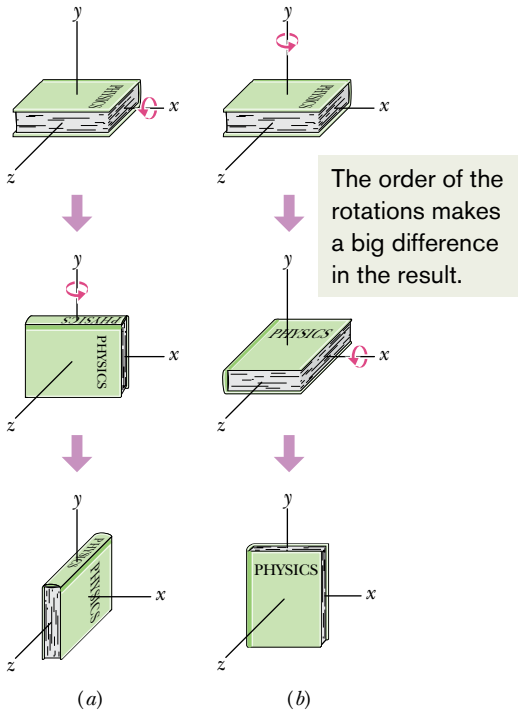

Figure 10-7 (*a*) From its initial position, at the top, the book is given two successive 90° rotations, first about the (horizontal) x axis and then about the (vertical) y axis. (*b*) The book is given the same rotations, but in the reverse order.

10.4 | ROTATION WITH CONSTANT ANGULAR ACCELERATION

Key Concept

- Constant angular acceleration (α = constant) is an important special case of rotational motion. The appropriate kinematic equations are

$$\omega = \omega_0 + \alpha t,$$
$$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2,$$
$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0),$$
$$\theta - \theta_0 = \frac{1}{2}(\omega_0 + \omega)t,$$
$$\theta - \theta_0 = \omega t - \frac{1}{2}\alpha t^2.$$

In pure translation, motion with a *constant linear acceleration* (for example, that of a falling body) is an important special case. In Table 2-1, we displayed a series of equations that hold for such motion.

In pure rotation, the case of *constant angular acceleration* is also important, and a parallel set of equations holds for this case also. We shall not derive them here, but simply write them from the corresponding linear equations, substituting equivalent angular quantities for the linear ones. This is done in Table 10-1, which lists both sets of equations (Eqs. 2-11 and 2-15 to 2-18; 10-12 to 10-16).

Recall that Eqs. 2-11 and 2-15 are basic equations for constant linear acceleration—the other equations in the Linear list can be derived from them. Similarly, Eqs. 10-12 and 10-13 are the basic equations for constant angular acceleration, and the other equations in the Angular list can be derived from them. To solve a simple problem involving constant angular acceleration, you can usually use an equation from the Angular list (*if* you have

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Table 10-1 Equations of Motion for Constant Linear Acceleration and for Constant Angular Acceleration

Equation Number	Linear Equation	Missing Variable		Angular Equation	Equation Number
(2-11)	$v = v_0 + at$	$x - x_0$	$\theta - \theta_0$	$\omega = \omega_0 + \alpha t$	(10-12)
(2-15)	$x - x_0 = v_0 t + \frac{1}{2}at^2$	v	ω	$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$	(10-13)
(2-16)	$v^2 = v_0^2 + 2a(x - x_0)$	t	t	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$	(10-14)
(2-17)	$x - x_0 = \frac{1}{2}(v_0 + v)t$	a	α	$\theta - \theta_0 = \frac{1}{2}(\omega_0 + \omega)t$	(10-15)
(2-18)	$x - x_0 = vt - \frac{1}{2}at^2$	v_0	ω_0	$\theta - \theta_0 = \omega t - \frac{1}{2}\alpha t^2$	(10-16)

the list). Choose an equation for which the only unknown variable will be the variable requested in the problem. A better plan is to remember only Eqs. 10-12 and 10-13, and then solve them as simultaneous equations whenever needed.

CHECKPOINT 2

In four situations, a rotating body has angular position $\theta(t)$ given by (a) $\theta = 3t - 4$, (b) $\theta = -5t^3 + 4t^2 + 6$, (c) $\theta = 2/t^2 - 4/t$, and (d) $\theta = 5t^2 - 3$. To which situations do the angular equations of Table 10-1 apply?

SAMPLE PROBLEM 10.03

Constant angular acceleration, grindstone

A grindstone (Fig. 10-8) rotates at constant angular acceleration $\alpha = 0.35$ rad/s^2. At time $t = 0$, it has an angular velocity of $\omega_0 = -4.6$ rad/s and a reference line on it is horizontal, at the angular position $\theta_0 = 0$.

(a) At what time after $t = 0$ is the reference line at the angular position $\theta = 5.0$ rev?

KEY IDEA

The angular acceleration is constant, so we can use the rotation equations of Table 10-1. We choose Eq. 10-13,

$$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2,$$

because the only unknown variable it contains is the desired time t.

Calculations: Substituting known values and setting $\theta_0 = 0$ and $\theta = 5.0$ rev $= 10\,\pi$ rad give us

$$10\,\pi \text{ rad} = (-4.6 \text{ rad/s})t + \frac{1}{2}(0.35 \text{ rad/s}^2)t^2.$$

(We converted 5.0 rev to 10 π rad to keep the units consistent.) Solving this quadratic equation for t, we find

$$t = 32 \text{ s.} \qquad \text{(Answer)}$$

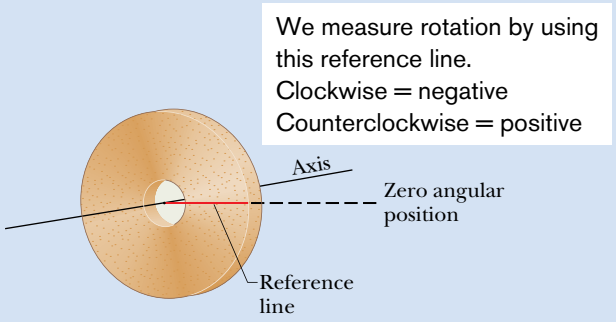

Figure 10-8 A grindstone. At $t = 0$ the reference line (which we imagine to be marked on the stone) is horizontal.

Now notice something a bit strange. We first see the wheel when it is rotating in the negative direction and through the $\theta = 0$ orientation. Yet, we just found out that 32 s later it is at the positive orientation of $\theta = 5.0$ rev. What happened in that time interval so that it could be at a positive orientation?

(b) Describe the grindstone's rotation between $t = 0$ and $t = 32$ s.

Reasoning: The wheel is initially rotating in the negative (clockwise) direction with angular velocity $\omega_0 = -4.6$ rad/s,

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but its angular acceleration α is positive. This initial opposition of the signs of angular velocity and angular acceleration means that the wheel slows in its rotation in the negative direction, stops, and then reverses to rotate in the positive direction. After the reference line comes back through its initial orientation of $\theta = 0$, the wheel turns an additional 5.0 rev by time $t = 32$ s.

(c) At what time t does the grindstone momentarily stop?

Calculation: We again go to the table of equations for constant angular acceleration, and again we need an equation that contains only the desired unknown variable t. However, now the equation must also contain the variable ω, so that we can set it to 0 and then solve for the corresponding time t. We choose Eq. 10-12, which yields

$$t = \frac{\omega - \omega_0}{\alpha} = \frac{0 - (-4.6 \text{ rad/s})}{0.35 \text{ rad/s}^2} = 13 \text{ s}. \qquad \text{(Answer)}$$

SAMPLE PROBLEM 10.04

Constant angular acceleration, riding a Rotor

While you are operating a Rotor (a large, vertical, rotating cylinder found in amusement parks), you spot a passenger in acute distress and decrease the angular velocity of the cylinder from 3.40 rad/s to 2.00 rad/s in 20.0 rev, at constant angular acceleration. (The passenger is obviously more of a "translation person" than a "rotation person.")

(a) What is the constant angular acceleration during this decrease in angular speed?

KEY IDEA

Because the cylinder's angular acceleration is constant, we can relate it to the angular velocity and angular displacement via the basic equations for constant angular acceleration (Eqs. 10-12 and 10-13).

Calculations: Let's first do a quick check to see if we can solve the basic equations. The initial angular velocity is $\omega_0 = 3.40$ rad/s, the angular displacement is $\theta - \theta_0 = 20.0$ rev, and the angular velocity at the end of that displacement is $\omega = 2.00$ rad/s. In addition to the angular acceleration α that we want, both basic equations also contain time t, which we do not necessarily want.

To eliminate the unknown t, we use Eq. 10-12 to write

$$t = \frac{\omega - \omega_0}{\alpha},$$

which we then substitute into Eq. 10-13 to write

$$\theta - \theta_0 = \omega_0\left(\frac{\omega - \omega_0}{\alpha}\right) + \frac{1}{2}\alpha\left(\frac{\omega - \omega_0}{\alpha}\right)^2.$$

Solving for α, substituting known data, and converting 20 rev to 125.7 rad, we find

$$\alpha = \frac{\omega^2 - \omega_0^2}{2(\theta - \theta_0)} = \frac{(2.00 \text{ rad/s})^2 - (3.40 \text{ rad/s})^2}{2(125.7 \text{ rad})}$$
$$= -0.0301 \text{ rad/s}^2. \qquad \text{(Answer)}$$

(b) How much time did the speed decrease take?

Calculation: Now that we know α, we can use Eq. 10-12 to solve for t:

$$t = \frac{\omega - \omega_0}{\alpha} = \frac{2.00 \text{ rad/s} - 3.40 \text{ rad/s}}{-0.0301 \text{ rad/s}^2}$$
$$= 46.5 \text{ s}. \qquad \text{(Answer)}$$

10.5 | RELATING THE LINEAR AND ANGULAR VARIABLES

Key Concepts

- A point in a rigid rotating body, at a perpendicular distance r from the rotation axis, moves in a circle with radius r. If the body rotates through an angle θ, the point moves along an arc with length s given by

$$s = \theta r \qquad \text{(radian measure)},$$

where θ is in radians.

- The linear velocity $\vec{v}$ of the point is tangent to the circle; the point's linear speed v is given by

$$v = \omega r \qquad \text{(radian measure)},$$

where ω is the angular speed (in radians per second) of the body, and thus also the point.

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- The linear acceleration $\vec{a}$ of the point has both tangential and radial components. The tangential component is

$$a_t = \alpha r \qquad \text{(radian measure)},$$

where α is the magnitude of the angular acceleration (in radians per second-squared) of the body. The radial component of $\vec{a}$ is

$$a_r = \frac{v^2}{r} = \omega^2 r \qquad \text{(radian measure)}.$$

- If the point moves in uniform circular motion, the period T of the motion for the point and the body is

$$T = \frac{2\pi r}{v} = \frac{2\pi}{\omega} \qquad \text{(radian measure)}.$$

In Section 7.4, we discussed uniform circular motion, in which a particle travels at constant linear speed v along a circle and around an axis of rotation. When a rigid body, such as a merry-go-round, rotates around an axis, each particle in the body moves in its own circle around that axis. Since the body is rigid, all the particles make one revolution in the same amount of time; that is, they all have the same angular speed ω.

However, the farther a particle is from the axis, the greater the circumference of its circle is, and so the faster its linear speed v must be. You can notice this on a merry-go-round. You turn with the same angular speed ω regardless of your distance from the center, but your linear speed v increases noticeably if you move to the outside edge of the merry-go-round.

We often need to relate the linear variables s, v, and a for a particular point in a rotating body to the angular variables θ, ω, and α for that body. The two sets of variables are related by r, the *perpendicular distance* of the point from the rotation axis. This perpendicular distance is the distance between the point and the rotation axis, measured along a perpendicular to the axis. It is also the radius r of the circle traveled by the point around the axis of rotation.

The Position

If a reference line on a rigid body rotates through an angle θ, a point within the body at a position r from the rotation axis moves a distance s along a circular arc, where s is given by Eq. 10-1:

$$s = \theta r \quad \text{(radian measure)}. \tag{10-17}$$

This is the first of our linear–angular relations.

Caution: The angle θ here must be measured in radians because Eq. 10-17 is itself the definition of angular measure in radians.

The Speed

Differentiating Eq. 10-17 with respect to time—with r held constant—leads to

$$\frac{ds}{dt} = \frac{d\theta}{dt} r.$$

However, ds/dt is the linear speed (the magnitude of the linear velocity) of the point in question, and $d\theta/dt$ is the angular speed ω of the rotating body. So

$$v = \omega r \quad \text{(radian measure)}. \tag{10-18}$$

Caution: The angular speed ω must be expressed in radian measure.

Equation 10-18 tells us that since all points within the rigid body have the same angular speed ω, points with greater radius r have greater linear speed v. Figure 10-9*a* reminds us that the linear velocity is always tangent to the circular path of the point in question.

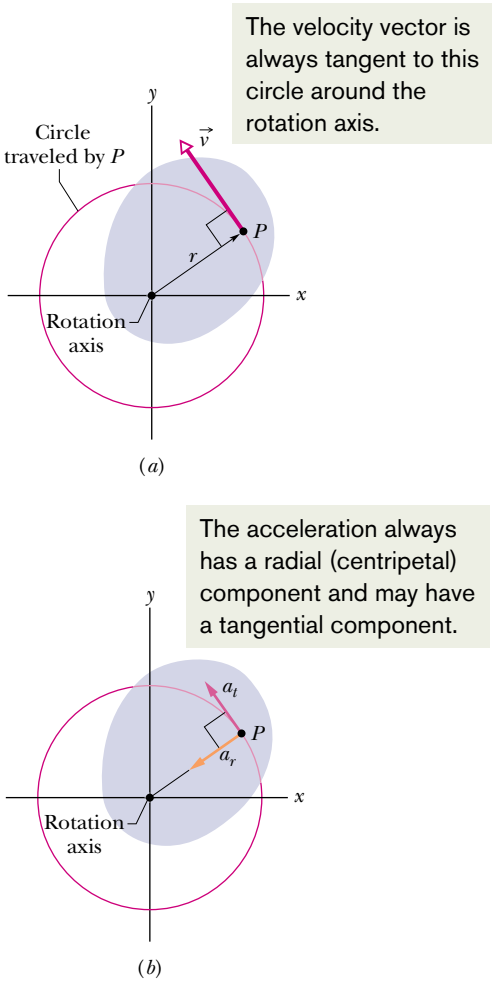

Figure 10-9 The rotating rigid body of Fig. 10-2, shown in cross section viewed from above. Every point of the body (such as P) moves in a circle around the rotation axis. (*a*) The linear velocity $\vec{v}$ of every point is tangent to the circle in which the point moves. (*b*) The linear acceleration $\vec{a}$ of the point has (in general) two components: tangential a_t and radial a_r.

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If the angular speed ω of the rigid body is constant, then Eq. 10-18 tells us that the linear speed v of any point within it is also constant. Thus, each point within the body undergoes uniform circular motion. The period of revolution T for the motion of each point and for the rigid body itself is given by Eq. 7-11:

$$T = \frac{2\pi r}{v}. \tag{10-19}$$

This equation tells us that the time for one revolution is the distance $2\pi r$ traveled in one revolution divided by the speed at which that distance is traveled. Substituting for v from Eq. 10-18 and canceling r, we find also that

$$T = \frac{2\pi}{\omega} \quad \text{(radian measure)}. \tag{10-20}$$

This equivalent equation says that the time for one revolution is the angular distance 2π rad traveled in one revolution divided by the angular speed (or rate) at which that angle is traveled.

The Acceleration

Differentiating Eq. 10-18 with respect to time—again with r held constant—leads to

$$\frac{dv}{dt} = \frac{d\omega}{dt} r. \tag{10-21}$$

Here we run up against a complication. In Eq. 10-21, dv/dt represents only the part of the linear acceleration that is responsible for changes in the *magnitude* v of the linear velocity $\vec{v}$. Like $\vec{v}$, that part of the linear acceleration is tangent to the path of the point in question. We call it the *tangential component* a_t of the linear acceleration of the point, and we write

$$a_t = \alpha r \quad \text{(radian measure)}, \tag{10-22}$$

where $\alpha = d\omega/dt$. *Caution:* The angular acceleration α in Eq. 10-22 must be expressed in radian measure.

In addition, as Eq. 7-10 tells us, a particle (or point) moving in a circular path has a *radial component* of linear acceleration, $a_r = v^2/r$ (directed radially inward), that is responsible for changes in the *direction* of the linear velocity $\vec{v}$. By substituting for v from Eq. 10-18, we can write this component as

$$a_r = \frac{v^2}{r} = \omega^2 r \quad \text{(radian measure)}. \tag{10-23}$$

Thus, as Fig. 10-9*b* shows, the linear acceleration of a point on a rotating rigid body has, in general, two components. The radially inward component a_r (given by Eq. 10-23) is present whenever the angular velocity of the body is not zero. The tangential component a_t (given by Eq. 10-22) is present whenever the angular acceleration is not zero.

CHECKPOINT 3

A cockroach rides the rim of a rotating merry-go-round. If the angular speed of this system (*merry-go-round + cockroach*) is constant, does the cockroach have (a) radial acceleration and (b) tangential acceleration? If ω is decreasing, does the cockroach have (c) radial acceleration and (d) tangential acceleration?

SAMPLE PROBLEM 10.05

Designing The Giant Ring, a large-scale amusement park ride

We are given the job of designing a large horizontal ring that will rotate around a vertical axis and that will have a radius of r = 33.1 m (matching that of Beijing's The Great Observation Wheel, the largest Ferris wheel in the world). Passengers will enter through a door in the outer wall of the ring and then stand next to that wall (Fig. 10-10*a*). We decide that for the time interval $t = 0$ to t = 2.30 s, the angular position $\theta(t)$ of a reference line on the ring will be given by

$$\theta = ct^3, \tag{10-24}$$

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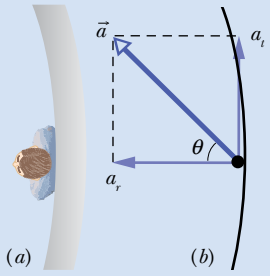

Figure 10-10 (*a*) Overhead view of a passenger ready to ride The Giant Ring. (*b*) The radial and tangential acceleration components of the (full) acceleration.

with $c = 6.39 \times 10^{-2}$ rad/s^3. After $t = 2.30$ s, the angular speed will be held constant until the end of the ride. Once the ring begins to rotate, the floor of the ring will drop away from the riders but the riders will not fall—indeed, they feel as though they are pinned to the wall. For the time $t = 2.20$ s, let's determine a rider's angular speed ω, linear speed v, angular acceleration α, tangential acceleration a_t, radial acceleration a_r, and acceleration $\vec{a}$.

KEY IDEAS

(1) The angular speed ω is given by Eq. 10-6 ($\omega = d\theta/dt$). (2) The linear speed v (along the circular path) is related to the angular speed (around the rotation axis) by Eq. 10-18 ($v = \omega r$). (3) The angular acceleration α is given by Eq. 10-8 ($\alpha = d\omega/dt$). (4) The tangential acceleration a_t (along the circular path) is related to the angular acceleration (around the rotation axis) by Eq. 10-22 ($a_t = \alpha r$). (5) The radial acceleration a_r is given Eq. 10-23 ($a_r = \omega^2 r$). (6) The tangential and radial accelerations are the (perpendicular) components of the (full) acceleration $\vec{a}$.

Calculations: Let's go through the steps. We first find the angular velocity by taking the time derivative of the given angular position function and then substituting the given time of $t = 2.20$ s:

$$\omega = \frac{d\theta}{dt} = \frac{d}{dt}(ct^3) = 3ct^2 \qquad (10\text{-}25)$$
$$= 3(6.39\times10^{-2}\,\text{rad/s}^3)(2.20\,\text{s})^2$$
$$= 0.928\ \text{rad/s}. \qquad \text{(Answer)}$$

From Eq. 10-18, the linear speed just then is

$$v = \omega r = 3ct^2 r \qquad (10\text{-}26)$$
$$= 3(6.39 \times 10^{-2}\ \text{rad/s}^3)(2.20\ \text{s})^2\ (33.1\ \text{m})$$
$$= 30.7\ \text{m/s}. \qquad \text{(Answer)}$$

Although this is fast (111 km/h or 68.7 mi/h), such speeds are common in amusement parks and not alarming because (as mentioned in Chapter 2) your body reacts to accelerations but not to velocities. (It is an accelerometer, not a speedometer.) From Eq. 10-26 we see that the linear speed is increasing as the square of the time (but this increase will cut off at $t = 2.30$ s).

Next, let's tackle the angular acceleration by taking the time derivative of Eq. 10-25:

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}(3ct^2) = 6ct$$
$$= 6(6.39\times10^{-2}\,\text{rad/s}^3)(2.20\,\text{s}) = 0.843\,\text{rad/s}^2. \qquad \text{(Answer)}$$

The tangential acceleration then follows from Eq. 10-22:

$$a_t = \alpha r = 6ctr \qquad (10\text{-}27)$$
$$= 6(6.39 \times 10^{-2}\ \text{rad/s}^3)(2.20\ \text{s})^2\ (33.1\ \text{m})$$
$$= 27.91\ \text{m/s}^2 \approx 27.9\ \text{m/s}^2, \qquad \text{(Answer)}$$

or 2.8g (which is reasonable and a bit exciting). Equation 10-27 tells us that the tangential acceleration is increasing with time (but it will cut off at $t = 2.30$ s). From Eq. 10-23, we write the radial acceleration as

$$a_r = \omega^2 r.$$

Substituting from Eq. 10-25 leads us to

$$a_r = (3ct^2)^2 r = 9c^2t^4 r \qquad (10\text{-}28)$$
$$= 9(6.39 \times 10^{-2}\ \text{rad/s}^3)^2(2.20\ \text{s})^4\ (33.1\ \text{m})$$
$$= 28.49\ \text{m/s}^2 \approx 28.5\ \text{m/s}^2, \qquad \text{(Answer)}$$

or 2.9g (which is also reasonable and a bit exciting).

The radial and tangential accelerations are perpendicular to each other and form the components of the rider's acceleration $\vec{a}$ (Fig. 10-10*b*). The magnitude of $\vec{a}$ is given by

$$a = \sqrt{a_r^2 + a_t^2} \qquad (10\text{-}29)$$
$$= \sqrt{(28.49\ \text{m/s}^2)^2 + (27.91\ \text{m/s}^2)^2}$$
$$\approx 39.9\ \text{m/s}^2. \qquad \text{(Answer)}$$

or 4.1g (which is really exciting!). All these values are acceptable.

To find the orientation of $\vec{a}$, we can calculate the angle θ shown in Fig. 10-10*b*:

$$\tan\theta = \frac{a_t}{a_r}.$$

However, instead of substituting our numerical results, let's use the algebraic results from Eqs. 10-27 and 10-28:

$$\theta = \tan^{-1}\left(\frac{6ctr}{9c^2t^4r}\right) = \tan^{-1}\left(\frac{2}{3ct^3}\right). \qquad (10\text{-}30)$$

The big advantage of solving for the angle algebraically is that we can then see that the angle (1) does not depend on the ring's radius and (2) decreases as t goes from 0 to 2.20 s. That is, the acceleration vector $\vec{a}$ swings toward being radially inward because the radial acceleration (which depends on t^4) quickly dominates over the tangential acceleration (which depends on only t). At our given time $t = 2.20$ s, we have

$$\theta = \tan^{-1}\frac{2}{3(6.39\times10^{-2}\,\text{rad/s}^3)(2.20\,\text{s})^3} = 44.4°. \qquad \text{(Answer)}$$

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10.6 | KINETIC ENERGY OF ROTATION

Key Concept

- The kinetic energy K of a rigid body rotating about a fixed axis is given by

$$K = \frac{1}{2} I\omega^2 \quad \text{(radian measure)},$$

in which I is the rotational inertia of the body, defined as

$$I = \sum m_i r_i^2$$

for a system of discrete particles.

The rapidly rotating blade of a table saw certainly has kinetic energy due to that rotation. How can we express the energy? We cannot apply the familiar formula $K = 1/2mv^2$ to the saw as a whole because that would give us the kinetic energy only of the saw's center of mass, which is zero.

Instead, we shall treat the table saw (and any other rotating rigid body) as a collection of particles with different speeds. We can then add up the kinetic energies of all the particles to find the kinetic energy of the body as a whole. In this way we obtain, for the kinetic energy of a rotating body,

$$\begin{aligned} K &= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + \cdots \\ &= \sum \frac{1}{2}m_i v_i^2, \end{aligned} \tag{10-31}$$

in which m_i is the mass of the ith particle and v_i is its speed. The sum is taken over all the particles in the body.

The problem with Eq. 10-31 is that v_i is not the same for all particles. We solve this problem by substituting for v from Eq. 10-18 ($v = \omega r$), so that we have

$$K = \sum \frac{1}{2}m_i(\omega r_i)^2 = \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2, \tag{10-32}$$

in which ω *is* the same for all particles.

The quantity in parentheses on the right side of Eq. 10-32 tells us how the mass of the rotating body is distributed about its axis of rotation. We call that quantity the **rotational inertia** (or **moment of inertia**) I of the body with respect to the axis of rotation. It is a constant for a particular rigid body and a particular rotation axis. (*Caution:* That axis must always be specified if the value of I is to be meaningful.)

We may now write

$$I = \sum m_i r_i^2 \quad \text{(rotational inertia)} \tag{10-33}$$

and substitute into Eq. 10-32, obtaining

$$K = \frac{1}{2} I\omega^2 \quad \text{(radian measure)} \tag{10-34}$$

as the expression we seek. Because we have used the relation $v = \omega r$ in deriving Eq. 10-34, ω must be expressed in radian measure. The SI unit for I is the kilogram–square meter ($kg \cdot m^2$).

The Plan. If we have a few particles and a specified rotation axis, we find mr^2 for each particle and then add the results as in Eq. 10-33 to get the total rotational inertia I. If we want the total rotational kinetic energy, we can then substitute that I into Eq. 10-34. That is the plan for a few particles, but suppose we have a huge number of particles such as in a rod. In the next module we shall see how to handle such *continuous bodies* and do the calculation in only a few minutes.

Equation 10-34, which gives the kinetic energy of a rigid body in pure rotation, is the angular equivalent of the formula $K = 1/2Mv_{com}^2$, which gives the kinetic energy of a rigid body in pure translation. In both formulas there is a factor of 1/2. Where mass M appears in one equation, I (which involves both mass and its distribution) appears in the other. Finally, each equation contains as a factor the square of a speed—translational or rotational as appropriate. The kinetic energies of translation and of rotation are not different kinds of energy. They are both kinetic energy, expressed in ways that are appropriate to the motion at hand.

We noted previously that the rotational inertia of a rotating body involves not only its mass but also how that mass is distributed. Here is an example that you can literally feel. Rotate a long, fairly heavy rod (a pole, a length

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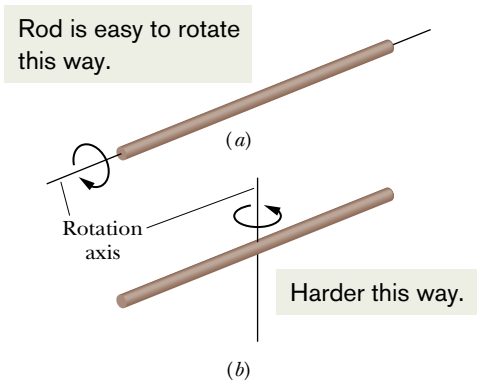
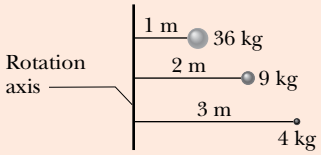


Figure 10-11 A long rod is much easier to rotate about (*a*) its central (longitudinal) axis than about (*b*) an axis through its center and perpendicular to its length. The reason for the difference is that the mass is distributed closer to the rotation axis in (*a*) than in (*b*).

of lumber, or something similar), first around its central (longitudinal) axis (Fig. 10-11*a*) and then around an axis perpendicular to the rod and through the center (Fig. 10-11*b*). Both rotations involve the very same mass, but the first rotation is much easier than the second. The reason is that the mass is distributed much closer to the rotation axis in the first rotation. As a result, the rotational inertia of the rod is much smaller in Fig. 10-11*a* than in Fig. 10-11*b*. In general, smaller rotational inertia means easier rotation.

CHECKPOINT 4

The figure shows three small spheres that rotate about a vertical axis. The perpendicular distance between the axis and the center of each sphere is given. Rank the three spheres according to their rotational inertia about that axis, greatest first.



10.7 | CALCULATING THE ROTATIONAL INERTIA

Key Concepts

- I is the rotational inertia of the body, defined as

$$I = \sum m_i r_i^2$$

for a system of discrete particles and defined as

$$I = \int r^2 dm$$

for a body with continuously distributed mass. The r and r_i in these expressions represent the perpendicular distance from the axis of rotation to each mass element in the body, and the integration is carried out over the entire body so as to include every mass element.

- The parallel-axis theorem relates the rotational inertia I of a body about any axis to that of the same body about a parallel axis through the center of mass:

$$I = I_{com} + Mh^2.$$

Here h is the perpendicular distance between the two axes, and I_{com} is the rotational inertia of the body about the axis through the com. We can describe h as being the distance the actual rotation axis has been shifted from the rotation axis through the com.

If a rigid body consists of a few particles, we can calculate its rotational inertia about a given rotation axis with Eq. 10-33 ($I = \Sigma m_i r_i^2$); that is, we can find the product mr^2 for each particle and then sum the products. (Recall that r is the perpendicular distance a particle is from the given rotation axis.)

If a rigid body consists of a great many adjacent particles (it is *continuous,* like a Frisbee), using Eq. 10-33 would require a computer. Thus, instead, we replace the sum in Eq. 10-33 with an integral and define the rotational inertia of the body as

$$I = \int r^2 dm \quad \text{(rotational inertia, continuous body)}. \tag{10-35}$$

Table 10-2 gives the results of such integration for nine common body shapes and the indicated axes of rotation.

Parallel-Axis Theorem

Suppose we want to find the rotational inertia I of a body of mass M about a given axis. In principle, we can always find I with the integration of Eq. 10-35. However, there is a neat shortcut if we happen to already know the rotational inertia I_{com} of the body about a *parallel* axis that extends through the body's center of mass. Let h be the

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Table 10-2 Some Rotational Inertias

Axis, R	Hoop about central axis	$I = MR^2$ (*a*)
Axis, R_1, R_2	Annular cylinder (or ring) about central axis	$I = \frac{1}{2}M(R_1^2 + R_2^2)$ (*b*)
Axis, R, L	Solid cylinder (or disk) about central axis	$I = \frac{1}{2}MR^2$ (*c*)
Axis, R, L	Solid cylinder (or disk) about central diameter	$I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$ (*d*)
Axis, L	Thin rod about axis through center perpendicular to length	$I = \frac{1}{12}ML^2$ (*e*)
Axis, $2R$	Solid sphere about any diameter	$I = \frac{2}{5}MR^2$ (*f*)
Axis, $2R$	Thin spherical shell about any diameter	$I = \frac{2}{3}MR^2$ (*g*)
Axis, R	Hoop about any diameter	$I = \frac{1}{2}MR^2$ (*h*)
Axis, a, b	Slab about perpendicular axis through center	$I = \frac{1}{12}M(a^2 + b^2)$ (*i*)

We need to relate the rotational inertia around the axis at P to that around the axis at the com.

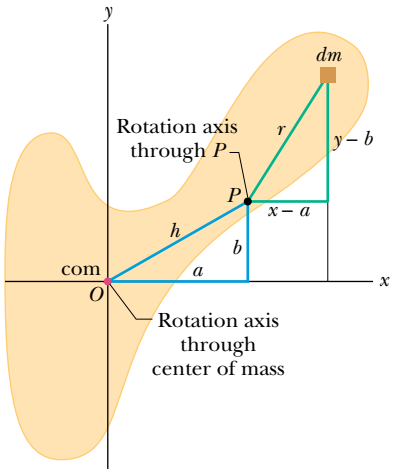

Figure 10-12 A rigid body in cross section, with its center of mass at O. The parallel axis theorem (Eq. 10-36) relates the rotational inertia of the body about an axis through O to that about a parallel axis through a point such as P, a distance h from the body's center of mass.

perpendicular distance between the given axis and the axis through the center of mass (remember these two axes must be parallel). Then the rotational inertia I about the given axis is

$$I = I_{com} + Mh^2 \quad \text{(parallel-axis theorem)}. \tag{10-36}$$

Think of the distance h as being the distance we have shifted the rotation axis from being through the com. This equation is known as the **parallel-axis theorem**. We shall now prove it.

Proof of the Parallel-Axis Theorem

Let O be the center of mass of the arbitrarily shaped body shown in cross section in Fig. 10-12. Place the origin of the coordinates at O. Consider an axis through O perpendicular to the plane of the figure, and another axis through point P parallel to the first axis. Let the x and y coordinates of P be a and b.

Let dm be a mass element with the general coordinates x and y. The rotational inertia of the body about the axis through P is then, from Eq. 10-35,

$$I = \int r^2\, dm = \int [(x-a)^2 + (y-b)^2]\, dm,$$

which we can rearrange as

$$I = \int (x^2 + y^2)\, dm - 2a\int x\, dm - 2b\int y\, dm + \int (a^2 + b^2)\, dm. \tag{10-37}$$

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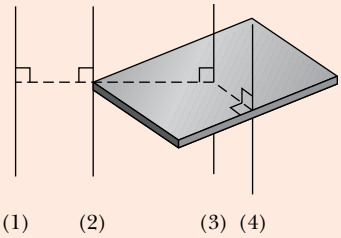
From the definition of the center of mass (Eq. 9-9), the middle two integrals of Eq. 10-37 give the coordinates of the center of mass (multiplied by a constant) and thus must each be zero. Because $x^2 + y^2$ is equal to R^2, where R is the distance from O to dm, the first integral is simply I_{com}, the rotational inertia of the body about an axis through its center of mass. Inspection of Fig. 10-12 shows that the last term in Eq. 10-37 is Mh^2, where M is the body's total mass. Thus, Eq. 10-37 reduces to Eq. 10-36, which is the relation that we set out to prove.

CHECKPOINT 5

The figure shows a book-like object (one side is longer than the other) and four choices of rotation axes, all perpendicular to the face of the object. Rank the choices according to the rotational inertia of the object about the axis, greatest first.



SAMPLE PROBLEM 10.06

Rotational inertia of a two-particle system

Figure 10-13*a* shows a rigid body consisting of two particles of mass m connected by a rod of length L and negligible mass.

(a) What is the rotational inertia I_{com} about an axis through the center of mass, perpendicular to the rod as shown?

KEY IDEA

Because we have only two particles with mass, we can find the body's rotational inertia I_{com} by using Eq. 10-33 rather than by integration. That is, we find the rotational inertia of each particle and then just add the results.

Calculations: For the two particles, each at perpendicular distance $1/2L$ from the rotation axis, we have

$$I = \sum m_i r_i^2 = (m)\left(\frac{1}{2}L\right)^2 + (m)\left(\frac{1}{2}L\right)^2$$
$$= \frac{1}{2}mL^2. \qquad \text{(Answer)}$$

(b) What is the rotational inertia I of the body about an axis through the left end of the rod and parallel to the first axis (Fig. 10-13*b*)?

KEY IDEAS

This situation is simple enough that we can find I using either of two techniques. The first is similar to the one used in part (a). The other, more powerful one is to apply the parallel-axis theorem.

Calculation method 1: We calculate I as in part (a), except here the perpendicular distance r_i is zero for the particle on the left and L for the particle on the right. Now Eq. 10-33 gives us

$$I = m(0)^2 + mL^2 = mL^2. \qquad \text{(Answer)}$$

Calculation method 2: Because we already know I_{com} about an axis through the center of mass and because the axis here is parallel to that "com axis," we can apply the parallel-axis theorem (Eq. 10-36). We find

$$I = I_{com} + Mh^2 = \frac{1}{2}mL^2 + (2m)\left(\frac{1}{2}L\right)^2$$
$$= mL^2. \qquad \text{(Answer)}$$

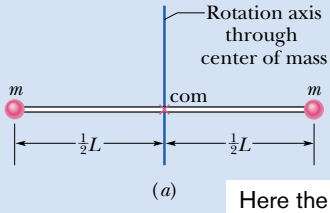


Figure 10-13 A rigid body consisting of two particles of mass m joined by a rod of negligible mass.

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SAMPLE PROBLEM 10.07

Rotational inertia of a uniform rod, integration

Figure 10-14 shows a thin, uniform rod of mass M and length L, on an x axis with the origin at the rod's center.

(a) What is the rotational inertia of the rod about the perpendicular rotation axis through the center?

KEY IDEAS

(1) The rod consists of a huge number of particles at a great many different distances from the rotation axis. We certainly do not want to sum their rotational inertias individually. So, we first write a general expression for the rotational inertia of a mass element dm at distance r from the rotation axis: $r^2\,dm$. (2) Then we sum all such rotational inertias by integrating the expression (rather than adding them up one by one). From Eq. 10-35, we write

$$I = \int r^2\,dm. \tag{10-38}$$

(3) Because the rod is uniform and the rotation axis is at the center, we are actually calculating the rotational inertia I_{com} about the center of mass.

Calculations: We want to integrate with respect to coordinate x (not mass m as indicated in the integral), so we must relate the mass dm of an element of the rod to its length dx along the rod. (Such an element is shown in Fig. 10-14.) Because the rod is uniform, the ratio of mass to length is the same for all the elements and for the rod as a whole. Thus, we can write

$$\frac{\text{element's mass } dm}{\text{element's length } dx} = \frac{\text{rod's mass } M}{\text{rod's length } L}$$

or

$$dm = \frac{M}{L}\,dx.$$

We can now substitute this result for dm and x for r in Eq. 10-38. Then we integrate from end to end of the rod (from $x = -L/2$ to $x = L/2$) to include all the elements. We find

$$I = \int_{x=-L/2}^{x=+L/2} x^2 \left(\frac{M}{L}\right) dx$$

$$= \frac{M}{3L}\left[x^3\right]_{-L/2}^{+L/2} = \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right]$$

$$= \frac{1}{12}ML^2. \qquad \text{(Answer)}$$

(b) What is the rod's rotational inertia I about a new rotation axis that is perpendicular to the rod and through the left end?

KEY IDEAS

We can find I by shifting the origin of the x axis to the left end of the rod and then integrating from $x = 0$ to $x = L$. However, here we shall use a more powerful (and easier) technique by applying the parallel-axis theorem (Eq. 10-36), in which we shift the rotation axis without changing its orientation.

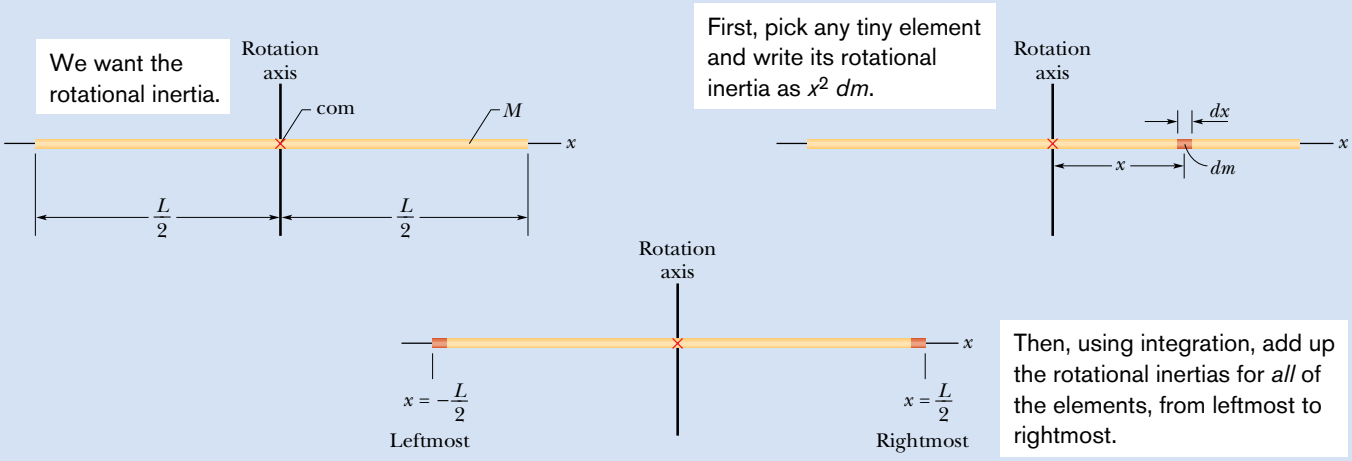

Figure 10-14 A uniform rod of length L and mass M. An element of mass dm and length dx is represented.

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Calculations: If we place the axis at the rod's end so that it is parallel to the axis through the center of mass, then we can use the parallel-axis theorem (Eq. 10-36). We know from part (a) that I_{com} is $1/12ML^2$. From Fig. 10-14, the perpendicular distance h between the new rotation axis and the center of mass is $1/2L$. Equation 10-36 then gives us

$$I = I_{com} + Mh^2 = \frac{1}{12}ML^2 + (M)\left(\frac{1}{2}L\right)^2$$
$$= \frac{1}{3}ML^2. \qquad \text{(Answer)}$$

Actually, this result holds for any axis through the left or right end that is perpendicular to the rod.

SAMPLE PROBLEM 10.08

Rotational kinetic energy, spin test explosion

Large machine components that undergo prolonged, high-speed rotation are first examined for the possibility of failure in a *spin test system.* In this system, a component is *spun up* (brought up to high speed) while inside a cylindrical arrangement of lead bricks and containment liner, all within a steel shell that is closed by a lid clamped into place. If the rotation causes the component to shatter, the soft lead bricks are supposed to catch the pieces for later analysis.

In 1985, Test Devices, Inc. (www.testdevices.com) was spin testing a sample of a solid steel rotor (a disk) of mass M = 272 kg and radius R = 38.0 cm. When the sample reached an angular speed ω of 14 000 rev/min, the test engineers heard a dull thump from the test system, which was located one floor down and one room over from them. Investigating, they found that lead bricks had been thrown out in the hallway leading to the test room, a door to the room had been hurled into the adjacent parking lot, one lead brick had shot from the test site through the wall of a neighbor's kitchen, the structural beams of the test building had been damaged, the concrete floor beneath the spin chamber had been shoved downward by about 0.5 cm, and the 900 kg lid had been blown upward through the ceiling and had then crashed back onto the test equipment (Fig. 10-15). The exploding pieces had not penetrated the room of the test engineers only by luck.

How much energy was released in the explosion of the rotor?

KEY IDEA

The released energy was equal to the rotational kinetic energy K of the rotor just as it reached the angular speed of 14 000 rev/min.

Courtesy Test Devices, Inc.

Figure 10-15 Some of the destruction caused by the explosion of a rapidly rotating steel disk.

Calculations: We can find K with Eq. 10-34 ($K = 1/2I\omega^2$), but first we need an expression for the rotational inertia I. Because the rotor was a disk that rotated like a merry-go-round, I is given in Table 10-2*c* ($I = 1/2MR^2$). Thus,

$$I = \frac{1}{2}MR^2 = \frac{1}{2}(272\text{ kg})(0.38\text{ m})^2 = 19.64\text{ kg}\cdot\text{m}^2.$$

The angular speed of the rotor was

$$\omega = (14\,000\text{ rev/min})(2\pi\text{ rad/rev})\left(\frac{1\text{ min}}{60\text{ s}}\right)$$
$$= 1.466\times10^3\text{ rad/s}.$$

Then, with Eq. 10-34, we find the (huge) energy release:

$$K = \frac{1}{2}I\omega^2 = \frac{1}{2}(19.64\text{ kg}\cdot\text{m}^2)(1.466\times10^3\text{ rad/s})^2$$
$$= 2.1\times10^7\text{ J}. \qquad \text{(Answer)}$$

CHECKPOINT 6

Find the moment of inertia of a half uniform ring (mass m, radius r) about its center.

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SAMPLE PROBLEM 10.09

Moment of inertia, uniform sphere

Find moment of inertia of hollow uniform sphere of mass M about its diameter.

KEY IDEA

The total mass M is given by the integral of the surface density σ over the area of the shell $M = \int \sigma\, dA$. Since the shell is homogeneous, the surface density σ is constant. Therefore,

$$M = \int \sigma\, dA = \sigma A = 4\pi R^2 \sigma.$$

This implies that

$$\sum = \frac{M}{4\pi r^2}.$$

Calculations: In the Fig. 10-16, the hoop shown has a radius $r = R \sin\theta$ and therefore a circumference of $2\pi R \sin\theta$. The width of the hoop is $Rd\theta$, so the area $dA = (2\pi R \sin\theta)\,(R\, d\theta)$, and the mass $dM = \sigma\, dA = (M/4\,\pi\, R^2)(2R^2 \sin\theta\, d\theta) = (M/2)\sin\theta\, d\theta$.

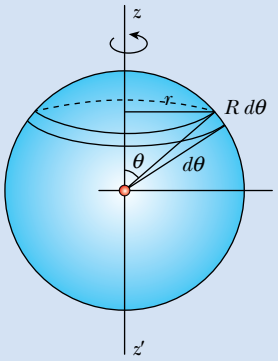

Figure 10-16 Hollow uniform sphere of mass M.

We use dI to denote the moment of inertia of the hoop about the axis zz'. The perpendicular distance from the axis to each part of the hoop is $R\sin\theta$. Therefore, $dI = dm(R\sin\theta)^2$ and $dM = (MR^2/2)\sin^3\theta\, d\theta$. The moment of inertia of the entire shell is the sum of moment of inertia of the various hoops.

$$dI = \frac{M}{2}\sin\theta\; d\theta R^2 \sin^2\theta.$$

Integrating over whole sphere, we get

$$I = \frac{MR^2}{2}\int_0^{\pi} \sin^3\theta\, d\theta.$$

We can write as

$$\sin^3\theta = \frac{-\sin 3\theta + 3\sin\theta}{4}.$$

Substituting in the above equation, we get

$$I = \frac{MR^2}{8}\left[\int_0^{\pi} \sin 3\theta d\theta + 3\int_0^{\pi} \sin\theta d\theta\right]$$

$$= \frac{MR^2}{8}\left[-\frac{2}{3} + 6\right]$$

$$= \frac{2}{3}MR^2. \qquad \text{(Answer)}$$

We have derived the result by adding moment of inertia of hoops.

SAMPLE PROBLEM 10.10

Moment of inertia, solid sphere

Find the moment of inertia of solid sphere of mass M about a diameter as shown in Fig. 10-17.

KEY IDEA

The sphere can be considered as made up of many concentric thin shells. From previous example, a spherical subshell of radius r has moment of inertia:

$$dI = \frac{2}{3}dMr^2.$$

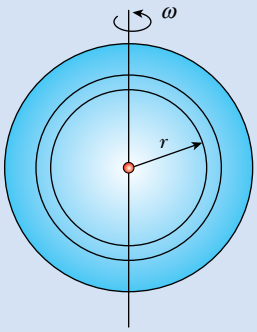

Figure 10-17 Solid spheres.

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Calculations: We know that

$$dM = \frac{M}{\frac{4}{3}\pi R^3} \times 4\pi r^2 dr$$

$$= \frac{3M}{R^3} \times r^2 dr.$$

Substituting for dM in the equation for moment of inertia, we get

$$dI = \frac{2}{3} \times \frac{3M}{R^3} r^2 (dr) r^2.$$

Integrating over the whole volume, we get

$$I = \frac{2M}{R^3} \int_0^R r^2 dr$$

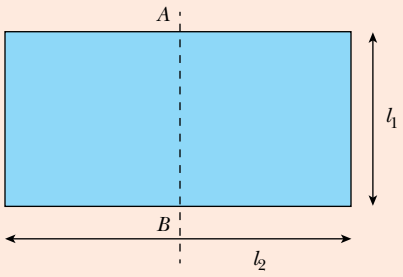
$$= \frac{2MR^2}{5}. \qquad \text{(Answer)}$$

CHECKPOINT 7

Find moment of inertia of a uniform solid sphere (mass m, radius r) about a tangent.

CHECKPOINT 8

Find moment of inertia of uniform rectangular plate (mass m) about axis AB lying in its plane and passing through center as shown in the figure.



Perpendicular-Axis Theorem

Before proceeding further, we demonstrate a useful theorem about moments of inertia for thin, slab-like objects. We may refer to Fig. 10-18 illustrating the rotating slab and its representative mass element m. The contribution of m_i to the moment of inertia I_z with respect to the z axis is $m_i r_i^2$, that is the mass of the particle times the square of its distance from the axis of rotation. If we consider rotation about the x axis, the mass element m_i would contribute $m_i y_i^2$ to the moment of inertia with respect to the x axis; similarly, it would contribute $m_i x_i^2$ with respect to the y axis. Thus,

$$I_x = \sum m_i y_i^2; \qquad (10\text{-}39)$$

$$I_y = \sum m_i x_i^2. \qquad (10\text{-}40)$$

By addition of Eq. (10-39) and Eq. (10-40), we obtain

$$I_x + I_y = \sum m_i (x_i^2 + y_i^2) = \sum m_i r_i^2 = I_z. \qquad (10\text{-}41)$$

This proves the perpendicular-axis theorem, true for plane, thin rigid bodies.

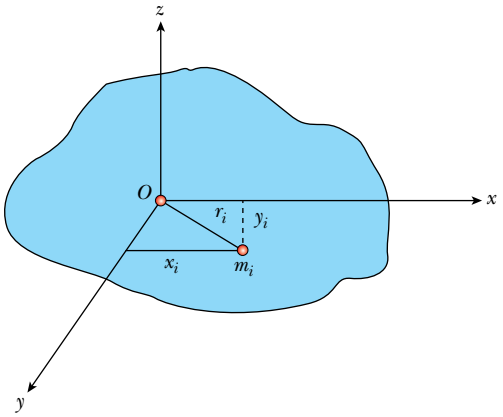


Figure 10-18 Rotating slab and its representative mass element.

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SAMPLE PROBLEM 10.11

Moment of inertia, uniform circular disk, central axis

Use the perpendicular-axis theorem to relate the moment of inertia of a thin uniform circular disk (about any diameter) to its moment of inertia about its central axis.

KEY IDEA

A thin uniform-density disk of radius R is confined to the xy plane, as shown in Fig. 10-19.

Calculations: According to perpendicular-axis theorem:

$$I_{xO} + I_{yO} = I_{zO}.$$

However, by symmetry, we have

$$I_{xO} = I_{yO} = I_d.$$

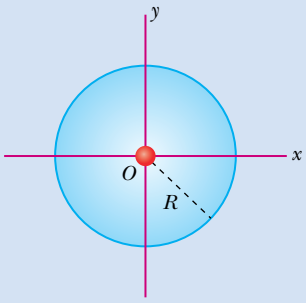


Figure 10-19 Thin uniform circular disk.

where I_d denotes the moment of inertia about any diameter. Therefore, we have

$$2I_d = I_{zO}.$$

The moment of inertia of a solid cylinder or disk of radius R about the symmetry axis is $1/2MR^2$. Thus,

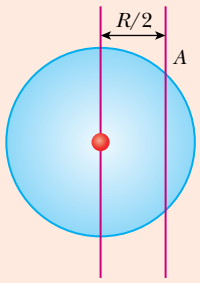
$$I_{zO} = \frac{1}{2}MR^2$$

and

$$I_d = \frac{1}{4}MR^2. \qquad \text{(Answer)}$$

CHECKPOINT 9

Find moment of inertia of a uniform ring (mass m, radius r) about chord AA' at distance $R/2$ from center.



SAMPLE PROBLEM 10.12

Moment of inertia, uniform rectangular plate

Find moment of inertia of a uniform rectangular plate about axis zz' passing through center and perpendicular to the plane of plate, as shown in Fig. 10-20*a*.

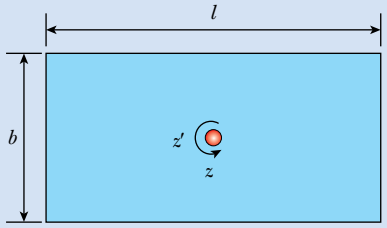


Figure 10-20 (*a*) Uniform rectangular plate in xy plane.

KEY IDEA

To understand the motion of plate, imagine twirling your note book about its center by keeping index finger of two hands on two faces of the note book.

Calculations: This is a simple problem that can be solved using theorem of perpendicular-axis.

Compute: From Fig. 10-20*b*, we see that

$$I_{xx'} = \frac{Mb^2}{12} \quad \text{and} \quad I_{xx'} = \frac{Ml^2}{12}.$$

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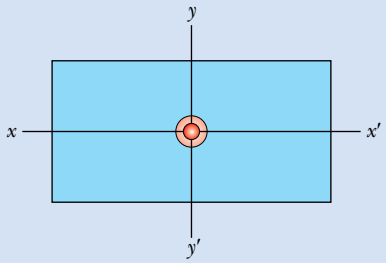

Figure 10-20 (*b*) Perpendicular axis in the plane of plate.

Hence, from theorem of perpendicular-axis

$$I_{zz'} = I_{xx'} + I_{yy'}$$

$$= M\left(\frac{B^2 + l^2}{12}\right). \qquad \text{(Answer)}$$

RADIUS OF GYRATION

When I is the moment of inertia about an axis, then K is radius of gyration about that axis, which is given by

$$K = \frac{I}{M}. \qquad (10\text{-}42)$$

10.8 | TORQUE

Key Concepts

- Torque is a turning or twisting action on a body about a rotation axis due to a force $\vec{F}$. If $\vec{F}$ is exerted at a point given by the position vector $\vec{r}$ relative to the axis, then the magnitude of the torque is

  $$\tau = rF_t = r_\perp F = rF \sin \phi,$$

  where F_t is the component of $\vec{F}$ perpendicular to $\vec{r}$ and ϕ is the angle between $\vec{r}$ and $\vec{F}$. The quantity $r_\perp$ is the perpendicular distance between the rotation axis and an extended line running through the $\vec{F}$ vector. This line is called the line of action of $\vec{F}$, and $r_\perp$ is called the moment arm of $\vec{F}$. Similarly, r is the moment arm of F_t.
- The SI unit of torque is the newton-meter (N·m). A torque τ is positive if it tends to rotate a body at rest counterclockwise and negative if it tends to rotate the body clockwise.

A doorknob is located as far as possible from the door's hinge line for a good reason. If you want to open a heavy door, you must certainly apply a force, but that is not enough. Where you apply that force and in what direction you push are also important. If you apply your force nearer to the hinge line than the knob, or at any angle other than 90° to the plane of the door, you must use a greater force than if you apply the force at the knob and perpendicular to the door's plane.

Figure 10-21*a* shows a cross section of a body that is free to rotate about an axis passing through O and perpendicular to the cross section. A force $\vec{F}$ is applied at point P, whose position relative to O is defined by a position vector $\vec{r}$. The directions of vectors $\vec{F}$ and $\vec{r}$ make an angle ϕ with each other. (For simplicity, we consider only forces that have no component parallel to the rotation axis; thus, $\vec{F}$ is in the plane of the page.)

To determine how $\vec{F}$ results in a rotation of the body around the rotation axis, we resolve $\vec{F}$ into two components (Fig. 10-21*b*). One component, called the *radial component* F_r, points along $\vec{r}$. This component does not cause rotation, because it acts along a line that extends through O. (If you pull on a door parallel to the plane of the door, you do not rotate the door.) The other component of $\vec{F}$, called the *tangential component* F_t, is perpendicular to $\vec{r}$ and has magnitude $F_t = F \sin \phi$. This component *does* cause rotation.

Calculating Torques. The ability of $\vec{F}$ to rotate the body depends not only on the magnitude of its tangential component F_t, but also on just how far from O the force is applied. To include both these factors, we define a quantity called **torque** τ as the product of the two factors and write it as

$$\tau = (r)(F \sin \phi). \qquad (10\text{-}43)$$

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Two equivalent ways of computing the torque are

$$\tau = (r)(F \sin \phi) = rF_t \qquad (10\text{-}44)$$

and

$$\tau = (r \sin \phi)(F) = r_\perp F, \qquad (10\text{-}45)$$

where $r_\perp$ is the perpendicular distance between the rotation axis at O and an extended line running through the vector $\vec{F}$ (Fig. 10-21*c*).This extended line is called the **line of action** of $\vec{F}$, and $r_\perp$ is called the **moment arm** of $\vec{F}$. Figure 10-21*b* shows that we can describe r, the magnitude of r, as being the moment arm of the force component F_t.

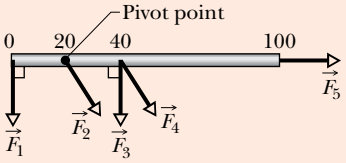
Torque, which comes from the Latin word meaning "to twist," may be loosely identified as the turning or twisting action of the force $\vec{F}$. When you apply a force to an object—such as a screwdriver or torque wrench—with the purpose of turning that object, you are applying a torque. The SI unit of torque is the newtonmeter (N·m). *Caution:* The newton-meter is also the unit of work. Torque and work, however, are quite different quantities and must not be confused. Work is often expressed in joules (1 J = 1 N·m), but torque never is.

Clocks Are Negative. We shall use vector notation for torques, but here, with rotation around a single axis, we use only an algebraic sign. If a torque would cause counterclockwise rotation, it is positive. If it would cause clockwise rotation, it is negative. (The phrase "clocks are negative" from Section 10.2 still works.)

Torques obey the superposition principle that we discussed in Chapter 5 for forces: When several torques act on a body, the **net torque** (or **resultant torque**) is the sum of the individual torques. The symbol for net torque is τ_{net}.

CHECKPOINT 10

The figure shows an overhead view of a meter stick that can pivot about the dot at the position marked 20 (for 20 cm). All five forces on the stick are horizontal and have the same magnitude. Rank the forces according to the magnitude of the torque they produce, greatest first.



Here, to recall the topic, we discuss once again about "vector product" that we discussed in Chapter 3.

10.9 | THE VECTOR PRODUCT REVISITED

Key Concepts

- The **vector product** of $\vec{a}$ and $\vec{b}$, written as $\vec{a} \times \vec{b}$ produces a third vector $\vec{c}$ whose magnitude is

$$c = ab \sin \phi,$$

  where ϕ is the *smaller* of the two angles between $\vec{a}$ and $\vec{b}$.
- The direction of $\vec{c}$ is perpendicular to the plane that contains $\vec{a}$ and $\vec{b}$.

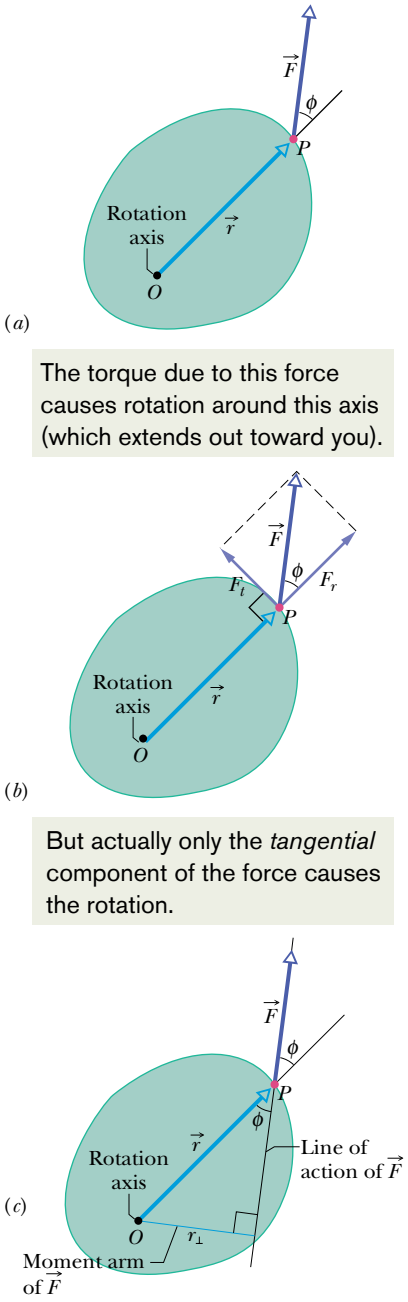


Figure 10-21 (*a*) A force $\vec{F}$ acts on a rigid body, with a rotation axis perpendicular to the page. The torque can be found with (*a*) angle ϕ, (*b*) tangential force component F_t, or (*c*) moment arm $r_\perp$.

The **vector product** of $\vec{a}$ and $\vec{b}$, written as $\vec{a} \times \vec{b}$ produces a third vector $\vec{c}$ whose magnitude is

$$c = ab \sin \phi, \qquad (10\text{-}46)$$

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where ϕ is the *smaller* of the two angles between $\vec{a}$ and $\vec{b}$. (You must use the smaller of the two angles between the vectors because sin ϕ and sin (360° – ϕ) differ in algebraic sign.) Because of the notation, $\vec{a}\times\vec{b}$ is also known as the **cross product**, and in speech it is "a cross b."

If $\vec{a}$ and $\vec{b}$ are parallel or antiparallel, $\vec{a}\times\vec{b}=0$. The magnitude of $\vec{a}\times\vec{b}$, which can be written as $|\vec{a}\times\vec{b}|$, is maximum when $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

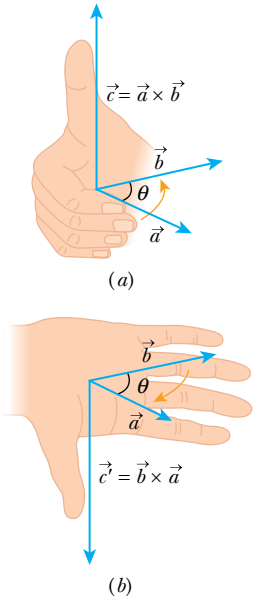

Figure 10-22 Illustration of the right-hand rule for vector products. (*a*) Sweep vector $\vec{a}$ into vector $\vec{b}$ with the fingers of your right hand. Your outstretched thumb shows the direction of vector $\vec{c}=\vec{a}\times\vec{b}$. (b) Showing that $\vec{b}\times\vec{a}$ is the reverse of $\vec{a}\times\vec{b}$.

The direction of $\vec{c}$ is perpendicular to the plane that contains $\vec{a}$ and $\vec{b}$. Figure 10-22*a* shows how to determine the direction of $\vec{c}=\vec{a}\times\vec{b}$ with what is known as a **right-hand rule**. Place the vectors $\vec{a}$ and $\vec{b}$ tail to tail without altering their orientations and imagine a line that is perpendicular to their plane where they meet. Pretend to place your *right* hand around that line in such a way that your fingers would sweep $\vec{a}$ into $\vec{b}$ through the smaller angle between them. Your outstretched thumb points in the direction of $\vec{c}$.

The order of the vector multiplication is important. In Fig. 10-22*b*, we are determining the direction of $\vec{c}'=\vec{a}\times\vec{b}$, so the fingers are placed to sweep $\vec{b}$ into $\vec{a}$ through the smaller angle. The thumb ends up in the opposite direction form previously, and so it must be that $\vec{c}'=-\vec{c}$, that is,

$$\vec{b}\times\vec{a}=(\vec{a}\times\vec{b}). \tag{10-47}$$

In other words, the commutative law does not apply to a vector product. In unit-vector notation, we write as

$$\vec{a}\times\vec{b}=(a_x\hat{\mathrm{i}}+a_y\hat{\mathrm{j}}+a_z\hat{\mathrm{k}}), \tag{10-48}$$

which can be expanded according to the distributive law, that is, each component of the first vector is to be crossed with each component of the second vector. The cross products of unit vectors are given in Appendix E (see "Products of Vectors"). For example, in the expansion of Eq. 10-48, we have

$$a_x\hat{\mathrm{i}}+b_x\hat{\mathrm{i}}=a_xb_x(\hat{\mathrm{i}}\times\hat{\mathrm{i}})=0$$

because the two unit vectors $\hat{\mathrm{i}}$ and $\hat{\mathrm{j}}$ are parallel and thus have a zero cross product. Similarly, we have

$$a_x\hat{\mathrm{i}}+b_y\hat{\mathrm{j}}=a_xb_y(\hat{\mathrm{i}}\times\hat{\mathrm{j}})=a_xb_y\hat{\mathrm{k}},$$

In the last step, we used Eq. 10-46 to evaluate the magnitude of $\hat{\mathrm{i}}\times\hat{\mathrm{j}}$ as unity. (These vectors $\hat{\mathrm{i}}$ and $\hat{\mathrm{j}}$ each have a magnitude of unity, and the angle between them is 90°.) Also, we used the right-hand rule to get the direction of $\hat{\mathrm{i}}\times\hat{\mathrm{j}}$ as being in the positive direction of the z axis (thus in the direction of $\hat{\mathrm{k}}$).

Continuing to expand Eq. 10-48, you can show that

$$\vec{a}\times\vec{b}=(a_yb_z-b_ya_z)\hat{\mathrm{i}}+(a_zb_x-b_za_x)\hat{\mathrm{j}}+(a_xb_y-b_xa_y)\hat{\mathrm{k}}. \tag{10-49}$$

A determinant (Appendix E) or a vector-capable calculator can also be used.

To check whether any xyz coordinate system is a right-handed coordinate system, use the right-hand rule for the cross product $\hat{\mathrm{i}}\times\hat{\mathrm{j}}=\hat{\mathrm{k}}$ with that system. If your fingers sweep $\hat{\mathrm{i}}$ (positive direction of x) into $\hat{\mathrm{j}}$ (positive direction of y) with the outstretched thumb pointing in the positive direction of z, then the system is right-handed.

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PROBLEM-SOLVING TACTICS

Tactic 1: Common Errors with Cross Products Several errors are common in finding a cross product. (1) Failure to arrange vectors tail to tail is tempting when an illustration presents them head to tail; you must mentally shift (or better, redraw) one vector to the proper arrangement without changing its orientation. (2) Failing to use the right hand in applying the right-hand rule is easy when the right hand is occupied with a calculator or pencil. (3) Failure to sweep the first vector of the product into the second vector can occur when the orientations of the vectors require an awkward twisting of your hand to apply the right-hand rule. Sometimes that happens when you try to make the sweep mentally rather than using your hand. (4) Failure to work with a right-handed coordinate system results when you forget how to draw such a system.

10.10 | VECTOR PRODUCT AND TORQUE

Key Concepts

- In three dimensions, torque $\vec{\tau}$ is a vector quality defined relative to a fixed point (usually an origin); it is

$$\vec{\tau} = \vec{r} \times \vec{F},$$

  where $\vec{F}$ is a force applied to a particle and $\vec{r}$ is a position vector locating the particle relative to the fixed point.

- The magnitude of $\vec{\tau}$ is given by

$$\tau = rF \sin \phi = rF_\perp = r_\perp F,$$

  where ϕ is the angle between $\vec{F}$ and $\vec{r}$, $F_\perp$ is the component of $\vec{F}$ perpendicular to $\vec{r}$, and $r_\perp$ is the moment arm of $\vec{F}$.

- The direction of $\vec{\tau}$ is given by the right-hand rule for cross products.

In the earlier section, we defined torque τ for a rigid body that can rotate around a fixed axis. We now expand the definition of torque to apply it to an individual particle that moves along any path relative to a fixed *point* (rather than a fixed axis). The path need no longer be a circle, and we must write the torque as a vector $\vec{\tau}$ that may have any direction. We can calculate the magnitude of the torque with a formula and determine its direction with the right-hand rule for cross products.

Figure 10-23*a* shows such a particle at point A in an xy plane. A single force $\vec{F}$ in that plane acts on the particle, and the particle's position relative to the origin O is given by position vector $\vec{r}$. The torque $\vec{\tau}$ acting on the particle relative to the fixed point O is a vector quantity defined as

$$\vec{\tau} = \vec{r} \times \vec{F} \quad \text{(torque defined)}. \tag{10-50}$$

We can evaluate the vector (or cross) product in this definition of $\vec{\tau}$ by using the rules in Section 10.9. To find the direction of $\vec{\tau}$, we slide the vector $\vec{F}$ (without changing its direction) until its tail is at the origin O, so that the two vectors in the vector product are tail to tail as in Fig. 10-23*b*. We then use the right-hand rule in Fig. 10.22*a*, sweeping the fingers of the right hand from $\vec{r}$ (the first vector in the product) into $\vec{F}$ (the second vector). The outstretched right thumb then gives the direction of $\vec{\tau}$. In Fig. 10-23*b*, it is in the positive direction of the z axis.

To determine the magnitude of $\vec{\tau}$, we apply the general result of Eq. 10-40 ($c = ab \sin \phi$), finding

$$\tau = rF \sin \phi, \tag{10-51}$$

where ϕ is the smaller angle between the directions of $\vec{r}$ and $\vec{F}$ when the vectors are tail to tail. From Fig. 10-23*b*, we see that Eq. 10-51 can be rewritten as

$$\tau = rF_\perp, \tag{10-52}$$

where $F_\perp$ ($= F \sin \phi$) is the component of $\vec{F}$ perpendicular to $\vec{r}$. From Fig. 10-23*c*, we see that Eq. 10-51 can also be rewritten as

$$\tau = r_\perp F, \tag{10-53}$$

where $r_\perp$ ($= r \sin \phi$) is the moment arm of $\vec{F}$ (the perpendicular distance between O and the line of action of $\vec{F}$).

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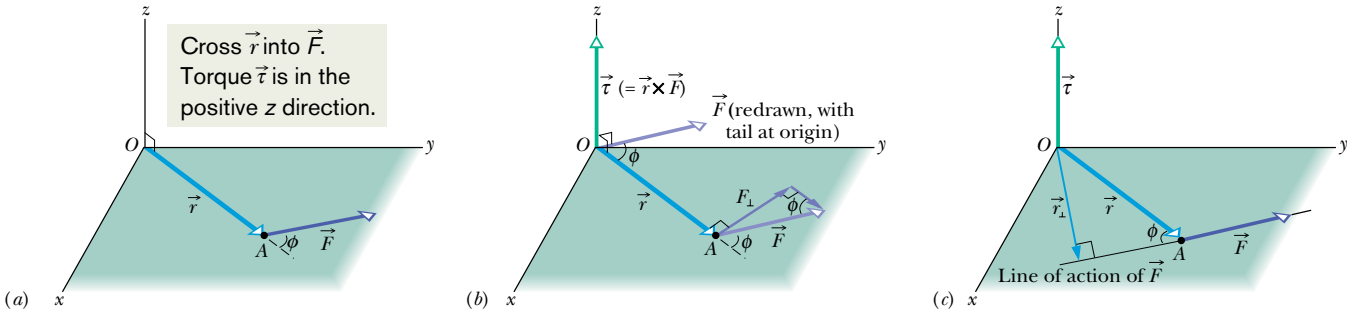
Cross $\vec{r}$ into $\vec{F}$. Torque $\vec{\tau}$ is in the positive z direction.
$\vec{\tau}$ (= $\vec{r}$ × $\vec{F}$)
$\vec{F}$ (redrawn, with tail at origin)
Line of action of $\vec{F}$
(a)
(b)
(c)

Figure 10-23 Defining torque. (*a*) A force $\vec{F}$, lying in an xy plane, acts on a particle at point A. (*b*) This force produces a torque $\vec{\tau} = (\vec{r} \times \vec{F})$, on the particle with respect to the origin O. By the right-hand rule for vector (cross) products, the torque vector points in the positive direction of z. Its magnitude is given by $rF_\perp$ in (*b*) and by $r_\perp F$ in (*c*).

CHECKPOINT 11

The position vector $\vec{r}$ of a particle points along the positive direction of a z axis. If the torque on the particle is (a) zero, (b) in the negative direction of x, and (c) in the negative direction of y, in what direction is the force causing the torque?

SAMPLE PROBLEM 10.13

Torque on a particle due to a force

In Fig. 10-24*a*, three forces, each of magnitude 2.0 N, act on a particle. The particle is in the xz plane at point A given by position vector $\vec{r}$, where $r = 3.0$ m and $\theta = 30°$. What is the torque, about the origin O, due to each force?

KEY IDEA

Because the three force vectors do not lie in a plane, we must use cross products, with magnitudes given by Eq. 10-51 ($\tau = rF \sin \phi$) and directions given by the right-hand rule.

Calculations: Because we want the torques with respect to the origin O, the vector $\vec{r}$ required for each cross product is the given position vector. To determine the angle ϕ between $\vec{r}$ and each force, we shift the force vectors of Fig. 10-24*a*, each in turn, so that their tails are at the origin. Figures 10-24*b*, *c*, and *d*, which are direct views of the xz plane, show the shifted force vectors $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$,

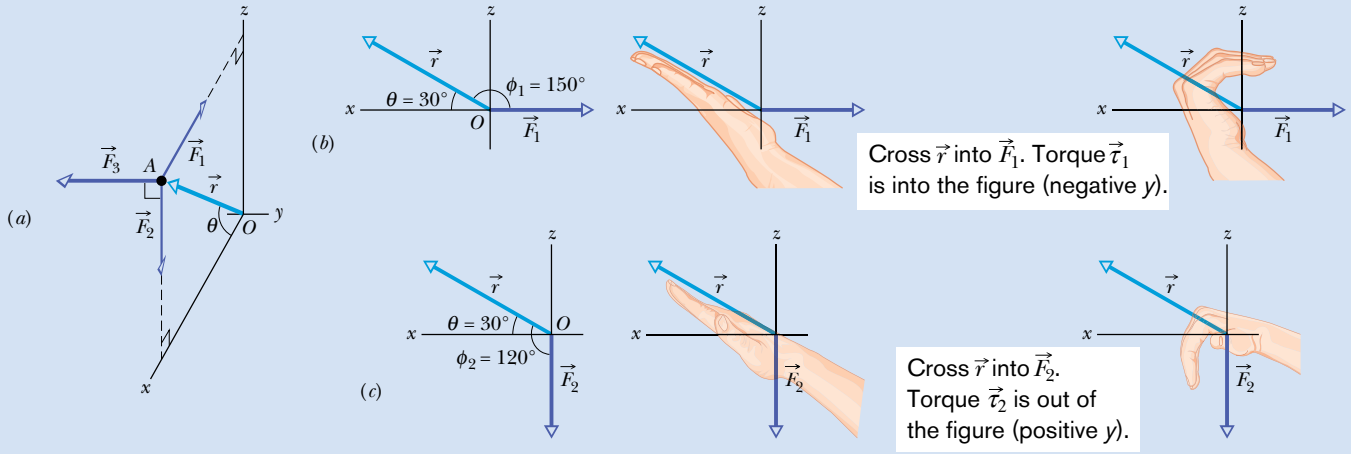
(a)
(b)
(c)
$\theta = 30°$
$\phi_1 = 150°$
$\phi_2 = 120°$
Cross $\vec{r}$ into $\vec{F}_1$. Torque $\vec{\tau}_1$ is into the figure (negative y).
Cross $\vec{r}$ into $\vec{F}_2$. Torque $\vec{\tau}_2$ is out of the figure (positive y).

Figure 10-24 (*a*) A particle at point A is acted on by three forces, each parallel to a coordinate axis. The angle ϕ (used in finding torque) is shown (*b*) for $\vec{F}_1$ and (*c*) for $\vec{F}_2$. (*Figure continues on following page*)

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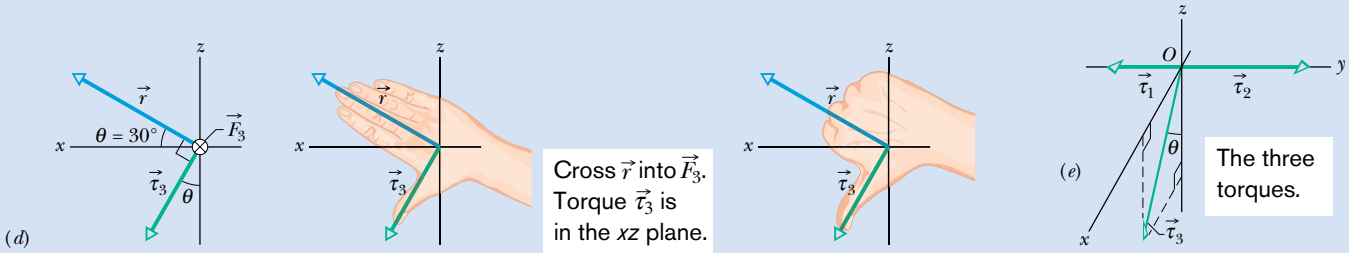

Figure 10-24 (*Continued from previous page*) (*d*) Torque $\vec{\tau}_3$ is perpendicular to both $\vec{r}$ and $\vec{F}_3$ (force $\vec{F}_3$ is directed into the plane of the figure). (*e*) The torques.

respectively. (Note how much easier the angles between the force vectors and the position vector are to see.) In Fig. 10-24*d*, the angle between the directions of $\vec{r}$ and $\vec{F}_3$ is 90° and the symbol ⊗ means $\vec{F}_3$ is directed into the page. (For out of the page, we would use ⊙.)

Now, applying Eq. 10-51, we find

$$\tau_1 = rF_1 \sin \phi_1 = (3.0\text{ m})(2.0\text{ N})(\sin 150°) = 3.0\text{ N}\cdot\text{m},$$

$$\tau_2 = rF_2 \sin \phi_2 = (3.0\text{ m})(2.0\text{ N})(\sin 120°) = 5.2\text{ N}\cdot\text{m},$$

and

$$\tau_3 = rF_3 \sin \phi_3 = (3.0\text{ m})(2.0\text{ N})(\sin 90°)$$
$$= 6.0\text{ N}\cdot\text{m}. \quad \text{(Answer)}$$

Next, we use the right-hand rule, placing the fingers of the right hand so as to rotate $\vec{r}$ into $\vec{F}$ through the *smaller* of the two angles between their directions. The thumb points in the direction of the torque. Thus $\vec{\tau}_1$ is directed into the page in Fig. 10-24*b*; $\vec{\tau}_2$ is directed out of the page in Fig. 10-24*c*; and $\vec{\tau}_3$ is directed as shown in Fig. 10-24*d*. All three torque vectors are shown in Fig. 10-24*e*.

PROBLEM-SOLVING TACTICS

Tactic 2: Vector Products and Torques Equation 10-51 for torques is our first application of the vector (or cross) product.

Remember that a torque is calculated *with respect to* (or *about*) a point, which must be known if the value of the torque is to be meaningful. Changing the point can change the torque in both magnitude and direction. For example, in Sample Problem 10.13, the torques due to the three forces are calculated about the origin *O*. You can show that the torques due to the same three forces are all zero if they are calculated about point *A* (at the position of the particle) because $r = 0$ for each force.

CHECKPOINT 12

Find torque due to gravity at any time t about point of projection, if a body of mass m is projected at an angle θ with speed v_0 at $t = 0$.

Let us consider the example of door. We are applying a force F_1 on the door in vertically upward direction (Fig. 10-25).

When we apply this force, we know that door has no tendency to rotate about its axis yy', but considering our definition of torque as $\vec{r} \times \vec{F}_1$ we can see that $\vec{\tau}_1$ is coming out to be non-zero. This contradiction is arising because the door (and almost all the rigid bodies in this chapter) is constrained to rotate only about one axis, that is, *y*-axis. The torque we are getting due to force $\vec{F}$ has component along this axis. Only those torques, which are parallel to a particular axis, have the ability to produce rotation about that axis. To further understand concept of multiple axes, consider rotation of your palm about wrist (Fig. 10-26*a*). This complicated motion is a result of rotation about mainly two orthogonal axes *x* and *z* as shown in Fig. 10-26*a*. However, these complicated rotations are not our focus

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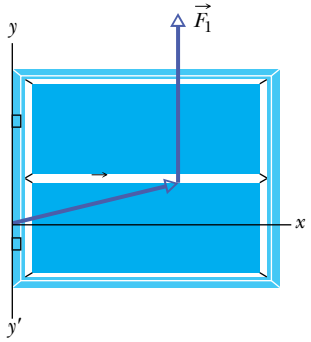

Figure 10-25 Application of upward force on a door.

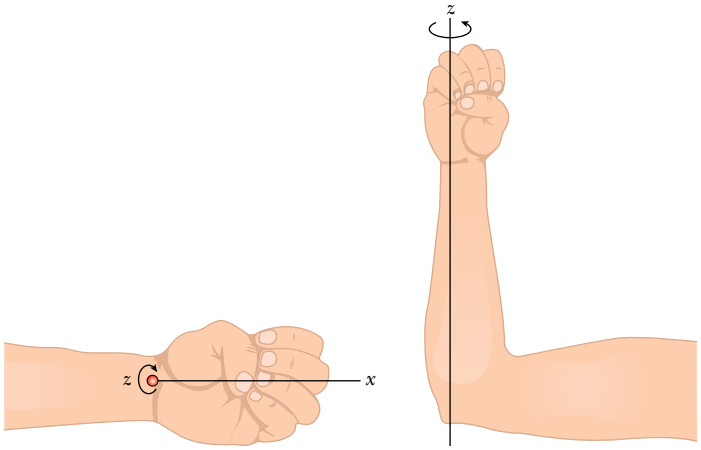

Figure 10-26 Rotation of (*a*) palm about wrist, (b) forearm about elbow axis.

at this level. We are going to restrict ourselves only to rotations about single axis, like rotation of one's forearm about elbow (Fig. 10-26*b*). To conclude, only those components of torque vector are useful in producing rotation which are parallel to the axis of rotation.

10.11 | THE CENTER OF GRAVITY

Key Concepts

- The center of gravity (cog) of a body is a point on which the gravitational force $\vec{F}_g$ acts (effectively) on the body.
- If $\vec{g}$ is the same for all elements of a body, then the body's center of gravity (cog) is coincident with the body's center of mass (com).

The gravitational force on an extended body is the vector sum of the gravitational forces acting on the individual elements (the atoms) of the body. Instead of considering all those individual elements, we can say the following:

The gravitational force $\vec{F}_g$ on a body effectively acts at a single point, called the **center of gravity** (cog) of the body.

Here, the word "effectively" means that if the forces on the individual elements were somehow turned off and force $\vec{F}_g$ at the center of gravity were turned on, the net force and the net torque (about any point) acting on the body would not change.

Until now, we have assumed that the gravitational force $\vec{F}_g$ acts at the center of mass (com) of the body. This is equivalent to assuming that the center of gravity is at the center of mass. Recall that, for a body of mass M, the force $\vec{F}_g$ is equal to $M\vec{g}$, where g is the acceleration that the force would produce if the body were to fall freely. In the proof that follows, we show that:

If $\vec{g}$ is the same for all elements of a body, then the body's center of gravity (cog) is coincident with the body's center of mass (com).

This is approximately true for everyday objects because $\vec{g}$ varies only a little along Earth's surface and decreases in magnitude only slightly with altitude. Thus, for objects like a mouse or a moose, we have been justified in assuming that the gravitational force acts at the center of mass. After the following proof, we shall resume that assumption.

Proof

First, we consider the individual elements of the body. Figure 10-27*a* shows an extended body, of mass M, and one of its elements, of mass m_i. A gravitational force $\vec{F}_{gi}$ acts on each such element and is equal to $m_i\vec{g}_i$. The subscript

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on $\vec{g}_i$ means $\vec{g}_i$ is the gravitational acceleration *at the location of the element i* (it can be different for other elements).

In Fig. 10-27*a*, each force $\vec{F}_{gi}$ produces a torque $\vec{\tau}_1$ on the element about the origin O, with moment arm x_i. Using Eq. 10-45 ($\tau = r_\perp F$), we can write torque τ_i as

$$\tau_i = x_i F_{gi}. \tag{10-54}$$

The net torque on all the elements of the body is then

$$\tau_{\text{net}} = \sum \tau_i = \sum x_i F_{gi}. \tag{10-55}$$

Next, we consider the body as a whole. Figure 10-27*b* shows the gravitational force $\vec{F}_g$ acting at the body's center of gravity. This force produces a torque t on the body about O, with moment arm x_{cog}. Again using Eq. 10-45 ($\tau = r_\perp F$), we can write this torque as

$$\tau = x_{\text{cog}} F_g. \tag{10-56}$$

The gravitational force $\vec{F}_g$ on the body is equal to the sum of the gravitational forces $\vec{F}_{gi}$ on all its elements, so we can substitute $\sum \vec{F}_{gi}$ for F_g in Eq. 10-56 to write

$$\tau = x_{\text{cog}} = \sum F_{gi}. \tag{10-57}$$

Now, recall that the torque due to force $\vec{F}_g$ acting at the center of gravity is equal to the net torque due to all the forces $\vec{F}_{gi}$ acting on all the elements of the body. (That is how we defined the center of gravity.) Thus, τ in Eq. 10-57 is equal to τ_{net} in Eq. 10-55. Putting those two equations together, we can write

$$x_{\text{cog}} \sum F_{gi} = \sum x_i F_{gi}.$$

Substituting $m_i g_i$ for F_{gi} gives us

$$x_{\text{cog}} \sum m_i g_i = \sum x_i m_i g_i. \tag{10-58}$$

Here is a key idea: If the accelerations g_i at all the locations of the elements are the same, we can cancel g_i from this equation to write

$$x_{\text{cog}} \sum m_i = \sum x_i m_i. \tag{10-59}$$

The sum $\sum m_i$ of the masses of all the elements is the mass M of the body. Therefore, we can rewrite Eq. 10-59 as

$$x_{\text{cog}} = \frac{1}{M} \sum x_i m_i. \tag{10-60}$$

The right side of this equation gives the coordinate x_{com} of the body's center of mass (Eq. 9-4). We now have what we sought to prove:

$$x_{\text{cog}} = x_{\text{com}}. \tag{10-61}$$

A practical use of this result is that to balance an object, the pivot point must be at the center of gravity.

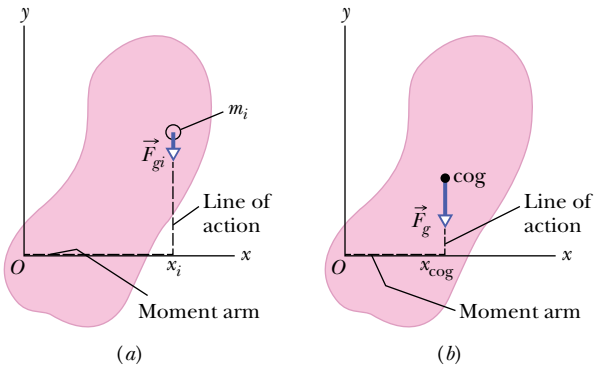

Figure 10-27 (*a*) An element of mass m_i in an extended body. The gravitational force $\vec{F}_{gi}$ on the element has moment arm x_i about the origin O of the coordinate system. (*b*) The gravitational force $\vec{F}_g$ on a body is said to act at the center of gravity (cog) of the body. Here, $\vec{F}_g$ has moment arm x_{cog} about origin O.

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EFFECTIVE POINT OF APPLICATION OF FORCES

Equivalent force: In the cases when the total torque of all external forces turns out to be perpendicular to the resultant force, that is, $\tau \perp F$, all external forces may be replaced by a F acting along a certain straight line. In fact, if the total torque relative to some point O is perpendicular to F, that is, $OM \perp F$, then we can always find a vector $\vec{r} \perp \tau$ (Fig. 10-28), such that with the given τ and F

$$\tau = \vec{r}_0 \times \vec{F}.$$

Here, the choice of $\vec{r}_0$ is not unambiguous: adding any vector $\vec{r}$ parallel to F, we do not violate the last equality. This means that this equality defines not a point of "application" of the force F, but a line along which it acts. Knowing the moduli τ and F of the corresponding vectors, we can find the arm l of the force F (Fig. 10-28):

$$l = \frac{\tau}{F}.$$

Thus, if $\tau \perp F$, then the system of forces acting on various points of a solid may be replaced by one equivalent force which is equal to the resultant force F and produces a torque equal to the total torque τ of all external forces.

This implies that the equivalent force mg of the gravitational forces passes through the center of inertia of the body. It is customary to say that the equivalent force of gravity is "applied" to the center of inertia of a body, or to its center of gravity. Clearly, the torque of this force relative to the center of inertia of a body is equal to zero.

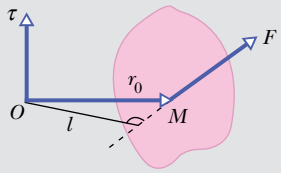

Figure 10-28 All forces are replaced by one force and one torque.

10.12 | THE RIGID OBJECT UNDER A NET TORQUE

Key Concept

- The rotational analog of Newton's second law is

$$\tau_{net} = I\alpha,$$

where τ_{net} is the net torque acting on a particle or rigid body, I is the rotational inertia of the particle or body about the rotation axis, and α is the resulting angular acceleration about that axis.

A torque can cause rotation of a rigid body, as when you use a torque to rotate a door. Here we want to relate the net torque τ_{net} on a rigid body to the angular acceleration α that torque causes about a rotation axis. We do so by analogy with Newton's second law ($F_{net} = ma$) for the acceleration a of a body of mass m due to a net force F_{net} along a coordinate axis. We replace F_{net} with τ_{net}, m with I, and a with α in radian measure, writing

$$\tau_{net} = I\alpha \quad \text{(Newton's second law for rotation)} \tag{10-62}$$

Proof of Eq. 10-62

We prove Eq. 10-62 by first considering the simple situation shown in Fig. 10-29. The rigid body there consists of a particle of mass m on one end of a massless rod of length r. The rod can move only by rotating about its other end, around a rotation axis (an axle) that is perpendicular to the plane of the page. Thus, the particle can move only in a circular path that has the rotation axis at its center.

A force $\vec{F}$ acts on the particle. However, because the particle can move only along the circular path, only the tangential component F_t of the force (the component that is tangent to the circular path) can accelerate the particle along the

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path. We can relate F_t to the particle's tangential acceleration a_t along the path with Newton's second law, writing

$$F_t = ma_t.$$

The torque acting on the particle is, from Eq. 10-44,

$$\tau = F_t r = ma_t r.$$

From Eq. 10-22 ($a_t = \alpha r$) we can write this as

$$\tau = m(\alpha r)r = (mr^2)\alpha. \tag{10-63}$$

The quantity in parentheses on the right is the rotational inertia of the particle about the rotation axis (see Eq. 10-33, but here we have only a single particle). Thus, using I for the rotational inertia, Eq. 10-63 reduces to

$$\tau = I\alpha \quad \text{(radian measure)}. \tag{10-64}$$

If more than one force is applied to the particle, Eq. 10-64 becomes

$$\tau_{net} = I\alpha \quad \text{(radian measure)}, \tag{10-65}$$

which we set out to prove. We can extend this equation to any rigid body rotating about a fixed axis, because any such body can always be analyzed as an assembly of single particles.

The torque due to the tangential component of the force causes an angular acceleration around the rotation axis.

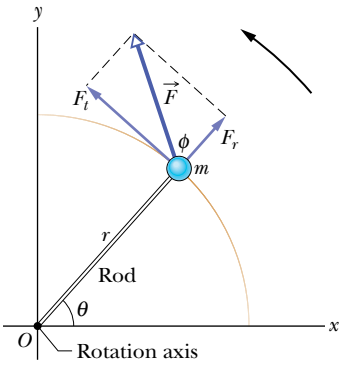
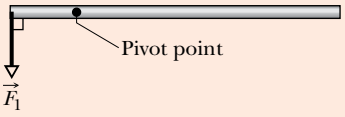


Figure 10-29 A simple rigid body, free to rotate about an axis through O, consists of a particle of mass m fastened to the end of a rod of length r and negligible mass. An applied force $\vec{F}_g$ causes the body to rotate.

CHECKPOINT 13

The figure shows an overhead view of a meter stick that can pivot about the point indicated, which is to the left of the stick's midpoint. Two horizontal forces, $\vec{F}_1$ and $\vec{F}_2$, are applied to the stick. Only $\vec{F}_1$ is shown. Force $\vec{F}_2$ is perpendicular to the stick and is applied at the right end. If the stick is not to turn, (a) what should be the direction of $\vec{F}_2$, and (b) should F_2 be greater than, less than,or equal to F_1?



Let's demonstrate that the total torque of all internal forces relative to any point is equal to zero. The internal forces refer to the forces of interaction between the particles of the given system. In accordance with Newton's third law these forces are pairwise, equal in magnitude and are opposite direction. Internal forces must be acting in accordance with third law and should be along the line joining the two particles in direction and lie on the same straight line as shown in Fig. 10-30. We can see that length of moment arm is same.

$$|\vec{\tau}_{12}| = F_{12}l \quad \text{and} \quad |\vec{\tau}_{21}| = F_{21}l$$

But

$$\hat{\tau}_{12} = -\hat{\tau}_{21}$$

Consequently, the torque of each pair of interaction are equal in magnitude and opposite in direction, that is, they cancel each other, and hence the total torque of all internal forces always equals zero.

Also note that $\tau_{net} = I\alpha$ has been derived assuming that the body is rigid. You should recall that taking out of summation in Eq. 10-63 implies that all particles have same angular acceleration which in turn implies that our collection of particles is a rigid body.

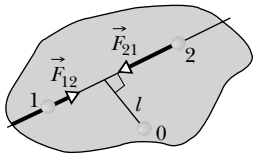


Figure 10-30 Length of moment arm for action and reaction is equal.

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SAMPLE PROBLEM 10.14

Using Newton's second law for rotation in a basic judo hip throw

To throw an 80 kg opponent with a basic judo hip throw, you intend to pull his uniform with a force $\vec{F}$ and a moment arm $d_1 = 0.30$ m from a pivot point (rotation axis) on your right hip (Fig. 10-31). You wish to rotate him about the pivot point with an angular acceleration α of -6.0 rad/s^2—that is, with an angular acceleration that is *clockwise* in the figure. Assume that his rotational inertia I relative to the pivot point is 15 kg · m^2.

(a) What must the magnitude of $\vec{F}$ be if, before you throw him, you bend your opponent forward to bring his center of mass to your hip (Fig. 10-31*a*)?

KEY IDEA

We can relate your pull $\vec{F}$ on your opponent to the given angular acceleration α via Newton's second law for rotation ($\tau_{net} = I\alpha$).

Calculations: As his feet leave the floor, we can assume that only three forces act on him: your pull $\vec{F}$, a force $\vec{N}$ on him from you at the pivot point (this force is not indicated in Fig. 10-31), and the gravitational force $\vec{F}_g$. To use ($\tau_{net} = I\alpha$, we need the corresponding three torques, each about the pivot point.

From Eq. 10-45 ($\tau = r_\perp F$), the torque due to your pull $\vec{F}$ is equal to $-d_1F$, where d_1 is the moment arm $r_\perp$ and the sign indicates the clockwise rotation this torque tends to cause. The torque due to $\vec{N}$ is zero, because $\vec{N}$ acts at the pivot point and thus has moment arm $r_\perp = 0$.

To evaluate the torque due to $\vec{F}_g$, we can assume that $\vec{F}_g$ acts at your opponent's center of mass. With the center of mass at the pivot point, $\vec{F}_g$ has moment arm $r_\perp = 0$ and thus the torque due to $\vec{F}_g$ is zero. So, the only torque on your opponent is due to your pull $\vec{F}$, and we can write $\tau_{net} = I\alpha$ as

$$-d_1F = I\alpha.$$

We then find

$$F = \frac{-I\alpha}{d_1} = \frac{-(15\ \text{kg}\cdot\text{m}^2)(-6.0\ \text{rad/s}^2)}{0.30\ \text{m}}$$
$$= 300\ \text{N}. \qquad \text{(Answer)}$$

(b) What must the magnitude of $\vec{F}$ be if your opponent remains upright before you throw him, so that $\vec{F}_g$ has a moment arm $d_2 = 0.12$ m (Fig. 10-31*b*)?

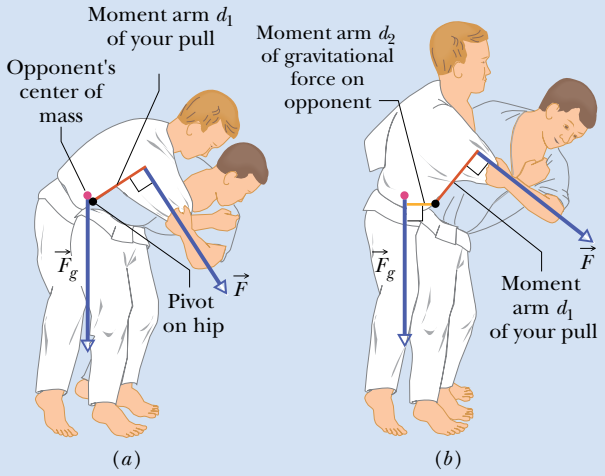

Figure 10-31 A judo hip throw (*a*) correctly executed and (*b*) incorrectly executed.

KEY IDEA

Because the moment arm for $\vec{F}_g$ is no longer zero, the torque due to $\vec{F}_g$ is now equal to d_2mg and is positive because the torque attempts counterclockwise rotation.

Calculations: Now we write $\tau_{net} = I\alpha$ as

$$-d_1F + d_2mg = I\alpha,$$

which gives

$$F = -\frac{I\alpha}{d_1} + \frac{d_2mg}{d_1}.$$

From (a), we know that the first term on the right is equal to 300 N. Substituting this and the given data, we have

$$F = 300\ \text{N} + \frac{(0.12\ \text{m})(80\ \text{kg})(9.8\ \text{m/s}^2)}{0.30\ \text{m}}$$
$$= 613.6\ \text{N} \approx 610\ \text{N}. \qquad \text{(Answer)}$$

The results indicate that you will have to pull much harder if you do not initially bend your opponent to bring his center of mass to your hip. A good judo fighter knows this lesson from physics. Indeed, physics is the basis of most of the martial arts, figured out after countless hours of trial and error over the centuries.

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SAMPLE PROBLEM 10.15

Newton's second law, rotation, torque, disk

Figure 10-32*a* shows a uniform disk, with mass $M = 2.5$ kg and radius $R = 20$ cm, mounted on a fixed horizontal axle. A block with mass $m = 1.2$ kg hangs from a massless cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the angular acceleration of the disk, and the tension in the cord.The cord does not slip, and there is no friction at the axle.

KEY IDEAS

(1) Taking the block as a system, we can relate its acceleration a to the forces acting on it with Newton's second law ($\vec{F}_{net} = m\vec{a}$). (2) Taking the disk as a system, we can relate its angular acceleration α to the torque acting on it with Newton's second law for rotation ($\tau_{net} = I\alpha$). (3) To combine the motions of block and disk, we use the fact that the linear acceleration a of the block and the (tangential) linear acceleration a_t of the disk rim are equal. (To avoid confusion about signs, let's work with acceleration magnitudes and explicit algebraic signs.)

Forces on block: The forces are shown in the block's freebody diagram in Fig. 10-32*b*: The force from the cord is $\vec{T}$, and the gravitational force is $\vec{F}_g$, of magnitude mg. We can now write Newton's second law for components along a vertical y axis ($F_{net,y} = ma_y$) as

$$T - mg = m(-a), \tag{10-66}$$

where a is the magnitude of the acceleration (down the y axis). However, we cannot solve this equation for a because it also contains the unknown T.

Torque on disk: Previously, when we got stuck on the y axis, we switched to the x axis. Here, we switch to the rotation of the disk and use Newton's second law in angular form. To calculate the torques and the rotational inertia I, we take the rotation axis to be perpendicular to the disk and through its center, at point O in Fig. 10-32*c*.

The torques are then given by Eq. 10-44 ($\tau = rF_t$). The gravitational force on the disk and the force on the disk from the axle both act at the center of the disk and thus at distance $r = 0$, so their torques are zero. The force $\vec{T}$ on the disk due to the cord acts at distance $r = R$ and is tangent to the rim of the disk. Therefore, its torque is $-RT$, negative because the torque rotates the disk clockwise from rest. Let α be the magnitude of the negative (clockwise) angular acceleration. From Table 10-2*c*, the rotational inertia I of the disk is $1/2MR^2$. Thus we can write the general equation $\tau_{net} = I\alpha$ as

$$-RT = \frac{1}{2}MR^2(-\alpha). \tag{10-67}$$

This equation seems useless because it has two unknowns, α and T, neither of which is the desired a. However, mustering physics courage, we can make it useful with this fact: Because the cord does not slip, the magnitude a of the block's linear acceleration and the magnitude a_t of the (tangential) linear acceleration of the rim of the disk are equal. Then, by Eq. 10-22 ($at = \alpha r$) we see that here $\alpha = a/R$. Substituting this in Eq. 10-67 yields

$$T = \frac{1}{2}Ma. \tag{10-68}$$

Combining results: Combining Eqs. 10-66 and 10-68 leads to

$$a = g\frac{2m}{M+2m} = (9.8\ \text{m/s}^2)\frac{(2)(1.2\ \text{kg})}{2.5\ \text{kg} + (2)(1.2\ \text{kg})}$$
$$= 4.8\ \text{m/s}^2. \qquad \text{(Answer)}$$

We then use Eq. 10-68 to find T:

$$T = \frac{1}{2}Ma = \frac{1}{2}(2.5\ \text{kg})(4.8\ \text{m/s}^2) = 6.0\ \text{N}. \qquad \text{(Answer)}$$

As we should expect, acceleration a of the falling block is less than g, and tension T in the cord (= 6.0 N) is less than the gravitational force on the hanging block (= mg = 11.8 N). We see also that a and T depend on the mass of the disk but not on its radius.

As a check, we note that the formulas derived above predict $a = g$ and $T = 0$ for the case of a massless disk ($M = 0$). This is what we would expect; the block simply falls as a free body. From Eq. 10-22, the magnitude of the angular acceleration of the disk is

$$\alpha = \frac{a}{R} = \frac{4.8\ \text{m/s}^2}{0.20\ \text{m}} = 24\ \text{rad/s}^2. \qquad \text{(Answer)}$$

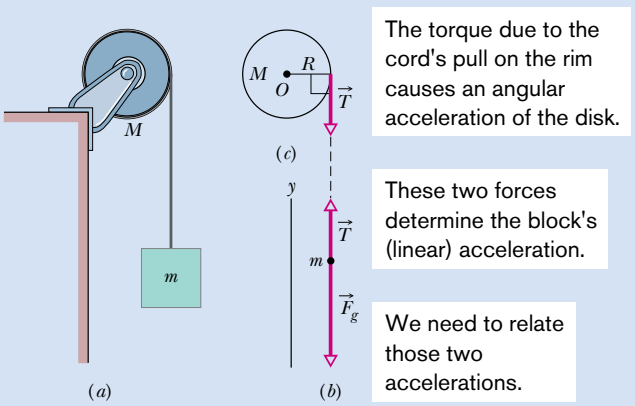

Figure 10-32 (*a*) The falling block causes the disk to rotate. (*b*) A free-body diagram for the block. (*c*) An incomplete free-body diagram for the disk.

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10.13 | WORK AND ROTATIONAL KINETIC ENERGY

Key Concepts

- The equations used for calculating work and power in rotational motion correspond to equations used for translational motion and are

$$W = \int_{\theta_i}^{\theta_f} \tau\, d\theta$$

and

$$P = \frac{dW}{dt} = \tau\omega.$$

- When τ is constant, the integral reduces to

$$W = \tau(\theta_f - \theta_i).$$

- The form of the work–kinetic energy theorem used for rotating bodies is

$$\Delta K = K_f - K_i = \frac{1}{2} I\omega_f^2 - \frac{1}{2} I\omega_i^2 = W.$$

As we discussed in Chapter 8, when a force F causes a rigid body of mass m to accelerate along a coordinate axis, the force does work W on the body. Thus, the body's kinetic energy ($K = 1/2mv^2$) can change. Suppose it is the only energy of the body that changes. Then we relate the change ΔK in kinetic energy to the work W with the work–kinetic energy theorem (Eq. 8-10), writing

$$\Delta K = K_f - K_i = \frac{1}{2} mv_f^2 - \frac{1}{2} mv_i^2 = W \quad \text{(work–kinetic energy theorem)}. \tag{10-69}$$

For motion confined to an x axis, we can calculate the work with Eq. 8-32,

$$W = \int_{x_i}^{x_f} F\, dx \quad \text{(work, one-dimensional motion)}. \tag{10-70}$$

This reduces to $W = Fd$ when F is constant and the body's displacement is d. The rate at which the work is done is the power, which we can find as,

$$P = \frac{dW}{dt} = Fv \quad \text{(power, one-dimensional motion)}. \tag{10-71}$$

Now, let us consider a rotational situation that is similar. When a torque accelerates a rigid body in rotation about a fixed axis, the torque does work W on the body. Therefore, the body's rotational kinetic energy ($K = 1/2I\omega^2$) can change. Suppose that it is the only energy of the body that changes. Then we can still relate the change ΔK in kinetic energy to the work W with the work–kinetic energy theorem, except now the kinetic energy is a rotational kinetic energy:

$$\Delta K = K_f - K_i = \frac{1}{2} I\omega_f^2 - \frac{1}{2} I\omega_i^2 = W \quad \text{(work–kinetic energy theorem)}. \tag{10-72}$$

Here, I is the rotational inertia of the body about the fixed axis and ω_i and ω_f are the angular speeds of the body before and after the work is done.

Also, we can calculate the work with a rotational equivalent of Eq. 10-70,

$$W = \int_{\theta_i}^{\theta_f} \tau\, d\theta \quad \text{(work, rotation about fixed axis)}, \tag{10-73}$$

where τ is the torque doing the work W, and θ_i and θ_f are the body's angular positions before and after the work is done, respectively. When τ is constant, Eq. 10-73 reduces to

$$W = \tau(\theta_f - \theta_i) \quad \text{(work, constant torque)}. \tag{10-74}$$

The rate at which the work is done is the power, which we can find with the rotational equivalent of Eq. 10-71,

$$P = \frac{dW}{dt} = \tau\omega \quad \text{(power, rotation about fixed axis)}. \tag{10-75}$$

Table 10-3 summarizes the equations that apply to the rotation of a rigid body about a fixed axis and the corresponding equations for translational motion.

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Proof of Eqs. 10-72 through 10-75

Let us again consider the situation of Fig. 10-29, in which force $\vec{F}$ rotates a rigid body consisting of a single particle of mass m fastened to the end of a massless rod. During the rotation, force $\vec{F}$ does work on the body. Let us assume that the only energy of the body that is changed by $\vec{F}$ is the kinetic energy. Then we can apply the work–kinetic energy theorem of Eq. 10-69:

$$\Delta K = K_f - K_i = W. \tag{10-76}$$

Using $K = 1/2mv^2$ and Eq. 10-18 ($v = \omega r$), we can rewrite Eq. 10-76 as

$$\Delta K = \frac{1}{2}mr^2\omega_f^2 - \frac{1}{2}mr^2\omega_i^2 = W. \tag{10-77}$$

From Eq. 10-33, the rotational inertia for this one-particle body is $I = mr^2$. Substituting this into Eq. 10-77 yields

$$\Delta K = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = W,$$

which is Eq. 10-72. We derived it for a rigid body with one particle, but it holds for any rigid body rotated about a fixed axis.

We next relate the work W done on the body in Fig. 10-29 to the torque τ on the body due to force $\vec{F}$. When the particle moves a distance ds along its circular path, only the tangential component F_t of the force accelerates the particle along the path. Therefore, only F_t does work on the particle. We write that work dW as $F_t ds$. However, we can replace ds with $rd\theta$, where $d\theta$ is the angle through which the particle moves. Thus, we have

$$dW = F_t r d\theta. \tag{10-78}$$

From Eq. 10-44, we see that the product $F_t r$ is equal to the torque τ, so we can rewrite Eq. 10-78 as

$$dW = \tau d\theta. \tag{10-79}$$

The work done during a finite angular displacement from θ_i to θ_f is then

$$W = \int_{\theta_i}^{\theta_f} \tau d\theta$$

which is Eq. 10-73. It holds for any rigid body rotating about a fixed axis. Equation 10-74 comes directly from Eq. 10-73.

We can find the power P for rotational motion from Eq. 10-79:

$$P = \frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\omega,$$

which is Eq. 10-75.

Table 10-3 Some Corresponding Relations for Translational and Rotational Motion

Pure Translation (Fixed Direction)		Pure Rotation (Fixed Axis)	
Position	x	Angular position	θ
Velocity	$v = dx/dt$	Angular velocity	$\omega = d\theta/dt$
Acceleration	$\alpha = dv/dt$	Angular acceleration	$\alpha = d\omega/dt$
Mass	m	Rotational inertia	I
Newton's second law	$F_{net} = ma$	Newton's second law	$\tau_{net} = I\alpha$
Work	$W = \int F\,dx$	Work	$W = \int \tau d\theta$
Kinetic energy	$K = 1/2mv^2$	Kinetic energy	$K = 1/2I\omega^2$
Power (constant force)	$P = Fv$	Power (constant torque)	$P = \tau\omega$
Work–kinetic energy theorem	$W = \Delta K$	Work–kinetic energy theorem	$W = \Delta K$

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SOME IMPORTANT EQUATIONS

Following equations are going to be very useful for solving problems of rotational dynamics about a fixed axis.

1. $\tau_{axis} = I_{axis}\alpha$.
2. $\vec{F}_{ext} = M\vec{a}_{com}$.
3. $\vec{P}_{sys} = M\vec{V}_{com}$ and $\vec{F}_{ext} = \frac{d\vec{P}_{sys}}{dt}$.
4. Rotational kinetic energy $= \frac{1}{2} I_{axis}\omega^2$.
5. $\tau = \frac{d\vec{L}_{sys}}{dt}$.
6. $a_t = \alpha r$.
7. $a_t = ar$ (where r is distance at any point from the axis).

PROBLEM-SOLVING TACTICS

Tactic 3: Following steps need to be followed for solving problems of rotational dynamics around fixed axis.

1. Select a body or bodies to be analyzed.
2. For each body, draw free-body diagram.
3. Identify axis of rotation.
4. Assume angular acceleration of the body about the axis of rotation.
5. Write torque about the axis of rotation.
6. Use $\tau_{axis} = \tau_{axis}\alpha$ with a correct sign.
7. If required, use other equations.
8. Continue to repeat steps 1–7 till you have equal number of unknowns and equations.

SAMPLE PROBLEM 10.16

Work, rotational kinetic energy, torque, disk

Let the disk in Fig. 10-19 start from rest at time $t = 0$ and also let the tension in the massless cord be 6.0 N and the angular acceleration of the disk be –24 rad/s^2. What is its rotational kinetic energy K at $t = 2.5$ s?

KEY IDEA

We can find K with Eq. 10-34 ($K = 1/2I\omega^2$). We already know that $I = 1/2MR^2$, but we do not yet know ω at $t = 2.5$ s. However, because the angular acceleration α has the constant value of –24 rad/s^2, we can apply the equations for constant angular acceleration in Table 10-1.

Calculations: Because we want ω and know α and $\omega_0 (= 0)$, we use Eq. 10-12:

$$\omega = \omega_0 + \alpha t = 0 + \alpha t = \alpha t.$$

Substituting $\omega = \alpha t$ and $I = 1/2MR^2$ into Eq. 10-34, we find

$$K = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(\frac{1}{2}MR^2\right)(\alpha t)^2 = \frac{1}{4}M(Rat)^2$$
$$= \frac{1}{4}(2.5\text{ kg})[(0.20\text{ m})(-24\text{ rad/s}^2)(2.5\text{ s})]^2$$
$$= 90\text{ J}. \qquad \text{(Answer)}$$

KEY IDEA

We can also get this answer by finding the disk's kinetic energy from the work done on the disk.

Calculations: First, we relate the *change* in the kinetic energy of the disk to the net work W done on the disk,

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using the work–kinetic energy theorem of Eq. 10-72 ($K_f - K_i = W$). With K substituted for K_f and 0 for K_i, we get

$$K = K_i + W = 0 + W = W. \qquad (10\text{-}80)$$

Next we want to find the work W. We can relate W to the torques acting on the disk with Eq. 10-78 or 10-79. The only torque causing angular acceleration and doing work is the torque due to force $\vec{T}$ on the disk from the cord, which is equal to $-TR$. Because α is constant, this torque also must be constant. Thus, we can use Eq. 10-74 to write

$$W = \tau(\theta_f - \theta_i) = -TR(\theta_f - \theta_i). \qquad (10\text{-}81)$$

Because α is constant, we can use Eq. 10-13 to find $\theta_f - \theta_i$. With $\omega_i = 0$, we have

$$\theta_f - \theta_i = \omega_i t + \frac{1}{2}\alpha t^2 = 0 + \frac{1}{2}\alpha t^2 = \frac{1}{2}\alpha t^2.$$

Now we substitute this into Eq. 10-81 and then substitute the result into Eq. 10-80. Inserting the given values $T = 6.0$ N and $\alpha = -24$ rad/s^2, we have

$$K = W = -TR(\theta_f - \theta_i) = -TR\left(\frac{1}{2}\alpha t^2\right) = -\frac{1}{2}TR\alpha t^2$$
$$= -\frac{1}{2}(6.0\text{ N})(0.20\text{ m})(-24\text{ rad/s}^2)(2.5\text{ s})^2$$
$$= 90\text{ J}. \qquad \text{(Answer)}$$

SAMPLE PROBLEM 10.17

Angular velocity and angle of rotation

A tall, cylindrical chimney starts falling over as shown in Fig. 10-33*a*. Treat the chimney as a thin rigid rod of length $L = 55.0$ m. At the instant it makes an angle of $\theta = 60.0°$ m with the vertical, what is its angular speed ω_f?

KEY IDEAS

(1) You can imagine keeping a pencil vertical and then releasing it. The path covered by the pencil shall be the same as that of the chimney in this problem. As the rod (chimney) tips over, its weight causes it to fall faster.

(2) During the rotation, the mechanical energy (the sum of the rotational kinetic energy K and the gravitational potential energy U) does not change.

The rotational kinetic energy is given by

$$K = \frac{1}{2}I\omega^2$$

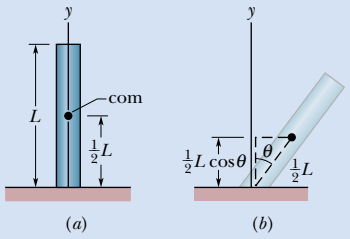

Figure 10-33 (*a*) A cylindrical chimney. (*b*) The height of its center of mass is determined with the right triangle.

Calculations:

Conservation of mechanical energy: As the center of mass of the chimney falls, energy is transferred from gravitational potential energy U to rotational kinetic energy K but the total amount does not change. We can write this fact as

$$K_f + U_f + K_i + U_i \qquad (10\text{-}82)$$

Rotational kinetic energy: The kinetic energy K is initially zero but its value ($1/2\ I\omega^2$) thereafter depends on the rotational inertia I. If we had a thin rod rotating around its center of mass (at its center), we know from Table 10-2 that $I_{com} = 1/12\,mL^2$, where m is the rod's mass and L is the rod's length. However, our rod-like chimney rotates around one end, at a distance of $L/2$ from the center, and so we use the parallel-axis theorem to find

$$I = \frac{1}{12}mL + m\left(\frac{L}{2}\right)^2 = \frac{1}{3}mL^2 \qquad (10\text{-}83)$$

Substituting this into $K = (1/2I\omega^2)$ tells us

$$K_f = \frac{1}{2}\left(\frac{1}{3}mL^2\right)\omega^2 \qquad (10\text{-}84)$$

Potential energy: The potential energy U ($= mgy$) depends on the height of each segment of the chimney. However, we can calculate U by assuming that all of the mass is concentrated at the chimney's com, which is initially at height $1/2L$. So, the initial potential energy is

$$U_i = \frac{1}{2}mgL \qquad (10\text{-}85)$$

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When the chimney has rotated through angle θ, Fig. 10-33*b* tells us that the center is at height $1/2L\cos\theta$. So, the potential energy is now

$$U_f = \frac{1}{2}mgL\cos\theta \qquad (10\text{-}86)$$

Angular speed: After substituting Eqs. 10-86, 10-85, and 10-84 into Eq. 10-82 and setting $K_i = 0$, we solve for and find

$$\omega = \sqrt{\frac{3g}{L}(1-\cos\theta)} = \sqrt{\frac{3(9.8\ \text{m/s}^2)}{55.0\ \text{m}}(1-\cos 60.0°)}$$
$$= 0.516\ \text{rad/s} \qquad \text{(Answer)}$$

Learn: We can see from the equation that angular velocity increases as angle increases. The bottom portion of a chimney tends to rotate around the base faster than the top portion, and the chimney is likely to break apart during the rotation, with the top half lagging behind the bottom half.

SAMPLE PROBLEM 10.18

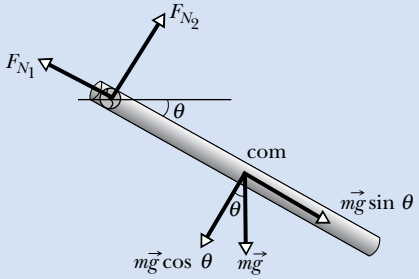
Acceleration and normal force for uniform rod free to rotate in the vertical plane about a pivot

A uniform rod of length l and mass M pivoted about its end as shown in Fig. 10-34*a*, and is free to rotate in the vertical plane about the pivot. The rod is released from rest in the horizontal position.

(a) What is the initial angular acceleration of the rod?

(*a*)

Figure 10-34 (*a*) A uniform rod in vertical plane with horizontal axis is released.



(*b*)

Figure 10-34 (*b*) Free-body diagram.

KEY IDEA

(1) It is easy to imagine that rod rotates clockwise around the pivot at the left end, when it is released. The rod will be experiencing two forces, one is the weight, effectively acting on center of mass and the other is the force due to hinge or pivot, acting through the hinge. (2) When we write torque about the pivot the torque due to second force will become zero because its moment arm is zero. Now the torque is due to only the gravitational force on the rod. For the rotation axis considered crossing the pivot, the rod is a rigid body under a net torque.

Calculation: We know the gravitational force acts effectively at the center of mass of the rod as shown in Fig. 10-34*b*.

The magnitude of the torque due to the weight about an axis through the pivot,

$$\tau = Mg\left(\frac{l}{2}\right)$$

Using Eq. 10-62 to obtain the angular acceleration of the rod, we get

$$\alpha = \frac{\tau}{I} = \frac{Mg(\frac{l}{2})}{\frac{1}{3}Ml^2} = \frac{3g}{2l}$$

(b) Find the initial acceleration of the right end of the rod?

Compute: Using Eq. 10-22 with $r = l$ for the initial acceleration of the right end of the rod, we get

$$\alpha_i = l\alpha = \frac{3}{2}g$$

Learn: These values are the initial values of the angular and translational accelerations. Once the rod begins to rotate, the gravitational force is no longer perpendicular to the rod and the values of the two acceleration changes as shown in next part of the problem.

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(c) Find normal contact force due to hinge when rod has rotated through angle θ as shown in Fig. 10-34*b*.

Calculation: We know the gravitational force effectively acts at the center of mass of the rod as shown in Fig. 10-34*b*. The magnitude of the torque due to the weight about an axis through the pivot is given by

$$mg\cos\theta\times\frac{l}{2}=\frac{ml^2}{3}\times\alpha$$
$$\alpha=\frac{3g\cos\theta}{2l}$$

By energy conservation,

$$mg\frac{1}{2}\sin\theta=\frac{1}{2}\frac{ml^2}{3}\omega^2$$
$$\omega=\sqrt{\frac{3g\sin\theta}{l}}$$

$$a_r=\omega^2\frac{l}{2}=\frac{3g\sin\theta}{l}\times\frac{l}{2}=\frac{3g\sin\theta}{2}$$
$$a_t=\alpha\frac{1}{2}=\frac{3g\cos\theta}{4}$$

$$F_{N_1}-mg\sin\theta=m\frac{3g\sin\theta}{2}$$

or
$$F_{N_1}=\frac{5mg\sin\theta}{2}$$

Also,
$$mg\cos\theta-F_{N_2}=m\frac{3g\cos\theta}{2}$$

so,
$$F_{N_2}=\frac{mg\cos\theta}{4}$$
$$F_N=\sqrt{F_{N_1}^2+F_{N_2}^2}$$ (Answer)

10.14 | ANGULAR MOMENTUM

Key Concepts

- The angular momentum $\vec{\ell}$ of a particle with linear momentum $\vec{p}$, mass m, and linear velocity $\vec{v}$ is a vector quantity defined relative to a fixed point (usually an origin) as
$$\vec{\ell}=\vec{r}\times\vec{p}=m(\vec{r}\times\vec{v}).$$
- The magnitude of $\vec{\ell}$ is given by
$$\ell=rmv\sin\phi$$
$$=rp_\perp=rmv_\perp$$
$$=r_\perp p=r_\perp mv,$$
where ϕ is the angle between $\vec{r}$ and $\vec{p}$, $p_\perp$ and $v_\perp$ are the components of $\vec{p}$ and $\vec{v}$ perpendicular to $\vec{r}$, and $r_\perp$ is the perpendicular distance between the fixed point and the extension of $\vec{p}$.
- The direction of $\vec{\ell}$ is given by the right-hand rule: Position your right hand so that the fingers are in the direction of $\vec{r}$. Then rotate them around the palm to be in the direction of $\vec{p}$. Your outstretched thumb gives the direction of $\vec{\ell}$.

Recall that the concept of linear momentum $\vec{p}$ and the principle of conservation of linear momentum are extremely powerful tools. They allow us to predict the outcome of, say, a collision of two cars without knowing the details of the collision. Here we begin a discussion of the angular counterpart of $\vec{p}$, winding up in Section 10.18 with the angular counterpart of the conservation principle, which can lead to beautiful (almost magical) feats in ballet, fancy diving, ice skating, and many other activities.

Figure 10-33 shows a particle of mass m with linear momentum $\vec{p}(=m\vec{v})$ as it passes through point A in an xy plane. The **angular momentum** $\vec{\ell}$ of this particle with respect to the origin O is a vector quantity defined as

$$\vec{\ell}=\vec{r}\times\vec{p}=m(\vec{r}\times\vec{v})\quad\text{(angular momentum defined)},\qquad(10\text{-}87)$$

where $\vec{r}$ is the position vector of the particle with respect to O. As the particle moves relative to O in the direction of its momentum $\vec{p}(=m\vec{v})$, position vector $\vec{r}$ rotates around O. Note carefully that to have angular momentum about O, the particle does *not* itself have to rotate around O. Comparison of Eqs. 10-50 and 10-87 shows that angular

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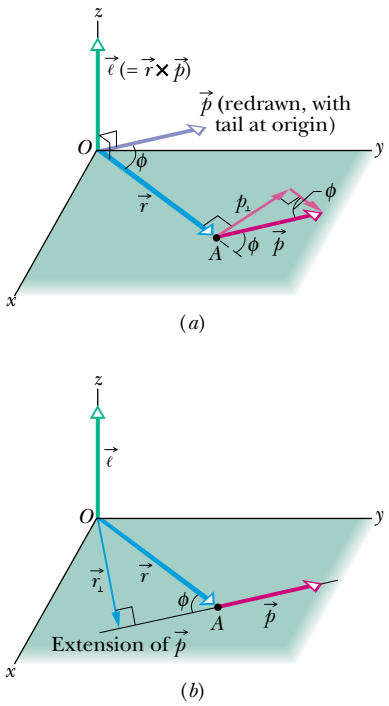


Figure 10-35 Defining angular momentum. A particle passing through point A has linear momentum $\vec{p}(=m\vec{v})$, with the vector $\vec{p}$ lying in an xy plane. The particle has angular momentum $\vec{\ell}\,(=\vec{r}\times\vec{p})$ with respect to the origin O. By the right-hand rule, the angular momentum vector points in the positive direction of z. (*a*) The magnitude of $\vec{\ell}$ is given by $\ell = rp_\perp = rmv_\perp$. (*b*) The magnitude of $\vec{\ell}$ is also given by $\ell = r_\perp p = r_\perp mv$.

momentum bears the same relation to linear momentum that torque does to force. The SI unit of angular momentum is the kilogrammeter-squared per second ($kg\cdot m^2/s$), equivalent to the joule-second ($J\cdot s$).

Direction. To find the direction of the angular momentum vector $\vec{\ell}$ in Fig. 10-35, we slide the vector $\vec{p}$ until its tail is at the origin O. Then we use the right-hand rule for vector products, sweeping the fingers from $\vec{r}$ into $\vec{p}$. The outstretched thumb then shows that the direction of $\vec{\ell}$ is in the positive direction of the z axis in Fig. 10-35. This positive direction is consistent with the counterclockwise rotation of position vector $\vec{r}$ about the z axis, as the particle moves. (A negative direction of $\vec{\ell}$ would be consistent with a clockwise rotation of $\vec{r}$ about the z axis.)

Magnitude. To find the magnitude of $\vec{\ell}$, we use the general result of Eq. 10-46 to write

$$\ell = rmv\sin\phi, \tag{10-88}$$

where ϕ is the smaller angle between $\vec{r}$ and $\vec{p}$ when these two vectors are tail to tail. From Fig. 10-35*a*, we see that Eq. 10-88 can be rewritten as

$$\ell = rp_\perp = rmv_\perp, \tag{10-89}$$

where $p_\perp$ is the component of $\vec{p}$ perpendicular to $\vec{r}$ and $v_\perp$ is the component of $\vec{v}$ perpendicular to $\vec{r}$. From Fig. 10-35*b*, we see that Eq. 10-88 can also be rewritten as

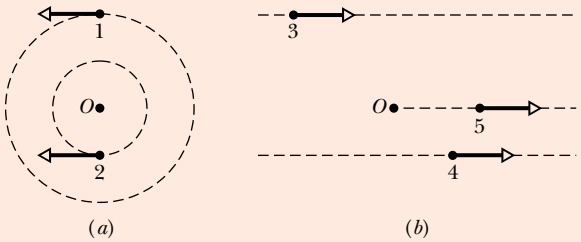
$$\ell = r_\perp p = r_\perp mv, \tag{10-90}$$

where $r_\perp$ is the perpendicular distance between O and the extension of $\vec{p}$.

Important. Note two features here: (1) angular momentum has meaning only with respect to a specified origin and (2) its direction is always perpendicular to the plane formed by the position and linear momentum vectors $\vec{r}$ and $\vec{p}$.

CHECKPOINT 14

In part *a* of the figure, particles 1 and 2 move around point O in circles with radii 2 m and 4 m. In part *b*, particles 3 and 4 travel along straight lines at perpendicular distances of 4 m and 2 m from point O. Particle 5 moves directly away from O. All five particles have the same mass and the same constant speed. (a) Rank the particles according to the magnitudes of their angular momentum about point O, greatest first. (b) Which particles have negative angular momentum about point O?



SAMPLE PROBLEM 10.19

Angular momentum of a two-particle system

Figure 10-36 shows an overhead view of two particles moving at constant momentum along horizontal paths. Particle 1, with momentum magnitude $p_1 = 5.0\ kg\cdot m/s$, has position vector $\vec{r}_1$ and will pass 2.0 m from point O.

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Particle 2, with momentum magnitude $p_2 = 2.0\ \text{kg}\cdot\text{m/s}$, has position vector $\vec{r}_2$ and will pass 4.0 m from point O. What are the magnitude and direction of the net angular momentum $\vec{L}$ about point O of the two particle system?

KEY IDEA

To find $\vec{L}$, we can first find the individual angular momenta $\vec{\ell}_1$ and $\vec{\ell}_2$ and then add them. To evaluate their magnitudes, we can use any one of Eqs. 10-87 through 10-89. However, Eq. 10-89 is easiest, because we are given the perpendicular distances $r_{\perp 1}$ (= 2.0 m) and $r_{\perp 2}$ (= 4.0 m) and the momentum magnitudes p_1 and p_2.

Calculations: For particle 1, Eq. 10-90 yields

$$\ell_1 = r_{\perp 1} p_1 = (2.0\ \text{m})(5.0\ \text{kg}\cdot\text{m/s}) = 10\ \text{kg}\cdot\text{m}^2/\text{s}.$$

To find the direction of vector $\vec{\ell}_1$, we use Eq. 10-87 and the right-hand rule for vector products. For $\vec{r}_1 \times \vec{p}_1$, the vector product is out of the page, perpendicular to the plane of Fig. 10-36. This is the positive direction, consistent with the counterclockwise rotation of the particle's position vector $\vec{r}_1$ around O as particle 1 moves. Thus, the angular momentum vector for particle 1 is

$$\ell_1 = +10\ \text{kg}\cdot\text{m}^2/\text{s}.$$

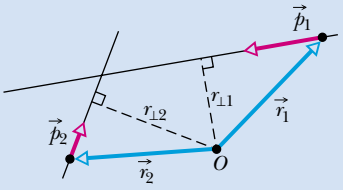

Figure 10-36 Two particles pass near point O.

Similarly, the magnitude of $\vec{\ell}_2$ is

$$\ell = r_{\perp 2} p_2 = (4.0\ \text{m})(2.0\ \text{kg}\cdot\text{m/s}) = 8.0\ \text{kg}\cdot\text{m}^2/\text{s},$$

and the vector product $\vec{r}_2 \times \vec{p}_2$ is into the page, which is the negative direction, consistent with the clockwise rotation of $\vec{r}_2$ around O as particle 2 moves. Thus, the angular momentum vector for particle 2 is

$$\ell_2 = -8.0\ \text{kg}\cdot\text{m}^2/\text{s}$$

The net angular momentum for the two particle system is

$$L = \ell_1 + \ell_2 = +10\ \text{kg}\cdot\text{m}^2/\text{s} + (-8.0\ \text{kg}\cdot\text{m}^2/\text{s}) = +2.0\ \text{kg}\cdot\text{m}^2/\text{s}. \quad \text{(Answer)}$$

The plus sign means that the system's net angular momentum about point O is out of the page.

10.15 | NEWTON'S SECOND LAW IN ANGULAR FORM

Key Concept

- Newton's second law for a particle can be written in angular form as

$$\vec{\tau}_{\text{net}} = \frac{d\vec{\ell}}{dt},$$

where $\vec{\tau}_{\text{net}}$ is the net torque acting on the particle and $\vec{\ell}$ is the angular momentum of the particle.

Newton's second law written in the form

$$\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} \quad \text{(single particle)} \qquad (10\text{-}91)$$

expresses the close relation between force and linear momentum for a single particle. We have seen enough of the parallelism between linear and angular quantities to be pretty sure that there is also a close relation between torque and angular momentum. Guided by Eq. 10-91, we can even guess that it must be

$$\vec{\tau}_{\text{net}} = \frac{d\vec{\ell}}{dt} \quad \text{(single particle)}. \qquad (10\text{-}92)$$

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Equation 10-92 is indeed an angular form of Newton's second law for a single particle:

The (vector) sum of all the torques acting on a particle is equal to the time rate of change of the angular momentum of that particle.

Equation 10-92 has no meaning unless the torques $\vec{\tau}$ and the angular momentum $\vec{\ell}$ are defined with respect to the same point, usually the origin of the coordinate system being used.

Proof of Eq. 10-92

We start with Eq. 10-87, the definition of the angular momentum of a particle:

$$\vec{\ell} = m(\vec{r} \times \vec{v}),$$

where $\vec{r}$ is the position vector of the particle and $\vec{v}$ is the velocity of the particle. Differentiating* each side with respect to time t yields

$$\frac{d\vec{\ell}}{dt} = m\left(\vec{r} \times \frac{d\vec{v}}{dt} + \frac{d\vec{r}}{dt} \times \vec{v}\right). \tag{10-93}$$

However, $d\vec{v}/dt$ is the acceleration $\vec{a}$ of the particle, and $d\vec{r}/dt$ is its velocity $\vec{v}$. Thus, we can rewrite Eq. 10-93 as

$$\frac{d\vec{\ell}}{dt} = m(\vec{r} \times \vec{a} + \vec{v} \times \vec{v}).$$

Now $\vec{v} \times \vec{v} = 0$ (the vector product of any vector with itself is zero because the angle between the two vectors is necessarily zero). Thus, the last term of this expression is eliminated and we then have

$$\frac{d\vec{\ell}}{dt} = m(\vec{r} \times \vec{a}) = \vec{r} \times m\vec{a}.$$

We now use Newton's second law ($\vec{F}_{net} = m\vec{a}$) to replace $m\vec{a}$ with its equal, the vector sum of the forces that act on the particle, obtaining

$$\frac{d\vec{\ell}}{dt} = \vec{r} \times \vec{F}_{net} = \sum(\vec{r} \times \vec{F}). \tag{10-94}$$

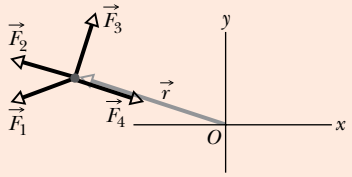
Here the symbol $\sum$ indicates that we must sum the vector products $\vec{r} \times \vec{F}$ for all the forces. However, from Eq. 10-50, we know that each one of those vector products is the torque associated with one of the forces. Therefore, Eq. 10-94 tells us that

$$\vec{\tau}_{net} = \frac{d\vec{\ell}}{dt}.$$

This is Eq. 10-92, the relation that we set out to prove.

CHECKPOINT 15

The figure shows the position vector $\vec{r}$ of a particle at a certain instant, and four choices for the direction of a force that is to accelerate the particle. All four choices lie in the xy plane. (a) Rank the choices according to the magnitude of the time rate of change ($d\vec{\ell}/dt$) they produce in the angular momentum of the particle about point O, greatest first. (b) Which choice results in a negative rate of change about O?



* In differentiating a vector product, be sure not to change the order of the two quantities (here $\vec{r}$ and $\vec{v}$) that form that product. (See Eq. 10-47.)

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SAMPLE PROBLEM 10.20

Torque and the time derivative of angular momentum

Figure 10-37*a* shows a freeze-frame of a 0.500 kg particle moving along a straight line with a position vector given by

$$\vec{r} = (-2.00t^2 - t)\hat{i} + 5.00\hat{j},$$

With $\vec{r}$ in meters and t in seconds, starting at $t = 0$. The position vector points from the origin to the particle. In unit-vector notation, find expressions for the angular momentum $\vec{\ell}$ of the particle and the torque $\vec{\tau}$ acting on the particle, both with respect to (or about) the origin. Justify their algebraic signs in terms of the particle's motion.

KEY IDEAS

(1) The point about which an angular momentum of a particle is to be calculated must always be specified. Here it is the origin. (2) The angular momentum $\vec{\ell}$ of a particle is given by Eq. 10-87 ($\vec{\ell} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$). (3) The sign associated with a particle's angular momentum is set by the sense of rotation of the particle's position vector (around the rotation axis) as the particle moves: clockwise is negative and counterclockwise is positive. (4) If the torque acting on a particle and the angular momentum of the particle are calculated around the *same* point, then the torque is related to angular momentum by Eq. 10-92 ($\vec{\tau} = d\vec{\ell}/dt$).

Calculations: In order to use Eq. 10-87 to find the angular momentum about the origin, we first must find an expression for the particle's velocity by taking a time derivative of its position vector. Following Eq. 4-10 ($\vec{v} = d\vec{r}/dt$), we write

$$\vec{v} = \frac{d}{dt}((-2.00t^2 - t)\hat{i} + 5.00\hat{j})$$

$$= (-4.00t - 1.00)\hat{i},$$

with $\vec{v}$ in meters per second.

Next, let's take the cross product of $\vec{r}$ and $\vec{v}$ using the template for cross products displayed in Eq. 10-49:

$$\vec{a} \times \vec{b} = (a_y b_z - b_y a_z)\hat{i} + (a_z b_x - b_z a_x)\hat{j} + (a_x b_y - b_x a_y)\hat{k}.$$

Here the generic $\vec{a}$ is $\vec{r}$ and the generic $\vec{b}$ is $\vec{v}$. However, because we really do not want to do more work than needed, let's first just think about our substitutions into the generic cross product. Because $\vec{r}$ lacks any z component and because $\vec{v}$ lacks any y or z component, the only nonzero term in the generic cross product is the very last

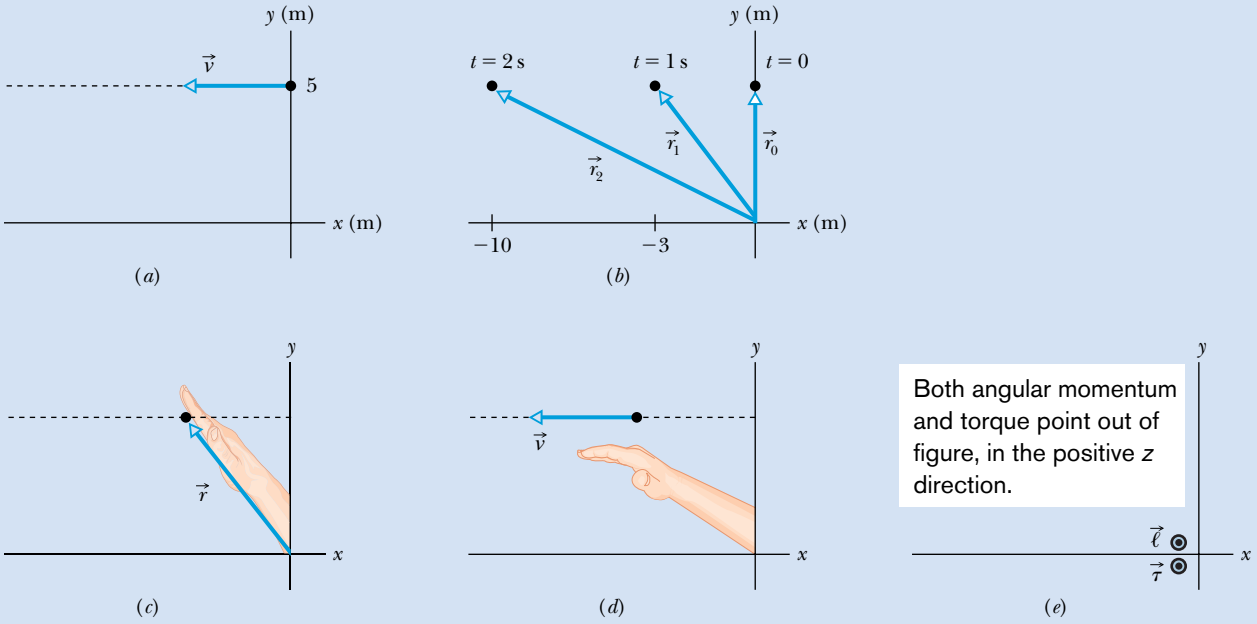

Figure 10-37 (*a*) A particle moving in a straight line, shown at time $t = 0$. (*b*) The position vector at $t = 0$, 1.00 s, and 2.00 s. (*c*) The first step in applying the right-hand rule for cross products. (*d*) The second step. (*e*) The angular momentum vector and the torque vector are along the z axis, which extends out of the plane of the figure.

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one $(-b_x a_y)\hat{k}$. So, let's cut to the (mathematical) chase by writing

$$\vec{r}\times\vec{v} = -(-4.00t - 1.00)(5.00)\hat{k} = (20.0t + 5.00)\hat{k}\ \text{m}^2/\text{s}.$$

Note that, as always, the cross product produces a vector that is perpendicular to the original vectors.

To finish up Eq. 10-87, we multiply by the mass, finding

$$\vec{\ell} = (0.500\ \text{kg})[(20.0t + 5.00)\hat{k}\ \text{m}^2/\text{s}]$$
$$= (10.0t + 2.50)\hat{k}\ \text{kg}\cdot\text{m}^2/\text{s}. \qquad \text{(Answer)}$$

The torque about the origin then immediately follows from Eq. 10-92:

$$\vec{\tau} = \frac{d}{dt}(10.0t + 2.50)\hat{k}\ \text{kg}\cdot\text{m}^2/\text{s}$$
$$= 10.0\hat{k}\ \text{kg}\cdot\text{m}^2/\text{s}^2 = 10.0\hat{k}\ \text{N}\cdot\text{m}, \qquad \text{(Answer)}$$

which is in the positive direction of the z axis.

Our result for $\vec{\ell}$ tells us that the angular momentum is in the positive direction of the z axis. To make sense of that positive result in terms of the rotation of the position vector, let's evaluate that vector for several times:

$$t = 0, \qquad \vec{r}_0 = 5.00\hat{j}\ \text{m}$$
$$t = 1.00\ \text{s}, \qquad \vec{r}_1 = -3.00\hat{i} + 5.00\hat{j}\ \text{m}$$
$$t = 2.00\ \text{s}, \qquad \vec{r}_2 = -10.0\hat{i} + 5.00\hat{j}\ \text{m}$$

By drawing these results as in Fig. 10-37*b*, we see that $\vec{r}$ rotates counterclockwise in order to keep up with the particle. That is the positive direction of rotation. Thus, even though the particle is moving in a straight line, it is still moving counterclockwise around the origin and thus has a positive angular momentum.

We can also make sense of the direction of $\vec{\ell}$ by applying the right-hand rule for cross products (here $\vec{r}\times\vec{v}$, or, if you like, $m\vec{r}\times\vec{v}$, which gives the same direction). For any moment during the particle's motion, the fingers of the right hand are first extended in the direction of the first vector in the cross product $(\vec{r})$ as indicated in Fig. 10-37*c*. The orientation of the hand (on the page or viewing screen) is then adjusted so that the fingers can be comfortably rotated about the palm to be in the direction of the second vector in the cross product $(\vec{v})$ as indicated in Fig. 10-37*d*. The outstretched thumb then points in the direction of the result of the cross product. As indicated in Fig. 10-37*e*, the vector is in the positive direction of the z axis (which is directly out of the plane of the figure), consistent with our previous result. Figure 10-37*e* also indicates the direction of $\vec{\tau}$, which is also in the positive direction of the z axis because the angular momentum is in that direction and is increasing in magnitude.

10.16 | THE ANGULAR MOMENTUM OF A SYSTEM OF PARTICLES

Key Concepts

- The angular momentum $\vec{L}$ of a system of particles is the vector sum of the angular momenta of the individual particles:

$$\vec{L} = \vec{\ell}_1 + \vec{\ell}_2 + \cdots + \vec{\ell}_n = \sum_{i=1}^{n}\vec{\ell}_i.$$

- The time rate of change of this angular momentum is equal to the net external torque on the system (the vector sum of the torques due to interactions of the particles of the system with particles external to the system):

$$\vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt} \qquad \text{(system of particles)}.$$

Now we turn our attention to the angular momentum of a system of particles with respect to an origin. The total angular momentum $\vec{L}$ of the system is the (vector) sum of the angular momenta $\vec{\ell}$ of the individual particles (here with label i):

$$\vec{L} = \vec{\ell}_1 + \vec{\ell}_2 + \vec{\ell}_3 + \cdots + \vec{\ell}_n = \sum_{i=1}^{n}\vec{\ell}_i. \qquad (10\text{-}95)$$

With time, the angular momenta of individual particles may change because of interactions between the particles or with the outside. We can find the resulting change in $\vec{L}$ by taking the time derivative of Eq. 10-95. Thus,

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{n}\frac{d\vec{\ell}_i}{dt}. \qquad (10\text{-}96)$$

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From Eq. 10-87, we see that $d\vec{\ell}_i/dt$ is equal to the net torque $\vec{\tau}_{\text{net},i}$ on the ith particle. We can rewrite Eq. 10-96 as

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{n} \vec{\tau}_{\text{net},i}. \tag{10-97}$$

That is, the rate of change of the system's angular momentum $\vec{L}$ is equal to the vector sum of the torques on its individual particles. Those torques include *internal torques* (due to forces between the particles) and *external torques* (due to forces on the particles from bodies external to the system). However, the forces between the particles always come in third-law force pairs so their torques sum to zero. Thus, the only torques that can change the total angular momentum $\vec{L}$ of the system are the external torques acting on the system.

Net External Torque. Let $\vec{\tau}_{\text{net}}$ represent the net external torque, the vector sum of all external torques on all particles in the system. Then we can write Eq. 10-97 as

$$\vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt} \quad \text{(system of particles)}, \tag{10-98}$$

which is Newton's second law in angular form. It says:

The net external torque $\vec{\tau}_{\text{net}}$ acting on a system of particles is equal to the time rate of change of the system's total angular momentum $\vec{L}$.

Equation 10-98 is analogous to $\vec{F}_{\text{net}} = d\vec{P}/dt$ (Eq. 9-29) but requires extra caution: Torques and the system's angular momentum must be measured relative to the same origin. If the center of mass of the system is not accelerating relative to an inertial frame, that origin can be any point. However, if it *is* accelerating, then it *must* be the origin. For example, consider a wheel as the system of particles. If it is rotating about an axis that is fixed relative to the ground, then the origin for applying Eq. 10-98 can be any point that is stationary relative to the ground. However, if it is rotating about an axis that is accelerating (such as when it rolls down a ramp), then the origin can be only at its center of mass.

10.17 | THE ANGULAR MOMENTUM OF A RIGID BODY ROTATING ABOUT A FIXED AXIS

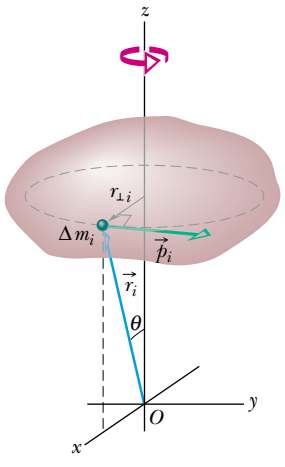
Key Concept

- For a rigid body rotating about a fixed axis, the component of its angular momentum parallel to the rotation axis is

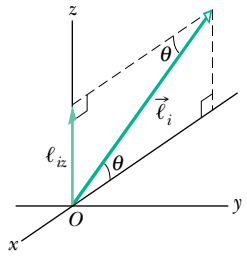
$$L = I\omega \quad \text{(rigid body, fixed axis)}.$$

We next evaluate the angular momentum of a system of particles that form a rigid body that rotates about a fixed axis. Figure 10-38*a* shows such a body. The fixed axis of rotation is a z axis, and the body rotates about it with constant angular speed ω. We wish to find the angular momentum of the body about that axis.

We can find the angular momentum by summing the z components of the angular momenta of the mass elements in the body. In Fig. 10-38*a*, a typical mass element, of mass Δm_i, moves around the z axis in a circular path. The position of the mass element is located relative to the origin O by position vector $\vec{r}_i$. The radius of the mass element's circular path is $r_{\perp i}$, the perpendicular distance between the element and the z axis.



(*a*)



(*b*)

Figure 10-38 (*a*) A rigid body rotates about a z axis with angular speed ω. A mass element of mass Δm_i within the body moves about the z axis in a circle with radius $r_{\perp i}$. The mass element has linear momentum $\vec{p}_i$, and it is located relative to the origin O by position vector $\vec{r}_i$. Here the mass element is shown when $r_{\perp i}$ is parallel to the x axis. (*b*) The angular momentum $\vec{\ell}_i$, with respect to O, of the mass element in (*a*). The z component l_{iz} is also shown.

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The magnitude of the angular momentum $\vec{\ell}_i$ of this mass element, with respect to O, is given by Eq. 10-88:

$$\ell_i = (r_i)(p_i)(\sin 90°) = (r_i)(\Delta m_i\, v_i),$$

where p_i and v_i are the linear momentum and linear speed of the mass element, and 90° is the angle between $\vec{r}_i$ and $\vec{p}_i$. The angular momentum vector $\vec{\ell}_i$ for the mass element in Fig. 10-38*a* is shown in Fig. 10-38*b*; its direction must be perpendicular to those of $\vec{r}_i$ and $\vec{p}_i$.

The z Components. We are interested in the component of $\vec{\ell}_i$ that is parallel to the rotation axis, here the z axis. That z component is

$$\ell_{iz} = \ell_i \sin\theta = (r_i \sin\theta)(\Delta m_i v_i) = r_{\perp i}\, \Delta m_i v_i.$$

The z component of the angular momentum for the rotating rigid body as a whole is found by adding up the contributions of all the mass elements that make up the body. Thus, because $v = \omega r_\perp$, we may write

$$L_z = \sum_{i=1}^{n} \ell_{iz} = \sum_{i=1}^{n} \Delta m_i\, v_i r_{\perp i} = \sum_{i=1}^{n} \Delta m_i\, (\omega r_{\perp i}) r_{\perp i}$$
$$= \omega \left(\sum_{i=1}^{n} \Delta m_i\, r_{\perp i}^2 \right). \qquad (10\text{-}99)$$

We can remove ω from the summation here because it has the same value for all points of the rotating rigid body.

The quantity $\sum \Delta m_i\, r_{\perp i}^2$ in Eq. 10-99 is the rotational inertia I of the body about the fixed axis (see Eq. 10-33). Thus Eq. 10-99 reduces to

$$L = I\omega \quad \text{(rigid body, fixed axis)}. \qquad (10\text{-}100)$$

We have dropped the subscript z, but you must remember that the angular momentum defined by Eq. 10-100 is the angular momentum about the rotation axis. Also, I in that equation is the rotational inertia about that same axis.

Table 10-1, which supplements Table 10-3, extends our list of corresponding linear and angular relations.

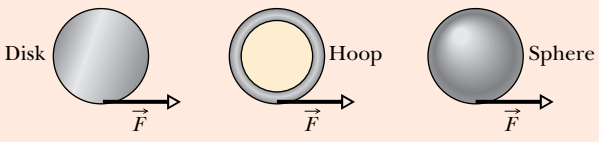
Table 10-4 More Corresponding Variables and Relations for Translational and Rotational Motion[a]

Translational		Rotational	
Force	$\vec{F}$	Torque	$\vec{\tau}(=\vec{r}\times\vec{F})$
Linear momentum	$\vec{p}$	Angular momentum	$\vec{\ell}(=\vec{r}\times\vec{p})$
Linear momentum[b]	$\vec{P}(=\sum\vec{p}_i)$	Angular momentum[b]	$\vec{L}(=\sum\vec{\ell}_i)$
Linear momentum[c]	$\vec{P} = M\vec{v}_{com}$	Angular momentum[c]	$L = I\omega$
Newton's second law[b]	$\vec{F}_{net} = \frac{d\vec{P}}{dt}$	Newton's second law[b]	$\vec{\tau}_{net} = \frac{d\vec{L}}{dt}$
Conservation law[d]	$\vec{P}$ = a constant	Conservation law[d]	$\vec{L}$ = a constant

[a] See also Table 10-3.
[b] For systems of particles, including rigid bodies.
[c] For a rigid body about a fixed axis, with L being the component along that axis.
[d] For a closed, isolated system.

CHECKPOINT 16

In the figure, a disk, a hoop, and a solid sphere are made to spin about fixed central axes (like a top) by means of strings wrapped around them, with the strings producing the same constant tangential force $\vec{F}$ on all three objects. The three objects have the same mass and radius, and they are initially stationary. Rank the objects according to (a) their angular momentum about their central axes and (b) their angular speed, greatest first, when the strings have been pulled for a certain time t.



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SAMPLE PROBLEM 10.21

Angular momentum and torque in Ferris wheel

The original Ferris wheel was built in 1893 by George Washington Gale Ferris, Jr., a civil engineering. The wheel, an amazing engineering construction at the time, carried 36 wooden cars, each holding as many as 60 passengers, around a circle of radius $R = 38$ m. The mass of each car was about 1.1×10^4 kg. The mass of the wheel's structure was about 6.0×10^5 kg, which was mostly in the circular grid from which the cars were suspended. The wheel made a complete rotation at an angular speed ω_F in about 2 min.

(a) Estimate the magnitude L of the angular momentum of the wheel and its passengers while the wheel rotated at ω_F.

KEY IDEA

We can treat the wheel, cars, and passengers as a rigid object rotating about a fixed axis, at the wheel's axle. Then Eq. 10-100 ($L = I\omega$) gives the magnitude of the angular momentum of that object. We need to find ω_F and the rotational inertia I of this object.

Calculations:

Rotational inertia: To find I, let us start with the loaded cars. Because we can treat them as particles, at distance R from the axis of rotation, we know from Eq. 10-23 that their rotational inertia is $I_{pc} = M_{pc}R^2$, where M_{pc} is their total mass. Let us assume that the 36 cars are each filled with 60 passengers, each of mass 70 kg. Then their total mass is

$$M_{pc} = 36[1.1 \times 10^4 \text{ kg} + 60(70 \text{ kg})] = 5.47 \times 10^5 \text{ kg}$$

and their rotational inertia is

$$I_{pc} = M_{pc}R^2 = (5.47 \times 10^5 \text{ kg})(38 \text{ m})^2 = 7.90 \times 10^8 \text{ kg} \cdot \text{m}^2$$

Next, we consider the structure of the wheel. Let us assume that the rotational inertia of the structure is due mainly to the circular grid suspending the cars.

Further, let us assume that the grid forms a hoop of radius R, with a mass M_{hoop} of 3.0×10^5 kg (half the wheel's mass). From Table 10-2*a*, the rotational inertia of the hoop is

$$I_{hoop} = M_{hoop}R^2 = (3.0 \times 10^5 \text{ kg})(38 \text{ m}^2) = 4.33 \times 10^8 \text{ kg} \cdot \text{m}^2$$

The combined rotational inertia I of the cars, passengers, and hoop is then

$$I = I_{pc} + I_{hoop} = 7.90 \times 10^8 \text{ kg} \cdot \text{m}^2 + 4.33 \times 10^8 \text{ kg} \cdot \text{m}^2 = 1.22 \times 10^9 \text{ kg} \cdot \text{m}^2$$

Angular speed: To find the rotational speed ω_F, we use $\omega_{\text{avg}} = \Delta\theta/\Delta t$. Here the wheel goes through an angular displacement of $\Delta\theta = 2\pi$ rad in a time period $\Delta t = 2$ min. Thus, we have

$$\omega_F = \frac{2\pi \text{ rad}}{(2 \text{ min})(60 \text{ s/min})} = 0.0524 \text{ rad/s}$$

Angular momentum: Now we can find the magnitude L of the angular momentum with Eq. 10-100:

$$L = I\omega_F = (1.22 \times 10^9 \text{ kg} \cdot \text{m}^2)(0.0524 \text{ rad/s}) = 6.39 \times 10^7 \text{ kg} \cdot \text{m}^2/\text{s} \approx 6.4 \times 10^7 \text{ kg} \cdot \text{m}^2/\text{s} \quad \text{(Answer)}$$

(b) If the fully loaded wheel is rotated from rest to ω_F in a time period $\Delta t_1 = 5.0$ s, what is the magnitude t_{avg} of the average net external torque acting on it?

Conceptualize/Classify: The average net external torque is related to the change ΔL in the angular momentum of the loaded wheel by Eq. 10-98 ($\vec{\tau}_{\text{net}} = d\vec{L}/dt$).

Compute: Because the wheel rotates about a fixed axis to reach angular speed ω_F in time period Δt_1, we can rewrite Eq. 10-98 as

$$\tau_{\text{avg}} = \Delta L/\Delta t_1$$

The change ΔL is from zero to the answer for part (a). Thus, we have

$$\tau_{\text{avg}} = \frac{\Delta L}{\Delta t_1} = \frac{6.39 \times 10^7 \text{ kg} \cdot \text{m}^2/\text{s} - 0}{5.0 \text{ s}} \approx 1.3 \times 10^7 \text{ N} \cdot \text{m} \quad \text{(Answer)}$$

SAMPLE PROBLEM 10.22

Angular momentum of a conical pendulum

The conical pendulum is in steady circular motion with constant angular velocity ω as shown in Fig. 10-39*a*. Mass of the particle is M and string makes angle α with vertical.

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(a) Find angular momentum of the particle about the center C of the circle and the hinge O about which the string is attached.

Conceptualize/Classify: We start by finding L_C, angular momentum of mass M about point C. We can see from Fig. 10-39*a* that velocity of the particle is always going to be perpendicular to the line joining particle with the center C. Also, both vectors r and v are lying on horizontal (xy) plane, so the angular momentum, L_C will be in the positive z direction. It has magnitude $|r_\perp||p| = |r||p| = rp$, where r is the radius of the circular motion as we can see from Fig. 10-39*a*.

Calculation: The magnitude of momentum is given by

$$|p| = Mv = Mr\omega$$

The angular momentum about the center C of the circle

$$\begin{aligned} L_C &= r \times p \\ &= Mr^2\omega\hat{k} \end{aligned}$$

Now let us evaluate the angular momentum L_O about the point O located at the hinge. From Fig. 10-39*b*,

$$\vec{L}_o = \vec{r} \times \vec{p}$$

Velocity of the particle is perpendicular to the line OA as shown in the figure. Therefore,

$$|L_O| = |r' \times p||r'||p| = l|p|$$

or

$$|L_O| = Mlr\omega$$

where $|r'| = l$, is the length of the string.

Learn: It can be again seen that the magnitude of L depends on the choice of the origin.

(b) Also check whether L is changing or is constant.

Calculation: We must note that L_C is constant, both in magnitude and direction. The magnitude of L_O is constant, but its direction is not constant as can be seen from Fig. 10-40*a*. L_O is always perpendicular to the plane containing r' and p. As the particle moves, the plane containing r' and p vector hinges, thus changing the direction of L_O. Figure 10-40*a* shows L_O at different times. As the bob swings around, L_O rotates as shown in Fig. 10-40*b*. The vertical component of L_O is not moving, but the horizontal component rotates in the circle with the bob.

We can draw the angular momentum at any position we choose, only the magnitude and direction of L have to be same.

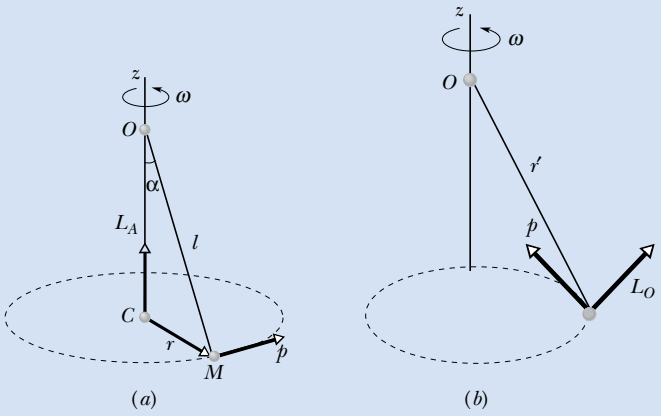

Figure 10-39 A conical pendulum in steady circular motion with a constant angular velocity. (*a*) Angular momentum about C. (*b*) Angular momentum about O.

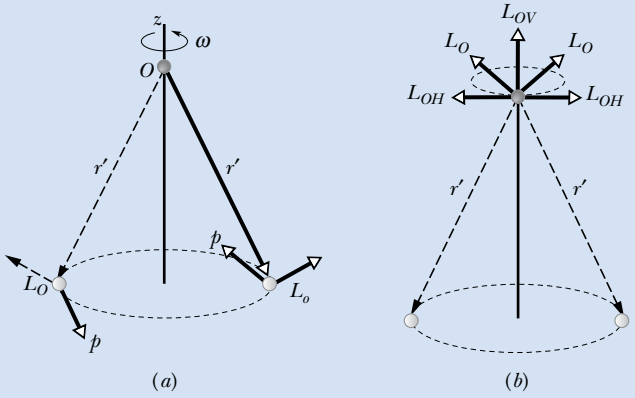

Figure 10-40 (*a*) Angular momentum vector of particle at opposite locations. (*b*) Horizontal and vertical components angular momentum.

SAMPLE PROBLEM 10.23

Acceleration of two masses connected by a string and passing over a pulley

A block of mass M_1 and a block of mass M_2 are connected by a string that passes over a pulley as shown in Fig. 10-41*a*. The pulley is a uniform disk of radius R and mass M. Find the acceleration of the two objects, using the concepts of angular momentum and torque.

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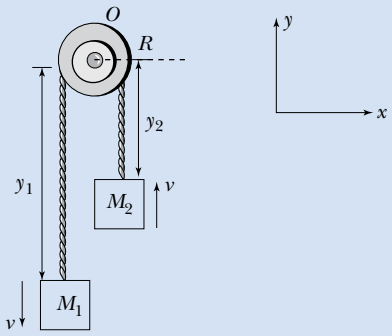

Figure 10-41 (*a*) Two masses M_1 and M_2 connected by a string that passes over a pulley.

KEY IDEAS

(1) If the block M_1 moves upward, the block M_2 moves downward, and the pulley will rotate counterclockwise. This problem is same as Sample Problem 10-15 we have already solved, except that now we have to use angular momentum and torque to find acceleration.

(2) We define the blocks and pulley as our system. We consider the axle of the pulley as our axis. We can write the angular momentum of the system considering the ball and the block that are moving translationally and the pulley which is undergoing pure rotation. The external torque on the system about the axis is due to the weights of the blocks.

Calculations: Due to constraint, the blocks have same speed n. We can find angular momentum using the definition of angular momentum about the axle, given by equation $\vec{L} = \vec{r} \times \vec{p}$.

The momentum and position vectors are given by

$$\vec{P}_1 = -M_1 v\hat{\mathrm{j}} \quad \text{and} \quad r_1 = -R\hat{\mathrm{i}} - y_1\hat{\mathrm{j}}$$

Using $\vec{L} = \vec{r} \times \vec{p}$, the angular momentum of block 1 is $M_1 vR\hat{\mathrm{k}}$.

Similarly, for block 2, momentum and position vectors are given by

$$\vec{P}_2 = +M_2 v\hat{\mathrm{j}} \quad \text{and} \quad r_1 = R\hat{\mathrm{i}} + y_2\hat{\mathrm{j}}$$

Using $\vec{L} = \vec{r} \times \vec{p}$, the angular momentum of block 2 is $M_2 vR\hat{\mathrm{k}}$ about our chosen axis.

The angular momentum of the pulley can be found using $L = I\omega$. It is given by $M_p R^2 \omega\hat{\mathrm{k}}/2$ about our chosen axis.

We calculate the external torque acting on the system about the pulley axle. The force is being exerted by the axle on the pulley, but it does not produce torque about the axle as the length of moment arm is zero. The gravitational force $M_1\vec{g}$ acting on block 1 produces a torque about the axle equal to $M_1 gR\hat{\mathrm{k}}$ and the gravitational force $M_2\vec{g}$ acting on block 2 produces a torque about the axle equal in magnitude to $-M_2 gR\hat{\mathrm{k}}$, where R is the moment arm of the force about the axle. Thus, the total external torque about the pulley axle is

$$\sum \tau_{\text{ext}} = M_1 gR\hat{\mathrm{k}} - M_2 gR\hat{\mathrm{k}}$$

Writing an expression for the total angular momentum of the system,

$$L = (M_1 vR + M_2 vR + MvR)\hat{\mathrm{k}} \qquad (10\text{-}101)$$

Substituting this expression and the total external torque in

$$\sum t_{\text{ext}} = \frac{dL}{dt}$$

we get,

$$M_1 gR\hat{\mathrm{k}} - M_2 gR\hat{\mathrm{k}} = \frac{d}{dt}\left[(M_1 + M_2 + M)vR\right]\hat{\mathrm{k}}$$

$$M_1 gR - M_2 gR = (M_1 + M_2 + M)R\frac{dv}{dt} \qquad (10\text{-}102)$$

Recognizing that $dv/dt = a$ and solving Eq. 10-102 for a, we get

$$a = \frac{(M_1 - M_2)g}{M_1 + M_2 + M} \qquad (10\text{-}103)$$

Learn: The answer is the same as for Sample Problem 10-15. Evaluating the net torque about the axle, we did not include the forces that the string exerts on the objects because these forces are internal to the system. Instead we analyzed the system. Only external torques contribute to the change in the system's angular momentum. While solving the same problem previously as Sample Problem 10-15 we could not have taken pulley and masses as one because the system is valid only for rigid systems. Here we can consider the pulley, ropes and blocks as one system although they are not a single rigid body because this equation is valid even for nonrigid system.

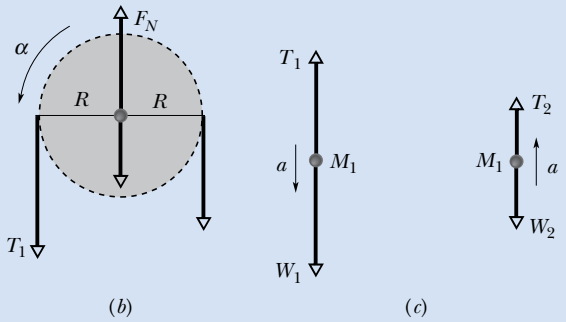

Figure 10-41 (*b*) and (*c*) Free-body diagrams.

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10.18 | CONSERVATION OF ANGULAR MOMENTUM

Key Concept

- The angular momentum $\vec{L}$ of a system remains constant if the net external torque acting on the system is zero:

$$\vec{L} = \text{a constant} \quad \text{(isolated system)}$$

or

$$\vec{L}_i = \vec{L}_f \quad \text{(isolated system).}$$

This is the law of conservation of angular momentum.

So far we have discussed two powerful conservation laws, the conservation of energy and the conservation of linear momentum. Now we meet a third law of this type, involving the conservation of angular momentum. We start from Eq. 10-98 ($\vec{\tau}_{net} = d\vec{L}/dt$), which is Newton's second law in angular form. If no net external torque acts on the system, this equation becomes $d\vec{L}/dt = 0$, or

$$\vec{L} = \text{a constant} \quad \text{(isolated system).} \tag{10-104}$$

This result, called the **law of conservation of angular momentum**, can also be written as

$$\begin{pmatrix}\text{net angular momentum}\\ \text{at some initial time } t_i\end{pmatrix} = \begin{pmatrix}\text{net angular momentum}\\ \text{at some later time } t_f\end{pmatrix},$$

or

$$\vec{L}_i = \vec{L}_f \quad \text{(isolated system).} \tag{10-105}$$

Equations 10-104 and 10-105 tell us:

If the net external torque acting on a system is zero, the angular momentum $\vec{L}$ of the system remains constant, no matter what changes take place within the system.

Equations 10-104 and 10-105 are vector equations; as such, they are equivalent to three component equations corresponding to the conservation of angular momentum in three mutually perpendicular directions. Depending on the torques acting on a system, the angular momentum of the system might be conserved in only one or two directions but not in all directions:

If the component of the net *external* torque on a system along a certain axis is zero, then the component of the angular momentum of the system along that axis cannot change, no matter what changes take place within the system.

This is a powerful statement: In this situation we are concerned with only the initial and final states of the system; we do not need to consider any intermediate state.

We can apply this law to the isolated body in Fig. 10-38, which rotates around the z axis. Suppose that the initially rigid body somehow redistributes its mass relative to that rotation axis, changing its rotational inertia about that axis. Equations 10-104 and 10-105 state that the angular momentum of the body cannot change. Substituting Eq. 10-100 (for the angular momentum along the rotational axis) into Eq. 10-105, we write this conservation law as

$$I_i\omega_i = I_f\omega_f. \tag{10-106}$$

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Here the subscripts refer to the values of the rotational inertia I and angular speed ω before and after the redistribution of mass.

Like the other two conservation laws that we have discussed, Eqs. 10-101 and 10-102 hold beyond the limitations of Newtonian mechanics. They hold for particles whose speeds approach that of light (where the theory of special relativity reigns), and they remain true in the world of subatomic particles (where qantum physics reigns). No exceptions to the law of conservation of angular momentum have ever been found.

We now discuss four examples involving this law.

1. ***The spinning volunteer*** Figure 10-42 shows a student seated on a stool that can rotate freely about a vertical axis. The student, who has been set into rotation at a modest initial angular speed ω_i, holds two dumbbells in his outstretched hands. His angular momentum vector $\vec{L}$ lies along the vertical rotation axis, pointing upward.

   The instructor now asks the student to pull in his arms; this action reduces his rotational inertia from its initial value I_i to a smaller value I_f because he moves mass closer to the rotation axis. His rate of rotation increases markedly, from ω_i to ω_f. The student can then slow down by extending his arms once more, moving the dumbbells outward.

   No net external torque acts on the system consisting of the student, stool, and dumbbells. Thus, the angular momentum of that system about the rotation axis must remain constant, no matter how the student maneuvers the dumbbells. In Fig. 10-42*a*, the student's angular speed ω_i is relatively low and his rotational inertia I_i is relatively high. According to Eq. 10-106, his angular speed in Fig. 10-42*b* must be greater to compensate for the decreased I_f.

2. ***The springboard diver*** Figure 10-43 shows a diver doing a forward one-and-a-half-somersault dive. As you should expect, her center of mass follows a parabolic path. She leaves the springboard with a definite angular momentum $\vec{L}$ about an axis through her center of mass, represented by a vector pointing into the plane of Fig. 10-43, perpendicular to the page. When she is in the air, no net external torque acts on her about her center of mass, so her angular momentum about her center of mass cannot change. By pulling her arms and legs into the closed *tuck position,* she can considerably reduce her rotational inertia about the same axis and thus, according to Eq. 10-106, considerably increase her angular speed. Pulling out of the tuck position (into the *open layout position*) at the

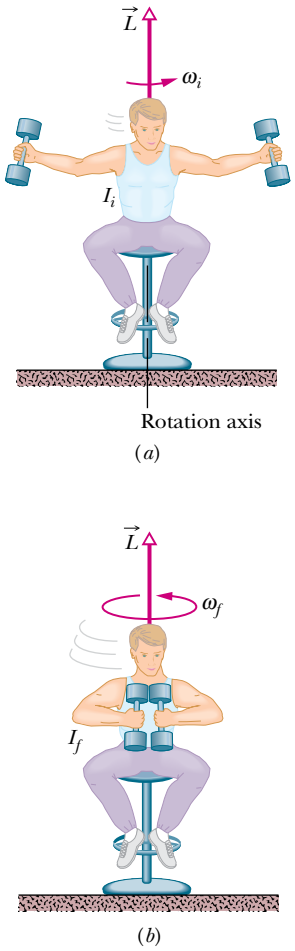

Figure 10-42 (*a*) The student has a relatively large rotational inertia about the rotation axis and a relatively small angular speed. (*b*) By decreasing his rotational inertia, the student automatically increases his angular speed. The angular momentum $\vec{L}$ of the rotating system remains unchanged.

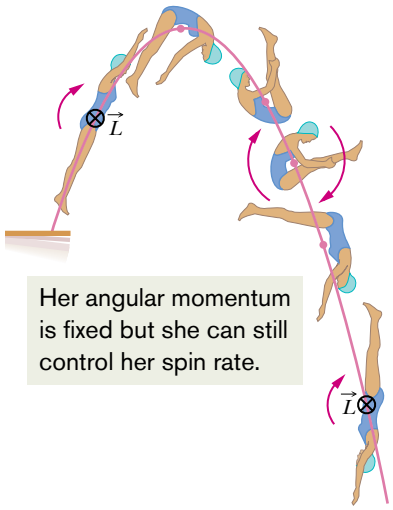

Figure 10-43 The diver's angular momentum $\vec{L}$ is constant throughout the dive, being represented by the tail ⊗ of an arrow that is perpendicular to the plane of the figure. Note also that her center of mass (see the dots) follows a parabolic path.

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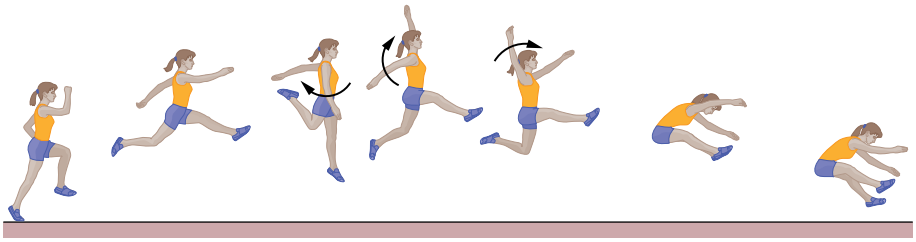

Figure 10-44 Windmill motion of the arms during a long jump helps maintain body orientation for a proper landing.

end of the dive increases her rotational inertia and thus slows her rotation rate so she can enter the water with little splash. Even in a more complicated dive involving both twisting and somersaulting, the angular momentum of the diver must be conserved, in both magnitude *and* direction, throughout the dive.

3. ***Long jump*** When an athlete takes off from the ground in a running long jump, the forces on the launching foot give the athlete an angular momentum with a forward rotation around a horizontal axis. Such rotation would not allow the jumper to land properly: In the landing, the legs should be together and extended forward at an angle so that the heels mark the sand at the greatest distance. Once airborne, the angular momentum cannot change (it is conserved) because no external torque acts to change it. However, the jumper can shift most of the angular momentum to the arms by rotating them in windmill fashion (Fig. 10-44). Then the body remains upright and in the proper orientation for landing.
4. ***Tour jeté*** In a tour jeté, a ballet performer leaps with a small twisting motion on the floor with one foot while holding the other leg perpendicular to the body (Fig. 10-45*a*). The angular speed is so small that it may not be perceptible to the audience. As the performer ascends, the outstretched leg is brought down and the other leg is brought up, with both ending up at angle θ to the body (Fig. 10-45*b*). The motion is graceful, but it also serves to increase the rotation because bringing in the initially outstretched leg decreases the performer's rotational inertia. Since no external torque acts on the airborne performer, the angular momentum cannot change. Thus, with a decrease in rotational inertia, the angular speed must increase. When the jump is well executed, the performer seems to suddenly begin to spin and rotates 180° before the initial leg orientations are reversed in preparation for the landing. Once a leg is again outstretched, the rotation seems to vanish.

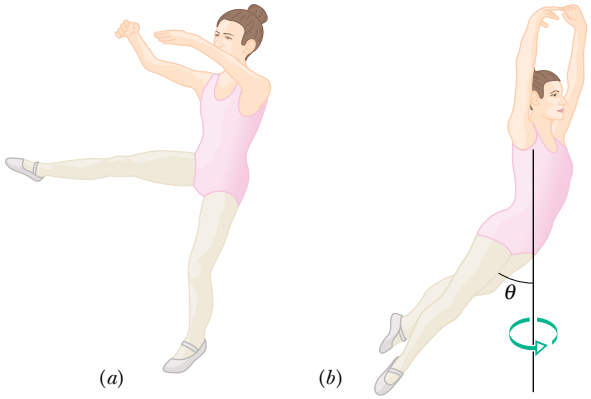

Figure 10-45 (*a*) Initial phase of a tour jeté: large rotational inertia and small angular speed. (*b*) Later phase: smaller rotational inertia and larger angular speed.

CHECKPOINT 17

A rhinoceros beetle rides the rim of a small disk that rotates like a merry-go-round. If the beetle crawls toward the center of the disk, do the following (each relative to the central axis) increase, decrease, or remain the same for the beetle–disk system: (a) rotational inertia, (b) angular momentum, and (c) angular speed?

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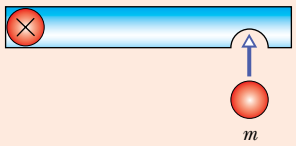
CHECKPOINT 18

A rod of mass 12 m and length l lying on a smooth horizontal plane can rotate freely about a stationary, vertical axis passing through the rod's end. A particle of mass m moving along the plane with speed v_0 hits the rod's end perpendicularly and gets stuck there. Find the angular velocity of the system.



SAMPLE PROBLEM 10.24

Conservation of angular momentum, rotating wheel demo

Figure 10-46*a* shows a student, again sitting on a stool that can rotate freely about a vertical axis. The student, initially at rest, is holding a bicycle wheel whose rim is loaded with lead and whose rotational inertia I_{wh} about its central axis is 1.2 kg·m^2. (The rim contains lead in order to make the value of I_{wh} substantial.)

The wheel is rotating at an angular speed ω_{wh} of 3.9 rev/s; as seen from overhead, the rotation is counterclockwise. The axis of the wheel is vertical, and the angular Momentum $\vec{L}_{wh}$ of the wheel points vertically upward.

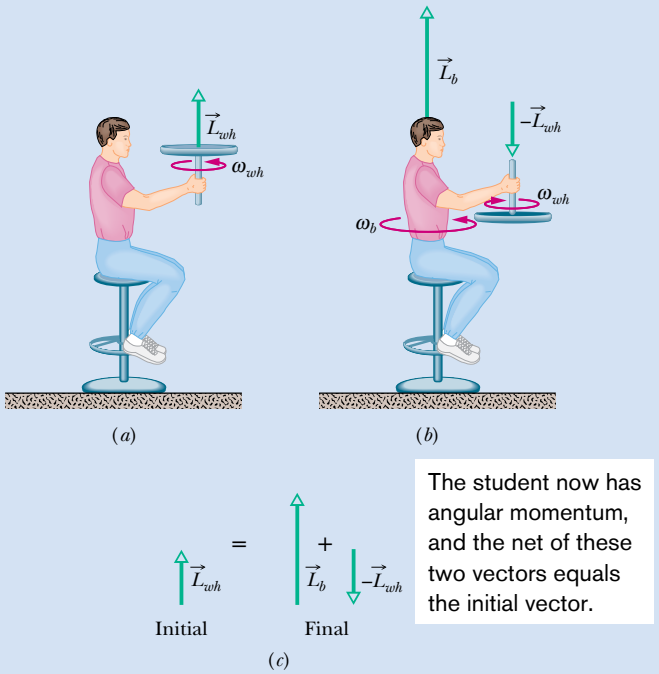


Figure 10-46 (*a*) A student holds a bicycle wheel rotating around a vertical axis. (*b*) The student inverts the wheel, setting himself into rotation. (*c*) The net angular momentum of the system must remain the same in spite of the inversion.

The student now inverts the wheel (Fig. 10-46*b*) so that, as seen from overhead, it is rotating clockwise. Its angular momentum is now $-\vec{L}_{wh}$. The inversion results in the student, the stool, and the wheel's center rotating together as a composite rigid body about the stool's rotation axis, with rotational inertia I_b = 6.8 kg·m^2. (The fact that the wheel is also rotating about its center does not affect the mass distribution of this composite body; thus, I_b has the same value whether or not the wheel rotates.) With what angular speed ω_b and in what direction does the composite body rotate after the inversion of the wheel?

KEY IDEAS

1. The angular speed ω_b we seek is related to the final angular momentum $\vec{L}_b$ of the composite body about the stool's rotation axis by Eq. 10-100 ($L = I\omega$).
2. The initial angular speed ω_{wh} of the wheel is related to the angular momentum $\vec{L}_{wh}$ of the wheel's rotation about its center by the same equation.
3. The vector addition of $\vec{L}_b$ and $\vec{L}_{wh}$ gives the total angular momentum $\vec{L}_{tot}$ of the system of the student, stool, and wheel.
4. As the wheel is inverted, no net *external* torque acts on that system to change $\vec{L}_{tot}$ about any vertical axis. (Torques due to forces between the student and the wheel as the student inverts the wheel are *internal* to the system.) So, the system's total angular momentum is conserved about any vertical axis, including the rotation axis through the stool.

Calculations: The conservation of $\vec{L}_{tot}$ is represented with vectors in Fig. 10-46*c*. We can also write this conservation in terms of components along a vertical axis as

$$L_{b,f} + L_{wh,f} = L_{b,i} + L_{wh,i}, \qquad (10\text{-}107)$$

where i and f refer to the initial state (before inversion of the wheel) and the final state (after inversion). Because

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inversion of the wheel inverted the angular momentum vector of the wheel's rotation, we substitute $-L_{wh,i}$ for $L_{wh,f}$. Then, if we set $L_{b,i} = 0$ (because the student, the stool, and the wheel's center were initially at rest), Eq. 10-104 yields

$$L_{b,f} = 2L_{wh,i}.$$

Using Eq. 10-100, we next substitute $I_b\omega_b$ for $L_{b,f}$ and $I_{wh}\omega_{wh}$ for $L_{wh,i}$ and solve for ω_b, finding

$$\omega_b = \frac{2I_{wh}}{I_b}\omega_{wh}$$
$$= \frac{(2)(1.2\ \text{kg}\cdot\text{m}^2)(3.9\ \text{rev/s})}{6.8\ \text{kg}\cdot\text{m}^2} = 1.4\ \text{rev/s}. \quad \text{(Answer)}$$

This positive result tells us that the student rotates counterclockwise about the stool axis as seen from overhead. If the student wishes to stop rotating, he has only to invert the wheel once more.

SAMPLE PROBLEM 10.25

Conservation of angular momentum, cockroach on disk

In Fig. 10-47, a cockroach with mass m rides on a disk of mass $6.00m$ and radius R. The disk rotates like a merry-go-round around its central axis at angular speed $\omega_i = 1.50$ rad/s. The cockroach is initially at radius $r = 0.800R$, but then it crawls out to the rim of the disk. Treat the cockroach as a particle. What then is the angular speed?

KEY IDEAS

(1) The cockroach's crawl changes the mass distribution (and thus the rotational inertia) of the cockroach–disk system. (2) The angular momentum of the system does not change because there is no external torque to change it. (The forces and torques due to the cockroach's crawl are internal to the system.) (3) The magnitude of the angular momentum of a rigid body or a particle is given by Eq. 10-100 ($L = I\omega$).

Calculations: We want to find the final angular speed. Our key is to equate the final angular momentum L_f to the initial angular momentum L_i, because both involve angular speed. They also involve rotational inertia I. So, let's start by finding the rotational inertia of the system of cockroach and disk before and after the crawl.

The rotational inertia of a disk rotating about its central axis is given by Table 10-2*c* as $1/2MR^2$. Substituting $6.00m$ for the mass M, our disk here has rotational inertia

$$I_d = 3.00mR^2 \qquad (10\text{-}108)$$

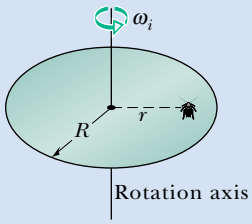

Figure 10-47 A cockroach rides at radius r on a disk rotating like a merry-go-round.

(We do not have values for m and R, but we shall continue with physics courage.)

From Eq. 10-33, we know that the rotational inertia of the cockroach (a particle) is equal to mr^2. Substituting the cockroach's initial radius ($r = 0.800R$) and final radius ($r = R$), we find that its initial rotational inertia about the rotation axis is

$$I_{ci} = 0.64mR^2 \qquad (10\text{-}109)$$

and its final rotational inertia about the rotation axis is

$$I_{cf} = mR^2. \qquad (10\text{-}110)$$

So, the cockroach–disk system initially has the rotational inertia

$$I_i = I_d + I_{ci} = 3.64mR^2, \qquad (10\text{-}111)$$

and finally has the rotational inertia

$$I_f = I_d + I_{cf} = 4.00mR^2. \qquad (10\text{-}112)$$

Next, we use Eq. 10-100 ($L = I\omega$) to write the fact that the system's final angular momentum L_f is equal to the system's initial angular momentum L_i:

$$I_f\omega_f = I_i\omega_i$$

or $$4.00mR^2\omega_f = 3.64mR^2(1.50\ \text{rad/s}).$$

After canceling the unknowns m and R, we come to

$$\omega_f = 1.37\ \text{rad/s}. \quad \text{(Answer)}$$

Note that ω decreased because part of the mass moved outward, thus increasing that system's rotational inertia.

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SAMPLE PROBLEM 10.26

Angular momentum of vertical cylinder fixed on a vertical plane

On a horizontal smooth plane there is a fixed vertical cylinder and a particle A connected to the cylinder by a thread AB (Fig. 10-48) (top view). The particle is set in motion with the initial velocity v as shown in the figure. Is there any fixed point relative to which the angular momentum of the particle is constant in the process of motion?

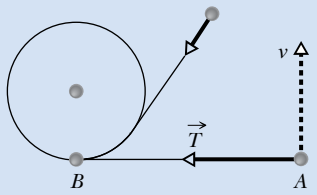

Figure 10-48 Fixed vertical cylinder on a smooth plane.

Reasoning: The only force acting on the particle A in this case is the tension $\vec{T}$ of the thread. It is easy to see that there is no point here relative to which the torque of the tension is always zero. Therefore, there is no point relative to which the angular momentum of the particle would be constant. The tension acting on the particle is along the string. Thus, any point lying on the line of string, will have zero torque due to tension, about itself. Since particle is moving, this line keeps on changing.

This example illustrates that sometimes a point relative to which the angular momentum of a particle is constant cannot be found at all.

SAMPLE PROBLEM 10.27

Rotational kinetic energy of two disks rotating about a vertical axis

Disk 1 (Fig. 10-49) rotates about a smooth vertical axis with the angular velocity ω_1. Disk 2 rotating with the angular velocity ω_2 is kept on disk 1 while rotating in the same direction. Find the change in the rotational kinetic energy of the system given that the moments of inertia of the disks relative to the rotation axis are equal to I_1 and I_2, respectively. The contact surfaces of the disks are rough.

KEY IDEA

(1) It is easy to imagine that finally disks will start rotating with same angular velocity. Friction between the two disks will slow down the faster disk and speed up the slower disk. (2) It is easy to understand that we can conserve angular momentum of the assembly about the axis as friction between the two disks will be providing internal torque.

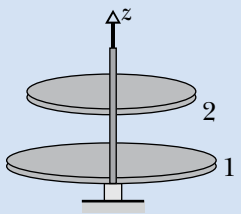

Figure 10-49 Two disks rotating about vertical axis.

Calculation: Let us say the final angular velocity of rotation of the assembly is ω. From the law of conservation of the angular momentum relative to the axis, we have

$$I_1\omega_1 + I_2\omega_2 = (I_1 + I_2)\omega$$

Thus,
$$\omega = \frac{(I_1\omega_1 + I_2\omega_2)}{(I_1 + I_2)} \qquad (10\text{-}113)$$

Note that ω_1, ω_2 and ω are algebraic quantities. If $\omega > 0$, the corresponding vector coincides with the positive direction of the z axis, and vice versa.

Initial kinetic energy $K_i = I_1\omega_1^2/2 + I_2\omega_2^2/2$

Final kinetic energy $K_2 = (I_1 + I_2)\omega_2/2$

$$\Delta K = K_f - K_i$$
$$= (I_1 + I_2)\omega^2/2 - (I_1\omega_1^2/2 + I_2\omega_2^2/2)$$
$$\Delta K = -\frac{I_1 + I_2}{2(I_1 + I_2)}(\omega_1 - \omega_2)^2 \qquad (10\text{-}114)$$

Learn: The minus sign shows that the kinetic energy of the system decreases. Note that the result in Eqs. 10-113 and 10-114 are quite similar in form and meaning to the case of inelastic collision.

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SAMPLE PROBLEM 10.28

Velocity of a ring tied to a freely rotating rod near its rotation axis

A smooth rod of length l rotates freely in a horizontal plane with the angular velocity about a stationary vertical axis O as shown in Fig. 10-50a. The moment of inertia of the rod is equal to I relative to the axis. A small ring of mass m is located on the rod close to the rotation axis and is tied to it by a thread. When the thread is burned, the ring starts sliding radially outwards along the rod. Find the velocity v of the ring relative to the rod as a function of its distance r from the rotation axis.

(a)

Figure 10-50 (a) Smooth rod rotating freely.

KEY IDEA

Consider the rod and the ring as our system. The system is experiencing no external torque about the axis hence, the angular momentum relative to the rotation axis does not vary. There is no nonconservative force or impulse acting so in the process of motion, the kinetic energy of the system does not vary.

Calculation: Since the angular momentum about the axis and kinetic energy of the system are conserved, we have

$$I\omega_0 = (I + mr^2)\omega$$

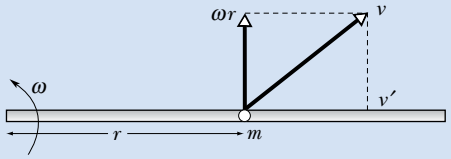
and

$$\frac{1}{2}I\omega_0^2 = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$$

where $v^2 = v'^2 + \omega^2 r^2$ as shown in Fig. 10-50b. From the above two equations we obtain

$$v = \omega_0 r/\sqrt{1 + mr^2/I}$$

Note: If $r = 0$, we get $v = 0$ which is the situation in the beginning.



(b)

Figure 10-50 (b) Ring at distance r moving with angular velocity ω.

SAMPLE PROBLEM 10.29

Collision of uniform rod hinged about one end with a particle moving with velocity v_0

A uniform rod of mass m and length l is hinged about one of its end. The hinge is smooth and rod lies on a smooth horizontal surface as shown in Fig. 10-51. A particle of mass m is coming towards the rod at speed v_0 perpendicular to the length of rod.

(a) Find angular velocity of the rod just after impact.

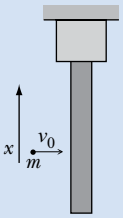


Figure 10-51 (a) Uniform rod hinged about one end and mass m moving with speed v_0.

KEY IDEAS

(1) When the particle will hit the rod, the hinge will restrict the motion of the end of the rod. We can imagine the rod starting to rotate about hinge and particle continuing to move on same line of motion. The impulse that particle will experience by rod is along the line of motion.

(2) We will consider rod and particle as our system. This system will experience external impulse due to hinge. So, the momentum of the system cannot be conserved. But the angular momentum of the system about hinge will remain conserved as torque due to impulse at hinge is zero.

Calculation: By conserving angular momentum about hinge

$$mv_0 x = mvx + \frac{ml^2\omega}{3} \qquad (10\text{-}115)$$

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Here v is the speed of particle after the collision and ω is the angular velocity of rod after the collision as shown in Fig. 10-50*b*. We can also write Newton's law of collision remembering that it is the velocity of contact points that is required. Thus,

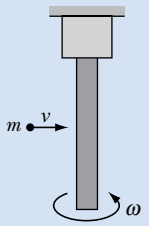

Figure 10-51 (*b*) Motion of mass and rod just after impact.

$$\frac{\omega x - v}{0 - v_0} = -e \qquad (10\text{-}116)$$

Solving Eqs. 10-115 and 10-116, we get

$$\omega = \frac{3v_0 x(1+e)}{(l^2 + 3x^2)}$$

(b) Also find impulse due to hinge on rod during collision.

KEY IDEAS

(1) Coefficient of restitution is e and perpendicular distance of line of motion from hinge is x. Consider $e = 1/3$, $v_0 = 3$ m/s, $l = 1$ m, $m = 2$ kg and $x = 1/3$ m.

(2) Now to find impulse J_1, due to the hinge, we will consider rod and particle as our system and write impulse and momentum equations.

Calculation: We should remember that impulse between rod and particle during the collision need not be considered since it will be in internal impulse, thus we can write

$$\vec{J}_1 = (\vec{P}_{\text{rod}\,f} + \vec{P}_{\text{particle}\,f}) - (\vec{P}_{\text{rod}\,i} + \vec{P}_{\text{particle}\,i})$$

$$\vec{P}_{\text{particle}\,i} = 6 \text{ Nm} \qquad \vec{P}_{\text{rod}\,i} = 0$$

For $\vec{P}_{\text{particle},\,f}$ we need to find speed of particle from part (a).

$$\omega = \frac{3 \times 3 \times (1/3)(1 + 1/3)}{[1^2 + 3(1/3)^2]}$$

Now solving for v by putting value of ω, we get

$$\frac{3 \times (1/3) - v}{0 - (3)} = -\frac{1}{3} \text{m/s}$$

Solving for $v = 0$, we get $\vec{P}_{\text{particle},\,f} = 0$.

For $\vec{P}_{\text{rod},\,f}$ we need velocity of center of mass of rod as $\vec{P}_{\text{rod},\,f} = M\vec{V}_{\text{com}}$.

Thus, $$V_{\text{com}} = \omega \times r_c = \omega \times \frac{l}{2} = \frac{3}{2} \text{m/s}$$

and $$P_{\text{rod}\,f} = 2 \times \frac{3}{2} = 3 \text{ kg m/s}$$

Hence, $J_1 = (3 + 0) - (0 + 6) = -3$ kg m/s

Learn: The negative sign on impulse means that impulse on the rod by the hinge is toward left. This means the end attached to the axis must have tried to move toward right and hinge opposed the motion by applying impulse toward left.

In Case of: What will be the sign of J_1 if particle hits the rod close to the other end?

Compute: The sign will be positive as the end attached to the hinge will have tendency to go left and hence the hinge will apply impulse toward right.

SAMPLE PROBLEM 10.30

Cylinder pivoted about its axis, rotational parameters

A horizontal, homogeneous cylinder of mass M and radius R is pivoted about its axis of symmetry. A string is wrapped several times around the cylinder and tied to a body of mass m resting on a support positioned so that the string has no slack (Fig. 10-52*a*). The body of mass m is then lifted vertically to a distance h, and then released.

(a) Evaluate the angular velocity ω_0 of the cylinder, the speed v_0 of the falling body of mass m, and the kinetic energy K_0 of the system, just before the string becomes taut.

KEY IDEA

Since string is slack (Fig 10-52*b*) we can expect that the block will fall freely for distance l. Till this time tension in string will be zero.

Calculation: Just before the string becomes taut, the cylinder is still at rest. Since the tension remained zero nothing has caused the cylinder to begin turning. That is, $\omega_0 = 0$. Since the string has exerted no force on the body of mass m, it has fallen freely with acceleration g thus $v_0 = 2gh$. Same result can be derived by writing

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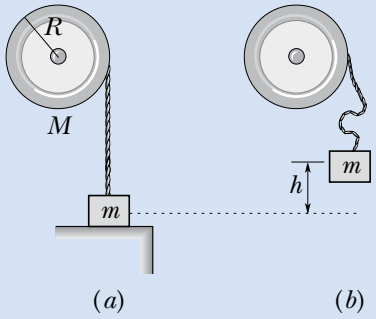

Figure 10-52 Cylinder pivoted about its axis.

work-kinetic energy theorem for the system. The kinetic energy of the system is given by

$$K_0 = \frac{1}{2}mv_0^2 + \frac{1}{2}I_C\omega_0^2 = mgh$$

where I_c is the inertia of the cylinder.

Putting $\omega_0 = 0$, we get

$$v_0 = \sqrt{2gh}$$

(b) Evaluate the corresponding quantities, ω_1, v_1, and K_1, for the instant just after the string becomes taut.

KEY IDEA

(1) The end attached to the block will be moving and it will pull the part of string touching the cylinder as the string becomes taut. (2) Thus, when string becomes tight an impulse/jerk will develop in the string. This jerk will start the motion of cylinder. There will be also an impulse due to axis of the cylinder. Here we can take cylinder, string and block as our system. We will take axle of cylinder as the axis for writing angular momentum.

Calculations: The impulse due to jerk in string will produce internal angular impulse, and the impulse due to the angular momentum will produce zero angular impulse about the axle. Further, the angular momentum of the system must be conserved, since the impulse due to the snap of the string is of very short duration and the weight mg contributes negligibly in that time. The string is assumed to be inextensible, so $v_1 = \omega_1 R$.

Thus, we have $L_1 = mv_1R + (1/2)MR^2\omega_1 = L_0$, where $L_0 = mv_0R = m\sqrt{2gh}R$. Solving for the angular speed, we get

$$\omega_1 = \frac{v_0}{R[1+((M/2)m)} = \frac{\sqrt{2gh}}{R[1+((M/2)m)]}$$

The corresponding speed is given by

$$v_1 = \omega_1 R = \frac{v_0}{1+((M/2)m)} = \frac{\sqrt{2gh}}{[1+((M/2)m)]}$$

The final kinetic energy K_1 is given by

$$K_1 = \frac{1}{2}mv_1^2 + \frac{1}{2}I_c\omega_1^2 = mgh$$

$$= \frac{1}{2}mv_1^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)$$

$$\left(\frac{v_1^2}{R^2}\right) = \frac{1}{2}\left(m + \frac{M}{2}\right)v_1^2$$

$$= \frac{1}{2}\left(\frac{mv_0^2}{1+((M/2)m)}\right)$$

Learn: The final kinetic energy is less than initial because whenever impulses occur in a system, energy does not remain conserved.

REVIEW AND SUMMARY

Angular Position To describe the rotation of a rigid body about a fixed axis, called the **rotation axis**, we assume a **reference line** is fixed in the body, perpendicular to that axis and rotating with the body. We measure the **angular position** θ of this line relative to a fixed direction. When θ is measured in **radians**,

$$\theta = \frac{s}{r} \quad \text{(radian measure)}, \tag{10-1}$$

where s is the arc length of a circular path of radius r and angle θ. Radian measure is related to angle measure in revolutions and degrees by

$$1 \text{ rev} = 360^\circ = 2\pi \text{ rad}. \tag{10-2}$$

Angular Displacement A body that rotates about a rotation axis, changing its angular position from θ_1 to θ_2, undergoes an **angular displacement**

$$\Delta\theta = \theta_2 = \theta_1, \tag{10-4}$$

where $\Delta\theta$ is positive for counterclockwise rotation and negative for clockwise rotation.

Angular Velocity and Speed If a body rotates through an angular displacement $\Delta\theta$ in a time interval Δt, its **average angular velocity** ω_{avg} is

$$\omega_{avg} = \frac{\Delta\theta}{\Delta t}. \tag{10-5}$$

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The **(instantaneous) angular velocity** ω of the body is

$$\omega = \frac{d\theta}{dt}. \tag{10-6}$$

Both ω_{avg} and ω are vectors, with directions given by the **right-hand rule** of Fig. 10-6. They are positive for counterclockwise rotation and negative for clockwise rotation. The magnitude of the body's angular velocity is the **angular speed**.

Angular Acceleration If the angular velocity of a body changes from ω_1 to ω_2 in a time interval $\Delta t = t_2 - t_1$, the **average angular acceleration** α_{avg} of the body is

$$\alpha_{avg} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}. \tag{10-7}$$

The **(instantaneous) angular acceleration** α of the body is

$$\alpha = \frac{d\omega}{dt}. \tag{10-8}$$

Both α_{avg} and α are vectors.

The Kinematic Equations for Constant Angular Acceleration *Constant angular acceleration* (α = constant) is an important special case of rotational motion. The appropriate kinematic equations, given in Table 10-1, are

$$\omega = \omega_0 + \alpha t, \tag{10-12}$$

$$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2, \tag{10-13}$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0), \tag{10-14}$$

$$\theta - \theta_0 = \frac{1}{2}(\omega_0 + \omega)t, \tag{10-15}$$

$$\theta - \theta_0 = \omega t - \frac{1}{2}\alpha t^2. \tag{10-16}$$

Linear and Angular Variables Related A point in a rigid rotating body, at a *perpendicular distance r* from the rotation axis, moves in a circle with radius r. If the body rotates through an angle θ, the point moves along an arc with length s given by

$$s = \theta r \quad \text{(radian measure)}, \tag{10-17}$$

where θ is in radians.

The linear velocity $\vec{v}$ of the point is tangent to the circle; the point's linear speed v is given by

$$v = \omega r \quad \text{(radian measure)}, \tag{10-18}$$

where ω is the angular speed (in radians per second) of the body.

The linear acceleration $\vec{a}$ of the point has both *tangential* and *radial* components. The tangential component is

$$a_t = \alpha r \quad \text{(radian measure)}, \tag{10-22}$$

where α is the magnitude of the angular acceleration (in radians per second-squared) of the body. The radial component of $\vec{a}$ is

$$a_r = \frac{v^2}{r} = \omega^2 r \quad \text{(radian measure)}. \tag{10-23}$$

If the point moves in uniform circular motion, the period T of the motion for the point and the body is

$$T = \frac{2\pi r}{v} = \frac{2\pi}{\omega} \quad \text{(radian measure)}. \tag{10-19, 10-20}$$

Rotational Kinetic Energy and Rotational Inertia The kinetic energy K of a rigid body rotating about a fixed axis is given by

$$K = \frac{1}{2}I\omega^2 \quad \text{(radian measure)}, \tag{10-34}$$

in which I is the **rotational inertia** of the body, defined as

$$I = \sum m_i r_i^2 \tag{10-33}$$

for a system of discrete particles and defined as

$$I = \int r^2\, dm \tag{10-35}$$

for a body with continuously distributed mass. The r and r_i in these expressions represent the perpendicular distance from the axis of rotation to each mass element in the body, and the integration is carried out over the entire body so as to include every mass element.

The Parallel-Axis Theorem The *parallel-axis theorem* relates the rotational inertia I of a body about any axis to that of the same body about a parallel axis through the center of mass:

$$I = I_{com} + Mh^2. \tag{10-36}$$

Here h is the perpendicular distance between the two axes, and I_{com} is the rotational inertia of the body about the axis through the com. We can describe h as being the distance the actual rotation axis has been shifted from the rotation axis through the com.

The Perpendicular-Axis Theorem The contribution of m_i to the moment of inertia I_z with respect to the z axis is $m_i r_i^2$, that is, the mass of the particle times the square of its distance from the axis of rotation. If we consider rotation about the x axis, the mass element mi would contribute $m_i y_i^2$ to the moment of inertia with respect to the x axis; similarly it would contribute $m_i x_i^2$ with respect to the y axis. Thus,

$$I_x + I_y = I_z \tag{10-41}$$

Torque *Torque* is a turning or twisting action on a body about a rotation axis due to a force $\vec{F}$. If $\vec{F}$ is exerted at a point given by the position vector $\vec{r}$ relative to the axis, then the magnitude of the torque is

$$\tau = rF_t = r_\perp F = rF\sin\phi, \tag{10-44, 10-45, 10-43}$$

where F_t is the component of $\vec{F}$ perpendicular to $\vec{r}$ and ϕ is the angle between $\vec{r}$ and $\vec{F}$. The quantity $r_\perp$ is the perpendicular distance between the rotation axis and an extended line running through the $\vec{F}$ vector. This line is called the **line of action** of $\vec{F}$, and $r_\perp$ is called the **moment arm** of $\vec{F}$. Similarly, r is the moment arm of F_t.

The SI unit of torque is the newton-meter (N · m). A torque τ is positive if it tends to rotate a body at rest counterclockwise and negative if it tends to rotate the body clockwise.

Torque as a Vector In three dimensions, *torque* $\vec{\tau}$ is a vector quantity defined relative to a fixed point (usually an origin); it is

$$\vec{\tau} = \vec{r} \times \vec{F}, \tag{10-50}$$

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where $\vec{F}$ is a force applied to a particle and $\vec{r}$ is a position vector locating the particle relative to the fixed point. The magnitude of $\vec{\tau}$ is

$$\tau = rF \sin\phi = rF_\perp = r_\perp F, \quad (10\text{-}51, 10\text{-}52, 10\text{-}53)$$

where ϕ is the angle between $\vec{F}$ and $\vec{r}$, $F_\perp$ is the component of $\vec{F}$ perpendicular to $\vec{r}$, and $r_\perp$ is the moment arm of $\vec{F}$. The direction of $\vec{\tau}$ is given by the right-hand rule.

The Center of Gravity The gravitational force $\vec{F}_g$ on a body effectively acts at a single point, called the **center of gravity** (cog) of the body.

If $\vec{g}$ is the same for all elements of a body, then the body's center of gravity (cog) is coincident with the body's center of mass (com).

The Rigid Object under a Net Torque The rotational analog of Newton's second law is

$$\tau_{net} = I\alpha, \quad (10\text{-}62)$$

where τ_{net} is the net torque acting on a particle or rigid body, I is the rotational inertia of the particle or body about the rotation axis, and α is the resulting angular acceleration about that axis.

Work and Rotational Kinetic Energy The equations used for calculating work and power in rotational motion correspond to equations used for translational motion and are

$$W = \int_{\theta_i}^{\theta_f} \tau\, d\theta \quad (10\text{-}73)$$

and

$$P = \frac{dW}{dt} = \tau\omega. \quad (10\text{-}75)$$

When τ is constant, Eq. 10-74 reduces to

$$W = \tau(\theta_f - \theta_i). \quad (10\text{-}74)$$

The form of the work–kinetic energy theorem used for rotating bodies is

$$\Delta K = K_f - K_i = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = W. \quad (10\text{-}77)$$

Angular Momentum of a Particle The *angular momentum* $\vec{\ell}$ of a particle with linear momentum $\vec{p}$, mass m, and linear velocity $\vec{v}$ is a vector quantity defined relative to a fixed point (usually an origin) as

$$\vec{\ell} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v}). \quad (10\text{-}87)$$

The magnitude of $\vec{\ell}$ is given by

$$\ell = rmv \sin\phi \quad (10\text{-}88)$$

$$= rp_\perp = rmv_\perp \quad (10\text{-}89)$$

$$= r_\perp p = r_\perp mv, \quad (10\text{-}90)$$

where ϕ is the angle between $\vec{r}$ and $\vec{p}$, $p_\perp$ and $v_\perp$ are the components of $\vec{p}$ and $\vec{v}$ perpendicular to $\vec{r}$, and $r_\perp$ is the perpendicular distance between the fixed point and the extension of $\vec{p}$. The direction of $\vec{\ell}$ is given by the right-hand rule for cross products.

Newton's Second Law in Angular Form Newton's second law for a particle can be written in angular form as

$$\vec{\tau}_{net} = \frac{d\vec{\ell}}{dt}, \quad (10\text{-}92)$$

where $\vec{\tau}_{net}$ is the net torque acting on the particle and $\vec{\ell}$ is the angular momentum of the particle.

Angular Momentum of a System of Particles The angular momentum $\vec{L}$ of a system of particles is the vector sum of the angular momenta of the individual particles:

$$\vec{L} = \vec{\ell}_1 + \vec{\ell}_2 + \cdots + \vec{\ell}_n = \sum_{i=1}^{n} \vec{\ell}_i. \quad (10\text{-}95)$$

The time rate of change of this angular momentum is equal to the net external torque on the system (the vector sum of the torques due to interactions with particles external to the system):

$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt} \quad \text{(system of particles)}. \quad (10\text{-}98)$$

Angular Momentum of a Rigid Body For a rigid body rotating about a fixed axis, the component of its angular momentum parallel to the rotation axis is

$$L = I\omega \quad \text{(rigid body, fixed axis)}. \quad (10\text{-}100)$$

Conservation of Angular Momentum The angular momentum $\vec{L}$ of a system remains constant if the net external torque acting on the system is zero:

$$\vec{L} = \text{a constant} \quad \text{(isolated system)} \quad (10\text{-}104)$$

or

$$\vec{L}_i = \vec{L}_f \quad \text{(isolated system)}. \quad (10\text{-}105)$$

This is the **law of conservation of angular momentum**.

PROBLEMS

1. A tractor has rear wheels with a radius of 1.00 m and front wheels with a radius of 0.250 m. The rear wheels are rotating at 100 rev/min. Find (a) the angular speed of the front wheels in revolutions per minute and (b) the distance covered by the tractor in 10.0 min.
2. Just as a helicopter is landing, its blades are turning at 30.0 rev/s and slowing at a constant rate. In the 2.00 min required for them to stop, how many revolutions do they make?
3. When a slice of buttered toast is accidentally pushed over the edge of a counter, it rotates as it falls. If the distance to the floor is 76 cm and for rotation less than 1 rev, what are the (a) smallest and (b) largest angular speeds that cause the toast to hit and then topple to be butter-side down?
4. The angular position of a point on a rotating wheel is given by $\theta = 2.0 + 4.0t^2 + 2.0t^3$, where θ is in radians and t is in seconds. At $t = 0$, what are (a) the point's angular position and (b) its angular velocity? (c) What is its angular

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velocity at $t = 3.0$ s? (d) Calculate its angular acceleration at $t = 4.0$ s. (e) Is its angular acceleration constant?

5. At time $t = 0$, a rotating bicycle wheel is thrown horizontally from a rooftop with a speed of 49 m/s. By the time its vertical speed is also 49 m/s, it has completed 40 revolutions. What has been its average angular speed to that point in the fall?

6. The wheel in Fig. 10-53 has eight equally spaced spokes and a radius of 30 cm. It is mounted on a fixed axle and is spinning at 2.5 rev/s. You want to shoot a 20 cm long arrow parallel to this axle and through the wheel without hitting any of the spokes. Assume that the arrow and the spokes are very thin. (a) What minimum speed must the arrow have? (b) Does it matter where between the axle and rim of the wheel you aim? If so, what is the best location?

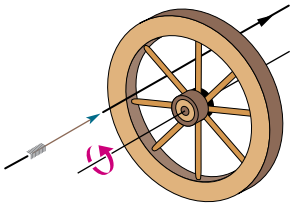

Figure 10-53 Problem 6.

7. The angular acceleration of a wheel is $\alpha = 6.0t^4 - 4.0t^2$, with α in radians per second-squared and t in seconds. At time $t = 0$, the wheel has an angular velocity of +2.5 rad/s and an angular position of +1.5 rad. Write expressions for (a) the angular velocity (rad/s) and (b) the angular position (rad) as functions of time (s).

8. In 5.00 s, a 2.00 kg stone moves in a horizontal circle of radius 2.00 m from rest to an angular speed of 4.00 rev/s. What are the stone's (a) average angular acceleration and (b) rotational inertia around the circle's center?

9. Starting from rest, a disk rotates about its central axis with constant angular acceleration. In 5.0 s, it rotates 20 rad. During that time, what are the magnitudes of (a) the angular acceleration and (b) the average angular velocity? (c) What is the instantaneous angular velocity of the disk at the end of the 5.0 s? (d) With the angular acceleration unchanged, through what additional angle will the disk turn during the next 5.0 s?

10. Two identical disks A and B can spin around vertical axes. Disk A is spinning with an initial angular speed of 40 rev/s when its rim touches initially stationary disk B, causing that disk to begin to spin. The rubbing at the contact point slows A while speeding up B. The rate at which both disks change their angular speeds is 2.0 rev/s^2. Find the time required for the two disks to reach the same angular speed.

11. The angular speed of an automobile engine is increased at a constant rate from 1200 rev/min to 3200 rev/min in 12 s. (a) What is its angular acceleration in revolutions per minute-squared? (b) How many revolutions does the engine make during this 12 s interval?

12. A flywheel turns through 40 rev as it slows from an angular speed of 1.5 rad/s to a stop. (a) Assuming a constant angular acceleration, find the time for it to come to rest. (b) What is its angular acceleration? (c) How much time is required for it to complete the first 20 of the 40 revolutions?

13. A disk rotates about its central axis starting from rest and accelerates with constant angular acceleration. At one time it is rotating at 10 rev/s; 60 revolutions later, its angular speed is 15 rev/s. Calculate (a) the angular acceleration, (b) the time required to complete the 60 revolutions, (c) the time required to reach the 10 rev/s angular speed, and (d) the number of revolutions from rest until the time the disk reaches the 10 rev/s angular speed.

14. Starting from rest, a wheel has constant $\alpha = 3.0$ rad/s^2. During a certain 4.0 s interval, it turns through 120 rad. How much time did it take to reach that 4.0 s interval?

15. At $t = 0$, a flywheel has an angular velocity of 4.7 rad/s, a constant angular acceleration of –0.25 rad/s^2, and a reference line at $\theta_0 = 0$. (a) Through what maximum angle θ_{max} will the reference line turn in the positive direction? What are the (b) first and (c) second times the reference line will be at $\theta = 1/2\,\theta_{max}$? At what (d) negative time and (e) positive time will the reference line be at $\theta = 10.5$ rad? (f) Graph θ versus t, and indicate your answers.

16. An open jar of water moves in a vertical circle of radius 0.50 m with a frequency that is small enough to put the water on the verge of falling out of the jar at the top of the circle. If the same demonstration were repeated on Mars, where the gravitational acceleration is only 3.7 m/s^2, what is the change in the circling frequency to again put the water on the verge of falling out at the top point?

17. An object rotates about a fixed axis, and the angular position of a reference line on the object is given by $\theta = 0.40e^{2t}$, where θ is in radians and t is in seconds. Consider a point on the object that is 6.0 cm from the axis of rotation. At $t = 0$, what are the magnitudes of the point's (a) tangential component of acceleration and (b) radial component of acceleration?

18. Between 1911 and 1990, the top of the leaning bell tower at Pisa, Italy, moved toward the south at an average rate of 1.2 mm/y. The tower is 55 m tall. In radians per second, what is the average angular speed of the tower's top about its base?

19. Starting from rest at time $t = 0$, a circus stunt man drives a motorbike on a horizontal circular track of radius 10.0 m. His speed is given by $v = ct^2$, where $c = 1.00$ m/s^3. At $t = 2.00$ s, what is the angle between his (total) acceleration vector and his radial acceleration vector?

20. A flywheel with a diameter of 1.20 m is rotating at an angular speed of 200 rev/min. (a) What is the angular speed of the flywheel in radians per second? (b) What is the linear speed of a point on the rim of the flywheel? (c) What constant angular acceleration (in revolutions per minute-squared) will increase the wheel's angular speed to 1000 rev/min in 60.0 s? (d) How many revolutions does the wheel make during that 60.0 s?

21. (a) What is the angular speed ω about the polar axis of a point on Earth's surface at latitude 40° N? (Earth rotates about that axis.) (b) What is the linear speed v of the point? What are (c) ω and (d) v for a point at the equator?

22. The flywheel of a steam engine runs with a constant angular velocity of 160 rev/min. When steam is shut off,

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the friction of the bearings and of the air stops the wheel in 2.2 h. (a) What is the constant angular acceleration, in revolutions per minute-squared, of the wheel during the slowdown? (b) How many revolutions does the wheel make before stopping? (c) At the instant the flywheel is turning at 75 rev/min, what is the tangential component of the linear acceleration of a flywheel particle that is 50 cm from the axis of rotation? (d)What is the magnitude of the net linear acceleration of the particle in (c)?

23. A seed is on a turntable rotating at 33 1/3 rev/min, 6.0 cm from the rotation axis. What are (a) the seed's acceleration and (b) the least coefficient of static friction to avoid slippage? (c) If the turntable had undergone constant angular acceleration from rest in 0.25 s, what is the least coefficient to avoid slippage?

24. In Fig. 10-54, wheel A of radius r_A = 10 cm is coupled by belt B to wheel C of radius r_C = 25 cm. The angular speed of wheel A is increased from rest at a constant rate of 2.0 rad/s^2. Find the time needed for wheel C to reach an angular speed of 100 rev/min, assuming the belt does not slip. (*Hint:* If the belt does not slip, the linear speeds at the two rims must be equal.)

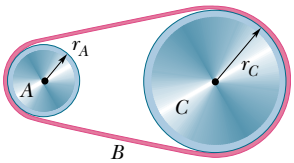

Figure 10-54 Problem 24.

25. Figure 10-55 shows an early method of measuring the speed of light that makes use of a rotating slotted wheel. A beam of light passes through one of the slots at the outside edge of the wheel, travels to a distant mirror, and returns to the wheel just in time to pass through the next slot in the wheel. One such slotted wheel has a radius of 5.0 cm and 500 slots around its edge. Measurements taken when the mirror is L = 500 m from the wheel indicate a speed of light of 3.0 × 10^5 km/s. (a) What is the (constant) angular speed of the wheel? (b) What is the linear speed of a point on the edge of the wheel?

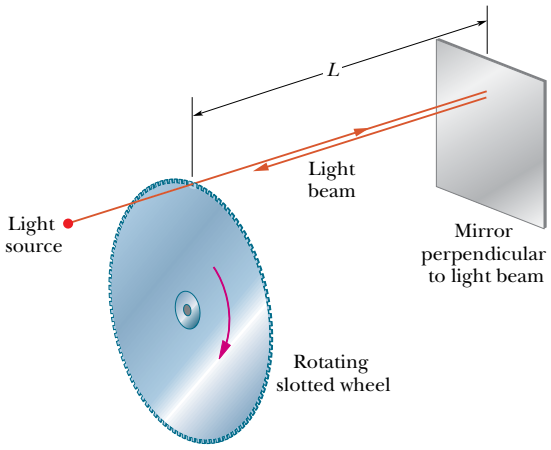

Figure 10-55 Problem 25.

26. A disk, with a radius of 0.25 m, is to be rotated like a merry-go-round through 800 rad, starting from rest, gaining angular speed at the constant rate α_1 through the first 400 rad and then losing angular speed at the constant rate $-\alpha_1$ until it is again at rest. The magnitude of the centripetal acceleration of any portion of the disk is not to exceed 400 m/s^2. (a) What is the least time required for the rotation? (b) What is the corresponding value of α_1?

27. A car starts from rest and moves around a circular track of radius 32.0 m. Its speed increases at the constant rate of 0.600 m/s^2. (a) What is the magnitude of its *net* linear acceleration 15.0 s later? (b) What angle does this net acceleration vector make with the car's velocity at this time?

28. (a) A uniform 2.00 kg disk of radius 0.300 m can rotate around its central axis like a merry-go-round. Beginning from rest at time t = 0, it undergoes a constant angular acceleration of 30.0 rad/s^2. When is the rotational kinetic energy equal to 2000 J? (b) Repeat the calculation but substitute a ring of the same mass and radius and assume that the spokes have negligible mass.

29. Figure 10-56 gives angular speed versus time for a thin rod that rotates around one end. The scale on the ω axis is set by ω_s = 60 rad/s. (a) What is the magnitude of the rod's angular acceleration? (b) At t = 4.0 s, the rod has a rotational kinetic energy of 1.60 J. What is its kinetic energy at t = 0?

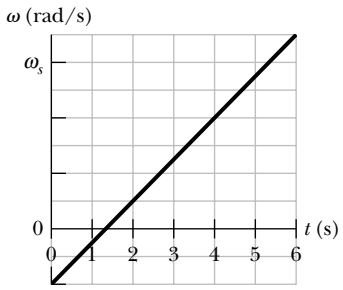

Figure 10-56 Problem 29.

30. A meter stick of negligible mass can rotate about a vertical axis through a point at distance x from the point marked "0". A small block of mass 0.100 kg is glued at the mark of "0" and a small block of mass 0.500 kg is glued at the opposite end, at the mark of "1." The stick and blocks are to rotate with an angular speed of 5.00 rad/min. (a) For what choice of x is the associated rotational kinetic energy least and (b) what is that least energy?

31. Figure 10-57*a* shows a disk that can rotate about an axis at a radial distance h from the center of the disk. Figure 10-57*b* gives the rotational inertia I of the disk about the axis as a function of that distance h, from the center out to the edge of the disk. The scale on the I axis is set by I_A = 0.050 kg·m^2 and I_B = 0.150 kg·m^2. What is the mass of the disk?

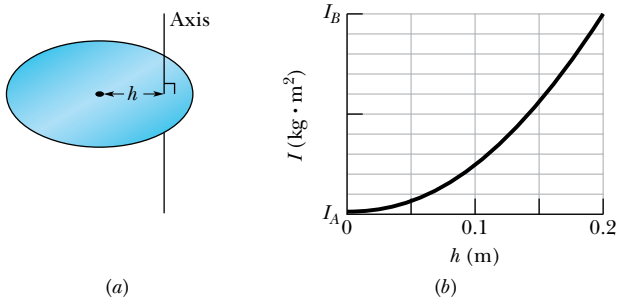

Figure 10-57 Problem 31.

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32. A 0.50 kg meter stick can rotate around an axis perpendicular to the stick. Find the difference in the stick's rotational inertia about the rotation axis if that axis is initially at the point marked "40 cm" and then at the point marked "10 cm."

33. Figure 10-58 shows three 0.0100 kg particles that have been glued to a rod of length $L = 8.00$ cm and negligible mass. The assembly can rotate around a perpendicular axis through point O at the left end. If we remove one particle (that is, 33% of the mass), by what percentage does the rotational inertia of the assembly around the rotation axis decrease when that removed particle is (a) the innermost one and (b) the outermost one?

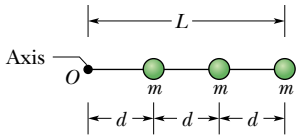

Figure 10-58 Problems 33

34. A wheel with a rotational inertia of 0.50 kg·m² about its central axis is initially rotating at an angular speed of 15 rad/s. At time $t = 0$, a man begins to slow it at a uniform rate until it stops at $t = 5.0$ s. (a) By time $t = 3.0$ s, how much work had the man done? (b) For the full 5.0 s, at what average rate did the man do work?

35. Figure 10-59 shows an arrangement of 15 identical disks that have been glued together in a rod-like shape of length $L = 1.0000$ m and (total) mass $M = 100.0$ mg. The disks are uniform, and the disk arrangement can rotate about a perpendicular axis through its central disk at point O. (a) What is the rotational inertia of the arrangement about that axis? (b) If we approximated the arrangement as being a uniform rod of mass M and length L, what percentage error would we make in using the formula in Table 10-2*e* to calculate the rotational inertia?

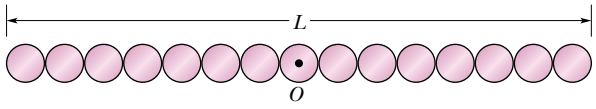

Figure 10-59 Problem 35.

36. In Fig. 10-60, two particles, each with mass $m = 0.85$ kg, are fastened to each other, and to a rotation axis at O, by two thin rods, each with length $d = 5.6$ cm and mass $M = 1.2$ kg. The combination rotates around the rotation axis with the angular speed $\omega = 0.30$ rad/s. Measured about O, what are the combination's (a) rotational inertia and (b) kinetic energy?

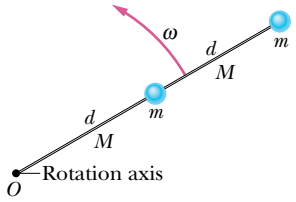

Figure 10-60 Problem 36.

37. Figure 10-61 is an overhead view of a rod of length 1.0 m and mass 1.0 kg that is lying stationary on a frictionless surface when three bullets hit it simultaneously. The bullets move along paths that are in the plane of the rod and perpendicular to the rod. Bullet 1 has mass 10 g and speed 2.0 m/s. Bullet 2 has mass 20 g and speed 3.0 m/s. Bullet 3 has mass 30 g and speed 5.0 m/s. The labelled distances are $\alpha = 10$ cm, $b = 60$ cm, and $c = 80$ cm. As a result of the impacts, the rod–bullets system rotates around its center of mass while the center of mass moves in a straight line over the frictionless surface. (a) What is the linear speed of the system's center of mass? (b) What is the distance between the rod's center and the system's center of mass? (c) What is the rotational inertia of the system about the system's center of mass?

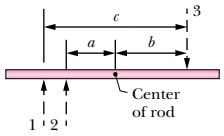

Figure 10-61 Problem 37.

38. The uniform solid block in Fig. 10-62 has mass 0.172 kg and edge lengths $a = 3.5$ cm, $b = 8.4$ cm, and $c = 1.4$ cm. Calculate its rotational inertia about an axis through one corner and perpendicular to the large faces.

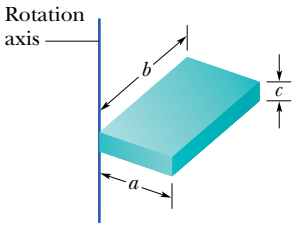

Figure 10-62 Problem 38.

39. Four identical particles of mass 0.75 kg each are placed at the vertices of a 2.0 m × 2.0 m square and held there by four massless rods, which form the sides of the square. What is the rotational inertia of this rigid body about an axis that (a) passes through the midpoints of opposite sides and lies in the plane of the square, (b) passes through the midpoint of one of the sides and is perpendicular to the plane of the square, and (c) lies in the plane of the square and passes through two diagonally opposite particles?

40. The body in Fig. 10-63 is pivoted at O. Three forces act on it: $F_A = 12$ N at point A, 8.0 m from O; $F_B = 14$ N at B, 4.0 m from O; and $F_C = 23$ N at C, 3.0 m from O. What is the net torque about O?

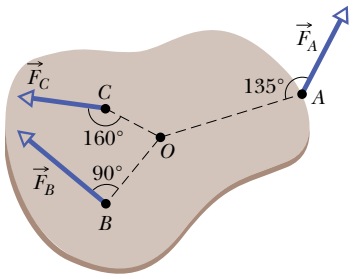

Figure 10-63 Problem 40.

41. A 60 kg father and 20 kg child sit on opposite ends of a seesaw consisting of a board of length 4.0 m and negligible mass. The pivot can be placed anywhere between the father and the child. At what distance from the child

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should it be placed so that the seesaw is balanced when the father and child are stationary?

42. A 100 kg cubical box lies on a floor. A child pushes horizontally at a top edge. What force magnitude puts the box on the verge of tipping over if there is sufficient friction between it and the floor to prevent sliding?

43. A cord with negligible mass is wrapped around a pulley that is a uniform disk of mass 5.00 kg and radius 0.300 m and that can rotate without friction about its central axis. A 1.0 kg bucket is attached at the free end of the cord hanging down from the pulley and then released at time $t = 0$. The cord begins to unwrap from the pulley as the bucket descends. At $t = 5.00$ s, through how many rotations has the pulley turned (the bucket is still descending)?

44. If a 42.0 N·m torque on a wheel causes angular acceleration 25.0 rad/s^2, what is the wheel's rotational inertia?

45. In Fig. 10-64, block 1 has mass $m_1 = 460$ g, block 2 has mass $m_2 = 500$ g, and the pulley, which is mounted on a horizontal axle with negligible friction, has radius $R = 5.00$ cm. When released from rest, block 2 falls 75.0 cm in 5.00 s without the cord slipping on the pulley. (a) What is the magnitude of the acceleration of the blocks? What are (b) tension T_2 and (c) tension T_1? (d) What is the magnitude of the pulley's angular acceleration? (e) What is its rotational inertia?

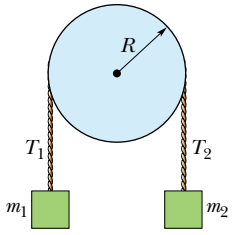

Figure 10-64 Problem 45.

46. In Fig. 10-65, a cylinder having a mass of 3.0 kg can rotate about its central axis through point O. Forces are applied as shown: $F_1 = 6.0$ N, $F_2 = 4.0$ N, $F_3 = 2.0$ N, and $F_4 = 5.0$ N. Also, $r = 5.0$ cm and $R = 12$ cm. Find the (a) magnitude and (b) direction of the angular acceleration of the cylinder. (During the rotation, the forces maintain their same angles relative to the cylinder.)

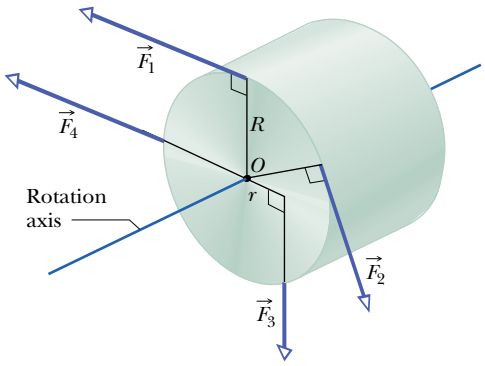

Figure 10-65 Problem 46.

47. Figure 10-66 shows a uniform disk that can rotate around its center like a merry-go-round. The disk has a radius of 2.00 cm and a mass of 20.0 grams and is initially at rest. Starting at time $t = 0$, two forces are to be applied tangentially to the rim as indicated, so that at time $t = 1.25$ s the disk has an angular velocity of 250 rad/s counterclockwise. Force $\vec{F}_1$ has a magnitude of 0.100 N. What is magnitude F_2?

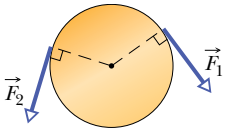

Figure 10-66 Problem 47.

48. In a judo foot-sweep move, you sweep your opponent's left foot out from under him while pulling on his gi (uniform) toward that side. As a result, your opponent rotates around his right foot and onto the mat. Figure 10-67 shows a simplified diagram of your opponent as you face him, with his left foot swept out. The rotational axis is through point O. The gravitational force $\vec{F}_g$ on him effectively acts at his center of mass, which is a horizontal distance $d = 28$ cm from point O. His mass is 75 kg, and his rotational inertia about point O is 65 kg m^2. What is the magnitude of his initial angular acceleration about point O if your pull $\vec{F}_a$ on his gi is (a) negligible and (b) horizontal with a magnitude of 300 N and applied at height $h = 1.4$ m?

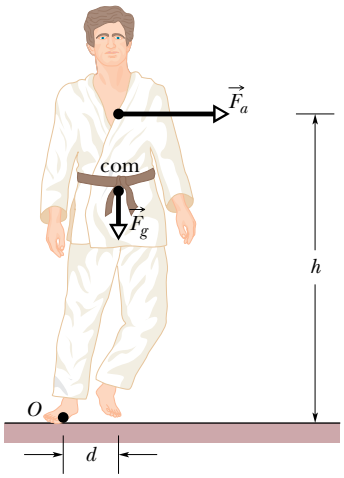

Figure 10-67 Problem 48.

49. Figure 10-68 shows particles 1 and 2, each of mass m, fixed to the ends of a rigid massless rod of length $L_1 + L_2$, with $L_1 = 20$ cm and $L_2 = 80$ cm. The rod is held horizontally on the fulcrum and then released. What are the magnitudes of the initial accelerations of (a) particle 1 and (b) particle 2?

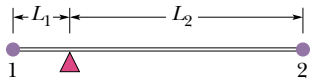

Figure 10-68 Problem 49.

50. A pulley, with a rotational inertia of 1.0×10^{-3} kg·m^2 about its axle and a radius of 10 cm, is acted on by a force applied tangentially at its rim. The force magnitude varies in time as $F = 0.50t + 0.30t^2$, with F in newtons and t in seconds. The pulley is initially at rest. At $t = 3.0$ s what are its (a) angular acceleration and (b) angular speed?

51. (a) If $R = 15$ cm, $M = 350$ g, and $m = 50$ g in Fig. 10-19, find the speed of the block after it has descended 50 cm starting from rest. Solve the problem using energy conservation principles. (b) Repeat (a) with $R = 5.0$ cm.

52. A uniform metal pole of height 30.0 m and mass 100 kg is initially standing upright but then falls over to one side without its lower end sliding or losing contact with the ground. What is the linear speed of the pole's upper end just before impact?

53. A thin rod of length 0.75 m and mass 0.42 kg is suspended freely from one end. It is pulled to one side and then allowed to swing like a pendulum, passing through its lowest position with angular speed 3.5 rad/s. Neglecting

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friction and air resistance, find (a) the rod's kinetic energy at its lowest position and (b) how far above that position the center of mass rises.

54. A meter stick is held vertically with one end on the floor and is then allowed to fall. Find the speed of the other end just before it hits the floor, assuming that the end on the floor does not slip. (*Hint:* Consider the stick to be a thin rod and use the conservation of energy principle.)

55. A uniform cylinder of radius 12 cm and mass 25 kg is mounted so as to rotate freely about a horizontal axis that is parallel to and 5.0 cm from the central longitudinal axis of the cylinder. (a) What is the rotational inertia of the cylinder about the axis of rotation? (b) If the cylinder is released from rest with its central longitudinal axis at the same height as the axis about which the cylinder rotates, what is the angular speed of the cylinder as it passes through its lowest position?

56. A tall, cylindrical chimney falls over when its base is ruptured. Treat the chimney as a thin rod of length 55.0 m. At the instant it makes an angle of 35.0° with the vertical as it falls, what are (a) the radial acceleration of the top, and (b) the tangential acceleration of the top. (*Hint:* Use energy considerations, not a torque.) (c) At what angle θ is the tangential acceleration equal to g?

57. A uniform spherical shell of mass $M = 4.5$ kg and radius $R = 8.5$ cm can rotate about a vertical axis on frictionless bearings (Fig. 10-69). A massless cord passes around the equator of the shell, over a pulley of rotational inertia $I = 3.0 \times 10^{-3}$ kg·m^2 and radius $r = 5.0$ cm, and is attached to a small object of mass $m = 0.60$ kg. There is no friction on the pulley's axle; the cord does not slip on the pulley. What is the speed of the object when it has fallen 82 cm after being released from rest? Use energy considerations.

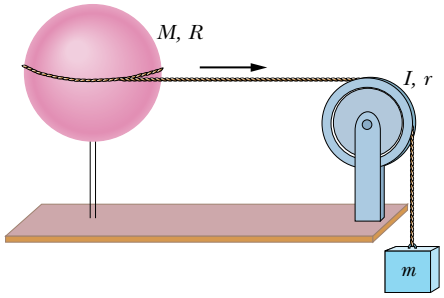

Figure 10-69 Problem 57.

58. Figure 10-70 shows a rigid assembly of a thin hoop (of mass m and radius $R = 0.150$ m) and a thin radial rod (of mass m and length $L = 2.00R$). The assembly is upright, but if we give it a slight nudge, it will rotate around a horizontal axis in the plane of the rod and hoop, through the lower end of the rod. Assuming that the energy given to the assembly in such a nudge is negligible, what would be the assembly's angular speed about the rotation axis when it passes through the upside-down (inverted) orientation?

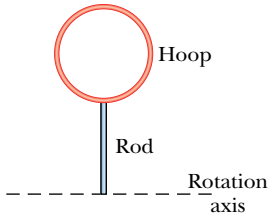

Figure 10-70 Problem 58.

59. In unit vector notation, find the net torque about the origin on a particle located at (2.0 m, –2.0 m, –6.0 m) when the three forces $\vec{F}_1 = (6.0\text{ N})\hat{j}$, $\vec{F}_2 = (1.0\text{ N})\hat{i} - (2.0\text{ N})\hat{j}$, and $\vec{F}_3 = (4.0\text{ N})\hat{i} + (2.0\text{ N})\hat{j} - (3.0\text{ N})\hat{k}$ act on the particle.

60. In unit vector notation, find the torque about a point at coordinates (2.0 m, 1.0 m, 3.0 m) on a particle located at (3.0 m, 1.0 m, 2.0 m) when a force $\vec{F} = (1.0\text{ N})\hat{i} - (3.0\text{ N})\hat{k}$ acts on the particle.

61. In unit vector notation, what is the torque about the origin on a particle located at coordinates (0, –4.0 m, 3.0 m) if that torque is due to (a) force $\vec{F}_1$ with components $F_{1x} = 2.0$ N, $F_{1y} = F_{1z} = 0$, and (b) force $\vec{F}_2$ with components $F_{2x} = 0$, $F_{2y} = 2.0$ N, $F_{2z} = 4.0$ N?

62. A particle moves through an xyz coordinate system while a force acts on the particle. When the particle has the position vector $\vec{r} = (2.00\text{ m})\hat{i} - (3.00\text{ m})\hat{j} + (2.00\text{ m})\hat{k}$, the force is given by $\vec{F} = F_x\hat{i} + (7.00\text{ N})\hat{j} - (6.00\text{ N})\hat{k}$ and the corresponding torque about the origin is $\vec{\tau} = (400\text{ N}\cdot\text{m})\hat{i} + (2.00\text{ N}\cdot\text{m})\hat{j} - (1.00\text{ N}\cdot\text{m})\hat{k}$. Determine F_x.

63. Force $\vec{F} = (2.0\text{ N})\hat{i} - (3.0\text{ N})\hat{k}$ acts on a pebble with position vector $\vec{r} = (0.50\text{ m})\hat{j} - (2.0\text{ m})\hat{k}$ relative to the origin. In unit-vector notation, what is the resulting torque on the pebble about (a) the origin and (b) the point (2.0 m, 0, –3.0 m)?

64. A force $\vec{F} = (6.00\,\hat{i} - 4.00\,\hat{j})$ N acts on a particle with position vector $\vec{r} = (-3.00\,\hat{i} + 1.00\,\hat{j})$ m. Find (a) the torque about the origin acting on the particle and (b) the angle between $\vec{r}$ and $\vec{F}$.

65. At the instant of Fig. 10-71, a 2.0 kg particle P has a position vector $\vec{r}$ of magnitude 5.0 m and angle $\theta_1 = 45°$ and a velocity vector $\vec{v}$ of magnitude 4.0 m/s and angle $\theta^2 = 30°$. Force $\vec{F}$, of magnitude 2.0 N and angle $\theta_3 = 30°$, acts on P. All three vectors lie in the xy plane. About the origin, what are the (a) magnitude and (b) direction of the angular momentum of P and the (c) magnitude and (d) direction of the torque acting on P?

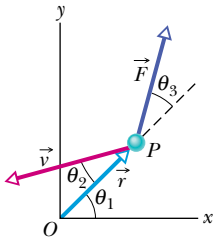

Figure 10-71 Problem 65.

66. A particle of mass m is shot from ground level at initial speed u and initial angle θ relative to the horizontal. When it reaches its highest point in its flight over the level ground, what are the magnitudes of (a) the torque acting on it from the gravitational force and (b) its angular momentum, both measured about the launch point?

67. A 3.0 kg particle-like object moves in a plane with velocity components $v_x = 30$ m/s and $v_y = 60$ m/s as it passes through

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the point with (x, y) coordinates of (3.0, –4.0) m. Just then, in unit vector notation, what is its angular momentum relative to (a) the origin and (b) the point located at (–2.0, –2.0) m?

68. Two particles, each of mass m and speed v, travel in opposite directions along parallel lines separated by a distance d. Show that the angular momentum of this system of particles is the same about any point taken as the origin.

69. At the instant the displacement of a 1.50 kg object relative to the origin is $\vec{d} = (2.00\text{ m})\hat{i} + (4.00\text{ m})\hat{j} - (3.00\text{ m/s})\hat{k}$, its velocity is $\vec{v} = -(6.00\text{ m/s})\hat{i} + (3.00\text{ m/s})\hat{j} + (3.00\text{ m})\hat{k}$ and it is subject to a force $\vec{F} = (6.00\text{ N})\hat{i} - (8.00\text{ N})\hat{j} + (4.00\text{ N})\hat{k}$. Find (a) the acceleration of the object, (b) the angular momentum of the object about the origin, (c) the torque about the origin acting on the object, and (d) the angle between the velocity of the object and the force acting on the object.

70. In Fig. 10-72, a 0.400 kg ball is shot directly upward at initial speed 40.0 m/s. What is its angular momentum about P, 2.00 m horizontally from the launch point, when the ball is (a) at maximum height and (b) halfway back to the ground? What is the torque on the ball about P due to the gravitational force when the ball is (c) at maximum height and (d) halfway back to the ground?

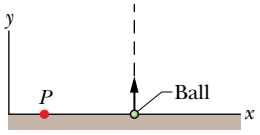

Figure 10-72 Problem 70.

71. A particle is acted on by two torques about the origin: $\vec{\tau}_1$ has a magnitude of 2.0 N·m and is directed in the positive direction of the x axis, and $\vec{\tau}_2$ has a magnitude of 3.0 N·m and is directed in the negative direction of the y axis. In unit-vector notation, find $d\vec{\ell}/dt$, where $\vec{\ell}$ is the angular momentum of the particle about the origin.

72. In Fig. 10-73, a particle of mass m is released from rest at point A, at distance b from the origin, and falls parallel to a vertical y axis. As a function of time t and with respect to the origin, find the magnitudes of (a) the torque τ on the particle due to the gravitational force and (b) the angular momentum L of the particle. (c) From those results, verify that $\tau = dL/dt$.

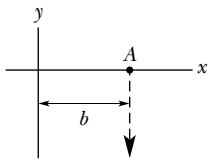

Figure 10-73 Problem 72.

73. Figure 10-74 is an overhead view of a rod of length d and negligible mass that rotates at angular speed ω in the horizontal plane about a vertical axis through point O. Particle A of mass m is attached at the center point and an identical particle B is attached at the far end. Find the angular momentum of particle B with respect to particle A.

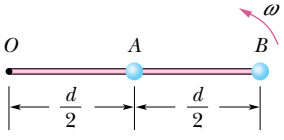

Figure 10-74 Problem 73.

74. A force $(2.00\hat{i} - 4.00\hat{j} + 2.00\hat{k})$ N acts on a particle located at $(3.00\hat{i} + 2.00\hat{j} - 4.00\hat{k})$ m. What is the magnitude of the torque on the particle as measured about the origin?

75. Figure 10-75 shows three rotating, uniform disks that are coupled by belts. One belt runs around the rims of disks A and C. Another belt runs around a central hub on disk A and the rim of disk B. The belts move smoothly without slippage on the rims and hub. Disk A has radius R; its hub has radius $0.5000R$; disk B has radius $0.2500R$; and disk C has radius $2.000R$. Disks B and C have the same density (mass per unit volume) and thickness. What is the ratio of the magnitude of the angular momentum of disk C to that of disk B?

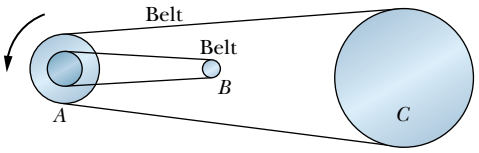

Figure 10-75 Problem 75.

76. In Fig. 10-76, three particles of mass m = 23 g are fastened to three rods of length d = 12 cm and negligible mass. The rigid assembly rotates around point O at the angular speed ω = 0.85 rad/s. About O, what are (a) the rotational inertia of the assembly, (b) the magnitude of the angular momentum of the middle particle, and (c) the magnitude of the angular momentum of the asssembly?

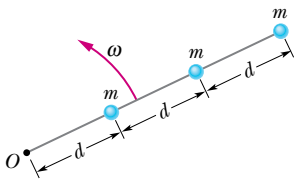

Figure 10-76 Problem 76.

77. A sanding disk with rotational inertia 8.6×10^{-3} kg·m² is attached to an electric drill whose motor delivers a torque of magnitude 16 N·m about the central axis of the initially stationary disk. About that axis and with the torque applied for 33 ms, what is the magnitude of the (a) angular momentum and (b) angular velocity of the disk?

78. The angular momentum of a flywheel having a rotational inertia of 0.140 kg·m² about its central axis decreases from 3.00 to 0.800 kg·m²/s in 1.50 s. (a) What is the magnitude of the average torque acting on the flywheel about its central axis during this period? (b) Assuming a constant angular acceleration, through what angle does the flywheel turn? (c) How much work is done on the wheel? (d) What is the average power of the flywheel?

79. A disk with a rotational inertia of 7.00 kg·m² rotates like a merry-go-round while undergoing a variable torque given by $\tau = (5.00 + 2.00t)$ N·m. At time t = 1.00 s, its angular momentum is 5.00 kg·m²/s. What is its angular momentum at t = 5.00 s?

80. Figure 10-77 shows a rigid structure consisting of a circular hoop of radius R and mass m, and a square made of four thin bars, each of length R and mass m. The rigid structure rotates at a constant speed about a vertical axis, with a period of rotation of 2.5 s. Assuming

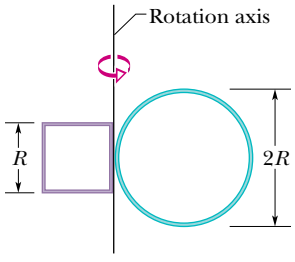

Figure 10-77 Problem 80.

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$R = 0.50$ m and $m = 2.0$ kg, calculate (a) the structure's rotational inertia about the axis of rotation and (b) its angular momentum about that axis.

81. In Fig. 10-78, two skaters, each of mass 50 kg, approach each other along parallel paths separated by 3.0 m. They have opposite velocities of 1.4 m/s each. One skater carries one end of a long pole of negligible mass, and the other skater grabs the other end as she passes. The skaters then rotate around the center of the pole. Assume that the friction between skates and ice is negligible. What are (a) the radius of the circle, (b) the angular speed of the skaters, and (c) the kinetic energy of the two-skater system? Next, the skaters pull along the pole until they are separated by 1.0 m. What then are (d) their angular speed and (e) the kinetic energy of the system? (f) What provided the energy for the increased kinetic energy?

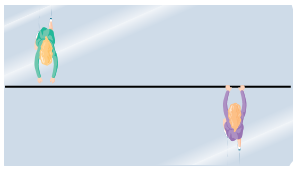

Figure 10-78 Problem 81.

82. A cockroach of mass 0.20 kg runs counterclockwise around the rim of a lazy Susan (a circular disk mounted on a vertical axle) that has radius 18 cm, rotational inertia 5.0×10^{-3} kg·m^2, and frictionless bearings. The cockroach's speed (relative to the ground) is 2.0 m/s, and the lazy Susan turns clockwise with angular speed $\omega_0 = 2.8$ rad/s. The cockroach finds a bread crumb on the rim and, of course, stops. (a) What is the angular speed of the lazy Susan after the cockroach stops? (b) Is mechanical energy conserved as it stops?

83. A man stands on a platform that is rotating (without friction) with an angular speed of 1.2 rev/s; his arms are outstretched and he holds a brick in each hand. The rotational inertia of the system consisting of the man, bricks, and platform about the central vertical axis of the platform is 6.0 kg·m^2. If by moving the bricks the man decreases the rotational inertia of the system to 2.0 kg·m^2, what are (a) the resulting angular speed of the platform and (b) the ratio of the new kinetic energy of the system to the original kinetic energy? (c) What source provided the added kinetic energy?

84. The rotational inertia of a collapsing spinning star drops to ¼ its initial value. What is the ratio of the new rotational kinetic energy to the initial rotational kinetic energy?

85. A track is mounted on a large wheel that is free to turn with negligible friction about a vertical axis (Fig. 10-79). A toy train of mass m is placed on the track and, with the system initially at rest, the train's electrical power is turned on. The train reaches speed 0.15 m/s with respect to the track. What is the wheel's angular speed if its mass is $1.1m$ and its radius is 0.43 m? (Treat it as a hoop, and neglect the mass of the spokes and hub.)

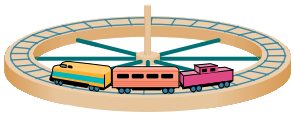

Figure 10-79 Problem 85.

86. Two disks are mounted (like a merry-go-round) on low-friction bearings on the same axle and can be brought together so that they couple and rotate as one unit. The first disk, with rotational inertia 3.30 kg·m^2 about its central axis, is set spinning counterclockwise at 450 rev/min. The second disk, with rotational inertia 6.60 kg·m^2 about its central axis, is set spinning counterclockwise at 900 rev/min. They then couple together. (a) What is their angular speed after coupling? If instead the second disk is set spinning clockwise at 900 rev/min, what are their (b) angular speed and (c) direction of rotation after they couple together?

87. The rotor of an electric motor has rotational inertia $I_m = 2.0 \times 10^{-3}$ kg·m^2 about its central axis. The motor is used to change the orientation of the space probe in which it is mounted. The motor axis is mounted along the central axis of the probe; the probe has rotational inertia $I_p = 10$ kg·m^2 about this axis. Calculate the number of revolutions of the rotor required to turn the probe through 30° about its central axis.

88. A wheel of rotational inertia 0.100 kg·m^2 is rotating about a shaft at an angular speed of 160 rev/min. A second wheel is set into rotation at 300 rev/min and mounted on the same shaft, so that the wheels couple together. Their final angular speed is 200 rev/min. Find (a) the rotational inertia of the second wheel and (b) the change in the rotational kinetic energy of the system due to the rubbing of wheel on wheel.

89. A cockroach of mass m lies on the rim of a uniform disk of mass $4.00m$ that can rotate freely about its center like a merry-goround. Initially the cockroach and disk rotate together with an angular velocity of 0.320 rad/s. Then the cockroach walks halfway to the center of the disk. (a) What then is the angular velocity of the cockroach–disk system? (b) What is the ratio K/K_0 of the new kinetic energy of the system to its initial kinetic energy? (c) What accounts for the change in the kinetic energy?

90. Figure 10-80 is an overhead view of a rod with length $2L$ and negligible mass and which lies on a frictionless surface. Two bullets, each with mass m and speed v and traveling parallel to the x axis, hit the ends of the rod simultaneously and are buried in the rod. After the collision, what are (a) the angular speed of the system and (b) the speed of the system's center of mass? (c) What is the change in the total kinetic energy because of the collisions?

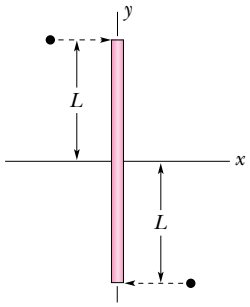

Figure 10-80 Problem 90.

91. Figure 10-81 shows an overhead view of a ring that can rotate about its center like a merry-go- round. Its outer radius R_2 is 0.800 m, its inner radius R_1 is $R_2/2.00$, its mass M is 6.00 kg, and the mass of the crossbars at its center is negligible. It initially rotates at an angular speed of 8.00 rad/s with a cat of mass $m = M/4.00$ on its outer edge, at radius R_2. By how much does the cat increase the kinetic energy of the cat–ring system if the cat crawls to the inner edge, at radius R_1?

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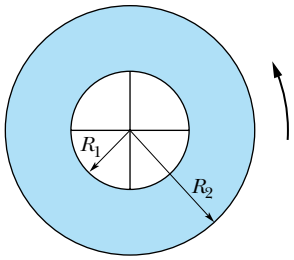

Figure 10-81 Problem 91.

92. Figure 10-82 is an overhead view of semicircular ring of mass m and radius R that lies on a frictionless surface. The ring is connected by elevated spokes (not shown) to a pivot at the center of the semicircle. The ring can rotate around that pivot but is initially stationary. A bead of mass m and speed v is shot into the ring, entering under the spokes and tangentially to the ring and then sliding along the ring. The ring and bead then rotate together around the pivot. Find their angular speed.

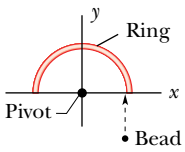

Figure 10-82 Problem 92.

93. A horizontal platform in the shape of a circular disk rotates on a frictionless bearing about a vertical axle through the center of the disk. The platform has a mass of 180 kg, a radius of 2.0 m, and a rotational inertia of 350 kg·m^2 about the axis of rotation. A 60 kg student walks slowly from the rim of the platform toward the center. If the angular speed of the system is 1.5 rad/s when the student starts at the rim, what is the angular speed when she is 0.50 m from the center?

94. Figure 10-83 is an overhead view of a spring lying on a frictionless surface and attached to a pivot at its right end. The spring has a relaxed length of $l_0 = 1.00$ m and negligible mass. A small 0.100 kg disk is attached to the free end at the left. That disk is then given a velocity $\vec{v}_0$ of magnitude 11.0 m/s perpendicular to the spring's length. The disk and spring then move around the pivot. (a) When the stretching of the spring reaches its maximum value of $0.100l_0$, what is the speed of the disk? (b) What is the spring constant?

Figure 10-83 Problem 94.

95. In Fig. 10-84, a 1.0 g bullet is fired into a 0.60 kg block attached to the end of a 0.75 m nonuniform rod of mass 0.50 kg. The block–rod–bullet system then rotates in the plane of the figure, about a fixed axis at A. The rotational inertia of the rod alone about that axis at A is 0.060 kg·m^2. Treat the block as a particle. (a) What then is the rotational inertia of the block–rod–bullet system about point A? (b) If the angular speed of the system about A just after impact is 4.5 rad/s, what is the bullet's speed just before impact?

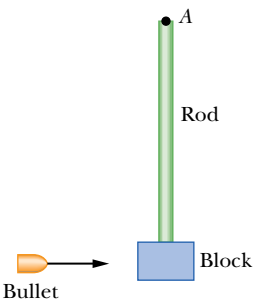

Figure 10-84 Problem 95.

96. The uniform rod (length 0.60 m, mass 1.0 kg) in Fig. 10-85 rotates in the plane of the figure about an axis through one end, with a rotational inertia of 0.12 kg·m^2. As the rod swings through its lowest position, it collides with a 0.20 kg putty wad that sticks to the end of the rod. If the rod's angular speed just before collision is 2.4 rad/s, what is the angular speed of the rod–putty system immediately after collision?

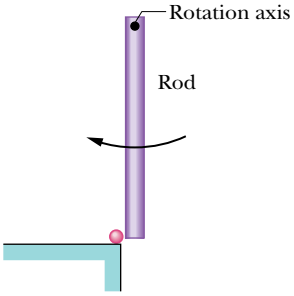

Figure 10-85 Problem 96.

97. In Fig. 10-86, a 30 kg child stands on the edge of a stationary merry-go-round of radius 2.0 m. The rotational inertia of the merry-go-round about its rotation axis is 150 kg·m^2. The child catches a ball of mass 1.0 kg thrown by a friend. Just before the ball is caught, it has a horizontal velocity $\vec{v}$ of magnitude 12 m/s, at angle $\phi = 37°$ with a line tangent to the outer edge of the merry-go-round, as shown. What is the angular speed of the merry-go-round just after the ball is caught?

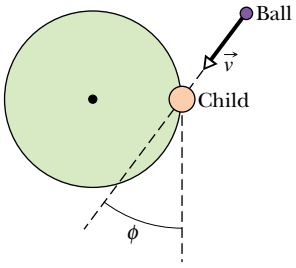

Figure 10-86 Problem 97.

98. Two 2.00 kg balls are attached to the ends of a thin rod of length 50.0 cm and negligible mass. The rod is free to rotate in a vertical plane without friction about a horizontal axis through its center. With the rod initially horizontal (Fig. 10-87), a 50.0 g wad of wet putty drops onto one of the balls, hitting it with a speed of 3.00 m/s and then sticking to it. (a)What is the angular speed of the system just after the putty wad hits? (b) What is the ratio of the kinetic energy of the system after the collision to that of the putty wad just before? (c) Through what angle will the system rotate before it momentarily stops?

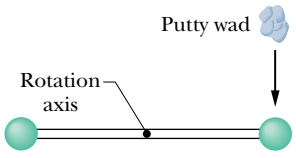

Figure 10-87 Problem 98.

99. In Fig. 10-88, a small 50 g block slides down a frictionless surface through height $h = 15$ cm and then sticks to a uniform rod of mass 100 g and length 35 cm. The rod pivots about point O through angle θ before momentarily stopping. Find θ.

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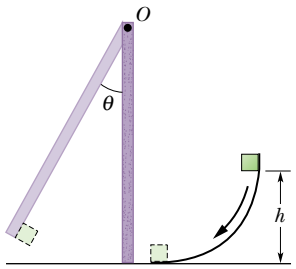

Figure 10-88 Problem 99.

100. A top spins at 25 rev/s about an axis that makes an angle of 30° with the vertical. The mass of the top is 0.50 kg, its rotational inertia about its central axis is 5.0×10^{-4} kg·m^2, and its center of mass is 4.2 cm from the pivot point. If the spin is clockwise from an overhead view, what are the (a) precession rate and (b) direction of the precession as viewed from overhead?

PRACTICE QUESTIONS

Single Correct Choice Type

43. A hoop of radius r rotating about its own axis with an angular velocity ω_0 is gently placed with its plane vertical onto a rough horizontal surface. Find the velocity of the center of the hoop when it ceases to slip. At the initial moment, the velocity of the center of the hoop was zero.
(a) $v = 0.5\omega_0 r$ **(b)** $v = 1.5\omega_0 r$
(c) $v = 2.0\omega_0 r$ **(d)** $v = 3.5\omega_0 r$

2. The Earth takes slightly less than one day to complete one rotation about the axis passing through its poles. The actual time is 8.616×10^4 s. With this given information, what is the angular speed of the Earth about its axis?
(a) 7.292×10^{-5} rad/s **(b)** 9.951×10^{-5} rad/s
(c) 2.321×10^{-6} rad/s **(d)** 6.334×10^{-4} rad/s

3. A marble and a cube are placed at the top of a ramp. Starting from rest at the same height, the marble rolls and the cube slides (no kinetic friction) down the ramp. Determine the ratio of the center-of-mass speed of the cube to the center-of-mass speed of the marble at the bottom of the ramp.
(a) 1.14 **(b)** 1.20
(c) 0.707 **(d)** 1.18

4. A wheel with a 0.10 m radius is rotating at 35 rev/s. It then slows uniformly to 15 rev/s over a 3.0 s interval. What is the angular acceleration of a point on the wheel?
(a) –2.0 rev/s^2 **(b)** –6.7 rev/s^2
(c) –17 rev/s^2 **(d)** 0.67 rev/s^2

5. A body is rotating non-uniformly about a vertical axis fixed in an inertial frame. The resultant force on a particle of the body not on the axis is
(a) vertical.
(b) horizontal and skew with the axis.
(c) horizontal and intersecting the axis.
(d) none of the above.

6. During the spin-dry cycle of a washing machine, the motor slows from 95 rad/s to 30 rad/s while the turning the drum through an angle of 402 rad. What is the magnitude of the angular acceleration of the motor?
(a) 64 rad/s^2 **(b)** 10 rad/s^2
(c) 32 rad/s^2 **(d)** 20 rad/s^2

7. Two people start at the same place and walk around a circular lake in opposite directions. One has an angular speed of 1.7×10^{-3} rad/s while the other has an angular speed of 3.4×10^{-3} rad/s. How long will it be before they meet?
(a) 600 s **(b)** 2400 s
(c) 1200 s **(d)** 1800 s

8. A fan rotating with an initial angular velocity of 1000 rev/min is switched off. In 2 s, the angular velocity decreases to 200 rev/min. Assuming the angular acceleration is constant, how many revolutions does the blade undergo during this time?
(a) 10 **(b)** 100
(c) 20 **(d)** 125

9. A basketball player is balancing a spinning basketball on the tip of his finger. The angular velocity of the ball slows down from 18.5 to 14.1 rad/s. During the slow-down, the angular displacement is 85.1 rad. Determine the time it takes for the ball to slow down.
(a) 4.50 s **(b)** 5.22 s
(c) 3.11 s **(d)** 4.76 s

10. An airplane engine starts from rest; and 2 s later, it is rotating with an angular speed of 300 rev/min. If the angular acceleration is constant, how many revolutions does the propeller undergo during this time?
(a) 5 **(b)** 50
(c) 10 **(d)** 300

11. In 9.5 s a fisherman winds 2.6 m of fishing line onto a reel whose radius is 3.0 cm (assumed to be constant as an approximation). The line is being reeled in at a constant speed. Determine the angular speed of the reel.
(a) 9.1 rad/s **(b)** 5.0 rad/s
(c) 3.7 rad/s **(d)** 7.4 rad/s

12. During the time a CD accelerates from rest to a constant rotational speed of 477 rev/min, it rotates through an angular displacement of 0.250 rev. What is the angular acceleration of the CD?
(a) 358 rad/s^2 **(b)** 901 rad/s^2
(c) 794 rad/s^2 **(d)** 866 rad/s^2

13. A person is riding a bicycle, and its wheels have an angular velocity of +20.0 rad/s. Then, the brakes are applied, and the bike is brought to a uniform stop. During braking,

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the angular displacement of each wheel is +15.92 rev. How much time does it take for the bike to come to rest?

(a) 5.00 s **(b)** 10.0 s
(c) 3.18 s **(d)** 6.36 s

14. A grindstone, initially at rest, is given a constant angular acceleration so that it makes 20.0 rev in the first 8.00 s. What is its angular acceleration?

(a) 0.313 rad/s² **(b)** 2.50 rad/s²
(c) 3.93 rad/s² **(d)** 1.97 rad/s²

15. A thin circular ring of mass M and radius R is rotating about its axis with a constant angular velocity ω. Two objects each of mass m are attached gently to the ring. The wheel now rotates with an angular velocity

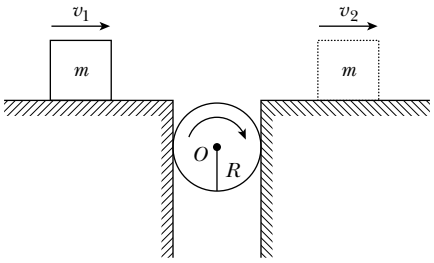
(a) $\dfrac{\omega M}{(m+M)}$ **(b)** $\dfrac{\omega(M-2m)}{(M+2m)}$

(c) $\dfrac{\omega M}{(M+2m)}$ **(d)** $\dfrac{\omega(M+2m)}{M}$

16. A wheel turns through an angle of 188 rad in 8.0 s; and its angular speed at the end of the period is 40 rad/s. If the angular acceleration is constant, what was the angular speed of the wheel at the beginning of the 8.0 s interval?

(a) 4.8 rad/s **(b)** 9.1 rad/s
(c) 7.0 rad/s **(d)** 23.5 rad/s

17. A uniform disc of mass M and radius R is initially at rest. Its axis is fixed through O. A block of mass m is moving with speed v_1 on a frictionless surface passes over the disc to the dotted position. On coming in contact with the disc, it slips on it. The slipping ceases before the block loses contact with the disc, due to the high friction. Now, the velocity of mass m is



(a) $v_2 = \dfrac{v_1}{1+(M/2m)}$ **(b)** $v_2 = \dfrac{v_1}{2M}$

(c) $v_2 = \dfrac{v_1}{1+(M/m)}$ **(d)** $v_2 = 2v_1$

18. A 1.0 m roulette wheel reaches a maximum angular speed of 18 rad/s before it begins decelerating. After reaching this maximum angular speed, it turns through 35 rev (220 rad) before it stops. How long did it take the wheel to stop after reaching its maximum angular speed?

(a) 12 s **(b)** 24 s
(c) 48 s **(d)** 88 s

19. A body is in pure rotation. The linear speed v of a particle, the distance r of the particle from the axis and the angular velocity ω of the body are related as $\omega = v/r$. Thus

(a) $\omega \propto \dfrac{1}{r}$ **(b)** $\omega \propto r$

(c) $\omega = r_0$ **(d)** ω is independent of r

20. A circular disk of radius 0.010 m rotates with a constant angular speed of 5.0 rev/s. What is the acceleration of a point on the edge of the disk?

(a) 0.31 m/s² **(b)** 9.9 m/s²
(c) zero m/s² **(d)** 1.6 m/s²

21. After 10.0 s, a spinning roulette wheel at a casino has slowed down to an angular velocity of +1.88 rad/s. During this time, the wheel has an angular acceleration of –5.04 rad/s². Determine the angular displacement of the wheel.

(a) +440 rad **(b)** +252 rad
(c) +271 rad **(d)** +188 rad

22. A circular disk of radius 2.0 m rotates, starting from rest, with a constant angular acceleration of 20.0 rad/s². What is the tangential acceleration of a point on the edge of the disk at the instant that its angular speed is 1.0 rev/s?

(a) 40 m/s² **(b)** 110 m/s²
(c) 79 m/s² **(d)** 120 m/s²

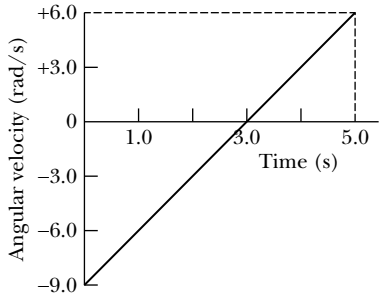
23. At the local swimming hole, a favorite trick is to run horizontally off a cliff that is 8.3 m above the water. One diver runs off the edge of the cliff, tucks into a "ball", and rotates on the way down with an average angular speed of 1.6 rev/s. Ignore air resistance and determine the number of revolutions she makes while on the way down.

(a) 1.6 rev **(b)** 2.4 rev
(c) 0.81 rev **(d)** 2.1 rev

24. The original Ferris wheel had a radius of 38 m and completed a full revolution (2π rad) every two minutes when operating at its maximum speed. If the wheel were uniformly slowed from its maximum speed to a stop in 35 s, what would be the magnitude of the tangential acceleration at the outer rim of the wheel during its deceleration?

(a) 0.0015 m/s² **(b)** 0.54 m/s²
(c) 0.057 m/s² **(d)** 1.6 m/s

25. The figure shows a graph of the angular velocity of a rotating wheel as a function of time. Although not shown in the graph, the angular velocity continues to increase at the same rate until t = 8.0 s. What is the angular displacement of the wheel from 0 to 8.0 s?



(a) 36 rad **(b)** 60 rad
(c) 24 rad **(d)** 48 rad

26. The figure shows the top view of a door that is 2 m wide. Two forces are applied to the door as indicated. What is

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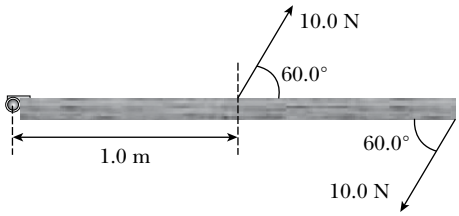
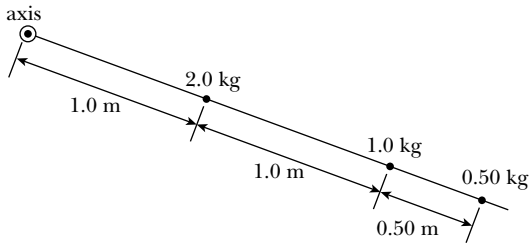
the magnitude of the net torque on the door with respect to the hinge?



(a) 0 N·m **(b)** 10.0 N·m
(c) 5.0 N·m **(d)** 8.7 N·m

27. Three objects are attached to a *massless* rigid rod that has an axis of rotation as shown. Assuming all of the mass of each object is located at the point shown for each, calculate the moment of inertia of this system.



(a) 5.3 kg·m² **(b)** 9.1 kg·m²
(c) 3.1 kg·m² **(d)** 7.2 kg·m²

28. A solid circular disk has a mass of 1.2 kg and a radius of 0.16 m. Each of three identical thin rods has a mass of 0.15 kg. The rods are attached perpendicularly to the plane of the disk at its outer edge to form a three-legged stool (see the figure). Find the moment of inertia of the stool with respect to an axis that is perpendicular to the plane of the disk at its center.

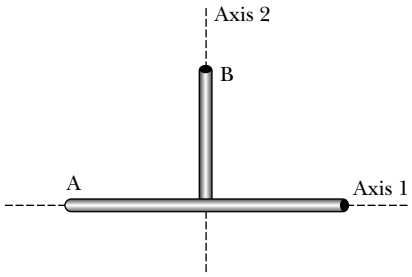
(***Hint:*** When considering the moment of inertia of each rod, note that all of the mass of each rod is located at the same perpendicular distance from the axis.)

(a) 0.027 kg·m² **(b)** 0.096 kg·m²
(c) 0.185 kg·m² **(d)** 0.072 kg·m²

29. A 50 kg rider on a moped of mass 75 kg is traveling with a speed of 20 m/s. Each of the two wheels of the moped has a radius of 0.2 m and a moment of inertia of 0.2 kg·m². What is the total *rotational* kinetic energy of the wheels?

(a) 500 J **(b)** 4000 J
(c) 100 J **(d)** 2000 J

30. A long thin rod is cut into two pieces, one being twice as long as the other. To the midpoint of piece *A* (the longer piece), piece *B* is attached perpendicularly, in order to form the inverted "T" as shown in the figure. The application of a net external torque causes this object to rotate about axis 1 with an angular acceleration of 4.6 rad/s². When the same net external torque is used to cause the object to rotate about axis 2, what is the angular acceleration?

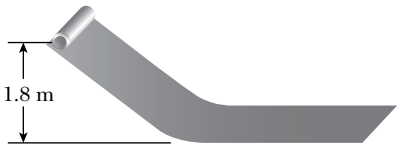


(a) 1.2 rad/s² **(b)** 2.3 rad/s²
(c) 3.5 rad/s² **(d)** 4.6 rad/s²

31. A 50 N m torque acts on a wheel with a moment of inertia 150 kg·m². If the wheel starts from rest, how long will it take the wheel to make 1 rev?

(a) 2.4 s **(b)** 10 s
(c) 0.66 s **(d)** 6.1 s

32. A solid cylinder of radius 0.35 m is released from rest from a height of 1.8 m and rolls down the incline as shown in the figure. What is the angular speed of the cylinder when it reaches the horizontal surface?



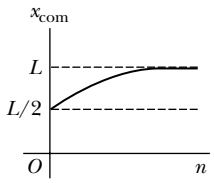
(a) 8.2 rad/s **(c)** 34 rad/s
(b) 14 rad/s **(d)** 67 rad/s

33. One end of a thin rod is attached to a pivot, about which it can rotate without friction. Air resistance is absent. The rod has a length of 0.80 m and is uniform. It is hanging vertically straight downward. The end of the rod nearest the floor is given a linear speed v_0, so that the rod begins to rotate upward about the pivot. What must be the value of v_0, such that the rod comes to a momentary halt in a straight-up orientation, exactly opposite to its initial orientation?

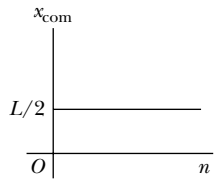
(a) 6.9 m/s **(b)** 5.6 m/s
(c) 4.7 m/s **(d)** 8.1 m/s

34. A thin rod of length L lying along the x-axis with its ends at $x = 0$ and $x = L$. Its linear density (mass/length) varies with x as $k(x/L)^n$, where n can be zero or any positive number. If the position x_{com} of the centre of mass of the rod is plotted against n, which of the graphs shown in the figure best approximates the dependence of x_{com} on n?

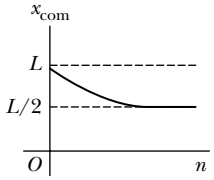
(a)



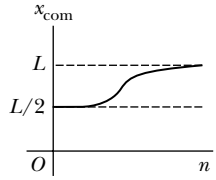
(b)



(c)



(d)



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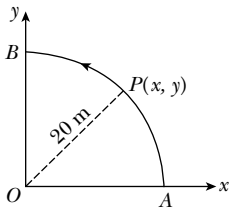
35. A solid cylinder with a mass m and radius r is mounted so that it can be rotated about an axis that passes through the center of both ends. At what angular speed, ω must the cylinder rotate to have the same total kinetic energy that it would have if it were moving horizontally with a speed v without rotation?

(a) $\omega = \frac{v}{r}$ **(b)** $\omega = \frac{v}{2r}$

(c) $\omega = \frac{v^2}{r^2}$ **(d)** $\omega = \frac{v}{r}\sqrt{2}$

36. A points P moves in counterclockwise direction on a circular path as shown in the figure. The movement of P is such that it sweeps out a length $s = t^3 + 5$, where s is in meters and t is in seconds. The radius of the path is 20 m. The acceleration of P when $t = 2$ s is nearly



(a) 14 m/s² **(b)** 13 m/s²

(c) 12 m/s² **(d)** 7.2 m/s²

37. A certain merry-go-round is accelerated uniformly from rest and attains an angular speed of 0.4 rad/s in the first 10 s. If the net applied torque is 2000 N·m, what is the moment of inertia of the merry-go-round?

(a) 400 kg·m² **(b)** 800 kg·m²

(c) 50 000 kg·m² **(d)** 5000 kg·m²

38. A body is rotating around a fixed axis with angular velocity 3 rad/s with constant angular acceleration of 1 rad/s² at some time. Find the magnitude of acceleration of a particle 5 m away from the axis after the body has turned by 90°.

(a) $5\sqrt{1+(9+\pi)^2}$ m/s² **(b)** $5(9+\pi)$ m/s²

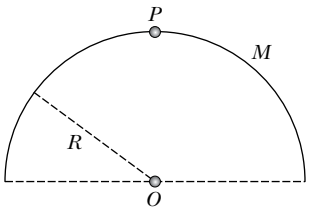
(c) 5 m/s² **(d)** $5\sqrt{82}$ m s^{-2}

39. A 1.0 kg wheel in the form of a solid disk rolls along a horizontal surface with a speed of 6.0 m/s. What is the total kinetic energy of the wheel?

(a) 9.0 J **(b)** 27 J

(c) 18 J **(d)** 36 J

40. Moment of inertia of a uniform semicircular ring of mass M and radius R about an axis passing through the middle point of the arc P and perpendicular to the plane of the ring, as shown in the figure, is



(a) $\frac{3}{2}MR^2$ **(b)** $2MR^2$

(c) $2MR^2\left(1-\frac{2}{\pi}\right)$ **(d)** $2MR^2\left(1+\frac{2}{\pi}\right)$

41. Which one of the following statements concerning the moment of inertia I is false?

(a) Moment of inertia I may be expressed in units of kg·m².

(b) Moment of inertia I depends on the angular acceleration of the object as it rotates.

(c) Moment of inertia I depends on the location of the rotation axis relative to the particles that make up the object.

(d) Moment of inertia I depends on the orientation of the rotation axis relative to the particles that make up the object

42. Consider the following four objects:

(i) a hoop **(iii)** a solid sphere

(ii) a flat disk **(iv)** a hollow sphere

Each of the objects has mass M and radius R. The axis of rotation passes through the center of each object and is perpendicular to the plane of the hoop and the plane of the flat disk. Which object requires the largest torque to give it the same angular acceleration?

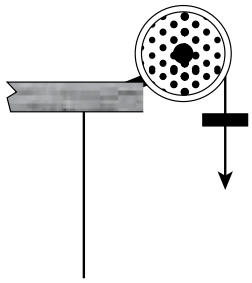
(a) The solid sphere

(b) The hollow sphere

(c) The hoop

(d) The flat disk

43. A 45 N brick is suspended by a light string from a 2.0 kg pulley. The brick is released from rest and falls to the floor below as the pulley rotates through 5.0 rad. The pulley may be considered a solid disk of radius 1.5 m. What is the angular speed of the pulley?



(a) 17 rad/s **(b)** 8.1 rad/s

(c) 15 rad/s **(d)** 7.3 rad/s

More than One Correct Choice Type

44. A rigid body is in pure rotation, that is, undergoing fixed-axis rotation. Then which of the following statement(s) are true:

(a) You can find two points in the body in a plane perpendicular to the axis of rotation having same velocity.

(b) You can find two points in the body in a plane perpendicular to the axis of rotation having same acceleration.

(c) Speed of all the particles lying on the curved surface of a cylinder whose axis coincides with the axis of rotation is same.

(d) Angular speed of the body is same as seen from any point in the body.

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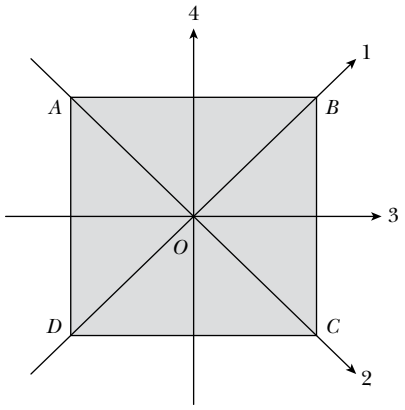
45. The axis of rotation of a purely rotating body
- **(a)** must pass through the center of mass
- **(b)** may pass through the center of mass
- **(c)** must pass through a particle of the body
- **(d)** may pass through a particle of the body

46. The moment of inertia of thin square plate $ABCD$ (see figure) of uniform thickness about an axis passing through the center O and perpendicular to the plane of the plate is (I_1, I_2, I_3, and I_4, respectively, are the moments of inertia about axes 1, 2, 3, and 4 which are in the plane of the plate)



- **(a)** $I_1 + I_2$
- **(b)** $I_3 + I_4$
- **(c)** $I_1 + I_3$
- **(d)** $I_1 + I_2 + I_3 + I_4$

47. When toque is acting on a system, then the
- **(a)** system may be in translational equilibrium but not in rotational equilibrium
- **(b)** system is neither in translation equilibrium nor in rotational equilibrium
- **(c)** angular momentum of the system remains conserved
- **(d)** angular momentum of the system is not conserved

48. When no external force acts on a system, then
- **(a)** the center of mass remains stationary
- **(b)** the center of mass accelerates
- **(c)** the momentum of the center of mass of the system remains conserved
- **(d)** the position of the center of mass of the system shifts

49. A body moves on a straight line with a uniform velocity. Its angular momentum
- **(a)** is zero about a point on the straight line
- **(b)** remains constant about any given point
- **(c)** is always zero
- **(d)** is not zero about a point away from the straight line

50. A sphere is rolled on a rough horizontal surface. It gradually slows down and finally stops. The force of friction tries to
- **(a)** decrease the angular velocity of the sphere
- **(b)** increase the angular velocity of the sphere
- **(c)** decrease the linear velocity of the sphere
- **(d)** increase the linear momentum of the sphere

Linked Comprehension

Paragraph for Questions 51 and 52: A grindstone of radius 4.0 m is initially spinning with an angular speed of 8.0 rad/s. The angular speed is then increased to 10 rad/s over the next 4.0 s. Assume that the angular acceleration is constant.

51. What is the magnitude of the angular acceleration of the grindstone?
- **(a)** 0.50 rad/s^2
- **(b)** 4.5 rad/s^2
- **(c)** 18 rad/s^2
- **(d)** 2.0 rad/s^2

52. Through how many revolutions does the grindstone turn during the 4.0 s interval?
- **(a)** 0.64
- **(b)** 4.0
- **(c)** 3.8
- **(d)** 5.7

Paragraph for Questions 53 and 54: A wheel of radius 0.5 m rotates with a constant angular speed about an axis perpendicular to its center. A point on the wheel that is 0.2 m from the center has a tangential speed of 2 m/s.

53. Determine the tangential speed of a point 0.4 m from the center of the wheel.
- **(a)** 0.4 m/s
- **(b)** 4 m/s
- **(c)** 2 m/s
- **(d)** 10 m/s

54. Determine the tangential acceleration of the point that is 0.2 m from the center.
- **(a)** 0.4 m/s^2
- **(b)** 4.0 m/s^2
- **(c)** zero m/s^2
- **(d)** 2.0 m/s^2

Paragraph for Questions 55—57: A long thin rod of length $2L$ rotates with a constant angular acceleration of 10 rad/s^2 about an axis that is perpendicular to the rod and passes through its center.

55. What is the ratio of the tangential acceleration of a point on the end of the rod to that of a point a distance $L/2$ from the end of the rod?
- **(a)** 1:1
- **(b)** 2:1
- **(c)** 1:4
- **(d)** 1:2

56. What is the ratio of the centripetal acceleration of a point on the end of the rod to that of a point a distance $L/2$ from the end of the rod?
- **(a)** 1:1
- **(b)** 2:1
- **(c)** 1:4
- **(d)** 1:2

57. What is the ratio of the tangential speed (at any instant) of a point on the end of the rod to that of a point a distance $L/2$ from the end of the rod?
- **(a)** 1:1
- **(b)** 1:4
- **(c)** 2:1
- **(d)** 1:2

Matrix-Match

58. Match the statement in Column I with the statements in Column II. One or more than one choice from Column II can match with a statement from Column I.

Column I	Column II
(a) When a sphere is in pure rolling on a fixed horizontal surface	**(p)** upward direction
(b) When a cylinder is in pure roiling on a fixed inclined plane in upward direction then friction force acts in	**(q)** $v_{com} > r\omega$

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Column I	Column II
(c) When a cylinder is in pure rolling down a fixed incline plane, friction force acts in	**(r)** $v_{com} < r\omega$
(d) When a sphere of radius R is rolling with slipping on a fixed horizontal surface, the relation between v_{com} and ω is	**(s)** no frictional force acts.
	(t) no frictional force acts.

Directions for Questions 59—60: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

59. In the given table, Column I gives masses of rigid bodies, Column II their length and Column III gives their accelaration.

Column I	Column II	Column III
(I) Mass = 12.5 kg	**(i)** Length = 10 m	**(J)** Acceleration = 11.3 m/s^2
(II) Mass = 18.8 kg	**(ii)** Length = 12.2 m	**(K)** Acceleration = 13 m/s^2
(III) Mass = 24.3 kg	**(iii)** Length = 10.1 m	**(L)** Acceleration = 14.5 m/s^2
(IV) Mass = 16.7 kg	**(iv)** Length = 11.3 m	**(M)** Acceleration = 18.2 m/s^2

(1) Which object has 2048.13 N as the net torque?
- **(a)** (I) (iv) (L)
- **(b)** (I) (iii) (L)
- **(c)** (II) (i) (M)
- **(d)** (I) (ii) (M)

(2) Which object has 1887.1 N as the net torque?
- **(a)** (I) (iii) (K)
- **(b)** (IV) (iii) (L)
- **(c)** (II) (iii) (L)
- **(d)** (IV) (i) (J)

(3) Which object has 2981.7 N as the net torque?
- **(a)** (III) (i) (J)
- **(b)** (I) (i) (L)
- **(c)** (II) (ii) (K)
- **(d)** (II) (iii) (M)

60. In the given table, Column I gives forces $(\vec{F})$ acting on rigid bodies.

Column I	Column II	Column III
(I) Force = 120 N	**(i)** Radius = 2.2 m	**(J)** $\phi = 90°$
(II) Force = 112 N	**(ii)** Radius = 3.6 m	**(K)** $\phi = 30°$
(III) Force = 99 N	**(iii)** Radius = 6.2 m	**(L)** $\phi = 45°$
(IV) Force = 125 N	**(iv)** Radius = 4 m	**(M)** $\phi = 60°$

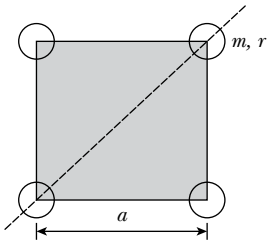
(1) Which object has 137.5 N·m as the resulting torque (as a vector)?
- **(a)** (I) (iii) (L)
- **(b)** (IV) (i) (M)
- **(c)** (II) (iv) (K)
- **(d)** (IV) (i) (K)

(2) Which object has $178.2\sqrt{3}$ N·m as the resulting torque (as a vector)?
- **(a)** (III) (ii) (L)
- **(b)** (III) (ii) (M)
- **(c)** (II) (iii) (K)
- **(d)** (I) (i) (M)

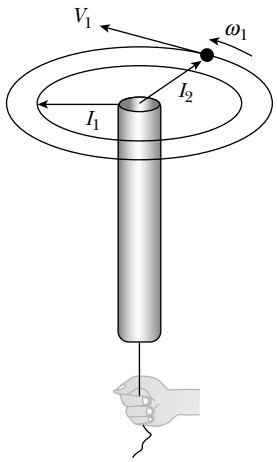
(3) Which object has 480 N·m as the resulting torque (as a vector)?
- **(a)** (I) (iv) (J)
- **(b)** (I) (i) (K)
- **(c)** (III) (iii) (L)
- **(d)** (I) (ii) (L)

Integer Type

61. Four solid spheres each of diameter $\sqrt{5}$ cm and mass 0.5 kg are placed with their centers at the corners of a square of side 4 cm. The moment of inertia of the system about the diagonal of the square is $N \times 10^{-4}$ kg·m^2, then N is _____.



62. A point mass is tied to one end of a cord whose other end passes through a vertical hollow tube, caught in one hand, the point mass is being rotated in a horizontal circle of radius 2 m with, speed of 4 m/s (see the figure). The cord is then pulled down so that the radius of the circle reduces to 1 m. Compute the ratio of kinetic energy under the initial and final states.



63. Two small balls A and B, each or mass m, are joined rigidly by a light horizontal rod of length L. The rod is clamped at the center such a way that it can rotate freely about a vertical axis through its center. The system is rotate with an angular speed 3 rad/s about the axis. A particle of mass m kept at rest sticks to the ball A as the ball collides with it. The new angular speed of the rod is _____.

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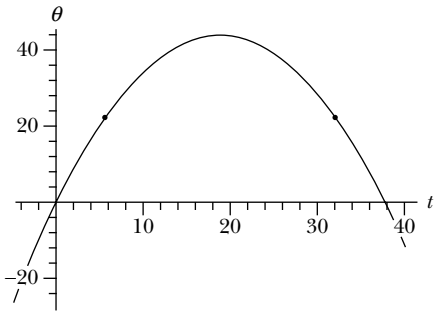
ANSWER KEY

Checkpoints

1. (b) and (c)
2. (a) and (d) ($a = d^2\theta/dt^2$) must be a constant.
3. (a) yes; (b) no; (c) yes; (d) yes.
4. All tie.
5. 1, 2, 4, 3
6. mr^2
7. $(7/5)mr^2$
8. $ml^2/12$
9. $(3/4)MR^2$
10. 1 and 3 tie, then 2 and 5 tie (zero).
11. (a) z or $-z$; (b) y; (c) $-x$
12. $mgv \cos \theta t$
13. (a) Downward in the figure ($\tau_{net} = 0$); (b) less (consider moment arms).
14. (a) 1 and 3 tie, then 2 and 4 tie, then 5 (zero); (b) 2 and 3
15. (a) 3, 1, then 2 and 4 tie (zero); (b) 3
16. (a) All tie (same t, thus same ΔL); (b) sphere, disk, hoop (reverse order of I).
17. (a) Decreases; (b) same (net = 0, so L is conserved); (c) increases.
18. $v_0/5I$

Problems

1. (a) 400 rev/min; (b) $\approx 6.28 \times 10^3$ m
2. 1800 rev
3. (a) 4.0 rad/s; (b) 11.9 rad/s
4. (a) 2.0 rad; (b) 0; (c) 78 rad/s; (d) 4.0 s; (e) Its angular acceleration is not constant.
5. $\approx$ 50 rad/s
6. (a) 4.0 m/s; (b) No – there is no dependence on radial position in the computation of part (a).
7. (a) $1.2t^5 - 1.3t^3 + 2.5$; (b) $0.20t^6 - 0.33t^4 + 2.0t + 1.5$
8. (a) 5.03 rad/s^2; (b) 8.00 kg·m^2
9. (a) 1.6 rad/s^2; (b) 4.0 rad/s; (c) 8.0 rad/s; (d) 60 rad
10. 10 s
11. (a) 1.0×10^4 rev/min; (b) 4.4×10^2 rev
12. (a) 3.4×10 s; (b) -4.5×10^{-3} rad/s^2; (c) 98 s
13. (a) 1.0 rev/s^2; (b) 4.8 s; (c) 9.6 s; (d) 48 rev
14. 8.0 s
15. (a) 44 rad; (b) $q_1 = 22$ rad is $t = 5.5$ s; (c) $q_1 = 22$ rad is $t = 32$ s; (d) $t = -2.1$ s; (e) $t = -2.1$ s; (f) With radians and seconds understood, the graph of q versus t is shown below (with the points found in the previous parts indicated as small dots).



16. $\approx$ 0.27 Hz
17. (a) 9.6 cm/s; (b) 3.8 cm/s^2
18. 6.9×10^{-13} rad/s
19. $\alpha = \tan^{-1}(1.50) = 68.2°$
20. (a) 20.9 rad/s; (b) 12.5 m/s; (c) 800 rev/min^2; (d) 600 rev
21. (a) 7.3×10^{-5} rad/s; (b) 3.5×10^2 m/s; (c) 7.3×10^{-5} rad/s; (d) 4.6×10^2 m/s
22. (a) -1.2 rev/min^2; (b) 1.1×10^4 rev; (c) $a_t = -1.1$ mm/s^2; this is independent of the given angular velocity; (d) 31 m/s^2
23. (a) 0.73 m/s^2; (b) .075; (c) 0.11
24. 13 s
25. (a) 3.8×10^3 rad/s; (b) 1.9×10^2 m/s
26. (a) 40 s; (b) 2.0 rad/s^2
27. (a) 2.60 m/s^2; (b) 76.7°
28. (a) $\approx$7.3 s; (b) $\approx$ 4.97 s
29. (a) 1.5 rad/s^2; (b) 0.40 J
30. (a) $x = 0.83$ m; (b) 2.9×10^{-4} J
31. 2.5 kg
32. 0.075 kg × m^2
33. (a) $\approx$71%; (b) $\approx$64%
34. (a) 47.25 J; (b) 11.25 W
35. (a) 8.352×10^{-3} kg·m^2; (b) 0.22%
36. (a) 0.023 kg·m^2; (b) 1.1 × 10 J
37. (a) 6.6×10^{-2} m/s; (b) $\approx 1.9 \times 10^{-3}$ m; (c) 8.8×10^{-2} kg·m^2
38. 4.7×10^{-4} kg·m^2
39. (a) 3.0 kg·m^2; (b) 9.0 kg·m^2; (c) 3.0 kg·m^2
40. 12 N·m
41. 3.0 m
42. 490 N
43. $\approx$ 19
44. 1.68 kg·m^2
45. (a) 6.00×10^{-2} m/s^2; (b) 4.87 N; (c) 4.54 N; (d) 1.20 rad/s^2; (e) 1.38×10^{-2} kg·m^2
46. (a) 6.5 rad/s^2; (b) The direction is counterclockwise (which is the positive sense of rotation).
47. 0.140 N
48. (a) $\approx$3.2 rad/s^2; (b) 9.6 rad/s^2
49. (a) 1.7 m/s; (b) 6.9 m/s^2
50. (a) 4.2×10^2 rad/s^2; (b) 5.0×10^2 rad/s
51. (a) 1.5×10^2 cm/s; (b) 1.5×10^2 cm/s = 1.5 m/s, since it is independent of R.
52. 29.7 m/s
53. (a) 0.48 J: (b) $\approx$ 0.12 m
54. 5.42 m/s
55. (a) 0.24 kg·m^2; (b) 10 rad/s

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56. (a) 5.32 m/s²; (b) 8.43 m/s²; (c) 41.8° **57.** 1.4 m/s **58.** 9.82 rad/s

59. $(42\hat{i} - 24\hat{j} + 22\hat{k})$ N·m **60.** $(2.0\ \text{N}\cdot\text{m})\hat{j}$ **61.** (a) $(6.0\ \text{N}\cdot\text{m})\hat{j} + (8.0\ \text{N}\cdot\text{m})\hat{k}$; (b) $(-22\ \text{N}\cdot\text{m})\hat{i}$

62. –5.00 N **63.** (a) $(-1.5\ \text{N}\cdot\text{m})\hat{i} - (4.0\ \text{N}\cdot\text{m})\hat{j} - (1.0\ \text{N}\cdot\text{m})\hat{k}$; (b) $(-1.5\ \text{N}\cdot\text{m})\hat{i} - (4.0\ \text{N}\cdot\text{m})\hat{j} - (1.0\ \text{N}\cdot\text{m})\hat{k}$

64. (a) $(6.00\ \text{N}\cdot\text{m})\hat{k}$; (b) 165°

65. (a) 20 kg·m²/s; (b) Using the right-hand rule for vector products, we find $\vec{r}\times\vec{p}$ points out of the page, or along the $+z$ axis, perpendicular to the plane of the figure; (c) 5.0 N·m; (d) Using the right-hand rule for vector products, we find $r\times\vec{F}$ is also out of the page, or along the $+z$ axis, perpendicular to the plane of the figure.

66. (a) $\frac{1}{2}mu^2\sin 2\theta$; (b) $\frac{mu^3\sin^2\theta\cos\theta}{2g}$ **67.** (a) $(9.0\times10^2\ \text{kg}\cdot\text{m}^2/\text{s})\hat{k}$; (b) $(11.0\times10^2\ \text{kg}\cdot\text{m}^2/\text{s})\hat{k}$

69. (a) $(4.00\ \text{m/s}^2)\hat{i} - (5.33\ \text{m/s}^2)\hat{j} + (2.67\ \text{m/s}^2)\hat{k}$; (b) $(31.5\ \text{kg}\cdot\text{m}^2/\text{s})\hat{i} + (18.0\ \text{kg}\cdot\text{m}^2/\text{s})\hat{j} + (45.0\ \text{kg}\cdot\text{m}^2/\text{s})\hat{k}$; (c) $(-8.00\ \text{N}\cdot\text{m})\hat{i} - (26.0\ \text{N}\cdot\text{m})\hat{j} - (40.0\ \text{N}\cdot\text{m})\hat{k}$; (d) 127°

70. (a) Zero; (b) –22.6 kg·m²/s; (c) –7.84 N·m; (d) Due to the way $r_\perp$ is defined, it does not matter how far up the ball is and hence the answer is the same as in part (c).

71. $(2.0\ \text{N}\cdot\text{m})\hat{i} + (-3.0\ \text{N}\cdot\text{m})\hat{j}$ **72.** (a) mgb; (b) $mgbt$; (c) $mgb = \tau$ **73.** $\frac{m\omega d^2}{4}$

74. 24.4 N·m **75.** 1024 **76.** (a) $4.6\times10^{-3}\ \text{kg}\cdot\text{m}^2$; (b) $1.1\times10^{-3}\ \text{kg}\cdot\text{m}^2/\text{s}$; (c) $3.9\times10^{-3}\ \text{kg}\cdot\text{m}^2/\text{s}$

77. (a) 61.4 rad/s; (b) $\approx 5.9\times10^2$ rev/min **78.** (a) 1.47 N·m; (b) 20.4 rad; (c) –29.9 J; (d) 19.9 W **79.** 49.0 kg·m²/s

80. (a) 1.6 kg·m²; (b) 4.0 kg·m²/s

81. (a) 1.5 m; (b) 0.93 rad/s; (c) 98 J; (d) 8.4 rad/s; (e) 8.8×10^2 J; (f) We account for the large increase in kinetic energy [part (e) minus part (c)] by noting that the skaters do a great deal of work (converting their internal energy into mechanical energy) as they pull themselves closer – "fighting" what appears to them to be large "centrifugal forces" trying to keep them apart.

82. (a) 5.1 rad/s; (b) Some of the initial kinetic energy is "lost" – that is, transferred to another form. And the culprit is the cockroach, who must find it difficult to stop (and "internalize" that energy).

83. (a) 3.6 rev/s; (b) 3.0; (c) The man did work in decreasing the rotational inertia by pulling the bricks closer to his body. This energy came from the man's internal energy.

84. 4 **85.** 0.17 rad/s

86. (a) 750 rev/min; (b) 450 rev/min; (c) In the direction of the second disk's initial angular velocity. **87.** 4.2×10^2 rev

88. (a) $4.00\times10^{-2}\ \text{kg}\cdot\text{m}^2$; (b) –3.07 J

89. (a) 0.427 rad/s; (b) 1.33; (c) The cockroach does positive work while walking toward the center of the disk, increasing the total kinetic energy of the system.

90. (a) $\omega = v/L$; (b) Zero; (c) Zero; hence, the kinetic energy of the system is conserved. **91.** 29.3 J

92. $v/2R$ **93.** 2.4 rad/s **94.** (a) 10 m/s; (b) 210 N/m

95. (a) 0.40 kg·m²; (b) 2.4×10^3 m/s **96.** 1.5 rad/s **97.** 0.070 rad/s

98. (a) 0.148 rad/s; (b) 0.0123; (c) 181° **99.** 29.4°

100. (a) ≈0.42 rev/s; (b) The direction of the precession is clockwise as viewed from overhead.

Practice Questions

Single Correct Choice Type

1. (a)	**2.** (a)	**3.** (d)	**4.** (b)	**5.** (b)
6. (b)	**7.** (c)	**8.** (c)	**9.** (b)	**10.** (a)
11. (a)	**12.** (c)	**13.** (b)	**14.** (c)	**15.** (c)
16. (c)	**17.** (a)	**18.** (b)	**19.** (d)	**20.** (b)
21. (c)	**22.** (c)	**23.** (d)	**24.** (c)	**25.** (c)
26. (d)	**27.** (b)	**28.** (a)	**29.** (d)	**30.** (b)
31. (d)	**32.** (b)	**33.** (a)	**34.** (a)	**35.** (d)
36. (a)	**37.** (c)	**38.** (a)	**39.** (b)	**40.** (c)
41. (b)	**42.** (c)	**43.** (a)		

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CHAPTER 11

Rigid Body Dynamics – II

11.1 | WHAT IS PHYSICS?

In Chapter 10 we discussed pure rotational motion of a rigid body. We learnt many new concepts and new physical quantities, namely, moment of inertia, torque, and angular momentum. In this chapter, we will discuss general plane motion of rigid body popularly known as combined rotation and translation. Our methods and tools for analyzing this type of motion are going to be similar as discussed in the previous chapter. We will also develop new understanding of equilibrium and stability.

11.2 | KINEMATICS OF COMBINED ROTATION AND TRANSLATION

Key Concepts

- For a wheel of radius R rolling smoothly,

$$v_{com} = \omega R,$$

where v_{com} is the linear speed of the wheel's center of mass and ω is the angular speed of the wheel about its center.

- The wheel may also be viewed as rotating instantaneously about any point P of the "road" that is in contact with the wheel. The angular speed of the wheel about this point is the same as the angular speed of the wheel about its center.

As we discussed in Chapter 10, physics includes the study of rotation. Arguably, the most important application of that physics is in the rolling motion of wheels and wheel-like objects. This applied physics has long been used. For example, when the prehistoric people of Easter Island moved their gigantic stone statues from the quarry and across the island, they dragged them over logs acting as rollers. Much later, when settlers moved westward across America in the 1800s, they rolled their possessions first by wagon and then later by train. Today, like it or not, the world is filled with cars, trucks, motorcycles, bicycles, and other rolling vehicles.

The physics and engineering of rolling have been around for so long that you might think no fresh ideas remain to be developed. However,

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11.2 Kinematics of Combined Rotation and Translation
11.3 The Kinetic Energy of Combined Translation and Rotation (Rolling)
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11.6 Angular Momentum of Rigid Body About a Point Lying Outside the Body
11.7 The Yo-Yo
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11.11 Some Examples of Static Equilibrium
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skateboards and inline skates were invented and engineered fairly recently, to become huge financial successes. Street luge is now catching on, and the self-righting Segway (Fig. 11-1) may change the way people move around in large cities. Applying the physics of rolling can still lead to surprises and rewards. Our starting point in exploring that physics is to simplify rolling motion.

Human contructions are supposed to be stable in spite of the forces that act on them. A building, for example, should be stable in spite of the gravitational force and wind forces on it, and a bridge should be stable in spite of the gravitational force pulling it downward and the repeated jolting it receives from cars and trucks.

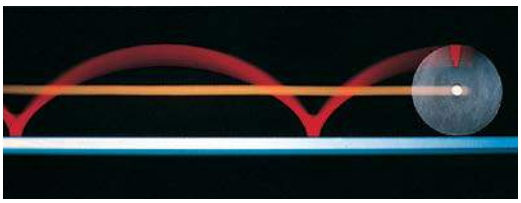
One focus of physics is on what allows an object to be stable in spite of any forces acting on it. In this chapter we examine the two main aspects of stability: the *equilibrium* of the forces and torques acting on rigid objects and the *elasticity* of nonrigid objects, a property that governs how such objects can deform. When this physics is done correctly, it is the subject of countless articles in physics and engineering journals; when it is done incorrectly, it is the subject of countless articles in newspapers and legal journals.

Justin Sullivan/Getty Images, Inc.

Figure 11-1 The self-righting Segway Human Transporter.

Rolling as Translation and Rotation Combined

Here we consider only objects that *roll smoothly* along a surface; that is, the objects roll without slipping or bouncing on the surface. Figure 11-2 shows how complicated smooth rolling motion can be: Although the center of the object moves in a straight line parallel to the surface, a point on the rim certainly does not. However, we can study this motion by treating it as a combination of translation of the center of mass and rotation of the rest of the object around that center.



Richard Megna/Fundamental Photographs

Figure 11-2 A time-exposure photograph of a rolling disk. Small lights have been attached to the disk, one at its center and one at its edge. The latter traces out a curve called a *cycloid.*

To see how we do this, pretend you are standing on a sidewalk watching the bicycle wheel of Fig. 11-3 as it rolls along a street. As shown, you see the center of mass O of the wheel move forward at constant speed v_{com}. The point P on the street where the wheel makes contact with the street surface also moves forward at speed v_{com}, so that P always remains directly below O.

During a time interval t, you see both O and P move forward by a distance s. The bicycle rider sees the wheel rotate through an angle θ about the center of the wheel, with the point of the wheel that was touching the street at the beginning of t moving through arc length s. Equation 10-17 relates the arc length s to the rotation angle θ:

$$s = \theta R, \tag{11-1}$$

where R is the radius of the wheel. The linear speed v_{com} of the center of the wheel (the center of mass of this uniform wheel) is ds/dt. The angular speed ω of the wheel about its center is $d\theta/dt$. Thus, differentiating Eq. 11-1 with respect to time (with R held constant) gives us

$$v_{com} = \omega R \quad \text{(smooth rolling motion)}. \tag{11-2}$$

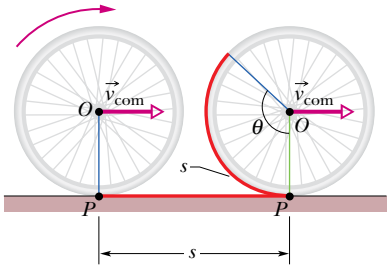


Figure 11-3 The center of mass O of a rolling wheel moves a distance s at velocity $\vec{v}_{com}$ while the wheel rotates through angle θ. The point P at which the wheel makes contact with the surface over which the wheel rolls also moves a distance s.

A Combination. Figure 11-4 shows that the rolling motion of a wheel is a combination of purely translational and purely rotational motions. Figure 11-4*a* shows the purely rotational motion (as if the rotation axis through the center were stationary): Every point on the wheel rotates about the center with angular speed ω. (This is the type of motion we considered in Chapter 10.) Every point on the outside edge of the wheel has linear speed v_{com} given by Eq. 11-2.

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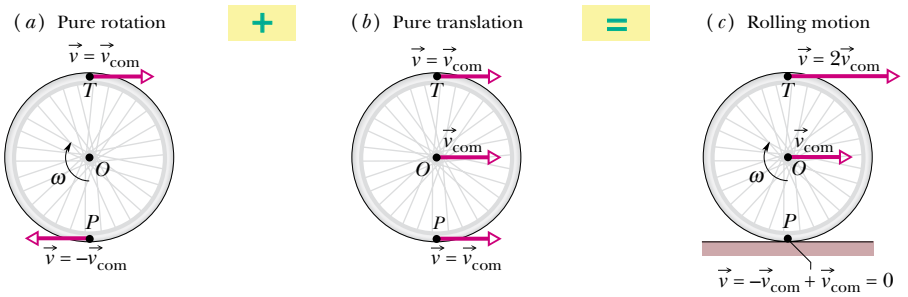

Figure 11-4 Rolling motion of a wheel as a combination of purely rotational motion and purely translational motion. (*a*) The purely rotational motion: All points on the wheel move with the same angular speed ω. Points on the outside edge of the wheel all move with the same linear speed $v = v_{com}$. The linear velocities $\vec{v}$ of two such points, at top (T) and bottom (P) of the wheel, are shown. (*b*) The purely translational motion: All points on the wheel move to the right with the same linear velocity $\vec{v}_{com}$. (*c*) The rolling motion of the wheel is the combination of (*a*) and (*b*).

Figure 11-4*b* shows the purely translational motion (as if the wheel did not rotate at all): Every point on the wheel moves to the right with speed v_{com}.

The combination of Figs. 11-4*a* and 11-4*b* yields the actual rolling motion of the wheel, Fig. 11-4*c*. Note that in this combination of motions, the portion of the wheel at the bottom (at point P) is stationary and the portion at the top (at point T) is moving at speed $2v_{com}$, faster than any other portion of the wheel. These results are demonstrated in Fig. 11-5, which is a time exposure of a rolling bicycle wheel. You can tell that the wheel is moving faster near its top than near its bottom because the spokes are more blurred at the top than at the bottom.

The motion of any round body rolling smoothly over a surface can be separated into purely rotational and purely translational motions, as in Figs. 11-4*a* and 11-4*b*.

Courtesy Alice Halliday

Figure 11-5 A photograph of a rolling bicycle wheel. The spokes near the wheel's top are more blurred than those near the bottom because the top ones are moving faster, as Fig. 11-4*c* shows.

Let's consider a bicycle going on a straight road in a straight line. The motion of any arbitrary point on wheel is quite complicated. To make the visualization of motion simple, we apply concept of net motion. We understand that motion of center of the wheel can be easily observed by us and in this case, it is a straight-line motion. If we put an observer on the bicycle and ask him to describe the motion of any point, on the wheel he will describe it as pure rotational motion about the axle. In other words, he will observe all the particles on the wheel undergoing circular motion with same angular velocity, angular acceleration, and undergoing same angular displacement about point lying on axle. Thus, we can say that

$$\vec{r}_P = \vec{r}_O + \vec{r}_{O/A};$$
$$\vec{s}_P = \vec{s}_{P/O} + \vec{s}_O;$$
$$\vec{v}_P = \vec{v}_{P/O} + \vec{v}_O;$$
$$\vec{a}_P = \vec{a}_{P/O} + \vec{a}_O.$$

where

$$\vec{s}_{P/O} = \theta_P \vec{r}_{P/O}; \quad \vec{v}_{P/O} = \omega \vec{r}_{P/O}; \quad \vec{a}_{P/O} = \alpha \vec{r}_{P/O}.$$

Thus, our strategy for solving problems in this chapter will involve finding angular parameters of motion of P about O and finding linear parameter of motion of O. If O is center of mass, we can find its acceleration using

$$\vec{F}_{ext} = \vec{M}a_{com}.$$

Later, we will prove that we will be able to use $\tau_{ext} = I\alpha$ about center of mass even if it is accelerating.

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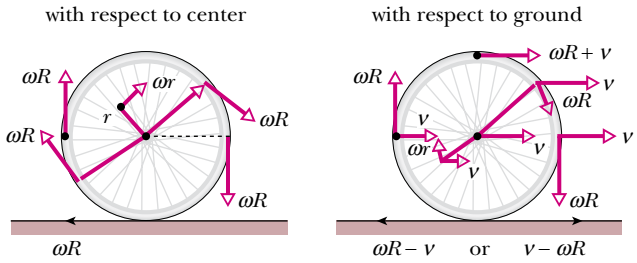
PROBLEM-SOLVING TACTICS

Tactic 1: Do not relate v and ω of wheel by writing $v = \omega R$. It is possible to have one without having the other. If we lock the cycle and pull it, we will have translational motion of the wheel and no rotation. If we keep it on stand and rotate the wheel we will have angular velocity but no linear velocity. These two parameters are completely independent of each other.

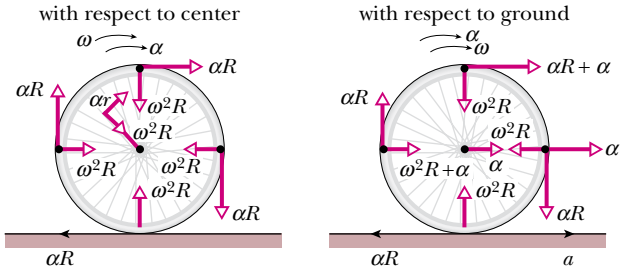
Let's say the bicycle is going at velocity v and wheel is rotating with angular velocity ω. The following figures help us finding velocity of any arbitrary point. All points on wheel must have a velocity perpendicular to the line joining the center in frame attached to the center.

Kinematics

1. Velocity



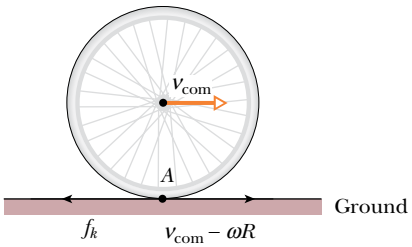
2. Acceleration



Rolling and Sliding

Case 1: $v_{com} > \omega R$

The friction will be kinetic in nature and its magnitude can be determined by using $f_k = \mu_k F_N$. Direction will be opposite to v_{com} (point of contact A is moving forward with respect to ground. This is how brakes stop a car).



Case 1

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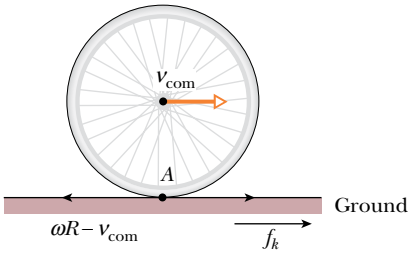
Case 2: $v_{com} < \omega R$

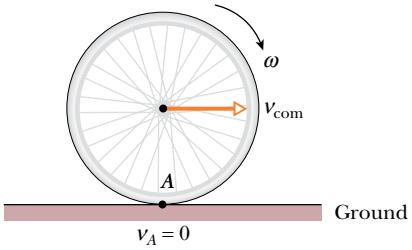
The friction will be kinetic in nature and its magnitude can be determined using $f_k = \mu_k F_N$. Direction will be same as v_{com} (point of contact A is moving backward with respect to ground. This is how a car accelerates because of friction).



Case 2

Case 3: $v_{com} = \omega R$

Known as perfect rolling. Maximum problems are faced when we come across this situation.



Case 3

1. The friction will be static in nature. Its magnitude cannot be determined, it will vary from 0 to $\mu_s F_N$.
2. Even its direction cannot be predicted. We can assume any direction and answer will come with sign. For example, if it is assumed that friction is acting toward left and the value comes to be –3 N, it means it is acting towards right.
3. The total work done by static friction is zero. Thus, mechanical energy of the system will remain conserved.

Rolling as Pure Rotation

Figure 11-6 suggests another way to look at the rolling motion of a wheel—namely, as pure rotation about an axis that always extends through the point where the wheel contacts the street as the wheel moves. We consider the rolling motion to be pure rotation about an axis passing through point P in Fig. 11-4*c* and perpendicular to the plane of the figure. The vectors in Fig. 11-6 then represent the instantaneous velocities of points on the rolling wheel.

Question: What angular speed about this new axis will a stationary observer assign to a rolling bicycle wheel?
Answer: The same ω that the rider assigns to the wheel as she or he observes it in pure rotation about an axis through its center of mass.

To verify this answer, let us use it to calculate the linear speed of the top of the rolling wheel from the point of view of a stationary observer. If we call the wheel's radius R, the top is a distance $2R$ from the axis through P in Fig. 11-6, so the linear speed at the top should be (using Eq. 11-2)

$$v_{top} = (\omega)(2R) = 2(\omega R) = 2v_{com},$$

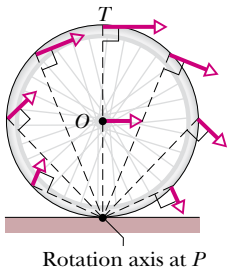


Figure 11-6 Rolling can be viewed as pure rotation, with angular speed ω, about an axis that always extends through P. The vectors show the instantaneous linear velocities of selected points on the rolling wheel. You can obtain the vectors by combining the translational and rotational motions as in Fig. 11-4.

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in exact agreement with Fig. 11-4*c*. You can similarly verify the linear speeds shown for the portions of the wheel at points *O* and *P* in Fig. 11-4*c*.

CHECKPOINT 1

The rear wheel on a clown's bicycle has twice the radius of the front wheel. (a) When the bicycle is moving, is the linear speed at the very top of the rear wheel greater than, less than, or the same as that of the very top of the front wheel? (b) Is the angular speed of the rear wheel greater than, less than, or the same as that of the front wheel?

11.3 | THE KINETIC ENERGY OF COMBINED TRANSLATION AND ROTATION (ROLLING)

Key Concept

- A smoothly rolling wheel has kinetic energy

$$K = \frac{1}{2} I_{com} \omega^2 + \frac{1}{2} M v_{com}^2,$$

where I_{com} is the rotational inertia of the wheel about its center of mass and M is the mass of the wheel.

Let us now calculate the kinetic energy of the rolling wheel as measured by the stationary observer. If we view the rolling as pure rotation about an axis through *P* in Fig. 11-6, then from Eq. 10-34 we have

$$K = \frac{1}{2} I_P \omega^2, \tag{11-3}$$

in which ω is the angular speed of the wheel and I_P is the rotational inertia of the wheel about the axis through *P*. From the parallel-axis theorem of Eq. 10-36 ($I = I_{com} + Mh^2$), we have

$$I_P = I_{com} + MR^2, \tag{11-4}$$

in which *M* is the mass of the wheel, I_{com} is its rotational inertia about an axis through its center of mass, and *R* (the wheel's radius) is the perpendicular distance *h*. Substituting Eq. 11-4 into Eq. 11-3, we obtain

$$K = \frac{1}{2} I_{com} \omega^2 + \frac{1}{2} MR^2 \omega^2,$$

and using the relation $v_{com} = \omega R$ (Eq. 11-2) yields

$$K = \frac{1}{2} I_{com} \omega^2 + \frac{1}{2} M v_{com}^2. \tag{11-5}$$

We can interpret the term $(1/2) I_{com} \omega^2$ as the kinetic energy associated with the rotation of the wheel about an axis through its center of mass (Fig. 11-4*a*), and the term $(1/2) M v_{com}^2$ as the kinetic energy associated with the translational motion of the wheel's center of mass (Fig. 11-4*b*). Thus, we have the following rule:

A rolling object has two types of kinetic energy: a rotational kinetic energy ($1/2 I_{com} \omega^2$) due to its rotation about its center of mass and a translational kinetic energy ($1/2 M v_{com}^2$) due to translation of its center of mass.

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SAMPLE PROBLEM 11.01

Setting the land-speed record, kinetic energy of car's each wheel

The current land-speed record was set in the Black Rock Desert of Nevada in 1997 by the jet-powered car *Thrust SSC*. The car's speed was 1222 km/h in one direction and 1233 km/h in the opposite direction. Both speeds exceeded the speed of sound at that location (1207 km/h).

Setting the land-speed record was obviously very dangerous for many reasons. One of them had to do with the car's wheels. Approximate each wheel on the car *Thrust SSC* as a disk of uniform thickness and mass $M = 170$ kg and assume smooth rolling. When the car's speed was 1233 km/h, what was the kinetic energy of each wheel?

KEY IDEAS

Equation 11-5 gives the kinetic energy of a rolling object, but we need three ideas to use it:

(1) When we speak of the speed of a rolling object, we always mean the speed of the center of mass, so here $v_{com} = 1233$ km/h $= 342.5$ m/s.

(2) Equation 11-5 requires the angular speed ω of the rolling object, which we can relate to v_{com} with Eq. 11-2, writing $\omega = v_{com}/R$, where R is the wheel's radius.

(3) Equation 11-5 also requires the rotational inertia I_{com} of the object about its center of mass. From Table 10-2*c*, we find that, for a uniform disk, $I_{com} = (1/2)MR^2$.

Calculations: Now Eq. 11-5 gives us

$$K = \frac{1}{2} I_{com}\omega^2 + \frac{1}{2} M v_{com}^2$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{2} MR^2\right)\left(\frac{v_{com}}{R}\right)^2 + \frac{1}{2} M v_{com}^2$$

$$= \frac{3}{4}(170 \text{ kg})(342.5 \text{ m/s})^2$$

$$= 1.50 \times 10^7 \text{ J}. \qquad \text{(Answer)}$$

(***Note:*** The wheel's radius R cancels out of the calculation.)

Reasoning: This answer gives one measure of the danger when the land-speed record was set by *Thrust SSC:* The kinetic energy of each (cast aluminum) wheel on the car was huge, almost as much as the kinetic energy (2.1×10^7 J) of the spinning steel disk that exploded in Sample Problem 10.8. Had a wheel hit any hard obstacle along the car's path, the wheel would have exploded the way the steel disk did, with the car and driver moving faster than sound!

11.4 | THE FORCES OF ROLLING

Key Concepts

- If the wheel is being accelerated but is still rolling smoothly, the acceleration of the center of mass $\vec{a}_{com}$ is related to the angular acceleration α about the center with
$$a_{com} = \alpha R.$$
- If the wheel rolls smoothly down a ramp of angle θ, its acceleration along an x axis extending up the ramp is
$$a_{com,x} = -\frac{g \sin\theta}{1 + I_{com}/MR^2}.$$

Friction and Rolling

If a wheel rolls at constant speed, as in Fig. 11-3, it has no tendency to slide at the point of contact P, and thus no frictional force acts there. However, if a net force acts on the rolling wheel to speed it up or to slow it, then that net force causes acceleration $\vec{a}_{com}$ of the center of mass along the direction of travel. It also causes the wheel to rotate faster or slower, which means it causes an angular acceleration α. These accelerations tend to make the wheel slide at P. Thus, a frictional force must act on the wheel at P to oppose that tendency.

If the wheel *does not* slide, the force is a *static* frictional force $\vec{f}_s$ and the motion is smooth rolling. We can then relate the magnitudes of the linear acceleration $\vec{a}_{com}$ and the angular acceleration α by differentiating Eq. 11-2 with respect to time (with R held constant). On the left side, dv_{com}/dt is a_{com}, and on the right side dw/dt is α. So, for smooth rolling we have

$$a_{com} = \alpha R \quad \text{(smooth rolling motion)}. \qquad (11\text{-}6)$$

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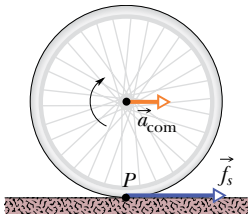

Figure 11-7 A wheel rolls horizontally without sliding while accelerating with linear acceleration $\vec{a}_{com}$, as on a bicycle at the start of a race. A static frictional force $\vec{f}_s$ acts on the wheel at P, opposing its tendency to slide.

If the wheel *does* slide when the net force acts on it, the frictional force that acts at P in Fig. 11-3 is a *kinetic* frictional force $\vec{f}_k$. The motion then is not smooth rolling, and Eq. 11-6 does not apply to the motion. In this chapter we discuss only smooth rolling motion.

Figure 11-7 shows an example in which a wheel is being made to rotate faster while rolling to the right along a flat surface, as on a bicycle at the start of a race. The faster rotation tends to make the bottom of the wheel slide to the left at point P. A frictional force at P, directed to the right, opposes this tendency to slide. If the wheel does not slide, that frictional force is a static frictional force $\vec{f}_s$ (as shown), the motion is smooth rolling, and Eq. 11-6 applies to the motion. (Without friction, bicycle races would be stationary and very boring.)

If the wheel in Fig. 11-7 were made to rotate slower, as on a slowing bicycle, we would change the figure in two ways: The directions of the center-of-mass acceleration $\vec{a}_{com}$ and the frictional force $\vec{f}_s$ at point P would now be to the left.

Rolling Down a Ramp

Figure 11-8 shows a round uniform body of mass M and radius R rolling smoothly down a ramp at angle θ, along an x axis. We want to find an expression for the body's acceleration $a_{com,x}$ down the ramp. We do this by using Newton's second law in both its linear version ($F_{net} = Ma$) and its angular version ($\tau_{net} = I\alpha$).

We start by drawing the forces on the body as shown in Fig. 11-8:

1. The gravitational force $\vec{F}_g$ on the body is directed downward. The tail of the vector is placed at the center of mass of the body. The component along the ramp is $F_g \sin\theta$, which is equal to $Mg \sin\theta$.
2. A normal force $\vec{F}_N$ is perpendicular to the ramp. It acts at the point of contact P, but in Fig. 11-8 the vector has been shifted along its direction until its tail is at the body's center of mass.
3. A static frictional force $\vec{f}_s$ acts at the point of contact P and is directed up the ramp. (Do you see why? If the body were to slide at P, it would slide *down* the ramp. Thus, the frictional force opposing the sliding must be *up* the ramp.)

We can write Newton's second law for components along the x axis in Fig. 11-8 ($F_{net,x} = ma_x$) as

$$f_s = Mg \sin\theta = Ma_{com,x}. \tag{11-7}$$

This equation contains two unknowns, f_s and $a_{com,x}$. (We should *not* assume that f_s is at its maximum value $f_{s,max}$. All we know is that the value of f_s is just right for the body to roll smoothly down the ramp, without sliding.)

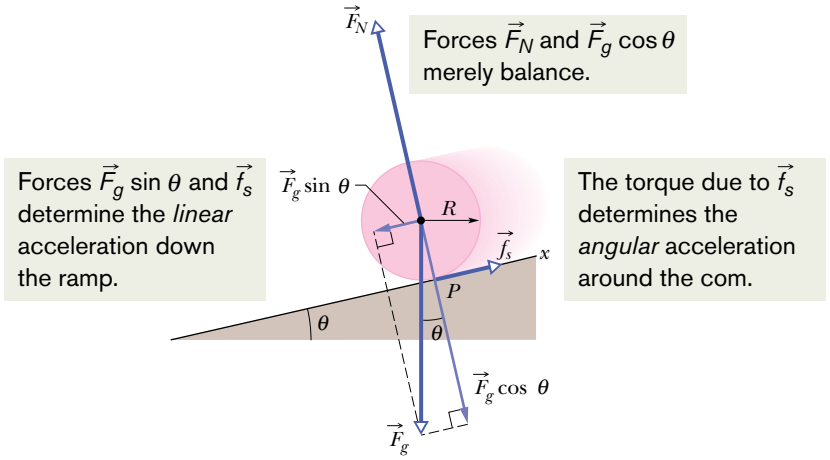

Figure 11-8 A round uniform body of radius R rolls down a ramp. The forces that act on it are the gravitational force $\vec{F}_g$, a normal force $\vec{F}_N$, and a frictional force $\vec{f}_s$ pointing up the ramp. (For clarity, vector $\vec{F}_N$ has been shifted in the direction it points until its tail is at the center of the body.)

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We now wish to apply Newton's second law in angular form to the body's rotation about its center of mass. First, we shall use Eq. 10-45 ($\tau = r_\perp F$) to write the torques on the body about that point. The frictional force $\vec{f}_s$ has moment arm R and thus produces a torque Rf_s, which is positive because it tends to rotate the body counterclockwise in Fig. 11-8. Forces $\vec{F}_g$ and $\vec{F}_N$ have zero moment arms about the center of mass and thus produce zero torques. So we can write the angular form of Newton's second law ($\tau_{net} = Ia$) about an axis through the body's center of mass as

$$Rf_s = I_{com}\alpha. \tag{11-8}$$

This equation contains two unknowns, f_s and α.

Because the body is rolling smoothly, we can use Eq. 11-6 ($a_{com} = \alpha R$) to relate the unknowns $a_{com,x}$ and α. But we must be cautious because here $a_{com,x}$ is negative (in the negative direction of the x axis) and α is positive (counterclockwise). Thus we substitute $-a_{com,x}/R$ for α in Eq. 11-8. Then, solving for f_s, we obtain

$$f_s = -I_{com}\left(\frac{a_{com,x}}{R^2}\right). \tag{11-9}$$

Substituting the right side of Eq. 11-9 for f_s in Eq. 11-7, we then find

$$a_{com,x} = -\frac{g\sin\theta}{1+(I_{com}/MR^2)}. \tag{11-10}$$

We can use this equation to find the linear acceleration $a_{com,x}$ of any body rolling along an incline of angle θ with the horizontal.

Note that the pull by the gravitational force causes the body to come down the ramp, but it is the frictional force that causes the body to rotate and thus roll. If you eliminate the friction (by, say, making the ramp slick with ice or grease) or arrange for $Mg\sin\theta$ to exceed $f_{s,max}$, then you eliminate the smooth rolling and the body slides down the ramp.

CHECKPOINT 2

Disks A and B are identical and roll across a floor with equal speeds. Then disk A rolls up an incline, reaching a maximum height h, and disk B moves up an incline that is identical except that it is frictionless. Is the maximum height reached by disk B greater than, less than, or equal to h?

SAMPLE PROBLEM 11.02

Ball rolling down a ramp

A uniform ball, of mass $M = 6.00$ kg and radius R, rolls smoothly from rest down a ramp at angle $\theta = 30.0°$ (Fig. 11-8).

(a) The ball descends a vertical height $h = 1.20$ m to reach the bottom of the ramp. What is its speed at the bottom?

KEY IDEAS

The mechanical energy E of the ball–Earth system is conserved as the ball rolls down the ramp. The reason is that the only force doing work on the ball is the gravitational force, a conservative force. The normal force on the ball from the ramp does zero work because it is perpendicular to the ball's path. The frictional force on the ball from the ramp does not transfer any energy to thermal energy because the ball does not slide (it *rolls smoothly*).

Thus, we conserve mechanical energy ($E_f = E_i$):

$$K_f + U_f = K_i + U_i, \tag{11-11}$$

where subscripts f and i refer to the final values (at the bottom) and initial values (at rest), respectively. The gravitational potential energy is initially $U_i = Mgh$ (where M is the ball's mass) and finally $U_f = 0$. The kinetic energy is initially $K_i = 0$. For the final kinetic energy K_f, we need an additional idea: Because the ball rolls, the kinetic energy involves both translation *and* rotation, so we include them both by using the right side of Eq. 11-5.

Calculations: Substituting into Eq. 11-11 gives us

$$\left(\frac{1}{2}I_{com}\omega^2 + \frac{1}{2}Mv_{com}^2\right) + 0 = 0 + Mgh, \tag{11-12}$$

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where I_{com} is the ball's rotational inertia about an axis through its center of mass, v_{com} is the requested speed at the bottom, and ω is the angular speed there.

Because the ball rolls smoothly, we can use Eq. 11-2 to substitute v_{com}/R for ω to reduce the unknowns in Eq. 11-12. Doing so, substituting $2/5MR^2$ for I_{com} (from Table 10-2*f*), and then solving for v_{com} give us

$$v_{com} = \sqrt{\left(\frac{10}{7}\right)gh} = \sqrt{\left(\frac{10}{7}\right)(9.8\ \text{m/s}^2)(1.20\ \text{m})}$$
$$= 4.10\ \text{m/s}. \qquad \text{(Answer)}$$

Note that the answer does not depend on M or R.

(b) What are the magnitude and direction of the frictional force on the ball as it rolls down the ramp?

KEY IDEA

Because the ball rolls smoothly, Eq. 11-9 gives the frictional force on the ball.

Calculations: Before we can use Eq. 11-9, we need the ball's acceleration $a_{com,x}$ from Eq. 11-10:

$$a_{com,x} = -\frac{g\sin\theta}{1+I_{com}/MR^2} = -\frac{g\sin\theta}{1+\frac{2}{5}MR^2/MR^2}$$
$$= -\frac{(9.8\ \text{m/s}^2)\sin 30.0°}{1+\frac{2}{5}} = -3.50\ \text{m/s}^2.$$

Note that we needed neither mass M nor radius R to find $a_{com,x}$. Thus, any size ball with any uniform mass would have this smoothly rolling acceleration down a 30.0° ramp.

We can now solve Eq. 11-9 as

$$f_s = -I_{com}\frac{a_{com,x}}{R^2} = -\frac{2}{5}MR^2\frac{a_{com,x}}{R^2} = -\frac{2}{5}Ma_{com,x}$$
$$= -\frac{2}{5}(6.00\ \text{kg})(-3.50\ \text{m/s}^2) = 8.40\ \text{N}. \qquad \text{(Answer)}$$

Note that we needed mass M but not radius R. Thus, the frictional force on any 6.00 kg ball rolling smoothly down a 30.0° ramp would be 8.40 N regardless of the ball's radius but would be larger for a larger mass.

11.5 | TORQUE ABOUT CENTER OF MASS

Key Concept

- Torque about center of mass of Pseudo force

$$\sum \tau_{pseudo,i} = 0$$

Some caution is needed in choosing the point to be used for writing the equation, $\tau_{net} = I\alpha$. Although we can use a point fixed in inertial space there is another convenient point about which $\tau_{net} = I\alpha$ can be used even if it is accelerated. As discussed in Chapter 6, where we derived

$$\sum \vec{F}_{ext} = M\vec{a}_{com} \quad \text{we will also derive} \quad \tau_{net} = I_{com}\alpha$$

$$\tau_{net} = \sum_{i/com} \vec{F}_{ext} = I_{com} \qquad \tau_{net} = I_{com}\alpha \qquad (11\text{-}13)$$

where the torques are taken about the center of mass. These two points, the fixed point and the center of mass, can always be used. Other points, particularly accelerated points, can be used only with great care, and sometimes a "fictitious force" must be introduced.

Torque of Pseudo Force about Center of Mass

Let's say that the center of mass is moving with acceleration $\vec{a}$. If we take center of mass as our reference frame, then we have to apply pseudo force $-M_i\vec{a}$ on all the particles. Now, torque not only involves torque due to external forces about center of mass but also involves torque due to pseudo forces about center of mass.

$$\tau_{pseudo,i} = -\vec{r}_i \times m_i\vec{a}$$

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Torque about center of mass

$$\sum \tau_{\text{pseudo},i} = \sum (\vec{r}_i \times m_i \vec{a}) = -\sum \vec{r}_i \times (m_i \vec{r}_i) \times \vec{a}.$$
$$= -M\left(\frac{\sum (m_i \vec{r}_i)}{M}\right) \times \vec{a}$$
$$= -M(\vec{r}_{\text{com}} \times \vec{a})$$
$$= -(\vec{r}_{\text{com}} \times (M\vec{a})) = 0$$
$$= 0$$

11.6 | ANGULAR MOMENTUM OF RIGID BODY ABOUT A POINT LYING OUTSIDE THE BODY

Key Concept

- Angular momentum of a body about any point or an axis is equal to the sum of angular momentum of the body about center of mass for an axis parallel to that axis and angular momentum of center of mass about that axis.

$$\vec{L}_{\text{body}} = \vec{L}_{\text{body-com}} + \vec{L}_{\text{com}}$$

***Theorem*:** Angular momentum of a body about any point or an axis is equal to the sum of angular momentum of the body about center of mass for an axis parallel to that axis and angular momentum of center of mass about that axis.

Angular momentum of a body about any point or an axis will be sum of angular momenta of individual particles $\vec{L} = \sum m_i(\vec{r}_i \times \vec{v}_i)$. From Fig. 11-9, we can write as

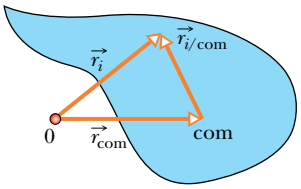

Figure 11-9 Position vectors of center of mass and a random point with respect to origin. Position vector of the random point with respect to center of mass.

$$\vec{r}_i = [\vec{r}_{i/\text{com}} + \vec{r}_{\text{com}}] \quad \text{and} \quad \vec{v}_i = [\vec{r}_{i/\text{com}} + \vec{v}_{\text{com}}].$$

$$\vec{L} = \sum m_i(\vec{r}_{i/\text{com}} + \vec{r}_{\text{com}}) \times (\vec{v}_{i/\text{com}} + \vec{v}_{\text{com}})$$
$$= \sum m_i(\vec{r}_{i/\text{com}} \times \vec{v}_{i/\text{com}}) + \sum m_i(\vec{r}_{\text{com}} \times \vec{v}_{i/\text{com}}) + \sum m_i(\vec{r}_{i/\text{com}} \times \vec{v}_{\text{com}}) + \sum m_i(\vec{r}_{\text{com}} \times \vec{v}_{\text{com}})$$
$$= \sum \vec{r}_{i/\text{com}} \times (m_i \vec{v}_{i/\text{com}}) + \vec{r}_{\text{com}} \times \sum m_i \vec{v}_{i/\text{com}} + \left(\sum m_i \vec{r}_{i/\text{com}}\right) \times \vec{v}_{\text{com}} + (\vec{r}_{\text{com}} \times \vec{v}_{\text{com}}) \sum m_i.$$

$\sum m_i \vec{v}_{i/\text{com}}$ means velocity of the body with respect to center of mass, which is zero. $(\sum m_i \vec{r}_{i/\text{com}})$ means position vector of the center of mass of the body with respect to center of mass, which is zero.

$$\vec{L}_{\text{body}} = \vec{L}_{\text{body-com}} + \vec{r}_{\text{com}} \times \vec{0} + \vec{0} \times \vec{v}_{\text{com}} + \vec{L}_{\text{com}}$$
$$= \vec{L}_{\text{body-com}} + \vec{L}_{\text{com}} \qquad (11\text{-}14)$$

11.7 | THE YO-YO

Key Concept

- A yo-yo, which travels vertically up or down a string, can be treated as a wheel rolling along an inclined plane at angle $\theta = 90°$.

A yo-yo is a physics lab that you can fit in your pocket. If a yo-yo rolls down its string for a distance h, it loses potential energy in amount mgh but gains kinetic energy in both translational $(1/2Mv_{\text{com}}^2)$ and rotational $(1/2I_{\text{com}}\omega^2)$ forms. As it climbs back up, it loses kinetic energy and regains potential energy.

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In a modern yo-yo, the string is not tied to the axle but is looped around it. When the yo-yo "hits" the bottom of its string, an upward force on the axle from the string stops the descent. The yo-yo then spins, axle inside loop, with only rotational kinetic energy. The yo-yo keeps spinning ("sleeping") until you "wake it" by jerking on the string, causing the string to catch on the axle and the yo-yo to climb back up. The rotational kinetic energy of the yo-yo at the bottom of its string (and thus the sleeping time) can be considerably increased by throwing the yo-yo downward so that it starts down the string with initial speeds v_{com} and ω instead of rolling down from rest.

To find an expression for the linear acceleration a_{com} of a yo-yo rolling down a string, we could use Newton's second law (in linear and angular forms) just as we did for the body rolling down a ramp in Fig. 11-8. The analysis is the same except for the following:

1. Instead of rolling down a ramp at angle θ with the horizontal, the yo-yo rolls down a string at angle $\theta = 90°$ with the horizontal.
2. Instead of rolling on its outer surface at radius R, the yo-yo rolls on an axle of radius R_0 (Fig. 11-10*a*).
3. Instead of being slowed by frictional force $\vec{f}_s$, the yo-yo is slowed by the force $\vec{T}$ on it from the string (Fig. 11-10*b*).

The analysis would again lead us to Eq. 11-10. Therefore, let us just change the notation in Eq. 11-10 and set $\theta = 90°$ to write the linear acceleration as

$$a_{com} = -\frac{g}{1 + (I_{com}/MR_0^2)}, \tag{11-15}$$

where I_{com} is the yo-yo's rotational inertia about its center and M is its mass. A yo-yo has the same downward acceleration when it is climbing back up.

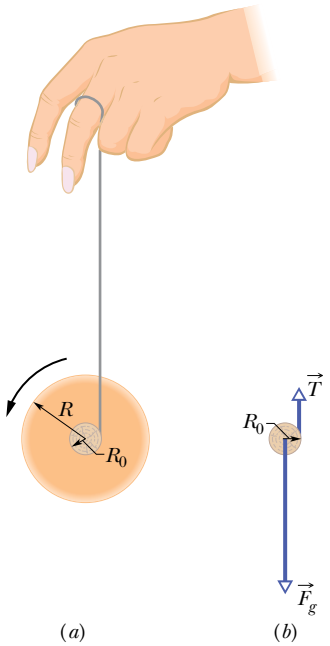

Figure 11-10 (*a*) A yo-yo, shown in cross section. The string, of assumed negligible thickness, is wound around an axle of radius R_0. (*b*) A free-body diagram for the falling yo-yo. Only the axle is shown.

SAMPLE PROBLEM 11.03

Conservation of mechanical energy of a yo-yo

A yo-yo has mass $M = 0.550$ kg and rotational inertia $I = 3.40 \times 10^{-4}$ kg·m^2. It rolls from rest down a string of length $L = 17.0$ cm. What is its angular speed at the bottom?

KEY IDEA

The mechanical energy of the yo-yo is conserved during the descent.

Calculations: Let the bottom position be the reference level for zero gravitational potential energy. Then at the initial point (at height $y = L$) the mechanical energy is

$$E_i = K_i + U_i = 0 + MgL.$$

The symbol K_i is the sum of the translational kinetic energy and rotational kinetic energy, both of which are zero. At the bottom (at height $y = 0$), the translational kinetic energy and gravitational potential energy are zero. So, the mechanical energy is just the rotational kinetic energy $1/2I\omega^2$ and we have

$$E_f = K_f + U_f = \frac{1}{2}I\omega^2.$$

Because the mechanical energy does not change, $E_f = E_i$:

$$\frac{1}{2}I\omega^2 = MgL$$

Solving for ω and substituting the given values lead to

$$\omega = 73.4 \text{ rad/s}. \qquad \text{(Answer)}$$

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SAMPLE PROBLEM 11.04

Forces and torques on a yo-yo

A yo-yo has mass $M = 0.550$ kg, axle radius $R_0 = 3.2$ mm, and rotational inertia $I = 3.4 \times 10^{-4}$ kg·m² about its rotation axis. Assume that the string has negligible mass and thickness.

(a) What is the linear acceleration of the yo-yo as it rolls down the string from rest?

KEY IDEA

The linear acceleration a_{com} is determined by Newton's second law in its linear and angular forms. When those two equations are solved simultaneously, they yield Eq. 11-16:

$$a_{com} = -\frac{g}{1+\frac{I_{com}}{MR_0^2}}. \qquad (11\text{-}16)$$

Calculation: Substituting the known data into Eq. 11-16, we find

$$a_{com} = -\frac{9.8\ \text{m/s}^2}{1+\frac{3.4\times10^{-4}\ \text{kg}\cdot\text{m}^2}{(0.550\ \text{kg})(0.0032\ \text{m})^2}} = -0.16\ \text{m/s}^2. \qquad \text{(Answer)}$$

The acceleration is directed downward and has this value whether the yo-yo is rolling down the string or climbing it.

Note that we could apply the constant-acceleration equations to the yo-yo's motion.

(b) What is the tension in the string as the yo-yo descends?

KEY IDEA

Note that the tension in the string is equal to the magnitude T of the force on the yo-yo from the string. This force and the gravitational force are the only forces acting on the yo-yo, and Newton's second law relates them to the acceleration we just calculated (see the free-body diagram of Fig. 11-10).

Calculations: Thus, we can write Newton's second law for components along a vertical y-axis ($F_{net,y} = ma_y$) as

$$T = Mg = Ma_{com}.$$

Solving for T and substituting $a = 0.16$ m/s², we have

$$T = M(a_{com} + g) = (0.550\ \text{kg})(-0.16\ \text{m/s}^2 + 9.8\ \text{m/s}^2) = 5.3\ \text{N}. \qquad \text{(Answer)}$$

This is the magnitude of the force on the yo-yo from the string, the tension in the string, and the magnitude of the force on your finger from the string, whether the yo-yo is rolling up or down the string.

11.8 | PROBLEM SOLVING ON ROLLING

Key Concepts

- Pure rotation about an axis–no translation.

$$L = I\omega;$$
$$\tau = I\alpha;$$
$$K = \frac{1}{2}I\omega^2.$$

- Rotation and translation.

$$L_z = I_{com}\omega + (\vec{R}\times m\vec{v});$$
$$\tau_z = \tau_{com} + (\vec{R}\times\vec{F});$$
$$\alpha_z = I_{com}\alpha;$$
$$K = \frac{1}{2}I_{com}\omega^2 + \frac{1}{2}Mv^2.$$

- In case of rolling with slipping of uniform circular bodies, a and a_{com} are independent of each other, but the magnitude of frictional force is known as it is kinetic friction.

$$f_k = \mu_k F_N.$$

Pure Rolling of Uniform Circular Bodies

First, we will take problems involving pure rolling only, that is, rolling without sliding. In this case, force of friction is static. The magnitude is unknown but it has to be less than $\mu_s F_N$. As far as direction is concerned, we can assume any direction, the answer will come with correct sign. We will have kinematic relation equating velocity of contact

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points and thus tangential accelerations of contact points. Many times, rather than explicitly saying that pure rolling is taking place, we will find problems stating that "friction is sufficient to prevent any slipping" or simply no slipping is taking place at any contact or just "friction is sufficient."

Plane Motion of Rigid Body

During plane motion of a rigid body, the center of mass C of a rigid body moves in a certain plane while the angular velocity vector ω of the body always remains perpendicular to that plane. This means that motion of the body can be interpreted as purely rotational motion about the axis passing through the center of mass of the body in the C frame. Thus, we have the following two equations describing plane motion of a rigid body:

$$\vec{F} = ma_c; \quad \tau_{com} = I_{com}\alpha.$$

where m is the mass of the body, $\vec{F}$ is the resultant of all external forces. I_{com} and τ_{com} are the moment of inertia and the total torque of all external forces, both quantities are being expressed relative to the axis passing through the center of mass of the body. It should be remembered that the torque τ_{com} includes only external forces.

Summary of dynamical formulas for fixed axis motion:

1. Pure rotation about an axis–no translation.

$$L = I\omega;$$
$$\tau = I\alpha;$$
$$K = \frac{1}{2}I\omega^2.$$

2. Rotation and translation.

$$L_z = I_{com}\omega + (\vec{R}\times m\vec{v});$$
$$\tau_z = \tau_{com} + (\vec{R}\times\vec{F});$$
$$\alpha_z = I_{com}\alpha;$$
$$K = \frac{1}{2}I_{com}\omega^2 + \frac{1}{2}Mv^2.$$

3. Relative split between translational and rotational energy.

The relative split between translational and rotational energy depends on the relative size of the rotational inertia of the rolling object. As Table 11-1 shows, the rolling object (the hoop) that has its mass farthest from the central axis of rotation (and so has the largest rotational inertia) has the largest share of its kinetic energy in rotational motion. The object (the sphere) that has its mass closest to the central axis of rotation (and so has the smallest rotational inertia) has the smallest share in rotational motion.

Rolling with Slipping

In case of rolling with slipping of uniform circular bodies, a and a_{com} are independent of each other, but the magnitude of frictional force is known as it is kinetic friction.

$$f_k = \mu_k F_N.$$

Table 11-1 Relative Splits between Rotational and Translation Energy for Rolling Objects

Object	Rotational Inertia I_{com}	Fraction of Energy in	
		Translation	Rotation
Hoop	$1MR^2$	0.05	0.50
Disk	$(1/2)MR^2$	0.67	0.33
Sphere	$(2/5)MR^2$	0.71	0.29
General	βMR^2	$1/(1+\beta)$	$1/(1+\beta)$

β may be computed for any rolling object as I_{com}/MR^2.

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Complications arise due to the sense of direction. If relative velocities of contact points are known, then direction of friction can be ascertained easily, otherwise we have to take cases and solve as in Chapter 6. If any quantity comes negative, we have to reject the solution and start again with opposite direction.

SAMPLE PROBLEM 11.05

Frictional torque on rolling wheel

A wheel rolls smoothly along a floor while being accelerating horizontally by a force $\vec{F}_a$ applied at its top. The wheel has radius $R = 0.500$ m, mass $m = 1.60$ kg, and acceleration magnitude $a = 2.00$ m/s^2. The frictional force on the wheel is in the forward direction and has magnitude $f = 1.80$ N. What is the rotational inertia of the wheel?

KEY IDEAS

(1) The (linear) acceleration a of a rolling object is the acceleration of the center, which is the center of mass com. (2) Newton's second law written for x components relates the forces on the wheel to the resulting acceleration. (3) Newton's second law written in angular form relates the torques on the wheel to the resulting angular acceleration. (4) The linear acceleration and the angular acceleration are related by the radius according to Eq. 11-6 ($a_{com} = \alpha R$).

Calculations: Newton's second law for motion ($F_{net,x} = ma_x$) along the x axis gives us

$$F_a + f = ma$$

or
$$F_a = ma - f = (1.60\text{ kg})(2.00\text{ m/s}^2) - 1.80\text{ N}$$
$$= 1.40\text{ N}.$$

Next, using Newton's second law for angular motion ($\tau_{net} = I\alpha$), with torques and the angular acceleration measured around the center of the wheel, we have

$$-RF + Rf = I\alpha.$$

On the left side of this equation we have the torque due to the applied force, at distance R from the wheel's center and perpendicular to that distance. We insert a minus sign because were that torque to act alone, it would cause a clockwise rotation. We also have the torque due to the frictional force. However, this torque acting alone would cause a counterclockwise rotation, and so we use a plus sign. We can relate the angular acceleration to the linear acceleration of the center with Eq. 11-6 ($a_{com} = \alpha R$). Substituting for α and solving for I, we find

$$I = \frac{R^2(-F + f)}{a} = \frac{(0.5)^2(-1.40\text{ N} + 1.80\text{ N})}{2.00\text{ m/s}^2}$$
$$= 0.050\text{ kg}\cdot\text{m}^2. \qquad \text{(Answer)}$$

SAMPLE PROBLEM 11.06

Conservation of mechanical energy rolling down a slope

A uniform hoop, disk, and sphere, with the same mass M and same radius R, are released simultaneously from rest at the top of a ramp of length $L = 2.5$ m and angle $\theta = 12°$ with the horizontal. The objects roll smoothly down the ramp.

(a) Which object wins the race to the bottom of the ramp?

Reasoning: First, the objects begin with the same mechanical energy E, because they start from rest and the same height. Second, E is conserved during the race to the bottom because the only force doing work on the objects is the gravitational force. (The normal force on them from the ramp and the frictional force at their point of contact with the ramp do not cause energy transfers.) Further, at any given point along the ramp, the objects must have the same kinetic energy K because the same amount of energy has been transferred from gravitational potential energy to kinetic energy.

If the objects were sliding down the ramp, this means they would have the same speed. However, they do not have the same speed v_{com} because each object shares its kinetic energy between its translational motion down the ramp and its rotational motion around its center of mass. The sphere has the greatest fraction as translational energy because its mass is close to its center, and so it has the greatest v_{com} and wins the race (Fig. 11-11).

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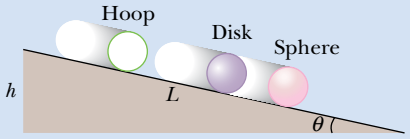

Figure 11-11 Order of objects during the race.

(b) What is v_{com} for each object at the bottom of the ramp?

KEY IDEA

Mechanical energy is conserved.

Calculations: Let us choose the bottom of the ramp as our reference height for zero gravitational potential energy, so at the finish each object–Earth system has $U_f = 0$. The final kinetic energy at the finish is given by Eq. 11-5 for each object. The initial kinetic energy for all three objects is $K_i = 0$. The initial potential energy is $U_i = mgh = mg(L\sin\theta)$. Now, we can write the conservation of mechanical energy ($E_f = E_i$) as

$$K_f + K_f = U_i + U_i$$

or
$$\left(\frac{1}{2}I_{com}\omega^2 + \frac{1}{2}Mv_{com}^2\right) + 0 = 0 + Mg(L\sin\theta).$$

Substituting $\omega = I_{com}/R$ and solving for v_{com} give us

$$v_{com} = \sqrt{\frac{2gL\sin\theta}{1 + I_{com}/MR^2}}. \qquad \text{(Answer)}$$

which is the symbolic answer to the question.

Note that the speed depends not on the mass or the radius of the rolling object, but only on the distribution of its mass about its central axis, which enters through the term I_{com}/MR^2. A marble and a bowling ball will have the same speed at the bottom of the ramp and will thus roll down the ramp in the same time. A bowling ball will beat a disk of any mass or radius, and almost anything that rolls will beat a hoop.

For the rolling hoop, we have $I_{com}/MR^2 = 1$, so

$$v_{com} = \sqrt{\frac{2gL\sin\theta}{1 + I_{com}/MR^2}}$$
$$= \sqrt{\frac{2(9.8\ \text{m/s}^2)(2.5\ \text{m})(\sin 12°)}{1+1}}$$
$$= 2.3\ \text{m/s}. \qquad \text{(Answer)}$$

From a similar calculation, we obtain $v_{com} = 2.6$ m/s for the disk ($I_{com}/MR^2 = 1/2$) and 2.7 m/s for the sphere ($I_{com}/MR^2 = 2/5$).

SAMPLE PROBLEM 11.07

Kinetic energy distribution in a rolling object

Kinetic energy distribution in a rolling object. Consider a hoop, a disk, and a sphere, each of mass M and radius R, that roll smoothly along a horizontal table. For each, what fraction of its kinetic energy is associated with the translation of its center of mass?

KEY IDEA

The kinetic energy of a smoothly rolling body is the sum of its translational kinetic energy ($1/2Mv_{com}^2$) and its rotational kinetic energy $1/2I_{com}\omega^2$, as stated by Eq. 11-5. Therefore, the fraction of the kinetic energy associated with translation is

$$\text{frac} = \frac{\frac{1}{2}Mv_{com}^2}{\frac{1}{2}Mv_{com}^2 + \frac{1}{2}I_{com}\omega^2}. \qquad (11\text{-}17)$$

Calculations: We can greatly simplify the right side of Eq. 11-17 by substituting v_{com}/R for (Eq. 11-2) and realizing that the expressions for rotational inertia in Table 10-2 are all of the form βMR^2, where β is a numerical coefficient (the "front number"). Here, β is 1 for a hoop, 1/2 for a disk, and 2/5 for a sphere. Thus, we can substitute βMR^2 for I_{com} in Eq. 11-17. After these substitutions and some cancellations, Eq. 11-17 becomes

$$\text{frac} = \frac{1}{1+\beta}. \qquad (11\text{-}18)$$

Now, substituting the β values for the hoop, disk, and sphere, we can generate Table 11-1 to show the fractional splits of translational and rotational kinetic energy. For example, 0.67 of the kinetic energy of the disk is associated with the translation.

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11.9 | RIGID BODY IN EQUILIBRIUM

Key Concept

- Objects are said to be in equilibrium if they fulfill these two requirements $\vec{P}$ = constant and $\vec{L}$ = constant.

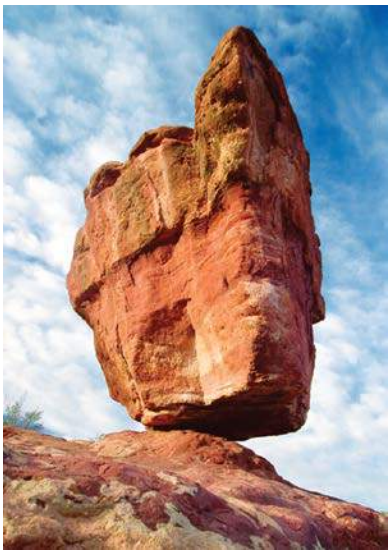
Consider these objects: (1) a book resting on a table, (2) a hockey puck sliding with constant velocity across a frictionless surface, (3) the rotating blades of a ceiling fan, and (4) the wheel of a bicycle that is traveling along a straight path at constant speed. For each of these four objects,

1. The linear momentum $\vec{P}$ of its center of mass is constant.
2. Its angular momentum $\vec{L}$ about its center of mass, or about any other point, is also constant.

We say that such objects are in **equilibrium**. The two requirements for equilibrium are then

$$\vec{P} = \text{a constant} \quad \text{and} \quad \vec{L} = \text{a constant.} \tag{11-19}$$

Our concern in this chapter is with situations in which the constants in Eq. 11-19 are zero; that is, we are concerned largely with objects that are not moving in any way—either in translation or in rotation—in the reference frame from which we observe them. Such objects are in **static equilibrium**. Of the four objects mentioned near the beginning of this module, only one—the book resting on the table—is in static equilibrium.



Kanwarjit Singh Boparai/Shutterstock

Figure 11-12 A balancing rock. Although its perch seems precarious, the rock is in static equilibrium.

The balancing rock of Fig. 11-12 is another example of an object that, for the present at least, is in static equilibrium. It shares this property with countless other structures, such as cathedrals, houses, filing cabinets, and taco stands, that remain stationary over time.

As we discussed in Section 8.18, if a body returns to a state of static equilibrium after having been displaced from that state by a force, the body is said to be in *stable* static equilibrium. A marble placed at the bottom of a hemispherical bowl is an example. However, if a small force can displace the body and end the equilibrium, the body is in *unstable* static equilibrium.

A Domino. For example, suppose we balance a domino with the domino's center of mass vertically above the supporting edge, as in Fig. 11-13*a*. The torque about the supporting edge due to the gravitational force $\vec{F}_g$ on the domino is zero because the line of action of $\vec{F}_g$ is through that edge. Thus, the domino is in equilibrium. Of course, even a slight force on it due to some chance disturbance ends the equilibrium. As the line of action of $\vec{F}_g$ moves to one side of the supporting edge (as in Fig. 11-13*b*), the torque due to $\vec{F}_g$ increases the rotation of the domino. Therefore, the domino in Fig. 11-13*a* is in unstable static equilibrium.

The domino in Fig. 11-13*c* is not quite as unstable. To topple this domino, a force would have to rotate it through and then beyond the balance position of Fig. 11-13*a*, in which the center of mass is above a supporting edge. A slight force will not topple this domino, but a vigorous flick of the finger against the domino certainly will. (If we arrange a chain of such upright dominos, a finger flick against the first can cause the whole chain to fall.)

A Block. The child's square block in Fig. 11-13*d* is even more stable because its center of mass would have to be moved even farther to get it to pass above a supporting edge. A flick of the finger may not topple the block. (This is why you never see a chain of toppling square blocks.) The worker in Fig. 11-14 is like both the domino and the square block: Parallel to the beam, his stance is wide and he is stable; perpendicular to the beam, his stance is narrow and he is unstable (and at the mercy of a chance gust of wind).

The analysis of static equilibrium is very important in engineering practice. The design engineer must isolate and identify all the external forces and torques that may act on a structure and, by good design and wise choice of

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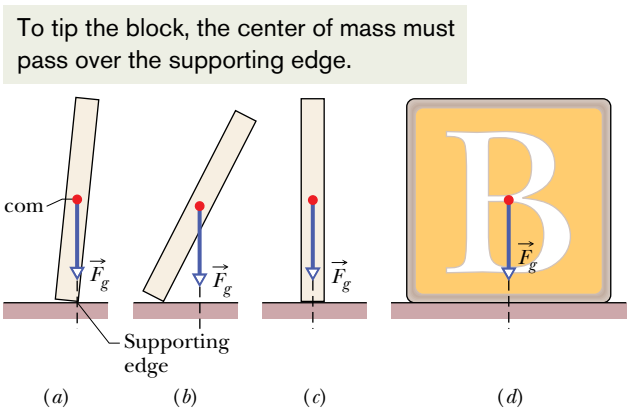

Figure 11-13 (*a*) A domino balanced on one edge, with its center of mass vertically above that edge. The gravitational force $\vec{F}_g$ on the domino is directed through the supporting edge. (*b*) If the domino is rotated even slightly from the balanced orientation, then $\vec{F}_g$ causes a torque that increases the rotation. (*c*) A domino upright on a narrow side is somewhat more stable than the domino in (*a*). (*d*) A square block is even more stable.

Robert Brenner/PhotoEdit

Figure 11-14 A construction worker balanced on a steel beam is in static equilibrium but is more stable parallel to the beam than perpendicular to it.

materials, ensure that the structure will remain stable under these loads. Such analysis is necessary to ensure, for example, that bridges do not collapse under their traffic and wind loads and that the landing gear of aircraft will function after the shock of rough landings.

11.10 | THE REQUIREMENTS OF EQUILIBRIUM

Key Concepts

- A rigid body at rest is said to be in static equilibrium. For such a body, the vector sum of the external forces acting on it is zero:

$$\vec{F}_{\text{net}} = 0 \quad \text{(balance of forces)}.$$

If all the forces lie in the *xy* plane, this vector equation is equivalent to two component equations:

$$F_{\text{net},x} = 0 \quad \text{and} \quad F_{\text{net},y} = 0 \quad \text{(balance of forces)}.$$

- Static equilibrium also implies that the vector sum of the external torques acting on the body about any point is zero, or

$$\vec{\tau}_{\text{net}} = 0 \quad \text{(balance of torques)}.$$

If the forces lie in the *xy* plane, all torque vectors are parallel to the *z* axis, and the balance-of-torques equation is equivalent to the single component equation

$$\tau_{\text{net},z} = 0 \quad \text{(balance of torques)}.$$

The translational motion of a body is governed by Newton's second law in its linear momentum form, given by Eq. 9-29 as

$$\vec{F}_{\text{net}} = \frac{d\vec{P}}{dt}. \tag{11-20}$$

If the body is in translational equilibrium—that is, if $\vec{P}$ is a constant—then $d\vec{P}/dt = 0$ and we must have

$$\vec{F}_{\text{net}} = 0 \quad \text{(balance of forces)}. \tag{11-21}$$

The rotational motion of a body is governed by Newton's second law in its angular momentum form, given by Eq. 10-93 as

$$\vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt}. \tag{11-22}$$

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If the body is in rotational equilibrium—that is, if $\vec{L}$ is a constant—then $d\vec{L}/dt = 0$ and we must have

$$\vec{\tau}_{net} = 0 \quad \text{(balance of torques).} \tag{11-23}$$

Thus, the two requirements for a body to be in equilibrium are as follows:

1. The vector sum of all the external forces that act on the body must be zero.
2. The vector sum of all external torques that act on the body, measured about *any* possible point, must also be zero.

These requirements obviously hold for *static* equilibrium. They also hold for the more general equilibrium in which $\vec{P}$ and $\vec{L}$ are constant but not zero.

Equations 11-21 and 11-23, as vector equations, are each equivalent to three independent component equations, one for each direction of the coordinate axes:

Balance of forces	Balance of torques
$F_{net,x} = 0$	$\tau_{net,x} = 0$
$F_{net,y} = 0$	$\tau_{net,y} = 0$
$F_{net,z} = 0$	$\tau_{net,z} = 0$

(11-24)

The Main Equations. We shall simplify matters by considering only situations in which the forces that act on the body lie in the xy plane. This means that the only torques that can act on the body must tend to cause rotation around an axis parallel to the z axis. With this assumption, we eliminate one force equation and two torque equations from Eq. 11-24, leaving

$$F_{net,x} = 0 \quad \text{(balance of forces),} \tag{11-25}$$

$$F_{net,y} = 0 \quad \text{(balance of forces),} \tag{11-26}$$

$$\tau_{net,z} = 0 \quad \text{(balance of torques),} \tag{11-27}$$

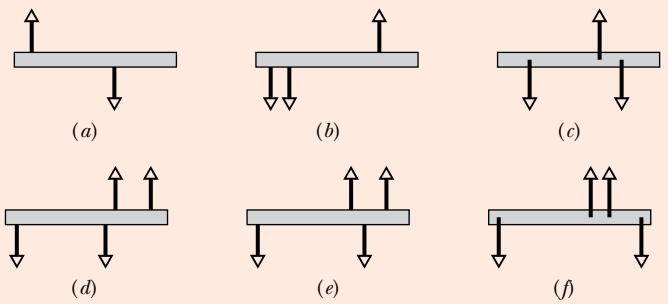
Here, $t_{net,z}$ is the net torque that the external forces produce either about the z axis or about *any* axis parallel to it.

A hockey puck sliding at constant velocity over ice satisfies Eqs. 11-25, 11-26, and 11-27 and is thus in equilibrium *but not in static equilibrium*. For static equilibrium, the linear momentum $\vec{P}$ of the puck must be not only constant but also zero; the puck must be at rest on the ice. Thus, there is another requirement for static equilibrium:

3. The linear momentum $\vec{P}$ of the body must be zero.

CHECKPOINT 3

The figure gives six overhead views of a uniform rod on which two or more forces act perpendicularly to the rod. If the magnitudes of the forces are adjusted properly (but kept nonzero), in which situations can the rod be in static equilibrium?



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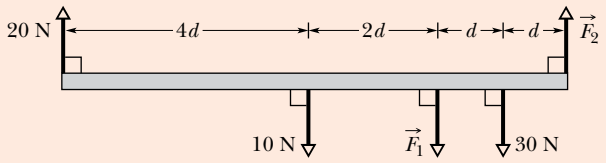
11.11 | SOME EXAMPLES OF STATIC EQUILIBRIUM

Here we examine several sample problems involving static equilibrium. In each, we select a system of one or more objects to which we apply the equations of equilibrium (Eqs. 11-25, 11-26, and 11-27). The forces involved in the equilibrium are all in the *xy* plane, which means that the torques involved are parallel to the *z* axis. Thus, in applying Eq. 11-27, the balance of torques, we select an axis parallel to the *z* axis about which to calculate the torques. Although Eq. 11-27 is satisfied for *any* such choice of axis, you will see that certain choices simplify the application of Eq. 11-27 by eliminating one or more unknown force terms.

CHECKPOINT 4

The figure gives an overhead view of a uniform rod in static equilibrium. (a) Can you find the magnitudes of unknown forces $\vec{F}_1$ and $\vec{F}_2$ by balancing the forces? (b) If you wish to find the magnitude of force $\vec{F}_2$ by using a balance of torques equation, where should you place a rotation axis to eliminate $\vec{F}_1$ from the equation? (c) The magnitude of $\vec{F}_2$ turns out to be 65 N. What then is the magnitude of $\vec{F}_1$?



SAMPLE PROBLEM 11.08

Balancing a horizontal beam

In Fig. 11-15*a*, a uniform beam, of length L and mass $m = 1.8$ kg, is at rest on two scales. A uniform block, with mass $M = 2.7$ kg, is at rest on the beam, with its center a distance $L/4$ from the beam's left end. What do the scales read?

KEY IDEAS

The first steps in the solution of *any* problem about static equilibrium are these: Clearly, define the system to be analyzed and then draw a free-body diagram of it, indicating all the forces on the system. Here, let us choose the system as the beam and block taken together. Then the forces on the system are shown in the free-body diagram of Fig. 11-15*b*. (Choosing the system takes experience, and often there can be more than one good choice.) Because the system is in static equilibrium, we can apply the balance of forces equations (Eqs. 11-25 and 11-26) and the balance of torques equation (Eq. 11-27) to it.

Calculations: The normal forces on the beam from the scales are $\vec{F}_l$ on the left and $\vec{F}_r$ on the right. The scale readings that we want are equal to the magnitudes of those forces. The gravitational force $\vec{F}_{g,\text{beam}}$ on the beam acts at the beam's center of mass and is equal to $m\vec{g}$. Similarly, the gravitational force $\vec{F}_{g,\text{block}}$ on the block acts at the block's center of mass and is equal to $M\vec{g}$.

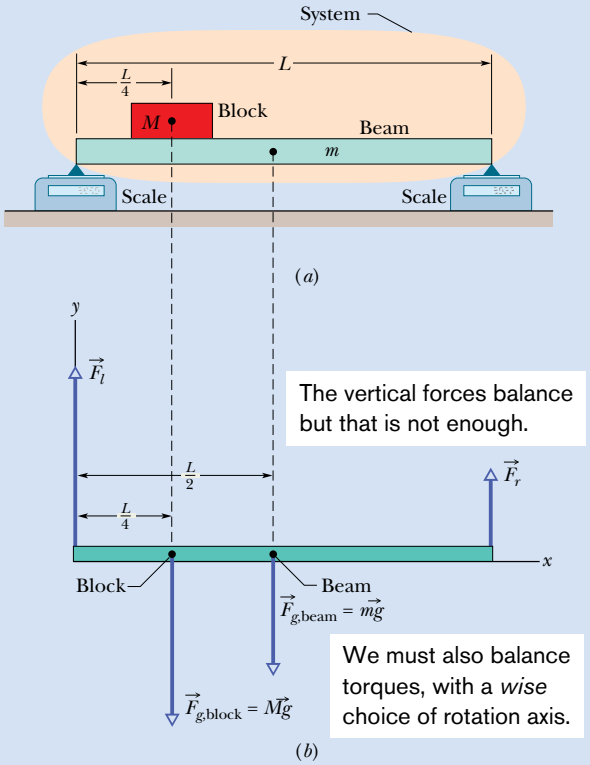


Figure 11-15 (*a*) A beam of mass m supports a block of mass M. (*b*) A free-body diagram, showing the forces that act on the system *beam + block*.

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However, to simplify Fig. 11-15*b*, the block is represented by a dot within the boundary of the beam and vector $\vec{F}_{g,\text{block}}$ is drawn with its tail on that dot. (This shift of the vector $\vec{F}_{g,\text{block}}$ along its line of action does not alter the torque due to $\vec{F}_{g,\text{block}}$ about any axis perpendicular to the figure.)

The forces have no x components, so Eq. 11-25 ($F_{\text{net},x} = 0$) provides no information. For the y components, Eq. 11-26 ($F_{\text{net},y} = 0$) gives us

$$F_l + F_r - Mg - mg = 0. \tag{11-28}$$

This equation contains two unknowns, the forces F_l and F_r, so we also need to use Eq. 11-27, the balance of torques equation. We can apply it to *any* rotation axis perpendicular to the plane of Fig. 11-15. Let us choose a rotation axis through the left end of the beam. We shall also use our general rule for assigning signs to torques: If a torque would cause an initially stationary body to rotate clockwise about the rotation axis, the torque is negative. If the rotation would be counterclockwise, the torque is positive. Finally, we shall write the torques in the form $r_\perp F$, where the moment arm $r_\perp$ is 0 for $\vec{F}_l$, $L/4$ for $M\vec{g}$, $L/2$ for $m\vec{g}$, and L for $\vec{F}_r$.

We now can write the balancing equation ($\tau_{\text{net},z} = 0$) as

$$(0)(F_l) - (L/4)(Mg) - (L/2)(mg) + (L)(F_r) = 0,$$

which gives us

$$\begin{aligned} F_r &= \frac{1}{4}Mg + \frac{1}{2}mg \\ &= \frac{1}{4}(2.7\text{ kg})(9.8\text{ m/s}^2) + \frac{1}{2}(1.8\text{ kg})(9.8\text{ m/s}^2) \\ &= 15.44\text{ N} \approx 15\text{ N}. \end{aligned} \tag{Answer}$$

Now, solving Eq. 11-28 for F_l and substituting this result, we find

$$\begin{aligned} F_l &= (M+m)g - F_r \\ &= (2.7\text{ kg} + 1.8\text{ kg})(9.8\text{ m/s}^2) - 15.44\text{ N} \\ &= 28.66\text{ N} \approx 29\text{ N}. \end{aligned} \tag{Answer}$$

Notice the strategy in the solution: When we wrote an equation for the balance of force components, we got stuck with two unknowns. If we had written an equation for the balance of torques around some *arbitrary* axis, we would have again gotten stuck with those two unknowns. However, because we chose the axis to pass through the point of application of one of the unknown forces, here $\vec{F}_l$, we did not get stuck. Our choice neatly eliminated that force from the torque equation, allowing us to solve for the other unknown force magnitude F_r. Then we returned to the equation for the balance of force components to find the remaining unknown force magnitude.

SAMPLE PROBLEM 11.09

Balancing a leaning boom

Figure 11-16*a* shows a safe (mass M = 430 kg) hanging by a rope (negligible mass) from a boom (a = 1.9 m and b = 2.5 m) that consists of a uniform hinged beam (m = 85 kg) and horizontal cable (negligible mass).

(a) What is the tension T_c in the cable? In other words, what is the magnitude of the force $\vec{T}_c$ on the beam from the cable?

KEY IDEAS

The system here is the beam alone, and the forces on it are shown in the free-body diagram of Fig. 11-16*b*. The force from the cable is $\vec{T}_c$. The gravitational force on the beam acts at the beam's center of mass (at the beam's center) and is represented by its equivalent $m\vec{g}$. The vertical component of the force on the beam from the hinge is $\vec{F}_v$, and the horizontal component of the force from the hinge is $\vec{F}_h$. The force from the rope supporting the safe is $\vec{T}_r$. Because beam, rope, and safe are stationary, the magnitude of $\vec{T}_r$ is equal to the weight of the safe: $T_r = Mg$. We place the origin O of an xy coordinate system at the hinge. Because the system is in static equilibrium, the balancing equations apply to it.

Calculations: Let us start with Eq. 11-27 ($\tau_{\text{net},z} = 0$). Note that we are asked for the magnitude of force $\vec{T}_c$ and not of forces $\vec{F}_h$ and $\vec{F}_v$ acting at the hinge, at point O. To eliminate $\vec{F}_h$ and $\vec{F}_v$ from the torque calculation, we should calculate torques about an axis that is perpendicular to the figure at point O. Then $\vec{F}_h$ and $\vec{F}_v$ will have moment arms of zero. The lines of action for $\vec{T}_c$, $\vec{T}_r$, and $m\vec{g}$ are dashed in Fig. 11-16*b*. The corresponding moment arms are a, b, and $b/2$.

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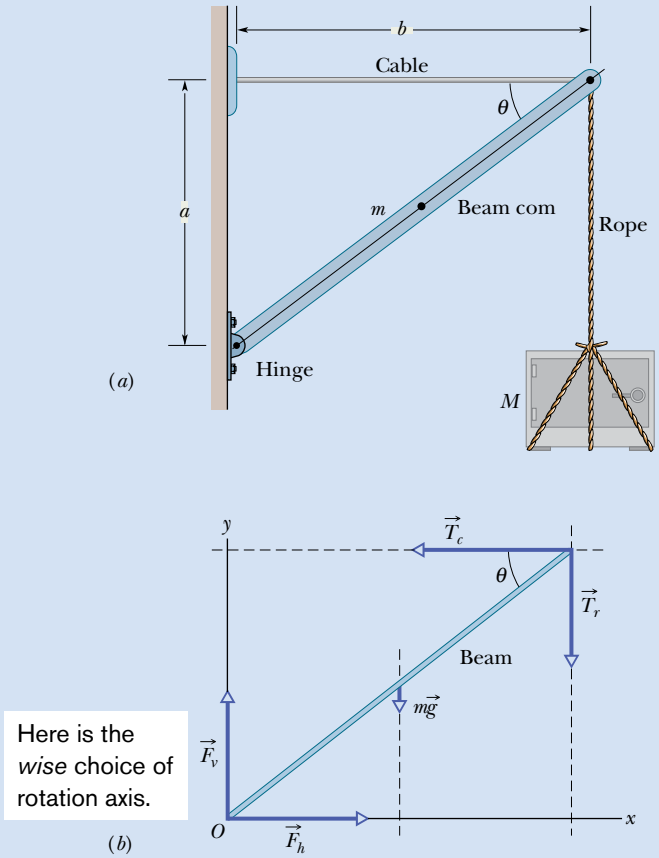

Figure 11-16 (*a*) A heavy safe is hung from a boom consisting of a horizontal steel cable and a uniform beam. (*b*) A free-body diagram for the beam.

Writing torques in the form of $r_\perp F$ and using our rule about signs for torques, the balancing equation $\tau_{net,z} = 0$ becomes

$$(a)(T_c) - (b)(T_r) - \left(\frac{1}{2}b\right)(mg) = 0. \qquad (11\text{-}29)$$

Substituting Mg for T_r and solving for T_c, we find that

$$T_c = \frac{gb(M + \frac{1}{2}m)}{a}$$
$$= \frac{(9.8\ \text{m/s}^2)(2.5\ \text{m})(430\ \text{kg} + 85/2\ \text{kg})}{1.9\ \text{m}}$$
$$= 6093\ \text{N} \approx 6100\ \text{N}. \qquad \text{(Answer)}$$

(b) Find the magnitude F of the net force on the beam from the hinge.

KEY IDEA

Now we want the horizontal component F_h and vertical component F_v so that we can combine them to get the magnitude F of the net force.Because we know T_c, we apply the force balancing equations to the beam.

Calculations: For the horizontal balance, we can rewrite $F_{net,x} = 0$ as

$$F_h - T_c = 0, \qquad (11\text{-}30)$$

and so $F_h = T_c = 6093\ \text{N}.$

For the vertical balance, we write $F_{net,y} = 0$ as

$$F_v - mg - T_r = 0.$$

Substituting Mg for T_r and solving for F_v, we find that

$$F_v = (m + M)g = (85\ \text{kg} + 430\ \text{kg})(9.8\ \text{m/s}^2)$$
$$= 5047\ \text{N}.$$

From the Pythagorean theorem, we now have

$$F = \sqrt{F_h^2 + F_v^2}$$
$$= \sqrt{(6093\ \text{N})^2 + (5047\ \text{N})^2} \approx 7900\ \text{N}. \qquad \text{(Answer)}$$

Note that F is substantially greater than either the combined weights of the safe and the beam, 5000 N, or the tension in the horizontal wire, 6100 N.

SAMPLE PROBLEM 11.10

Balancing a leaning ladder

In Fig. 11-17*a*, a ladder of length $L = 12$ m and mass $m = 45$ kg leans against a slick wall (that is, there is no friction between the ladder and the wall). The ladder's upper end is at height $h = 9.3$ m above the pavement on which the lower end is supported (the pavement is not frictionless). The ladder's center of mass is $L/3$ from the lower end, along the length of the ladder. A firefighter of mass $M = 72$ kg climbs the ladder until her center of mass is $L/2$ from the lower end. What then are the magnitudes of the forces on the ladder from the wall and the pavement?

KEY IDEA

First, we choose our system as being the firefighter and ladder, together, and then we draw the free-body diagram of Fig. 11-17*b* to show the forces acting on the system. Because the system is in static equilibrium, the balancing

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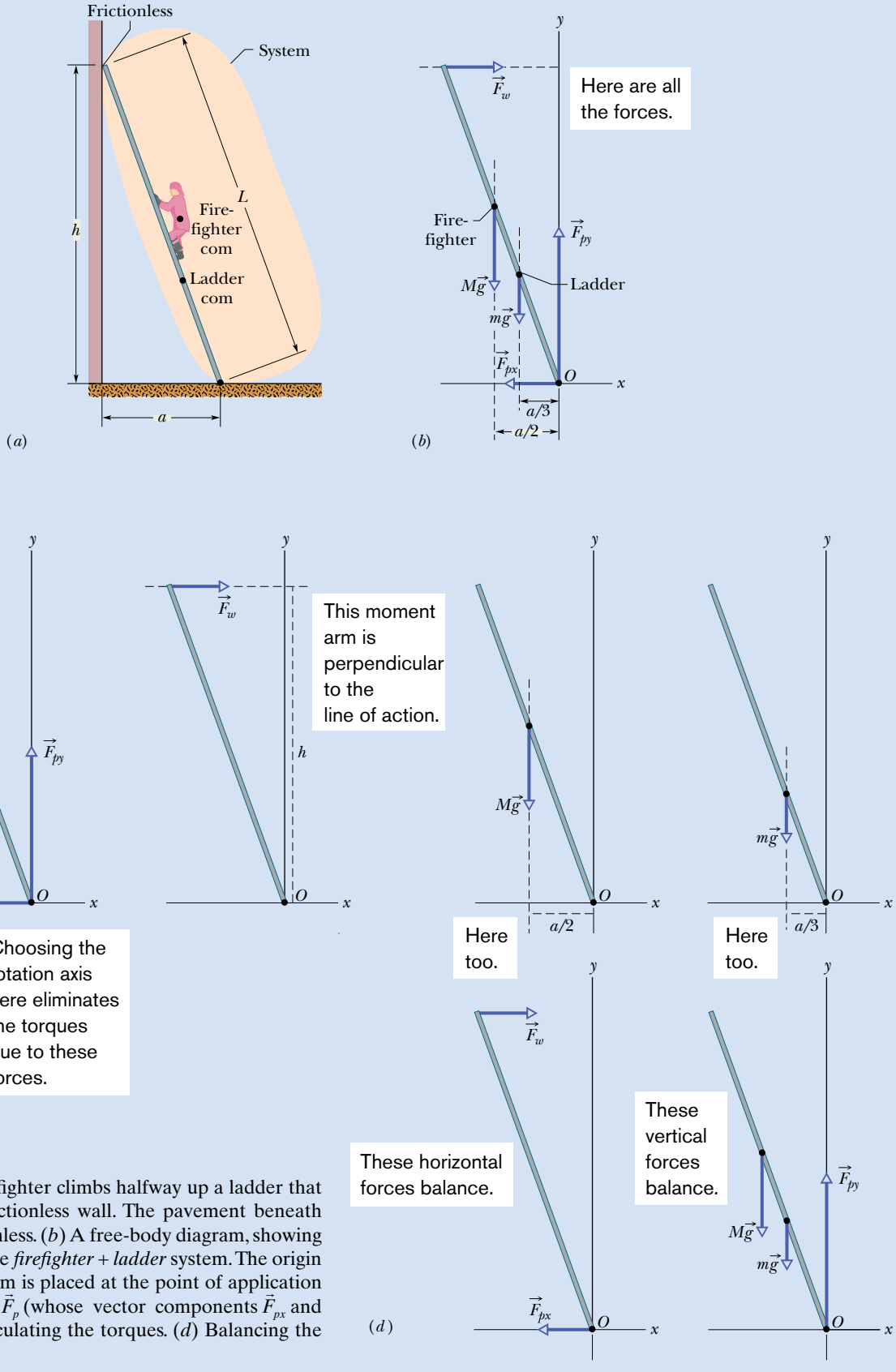

Figure 11-17 (*a*) A firefighter climbs halfway up a ladder that is leaning against a frictionless wall. The pavement beneath the ladder is not frictionless. (*b*) A free-body diagram, showing the forces that act on the *firefighter + ladder* system. The origin *O* of a coordinate system is placed at the point of application of the unknown force $\vec{F}_p$ (whose vector components $\vec{F}_{px}$ and $\vec{F}_{py}$ are shown). (*c*) Calculating the torques. (*d*) Balancing the forces.

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equations for both forces and torques (Eqs. 11-25 through 11-27) can be applied to it.

Calculations: In Fig. 11-17*b*, the firefighter is represented with a dot within the boundary of the ladder. The gravitational force on her is represented with its equivalent expression $M\vec{g}$, and that vector has been shifted along its line of action (the line extending through the force vector), so that its tail is on the dot. (The shift does not alter a torque due to $M\vec{g}$ about any axis perpendicular to the figure. Thus, the shift does not affect the torque balancing equation that we shall be using.)

The only force on the ladder from the wall is the horizontal force $\vec{F}_w$ (there cannot be a frictional force along a frictionless wall, so there is no vertical force on the ladder from the wall). The force $\vec{F}_p$ on the ladder from the pavement has two components: a horizontal component $\vec{F}_{px}$ that is a static frictional force and a vertical component $\vec{F}_{py}$ that is a normal force.

To apply the balancing equations, let's start with the torque balancing of Eq. 11-27 ($t_{\text{net},z} = 0$). To choose an axis about which to calculate the torques, note that we have unknown forces ($\vec{F}_w$ and $\vec{F}_p$) at the two ends of the ladder. To eliminate, say, $\vec{F}_p$ from the calculation, we place the axis at point O, perpendicular to the figure (Fig. 11-17*b*). We also place the origin of an xy coordinate system at O. We can find torques about O with any of Eqs. 10-43 through 10-45, but Eq. 10-45 ($\tau = r_\perp F$) is easiest to use here. *Making a wise choice about the placement of the origin can make our torque calculation much easier.*

To find the moment arm $r_\perp$ of the horizontal force $\vec{F}_w$ from the wall, we draw a line of action through that vector (it is the horizontal dashed line shown in Fig. 11-17*c*). Then $r_\perp$ is the perpendicular distance between O and the line of action. In Fig. 11-17*c*, $r_\perp$ extends along the y axis and is equal to the height h. We similarly draw lines of action for the gravitational force vectors $M\vec{g}$ and $m\vec{g}$ and see that their moment arms extend along the x axis. For the distance a shown in Fig. 11-17*a*, the moment arms are $a/2$ (the firefighter is halfway up the ladder) and $a/3$ (the ladder's center of mass is one-third of the way up the ladder), respectively. The moment arms for $\vec{F}_{px}$ and $\vec{F}_{py}$ are zero because the forces act at the origin.

Now, with torques written in the form $r_\perp F$, the balancing equation $\tau_{\text{net},z} = 0$ becomes

$$-(h)(F_w) + (a/2)(Mg) + (a/3)(mg) + (0)(F_{px}) + (0)(F_{py}) = 0. \tag{11-31}$$

(A positive torque corresponds to counterclockwise rotation and a negative torque corresponds to clockwise rotation.)

Using the Pythagorean theorem for the right triangle made by the ladder in Fig. 11-17*a*, we find that

$$a = \sqrt{L^2 - h^2} = 7.58 \text{ m}.$$

Then Eq. 11-35 gives us

$$F_w = \frac{ga(M/2 + m/3)}{h} = \frac{(9.8 \text{ m/s}^2)(7.58 \text{ m})(72/2 \text{ kg} + 45/3 \text{ kg})}{9.3 \text{ m}} = 407 \text{ N} \approx 410 \text{ N}. \quad \text{(Answer)}$$

Now we need to use the force balancing equations and Fig. 11-17*d*. The equation $F_{\text{net},x} = 0$ gives us

$$F_w - F_{px} = 0,$$

so

$$F_{px} = F_w = 410 \text{ N}. \quad \text{(Answer)}$$

The equation $F_{\text{net},y} = 0$ gives us

$$F_{py} - Mg - mg = 0,$$

so

$$F_{py} = (M + m)g = (72 \text{ kg} + 45 \text{ kg})(9.8 \text{ m/s}^2) = 1146.6 \text{ N} \approx 1100 \text{ N}. \quad \text{(Answer)}$$

SAMPLE PROBLEM 11.11

Balancing the leaning Tower of Pisa

Let's assume that the Tower of Pisa is a uniform hollow cylinder of radius $R = 9.8$ m and height $h = 60$ m. The center of mass is located at height $h/2$, along the cylinder's central axis. In Fig. 11-18*a*, the cylinder is upright. In Fig. 11-18*b*, it leans rightward (toward the tower's southern wall) by $\theta = 5.5°$, which shifts the com by a distance d. Let's assume that the ground exerts only two forces on the tower. A normal force $\vec{F}_{NL}$ acts on the left (northern) wall, and a normal force $\vec{F}_{NR}$ acts on the right (southern) wall. By what percent does the magnitude F_{NR} increase because of the leaning?

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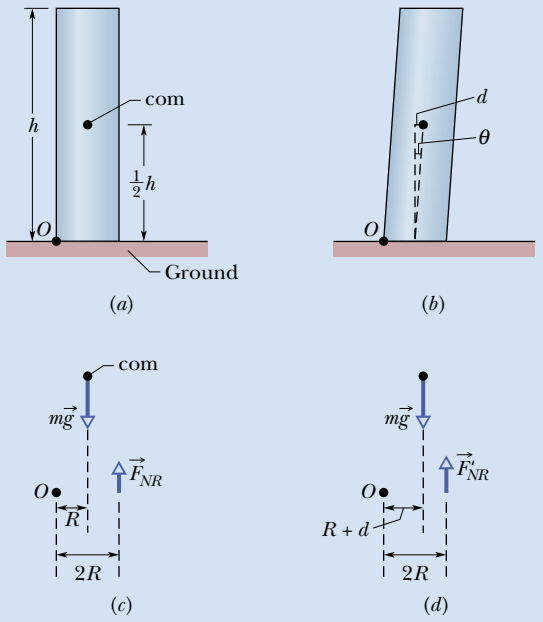

Figure 11-18 A cylinder modeling the Tower of Pisa: (*a*) upright and (*b*) leaning, with the center of mass shifted rightward. The forces and moment arms to find torques about a pivot at point *O* for the cylinder (*c*) upright and (*d*) leaning.

KEY IDEA

Because the tower is still standing, it is in equilibrium and thus the sum of torques calculated around any point must be zero.

Calculations: Because we want to calculate F_{NR} on the right side and do not know or want F_{NL} on the left side, we use a pivot point on the left side to calculate torques. The forces on the upright tower are represented in Fig. 11-18*c*. The gravitational force $m\vec{g}$, taken to act at the com, has a vertical line of action and a moment arm of R (the perpendicular distance from the pivot to the line of action). About the pivot, the torque associated with this force would tend to create clockwise rotation and thus is negative. The normal force $\vec{F}_{NR}$ on the southern wall also has a vertical line of action, and its moment arm is $2R$. About the pivot, the torque associated with this force would tend to create counterclockwise rotation and thus is positive. We now can write the torque-balancing equation ($\tau_{\text{net},z} = 0$) as

$$-(R)(mg) + (2R)(F_{NR}) = 0,$$

which yields

$$F_{NR} = \frac{1}{2}mg.$$

We should have been able to guess this result: With the center of mass located on the central axis (the cylinder's line of symmetry), the right side supports half the cylinder's weight.

In Fig. 11-18*b*, the com is shifted rightward by distance

$$d = \frac{1}{2}h\tan\theta.$$

The only change in the balance of torques equation is that the moment arm for the gravitational force is now $R + d$ and the normal force at the right has a new magnitude F'_{NR} (Fig. 11-18*d*). Thus, we write

$$-(R+d)(mg) + (2R)(F'_{NR}) = 0,$$

which gives us

$$F'_{NR} = \frac{(R+d)}{2R}mg.$$

Dividing this new result for the normal force at the right by the original result and then substituting for d, we obtain

$$\frac{F'_{NR}}{F_{NR}} = \frac{R+d}{R} = 1 + \frac{d}{R} = 1 + \frac{0.5h\tan\theta}{R}.$$

Substituting the values of $h = 60$ m, $R = 9.8$ m, and $\theta = 5.5°$ leads to

$$\frac{F'_{NR}}{F_{NR}} = 1.29.$$

Thus, our simple model predicts that, although the tilt is modest, the normal force on the tower's southern wall has increased by about 30%. One danger to the tower is that the force may cause the southern wall to buckle and explode outward. The cause of the leaning is the compressible soil beneath the tower, which worsened with each rainfall. Recently engineers have stabilized the tower and partially reversed the leaning by installing a drainage system.

PROBLEM-SOLVING TACTICS

Tactic 2: Static Equilibrium Problems Here is a list of steps for solving static equilibrium problems:

1. Draw a *sketch* of the problem.
2. Select the *system* to which you will apply the laws of equilibrium, drawing a closed curve around the system on your sketch to fix it clearly in your mind. In some situations, you can select a single object as the system; it is the object you wish to be in equilibrium. In other situations, you might include additional objects in the system *if*

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their inclusion simplifies the calculations for equilibrium. For example, suppose in Sample Problem 11.10 you select only the ladder as the system. Then in Fig. 11-17*b* you will have to account for additional unknown forces exerted on the ladder by the hands and feet of the firefighter. These additional unknowns complicate the equilibrium calculations. The system of Fig. 11-17 was chosen to include the firefighter so that those unknown forces are *internal* to the system and thus need not be found in order to solve Sample Problem 11.10.

3. Draw a *free-body diagram* of the system. Show all the forces that act on the system, labeling them and making sure that their points of application and lines of action are correctly shown.
4. Draw in the *x and y axes* of a coordinate system with at least one axis parallel to one or more unknown force. Resolve into components the forces that do not lie along one of the axes. In all our Sample Problems it made sense to choose the *x* axis horizontal and the *y* axis vertical.
5. Write the two *balance of forces equations*, using symbols throughout.
6. Choose one or more rotation axes perpendicular to the plane of the figure and write the *balance of torques equation* for each axis. If you choose an axis that passes through the line of action of an unknown force, the equation will be simplified because that force will not appear in it.
7. *Solve* your equations *algebraically* for the unknowns. Some students feel more confident in substituting numbers with units in the independent equations at this stage, especially if the algebra is particularly involved. However, experienced problem solvers prefer the algebra, which reveals the dependence of solutions on the various variables.
8. Finally, *substitute numbers* with units in your algebraic solutions, obtaining numerical values for the unknowns.
9. Look at your answer—does it make sense? Is it obviously too large or too small? Is the sign correct? Are the units appropriate?

11.12 | TOPPLING

Key Concept

- The normal forces are not uniformly distributed across the base, but we can find an effective point of application by taking $\tau_{net} = 0$, if the body is not toppling.

Toppling of a Box on an Inclined Plane

The normal contact force's line of action is not always predictable. For example, the forces acting on the box in Fig. 11-19 are its weight $\vec{W}$, friction $\vec{f}$, and the normal force exerted by the ground, $\vec{F}_N$. Weight acts at the body's center of mass. Friction acts along the surface where the crate and the plane are in contact. Normal reaction is a contact force. It does not act at a single point but is distributed on the surface. Actually, it is the resultant of a large number of small normal forces acting all over the base of the crate. The normal forces are not uniformly distributed across the base, but we can find an effective point of application by taking $\tau_{net} = 0$, if the body is not toppling.

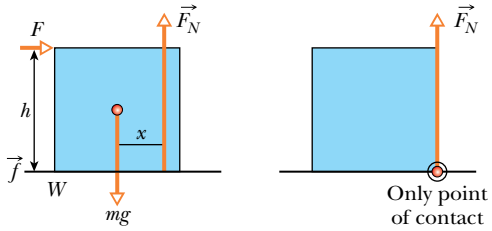

Figure 11-19 A box being pushed by horizontal force and shifting of normal.

SAMPLE PROBLEM 11.12

Crate and incline

The crate in Fig. 11-20 has dimensions *l* by *h* and the ramp is inclined at θ to the horizontal. Find the maximum θ for which it does not topple. Coefficient of friction between ramp and crate is μ.

KEY IDEA

When we keep a box on a very steep incline it topples downward. We can also conclude that the leftmost part

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of the box lifts before it topples. This means that normal contact force must have shifted toward right. The crate is in equilibrium. By applying force and torque balance, we may find where the normal force acts.

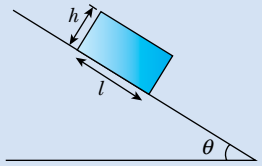

Figure 11-20 A crate resting on an incline.

Calculations: We choose the x axis to be along the slope and the y axis to be perpendicular to it. The sum of force components in each direction must be zero.

$$0 = \sum F_x = f - W\sin\theta$$

and

$$0 = \sum F_y = F_N - W\cos\theta.$$

We consider the net torque about the center of mass of the crate. Three forces exert torque:

1. The weight acts at the center of mass and so exerts no torque about it.
2. The friction force acts along the surface between crate and ramp and exerts a torque $fh/2$, out of the paper (Fig. 11-21).

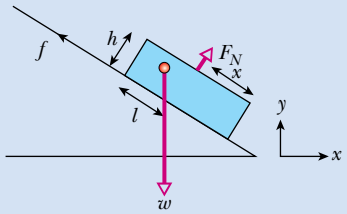

Figure 11-21 Free-body diagram.

3. The normal force acts at an unknown distance x from the downhill edge of the crate and exerts a torque $FN[(l/2) - x]$ into the page.

These torques sum to zero:

$$0 = \sum \vec{\tau} = [f(h/2) - n\{(l/2) - x\}].$$

From force balance, we have $f = W\sin\theta$ and $N = W\cos\theta$, so

$$W\sin\theta(h/2) = W\cos\theta[(l/2) - x].$$

When the crate is about to topple, x will become zero, as the effective point of application of the normal will be the right edge.

$$\theta = \tan^{-1}\left(\frac{l}{h}\right).$$

Learn: We can see that a taller box will overturn more easily than a shorter box. If $\tan\phi = \mu$, if $\phi > \theta$; first slip $\theta > \phi$; first topple.

SAMPLE PROBLEM 11.13

Car, horizontal track, overturn

A car of mass m traveling at speed v moves on a horizontal track. The center of mass of the car describes a circle of radius r. If $2a$ is the separation of the inner and outer wheels and h is the height of the center of mass above the ground, show that the limiting speed beyond which the car will overturn is given by

$$v^2 - \frac{gra}{h}.$$

KEY IDEA

Figure 11-22*a* shows the rear view of the car that is turning toward left. If you have watched any action movie where car overturns, you may know that it overturns outward of the curve when traveling at a high speed.

Frictional force on the tires provides a centripetal force while turning. When car is going at a high speed, the inner tires will lose contact with the ground. We can consider car in equilibrium in its own frame (Fig. 11-22*b*).

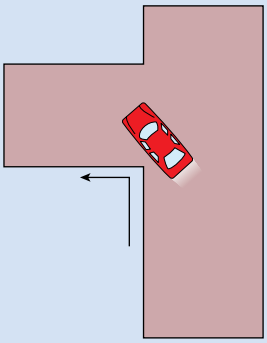

Figure 11-22 (a) Car turning at a crossing.

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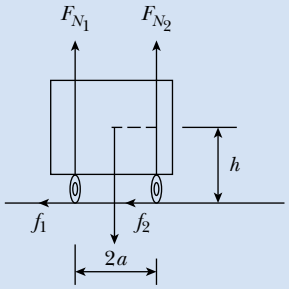

Figure 11-22 (b) Free-body diagram.

Calculations: As it will be a non-inertial reference frame, we need to apply pseudo centrifugal force

$$\sum F_x = f_1 + f_2 - \frac{mv^2}{r} = 0 \qquad (11\text{-}32)$$

$$\sum F_x = F_{N_1} + F_{N_2} - mg = 0 \qquad (11\text{-}33)$$

From condition of rotational equilibrium, sum of moments about center of mass G must be zero:

$$\sum \tau = (f_1 + f_2)h + F_{N_1}a - F_{N_2}a = 0 \qquad (11\text{-}34)$$

$$f_1 + f_2 = (F_{N_1} - F_{N_2})\frac{a}{h} \qquad (11\text{-}35)$$

On combining Eqs. 11-32 and 11-35, we can eliminate $f_1 + f_2$ to obtain

$$F_{N_1} - F_{N_2} = \frac{mv^2 h}{ar}. \qquad (11\text{-}36)$$

Now, we can solve Eqs. 11-33 and 11-36 simultaneously to obtain

$$2F_{N_1} = mg - \frac{mv^2 h}{r} \quad \text{and} \quad 2F_{N_2} = mg + \frac{mv^2 h}{r}.$$

The inner wheels will leave the ground when N_1 becomes zero, that is,

$$mg = \frac{mv^2 h}{ar}$$

or

$$v^2 = \frac{gra}{h}.$$

Learn: As expected in Eq. 11-36, we can see that as v increases the F_{N_1} will decrease and car will overturn outwardly.

If frictional force is not sufficient to provide centripetal acceleration, sideways skidding will occur instead of overturning.

$$f_1 + f_2 \le \mu(F_{N_1} - F_{N_2}).$$

Equality sign holds when skidding starts:

$$F_{N_1} - F_{N_2} = mg$$

and

$$f_1 + f_2 = \frac{mv^2}{r}.$$

Therefore, the limiting speed at which skidding occurs is given by $v^2 + \mu gr$. If the car is to skid rather than overturn,

$$\mu gr < \frac{gra}{h} \quad \text{or} \quad \mu < \frac{a}{h}$$

11.13 | INSTANTANEOUS CENTER OF ZERO VELOCITY

Key Concepts

- The instantaneous center of zero velocity for the body can only be used for an instant of time.
- Since the body changes its position from one instant to another, then for each position of the body a unique instantaneous center must be determined.

The velocity of an arbitrary point, B on a rigid body, at a given instant can be obtained easily if we can find the reference point, A that has zero velocity at that instant. In this case, $\vec{v}_A = 0$, and using the net motion equation, $\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$ and $\vec{v}_{B/A} = \omega + \vec{r}_{BA}$. Point A is called the instantaneous center of zero velocity (IC) and it lies on the instantaneous axis of zero velocity. This axis is perpendicular to the plane of the motion. The body appears to rotate about this axis instantaneously. Position vector of point B with respect to the IC is $\vec{r}_{BC}$. Then, the magnitude of $\vec{v}_B$ is simply $\vec{v}_B = \omega\vec{r}_{B/\text{IC}}$, where ω is the angular velocity of the body. Since all the particles are undergoing circular motion, the direction of $\vec{v}_B$ is always perpendicular to $\vec{r}_{B/\text{IC}}$.

For example, consider a wheel which rolls without slipping, Fig. 11-23*a*. In this case, the point of contact with the ground has zero velocity. Hence, this point represents the IC for the wheel, Fig. 11-23*b*. If it is imagined that the wheel is momentarily pinned at this point, the velocities of points A, B, O, and so on, can be found using $= \omega r$.

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Here, the radial distances $\vec{r}_{A/IC}$, $\vec{r}_{O/IC}$ and $\vec{r}_{B/IC}$, as shown in Fig. 11-23*b*, must be determined from the geometry of the wheel.

The instantaneous center of zero velocity for the body can only be used for an instant of time. Since the body changes its position from one instant to another, then for each position of the body a unique instantaneous center must be determined.

Although the IC may be conveniently used to determine the velocity of any point in a body, it generally does not have zero acceleration and therefore it should not be used for finding the acceleration of points in a body.

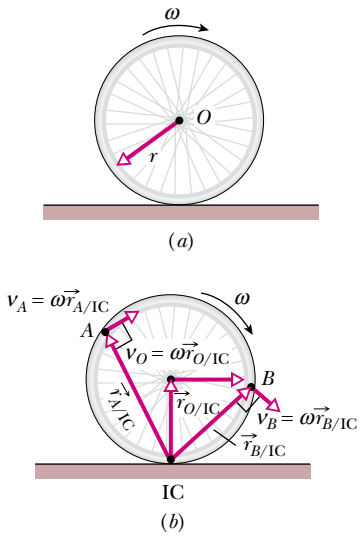

Figure 11-23 A wheel undergoing rolling.

Location of the IC

If the location of the IC is unknown, it may be determined by using the fact that the relative position vector extending from the IC to a point is always perpendicular to the velocity of the point. Several possibilities exist.

Example 1

Suggest a method to locate IC for the following situations:

(a) Given the velocity of a point on the body and the angular velocity of the body.
(b) Given the lines of action of two nonparallel velocities.
(c) Given the magnitude and direction of two parallel velocities.

Solution

(a) In this case, if $\vec{v}_B$ and ω are known, the IC is located along the line drawn perpendicular to at $\vec{v}_B$ at B, such that the distance from B to the IC is $\vec{r}_{B/C} = v_B/\omega$, Fig. 11-24*a*. Note that the IC lies on that side of B which causes rotation about the IC, which is consistent with the direction of motion caused by ω and $\vec{v}_B$.

(b) Given the lines of action of two nonparallel velocities. Consider the body in Fig. 11-24*b*, where the lines of action of the velocities $\vec{v}_A$ and $\vec{v}_B$ are known. Draw two lines from points A and B that are perpendicular to v_A and v_B. Extending these perpendiculars to their point of intersection as shown locates the IC at the instant considered. The magnitudes $\vec{r}_{A/IC}$ and $\vec{r}_{B/IC}$ are generally determined from the geometry of the body and trigonometry. Furthermore, if the magnitude and sense of $\vec{v}_A$ are known, then the angular velocity of the body is determined from $\vec{v}_A = \omega\vec{r}_{A/IC}$. Once computed, ω can then be used to determine $\vec{v}_B = \omega\vec{r}_{B/C}$.

(c) When the velocities of points A and B are parallel and have known magnitudes v_A and v_B, then the location of the IC is determined by proportional triangles. Examples are shown in Fig. 11-24*c* and *d*. In both cases, $\vec{r}_{A/IC} = \vec{v}_A/\omega$

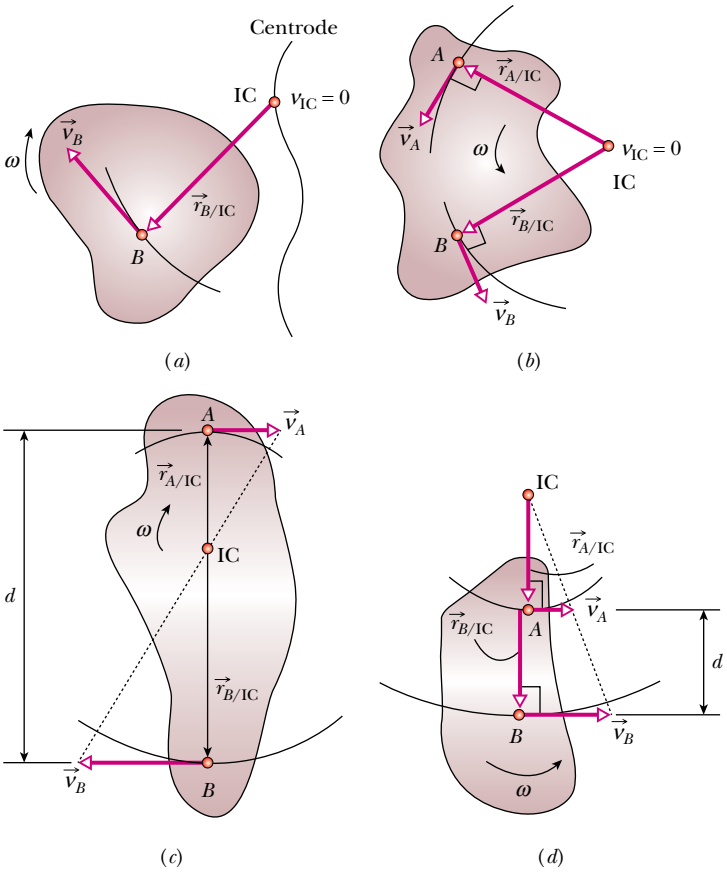

Figure 11-24 Location of IC in different situations.

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and $\vec{r}_{B/IC} = \vec{v}_B/\omega$. If d is a known distance between points A and B, then in Fig. 11-24*c*, $\vec{r}_{A/IC} + \vec{r}_{B/IC} = d$ and in Fig. 11-24*d*, $\vec{r}_{B/IC} - \vec{r}_{A/IC} = d$. As a special case, note that if the body is translating, $\vec{v}_A = \vec{v}_B$ and the IC would be located at infinity, in which case $\vec{r}_{A/IC} = \vec{r}_{B/IC} \to \infty$. This being the case, $\omega = v_A/\infty = v_B/\infty = 0$ as expected.

SAMPLE PROBLEM 11.14

A rolling cylinder, velocity of particles of cylinder

A cylinder of radius r is rolling purely on ground, find out if the number of particles having velocity greater than v_c are more than those having velocity less than v_c.

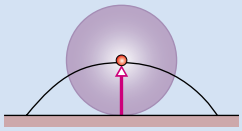

Figure 11-25 A cylinder rolling without sliding on ground.

Reasoning: The points lying below the arc in Fig. 11-25 are the ones having velocity less than v_c as their distance from the IC which is in contact with the ground is less than r. Thus, it is clear that these points are less in number than those having distance larger than r and thus having velocity greater than v_{com}.

REVIEW AND SUMMARY

Rolling Bodies For a wheel of radius R rolling smoothly,

$$v_{com} = \omega R, \tag{11-2}$$

where v_{com} is the linear speed of the wheel's center of mass and ω is the angular speed of the wheel about its center. The wheel may also be viewed as rotating instantaneously about the point P of the "road" that is in contact with the wheel. The angular speed of the wheel about this point is the same as the angular speed of the wheel about its center. The rolling wheel has kinetic energy

$$K = \frac{1}{2} I_{com}\omega^2 + \frac{1}{2} M v_{com}^2, \tag{11-5}$$

where I_{com} is the rotational inertia of the wheel about its center of mass and M is the mass of the wheel. If the wheel is being accelerated but is still rolling smoothly, the acceleration of the center of mass $\vec{a}_{com}$ is related to the angular acceleration α about the center with

$$a_{com} = \alpha R. \tag{11-6}$$

If the wheel rolls smoothly down a ramp of angle θ, its acceleration along an x axis extending up the ramp is

$$a_{com,x} = -\frac{g\sin\theta}{1 + (I_{com}/MR^2)}. \tag{11-10}$$

Torque about Center of Mass Torque will not only involve torque due to external forces about center of mass but it will also involve torque due to psuedo forces about center of mass.

$$\tau_{pseudo,i} = -r_i \times m_i \vec{a}$$

Torque about center of mass

$$\sum \tau_{pseudo,i} = 0$$

Angular Momentum of Rigid Body about a Point Lying Outside the Body Angular momentum of a body about any point or an axis is equal to the sum of angular momentum of the body about center of mass for an axis parallel to that axis and angular momentum of center of mass about that axis.

$$\vec{L}_{body} = \vec{L}_{body\text{-}com} + \vec{L}_{com} \tag{11-14}$$

The yo-yo The linear acceleration a_{com} of a yo-yo rolling down a string

$$a_{com} = -\frac{g}{1 + \left(\frac{I_{com}}{MR_0^2}\right)} \tag{11-15}$$

Problem Solving on Rolling Summary of dynamical formulas for fixed axis motion:

- Pure rotation about an axis–no translation.

$$L = I\omega;$$
$$\tau = I\alpha;$$
$$K = \frac{1}{2} I\omega^2.$$

- Rotation and translation.

$$L_z = I_{com}\omega + (\vec{R} \times m\vec{v});$$
$$\tau_z = \tau_{com} + (\vec{R} \times \vec{F});$$
$$\alpha_z = I_{com}\alpha;$$
$$K = \frac{1}{2} I_{com}\omega^2 + \frac{1}{2} M v^2.$$

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In case of rolling with slipping of uniform circular bodies, a and a_{com} are independent of each other, but the magnitude of frictional force is known as it is kinetic friction.

$$f_k = \mu_k F_N.$$

Static Equilibrium A rigid body at rest is said to be in **static equilibrium**. For such a body, the vector sum of the external forces acting on it is zero:

$$\vec{F}_{net} = 0 \quad \text{(balance of forces)}. \tag{11-21}$$

If all the forces lie in the xy plane, this vector equation is equivalent to two component equations:

$$F_{net,x} = 0 \quad \text{and} \quad F_{net,y} = 0 \quad \text{(balance of forces)}. \tag{11-25, 11-26}$$

Static equilibrium also implies that the vector sum of the external torques acting on the body about *any* point is zero, or

$$\vec{\tau}_{net} = 0 \quad \text{(balance of torques)}. \tag{11-23}$$

If the forces lie in the xy plane, all torque vectors are parallel to the z axis, and Eq. 11-23 is equivalent to the single component equation

$$\tau_{net,z} = 0 \quad \text{(balance of torques)}. \tag{11-27}$$

Instantaneous Center of Zero Velocity The instantaneous center of zero velocity (IC) lies on the instantaneous axis of zero velocity. This axis is perpendicular to the plane of the motion. The body appears to rotate about this axis instantaneously.

If the location of the IC is unknown, it may be determined by using the fact that the relative position vector extending from the IC to a point is always perpendicular to the velocity of the point. Several possibilities exist.

PROBLEMS

1. A car travels at 80 km/h on a level road in the positive direction of an x axis. Each tyre has a diameter of 66 cm. Relative to a woman riding in the car and in unit-vector notation, what are the velocity $\vec{v}$ at the (a) center, (b) top, and (c) bottom of the tyre and the magnitude a of the acceleration at the (d) center, (e) top, and (f) bottom of each tyre? Relative to a hitchhiker sitting next to the road and in unit-vector notation, what are the velocity $\vec{v}$ at the (g) center, (h) top, and (i) bottom of the tyre and the magnitude a of the acceleration at the (j) center, (k) top, and (l) bottom of each tyre?
2. An automobile traveling at 80.0 km/h has tyres of 70.0 cm diameter. (a) What is the angular speed of the tires about their axles? (b) If the car is brought to a stop uniformly in 30.0 complete turns of the tyres (without skidding), what is the magnitude of the angular acceleration of the wheels? (c) How far does the car move during the braking?
3. A solid ball of radius 6.0 cm is initially rolling smoothly at 10 m/s along a horizontal floor. It then rolls smoothly up a ramp until it momentarily stops. What maximum height above the floor does it reach?
4. A uniform solid sphere rolls down an incline. (a) What must be the incline angle if the linear acceleration of the center of the sphere is to have a magnitude of 0.15 g? (b) If a frictionless block were to slide down the incline at that angle, would its acceleration magnitude be more than, less than, or equal to 0.15 g? Why?
5. From an initial height h, a solid ball rolls smoothly down one side of a U-shaped ramp and then moves up the other side, which is frictionless. What maximum height does the ball reach?
6. Figure 11-26 gives the speed v versus time t for a 0.500 kg object of radius 5.75 cm that rolls smoothly down a 30° ramp. The scale on the velocity axis is set by $v_s = 4.0$ m/s. What is the rotational inertia of the object?

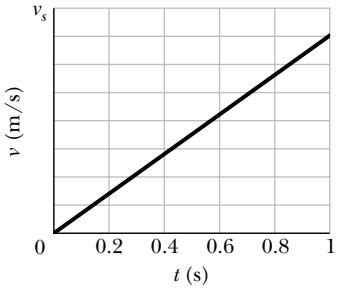

Figure 11-26 Problem 6.

7. In Fig. 11-27, a solid cylinder of radius 10 cm and mass 12 kg starts from rest and rolls without slipping a distance $L = 6.0$ m down a roof that is inclined at angle $\theta = 30°$. (a) What is the angular speed of the cylinder about its center as it leaves the roof? (b) The roof's edge is at height $H = 5.0$ m. How far horizontally from the roof's edge does the cylinder hit the level ground?

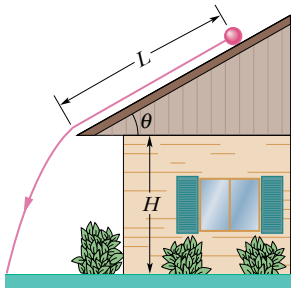

Figure 11-27 Problem 7.

8. Figure 11-28 shows the potential energy $U(x)$ of a solid ball that can roll along an x axis. The scale on the U axis is set by $U_s = 100$ J. The ball is uniform, rolls smoothly, and has a mass of 0.400 kg. It is released at $x = 7.0$ m headed in the negative direction of the x axis with a mechanical energy of 75 J. (a) If the ball can reach $x = 0$ m, what is

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its speed there, and if it cannot, what is its turning point? Suppose, instead, it is headed in the positive direction of the x axis when it is released at $x = 7.0$ m with 75 J. (b) If the ball can reach $x = 13$ m, what is its speed there, and if it cannot, what is its turning point?

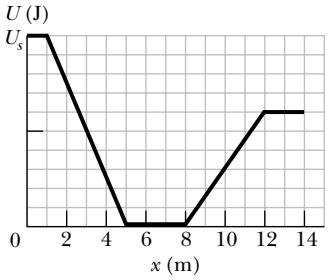

Figure 11-28 Problem 8.

9. In Fig. 11-29, a solid ball rolls smoothly from rest (starting at height $H = 6.0$ m) until it leaves the horizontal section at the end of the track, at height $h = 2.0$ m. How far horizontally from point A does the ball hit the floor?

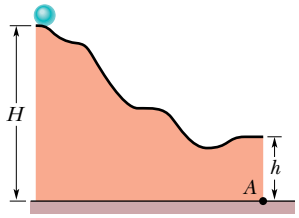

Figure 11-29 Problem 9.

10. A hollow sphere of radius 0.15 m, with rotational inertia $I = 0.048$ kg·m^2 about a line through its center of mass, rolls without slipping up a surface inclined at 30° to the horizontal. At a certain initial position, the sphere's total kinetic energy is 20 J. (a) How much of this initial kinetic energy is rotational? (b) What is the speed of the center of mass of the sphere at the initial position? When the sphere has moved 1.0 m up the incline from its initial position, what are (c) its total kinetic energy and (d) the speed of its center of mass?

11. In Fig. 11-30, a constant horizontal Force $\vec{F}_{app}$ of magnitude 10 N is applied to a wheel of mass 10 kg and radius 0.30 m. The wheel rolls smoothly on the horizontal surface, and the acceleration of its center of mass has magnitude 0.60 m/s^2. (a) In unit-vector notation, what is the frictional force on the wheel? (b) What is the rotational inertia of the wheel about the rotation axis through its center of mass?

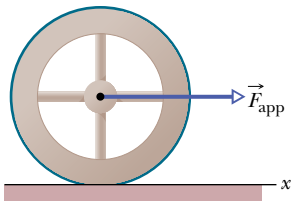

Figure 11-30 Problem 11.

12. In Fig. 11-31, a solid brass ball of mass 0.320 g will roll smoothly along a loop-the-loop track when released from rest along the straight section. The circular loop has radius $R = 12.0$ cm, and the ball has radius $r \ll R$. (a) What is h if the ball is on the verge of leaving the track when it reaches the top of the loop? If the ball is released at height $h = 6.00R$, what are the (b) magnitude and (c) direction of the horizontal force component acting on the ball at point Q?

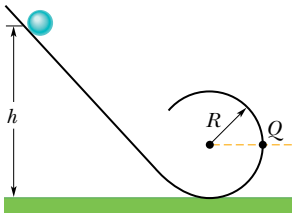

Figure 11-31 Problem 12.

13. *Nonuniform ball.* In Fig. 11-32, a ball of mass M and radius R rolls smoothly from rest down a ramp and onto a circular loop of radius 0.48 m. The initial height of the ball is $h = 0.36$ m. At the loop bottom, the magnitude of the normal force on the ball is $2.00Mg$. The ball consists of an outer spherical shell (of a certain uniform density) that is glued to a central sphere (of a different uniform density). The rotational inertia of the ball can be expressed in the general form $I = \beta MR^2$, but β is not 0.4 as it is for a ball of uniform density. Determine β.

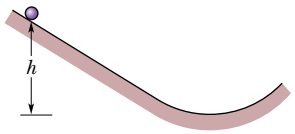

Figure 11-32 Problem 13.

14. In Fig. 11-33, a small, solid, uniform ball is to be shot from point P so that it rolls smoothly along a horizontal path, up along a ramp, and onto a plateau. Then it leaves the plateau horizontally to land on a game board, at a horizontal distance d from the right edge of the plateau. The vertical heights are $h_1 = 5.00$ cm and $h_2 = 1.60$ cm. With what speed must the ball be shot at point P for it to land at $d = 7.50$ cm?

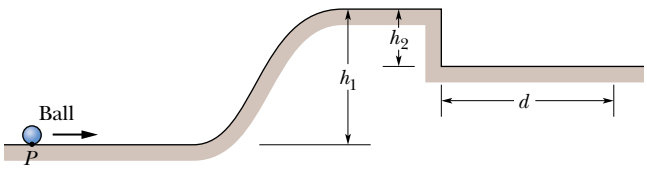

Figure 11-33 Problem 14.

15. A bowler throws a bowling ball of radius $R = 11$ cm along a lane. The ball (Fig. 11-34) slides on the lane with initial speed $v_{com,0} = 8.5$ m/s and initial angular speed $\omega_0 = 0$. The coefficient of kinetic friction between the ball and the lane is 0.21. The kinetic frictional force $\vec{f}_k$ acting on the ball causes a linear acceleration of the ball while producing a torque that causes an angular acceleration of the ball. When speed v_{com} has decreased enough and angular speed ω has increased enough, the ball stops sliding and then rolls smoothly. (a) What then is v_{com} in terms of ω? During the sliding, what are the ball's (b) linear acceleration and (c) angular acceleration? (d) How long does the ball slide? (e) How far does the ball slide? (f) What is the linear speed of the ball when smooth rolling begins?

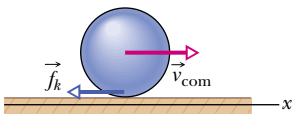

Figure 11-34 Problem 15.

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16. *Nonuniform cylindrical object.* In Fig. 11-35, a cylindrical object of mass M and radius R rolls smoothly from rest down a ramp and onto a horizontal section. From there it rolls off the ramp and onto the floor, landing a horizontal distance $d = 0.506$ m from the end of the ramp. The initial height of the object is $H = 0.90$ m; the end of the ramp is at height $h = 0.10$ m. The object consists of an outer cylindrical shell (of a certain uniform density) that is glued to a central cylinder (of a different uniform density). The rotational inertia of the object can be expressed in the general form $I = \beta MR^2$, but β is not 0.5 as it is for a cylinder of uniform density. Determine β.

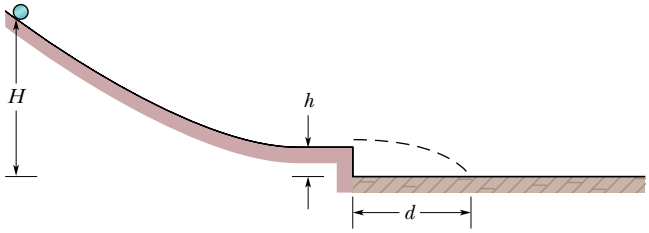

Figure 11-35 Problem 16.

17. A yo-yo has a rotational inertia of 950 g cm^2 and a mass of 120 g. Its axle radius is 3.2 mm, and its string is 120 cm long. The yo-yo rolls from rest down to the end of the string. (a) What is the magnitude of its linear acceleration? (b) How long does it take to reach the end of the string? As it reaches the end of the string, what are its (c) linear speed, (d) translational kinetic energy, (e) rotational kinetic energy, and (f) angular speed?

18. In 1980, over San Francisco Bay, a large yo-yo was released from a crane. The 116 kg yo-yo consisted of two uniform disks of radius 32 cm connected by an axle of radius 3.2 cm. What was the magnitude of the acceleration of the yo-yo during (a) its fall and (b) its rise? (c) What was the tension in the cord on which it rolled? (d) Was that tension near the cord's limit of 52 kN? Suppose you build a scaled-up version of the yo-yo (same shape and materials but larger). (e) Will the magnitude of your yo-yo's acceleration as it falls be greater than, less than, or the same as that of the San Francisco yo-yo? (f) How about the tension in the cord?

19. A thief intends to enter an apartment by climbing a ladder but foolishly places the upper end against a window. When he is 3.00 m up the ladder, the window is on the verge of shattering. His mass is 90.0 kg, the ladder's mass is 20.0 kg, the ladder's length is 5.00 m, and the foot of the ladder is 2.50 m from the base of the wall, on a non-slip ground surface. What are (a) the magnitude of the force on the glass from the ladder, (b) the magnitude of the force on the ladder from the ground, and (c) the angle between that ground force and the horizontal?

20. In Fig. 11-36, a 50.0 kg uniform square sign, of edge length $L = 2.50$ m, is hung from a horizontal rod of length $d_h = 3.00$ m and negligible mass. A cable is attached to the end of the rod and to a point on the wall at distance $d_v = 4.00$ m above the point where the rod is hinged to the wall. (a) What is the tension in the cable? What are the (b) magnitude and (c) direction (left or right) of the horizontal component of the force on the rod from the wall, and the (d) magnitude and (e) direction (up or down) of the vertical component of this force?

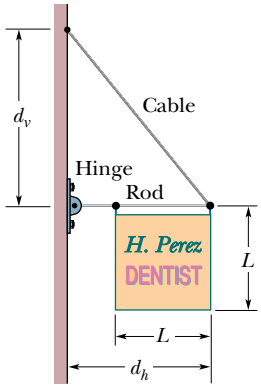

Figure 11-36 Problem 20.

21. The system in Fig. 11-37 is in equilibrium, with the string in the center exactly horizontal. Block A weighs 35 N, block B weighs 45 N, and angle ϕ is 35°. Find (a) tension T_1, (b) tension T_2, (c) tension T_3, and (d) angle θ.

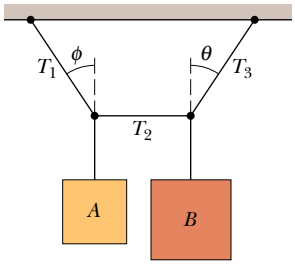

Figure 11-37 Problem 21.

22. In Fig. 11-38, suppose the length L of the uniform bar is 2.75 m and its weight is 185 N. Also, let the block's weight $W = 300$ N and the angle $\theta = 30.0°$. The wire can withstand a maximum tension of 500 N. (a) What is the maximum possible distance x before the wire breaks? With the block placed at this maximum x, what are the (b) horizontal and (c) vertical components of the force on the bar from the hinge at A?

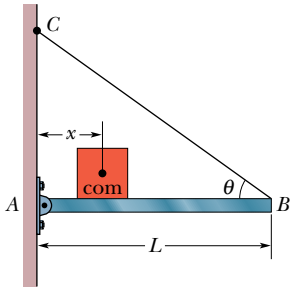

Figure 11-38 Problems 22.

23. Figure 11-39*a* shows a horizontal uniform beam of mass m_b and length L that is supported on the left by a hinge attached to a wall and on the right by a cable at angle θ with the horizontal. A package of mass m_p is positioned on

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the beam at a distance x from the left end. The total mass is $m_b + m_p = 79.30$ kg. Figure 11-39*b* gives the tension T in the cable as a function of the package's position given as a fraction x/L of the beam length. The scale of the T axis is set by $T_a = 500$ N and $T_b = 700$ N. Evaluate (a) angle θ, (b) mass m_b, and (c) mass m_p.

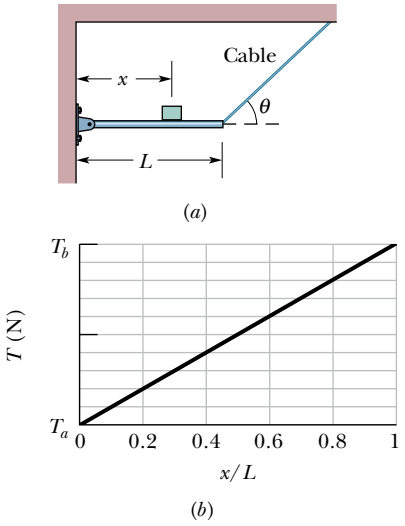

Figure 11-39 Problem 23.

24. In Fig. 11-40, a 25 kg block is held in place via a pulley system. The person's upper arm is vertical; the forearm is at angle $\theta = 30°$ with the horizontal. Forearm and hand together have a mass of 2.0 kg, with a center of mass at distance $d_1 = 15$ cm from the contact point of the forearm bone and the upper-arm bone (humerus). The triceps muscle pulls vertically upward on the forearm at distance $d_2 = 2.5$ cm behind that contact point. Distance d_3 is 35 cm. What are the (a) magnitude and (b) direction (up or down) of the force on the forearm from the triceps muscle and the (c) magnitude and (d) direction (up or down) of the force on the forearm from the humerus?

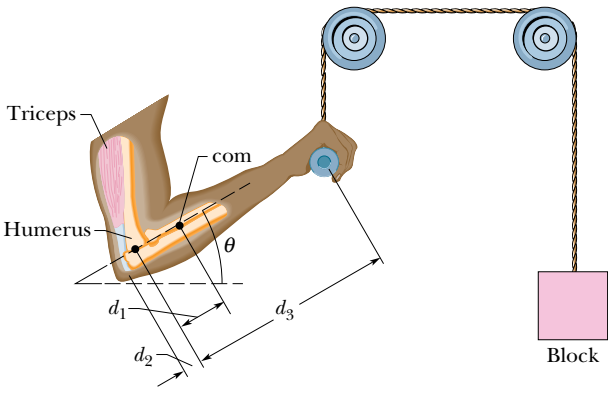

Figure 11-40 Problem 24.

25. A crate, in the form of a cube with edge lengths of 1.2 m, contains a piece of machinery; the center of mass of the crate and its contents is located 0.30 m above the crate's geometrical center. The crate rests on a ramp that makes an angle θ with the horizontal. As θ is increased from zero, an angle will be reached at which the crate will either tip over or start to slide down the ramp. If the coefficient of static friction μ_s between ramp and crate is 0.30, (a) does the crate tip or slide and (b) at what angle θ does this occur? If $\mu_s = 0.70$, (c) does the crate tip or slide and (d) at what angle θ does this occur? (*Hint:* At the onset of tipping, where is the normal force located?)

26. Figure 11-41 shows a diver of weight 580 N standing at the end of a diving board with a length of L = 5.70 m and negligible mass. The board is fixed to two pedestals (supports) that are separated by distance d = 1.50 m. Of the forces acting on the board, what are the (a) magnitude and (b) direction (up or down) of the force from the left pedestal and the (c) magnitude and (d) direction (up or down) of the force from the right pedestal? (e) Which pedestal (left or right) is being stretched, and (f) which pedestal is being compressed?

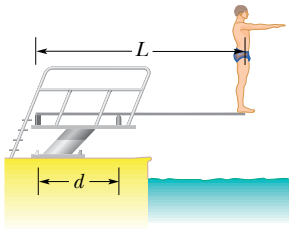

Figure 11-41 Problem 26.

27. A scaffold of mass 65 kg and length 5.2 m is supported in a horizontal position by a vertical cable at each end. A window washer of mass 80 kg stands at a point 1.5 m from one end. What is the tension in (a) the nearer cable and (b) the farther cable?

28. In Fig. 11-42, a nonuniform bar is suspended at rest in a horizontal position by two massless cords. One cord makes the angle $\theta = 30.0°$ with the vertical; the other makes the angle $\phi = 60.0°$ with the vertical. If the length L of the bar is 9.50 m, compute the distance x from the left end of the bar to its center of mass.

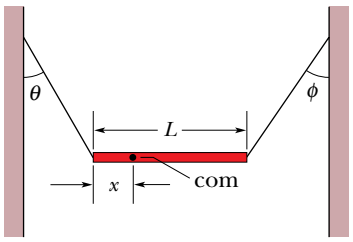

Figure 11-42 Problem 28.

29. In Fig. 11-43, a climber leans out against a vertical ice wall that has negligible friction. Distance a is 0.914 m and distance L is 2.10 m. His center of mass is distance $d = 0.948$ m from the feet–ground contact point. If he is on the verge of sliding, what is the coefficient of static friction between feet and ground?

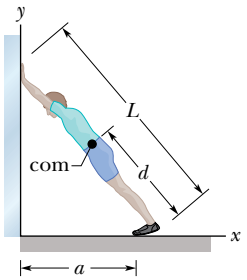

Figure 11-43 Problem 29.

30. In Fig. 11-44, trying to get his car out of mud, a man ties one end of a rope around the

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front bumper and the other end tightly around a utility pole 15 m away. He then pushes sideways on the rope at its midpoint with a force of 520 N, displacing the center of the rope 0.30 m, but the car barely moves. What is the magnitude of the force on the car from the rope? (The rope stretches somewhat.)

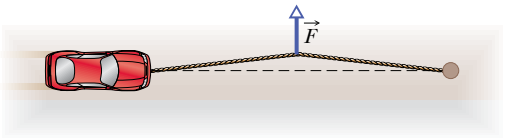

Figure 11-44 Problem 30.

31. In Fig. 11-17 and the associated sample problem, let the coefficient of static friction μ_s between the ladder and the pavement be 0.40. How far (in percent) up the ladder must the firefighter go to put the ladder on the verge of sliding?

32. A 14.0 m cord of negligible mass is stretched horizontally between two supports. When a rock climber of weight 900 N hangs from the cord's midpoint, that midpoint sags by 0.85 m. What then is the tension in the cord?

33. In Fig. 11-45, the driver of a car on a horizontal road makes an emergency stop by applying the brakes so that all four wheels lock and skid along the road. The coefficient of kinetic friction between tires and road is 0.37. The separation between the front and rear axles is $L = 4.0$ m, and the center of mass of the car is located at distance $d = 1.9$ m behind the front axle and distance $h = 0.75$ m above the road. The car weighs 11 kN. Find the magnitude of (a) the braking acceleration of the car, (b) the normal force on each rear wheel, (c) the normal force on each front wheel, (d) the braking force on each rear wheel, and (e) the braking force on each front wheel. (*Hint:* Although the car is not in translational equilibrium, it *is* in rotational equilibrium.)

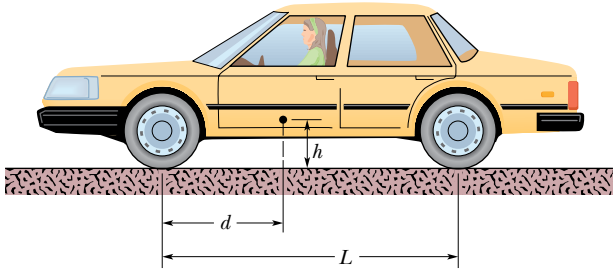

Figure 11-45 Problem 33.

34. Draw the string of a simple archery bow back at its midpoint until the string tension is twice the magnitude of your force on the string. What is the angle between the two halves of the string?

35. In Fig. 11-46, what magnitude of (constant) force $\vec{F}$ applied horizontally at the axle of the wheel is necessary to raise the wheel over a step obstacle of height $h = 3.00$ cm? The wheel's radius is $r = 8.00$ cm, and its mass is $m = 0.600$ kg.

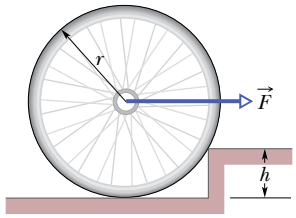

Figure 11-46 Problem 35.

36. Figure 11-47*a* shows a vertical uniform beam of length L that is hinged at its lower end. A horizontal force $\vec{F}_a$ is applied to the beam at distance y from the lower end. The beam remains vertical because of a cable attached at the upper end, at angle θ with the horizontal. Figure 11-47*b* gives the tension T in the cable as a function of the position of the applied force given as a fraction y/L of the beam length. The scale of the T axis is set by $T_s = 800$ N. Figure 11-47*c* gives the magnitude F_h of the horizontal force on the beam from the hinge, also as a function of y/L. Evaluate (a) angle θ and (b) the magnitude of $\vec{F}_a$.

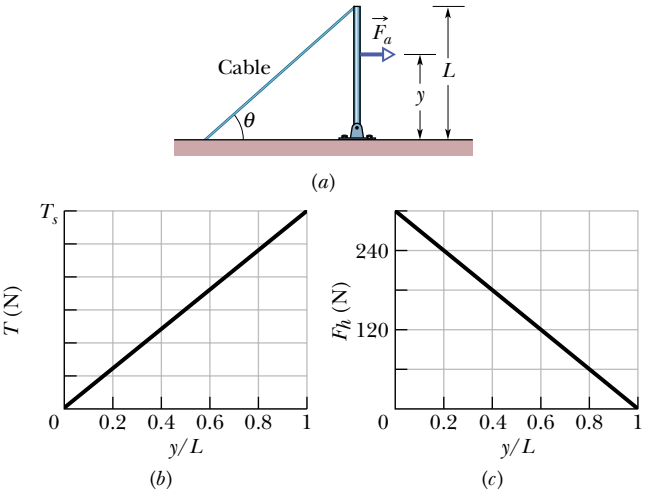

Figure 11-47 Problem 36.

37. In Fig. 11-48, a climber with a weight of 533.8 N is held by a belay rope connected to her climbing harness and belay device; the force of the rope on her has a line of action through her center of mass. The indicated angles are $\theta = 38.0°$ and $\phi = 30.0°$. If her feet are on the verge of sliding on the vertical wall, what is the coefficient of static friction between her climbing shoes and the wall?

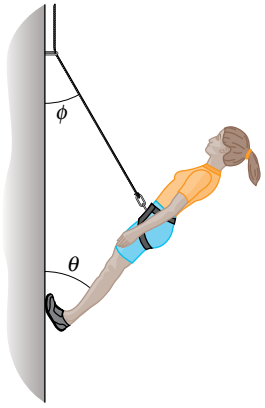

Figure 11-48 Problem 37.

38. In Fig. 11-49, a lead brick rests horizontally on cylinders A and B. The areas of the top faces of the cylinders are related by $A_A = 2.2A_B$; the Young's moduli of the cylinders are

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related by $E_A = 2.2E_B$. The cylinders had identical lengths before the brick was placed on them. What fraction of the brick's mass is supported (a) by cylinder A and (b) by cylinder B? The horizontal distances between the center of mass of the brick and the centerlines of the cylinders are d_A for cylinder A and d_B for cylinder B. (c) What is the ratio d_A/d_B?

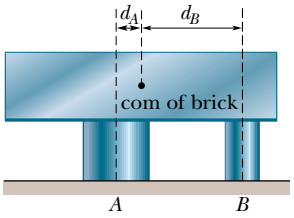

Figure 11-49 Problem 38.

39. In Fig. 11-50, one end of a uniform beam of mass 40.0 kg is hinged to a wall; the other end is supported by a wire that makes angles $\theta = 30.0°$ with both wall and beam. Find (a) the tension in the wire and the (b) magnitude and (c) angle from the horizontal of the force of the hinge on the beam.

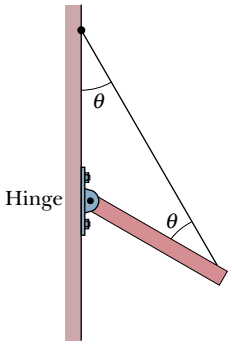

Figure 11-50 Problem 39.

40. A cubical box is filled with sand and weighs 520 N. We wish to "roll" the box by pushing horizontally on one of the upper edges. (a) What minimum force is required? (b) What minimum coefficient of static friction between box and floor is required? (c) If there is a more efficient way to roll the box, find the smallest possible force that would have to be applied directly to the box to roll it. (*Hint:* At the onset of tipping, where is the normal force located?)

41. Forces $\vec{F}_1, \vec{F}_2$, and $\vec{F}_3$ act on the structure of Fig. 11-51, shown in an overhead view. We wish to put the structure in equilibrium by applying a fourth force, at a point such as P. The fourth force has vector components $\vec{F}_h$ and $\vec{F}_v$. We are given that $a = 2.0$ m, $b = 3.0$ m, $c = 1.0$ m, $F_1 = 30$ N, $F_2 = 10$ N, and $F_3 = 5.0$ N. Find (a) F_h, (b) F_v, and (c) d.

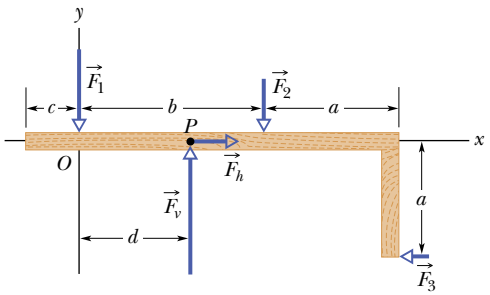

Figure 11-51 Problem 41.

42. In Fig. 11-52, a 55 kg rock climber is in a lie-back climb along a fissure, with hands pulling on one side of the fissure and feet pressed against the opposite side. The fissure has width $w = 0.20$ m, and the center of mass of the climber is a horizontal distance $d = 0.40$ m from the fissure. The coefficient of static friction between hands and rock is $\mu_1 = 0.45$, and between boots and rock it is $\mu_2 = 1.2$. (a) What is the least horizontal pull by the hands and push by the feet that will keep the climber stable? (b) For the horizontal pull of (a), what must be the vertical distance h between hands and feet? If the climber encounters wet rock, so that μ_1 and μ_2 are reduced, what happens to (c) the answer to (a) and (d) the answer to (b)?

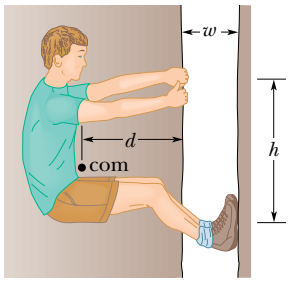

Figure 11-52 Problem 42.

43. Figure 11-53 shows a 75 kg climber hanging by only the *crimp hold* of one hand on the edge of a shallow horizontal ledge in a rock wall. (The fingers are pressed down to gain purchase.) Her feet touch the rock wall at distance $H = 2.0$ m directly below her crimped fingers but do not provide any support. Her center of mass is distance $a = 0.18$ m from the wall. Assume that the force from the ledge supporting her fingers is equally shared by the four fingers. What are the values of the (a) horizontal component F_h and (b) vertical component F_v of the force on *each* fingertip?

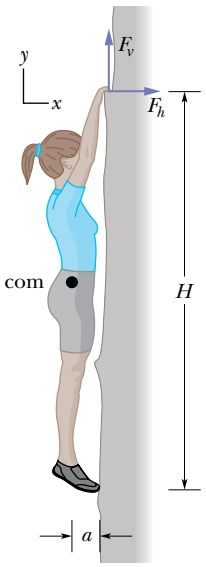

Figure 11-53 Problem 43.

44. A uniform cubical crate is 0.850 m on each side and weighs 500 N. It rests on a floor with one edge against a very small, fixed obstruction. At what least height above the floor must a horizontal force of magnitude 400 N be applied to the crate to tip it?

45. The system in Fig. 11-54 is in equilibrium. A concrete block of mass 300 kg hangs from the end of the uniform strut of mass 45.0 kg. A cable runs from the ground, over the top of the strut, and down to the block, holding the block in place. For angles $\phi = 30.0°$ and $\theta = 45.0°$, find (a) the tension T in the cable and the (b) magnitude and (c) angle (relative to the horizontal) of the force on the strut from the hinge.

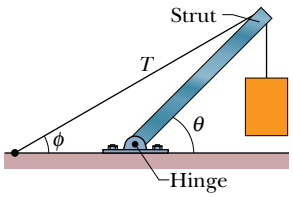

Figure 11-54 Problem 45.

46. In Fig. 11-55, a uniform plank, with a length L of 10.5 m and a weight of 445 N, rests on the ground and against a frictionless roller at the top of a wall of height $h = 3.05$ m. The plank remains in equilibrium for any value of $\theta \geq 70°$

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but slips if $\theta < 70°$. Find the coefficient of static friction between the plank and the ground.

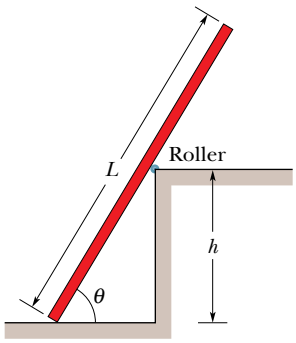

Figure 11-55 Problem 46.

47. In a classroom demonstration, a meter stick is balanced at its midpoint on a narrow support. When a 23.0 g candy piece is placed at the 25.0 cm mark, the support must be moved to the 45.5 cm mark to rebalance the stick. What is the stick's mass?

48. A uniform wood panel of mass 35.0 kg is hinged vertically on a wall by two hinges, one 0.300 m from its top and the other 0.300 m from its bottom. The panel is 2.10 m tall and 0.910 m wide. A y axis extends upward through the hinges and an x axis extends outward along the panel's width. If each hinge supports half the panel's weight, what are the forces on the panel at (a) the top hinge and (b) the bottom hinge, both in unit-vector notation?

49. In Fig. 11-56, uniform beams A and B are attached to a wall with hinges and loosely bolted together (there is no torque of one on the other). Beam A has length $L_A = 2.40$ m and mass 58.0 kg; beam B has mass 70.0 kg. The two hinge points are separated by distance $d = 1.80$ m. In unit-vector notation, what is the force on (a) beam A due to its hinge, (b) beam A due to the bolt, (c) beam B due to its hinge, and (d) beam B due to the bolt?

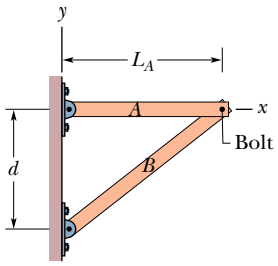

Figure 11-56 Problem 49.

50. For the stepladder shown in Fig. 11-57, sides AC and CE are each 2.44 m long and hinged at C. Bar BD is a tie-rod 0.762 m long, halfway up. A man weighing 1045 N climbs 1.80 m along the ladder. Assuming that the floor is frictionless and neglecting the mass of the ladder, find (a) the tension in the tie-rod and the magnitudes of the forces on the ladder from the floor at (b) A and (c) E. (*Hint:* Isolate parts of the ladder in applying the equilibrium conditions.)

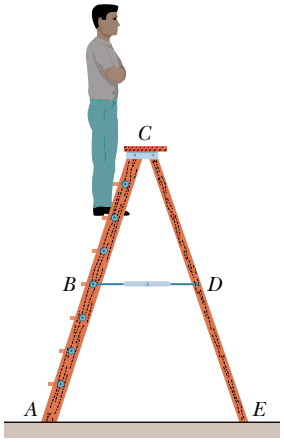

Figure 11-57 Problem 50.

51. A tunnel of length $L = 200$ m, height $H = 7.2$ m, and width 6.3 m (with a flat roof) is to be constructed at distance $d = 70$ m beneath the ground (Fig. 11-58). The tunnel roof is to be supported entirely by square steel columns, each with a cross-sectional area of 960 cm^2. The mass of 1.0 cm^3 of the ground material is 2.8 g. (a) What is the total weight of the ground material the columns must support? (b) How many columns are needed to keep the compressive stress on each column at one-half its ultimate strength?

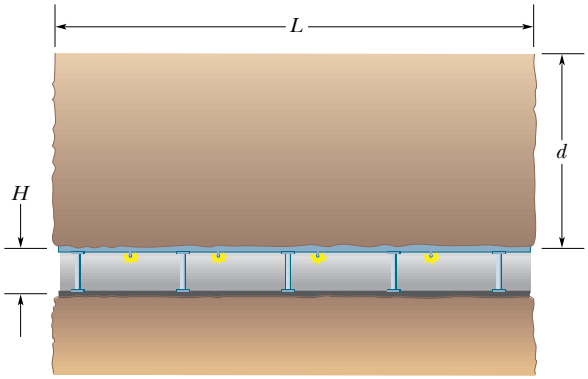

Figure 11-58 Problem 51.

52. In Fig. 11-59, horizontal scaffold 2, with uniform mass $m_2 = 30.0$ kg and length $L_2 = 2.00$ m, hangs from horizontal scaffold 1, with uniform mass $m_1 = 50.0$ kg. A 18.0 kg box of nails lies on scaffold 2, centered at distance $d = 0.500$ m from the left end. What is the tension T in the cable indicated?

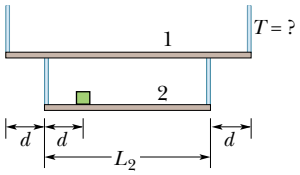

Figure 11-59 Problem 52.

53. A physics Brady Bunch, whose weights in newtons are indicated in Fig. 11-60, is balanced on a seesaw. What is the number of the person who causes the smallest torque

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about the rotation axis at *fulcrum f* directed (a) out of the page and (b) into the page?

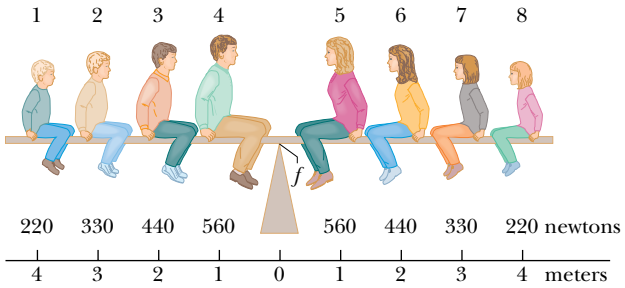

Figure 11-60 Problem 53.

54. A bowler holds a bowling ball ($M = 7.6$ kg) in the palm of his hand (Fig. 11-61). His upper arm is vertical; his lower arm (1.9 kg) is horizontal. What is the magnitude of (a) the force of the biceps muscle on the lower arm and (b) the force between the bony structures at the elbow contact point?

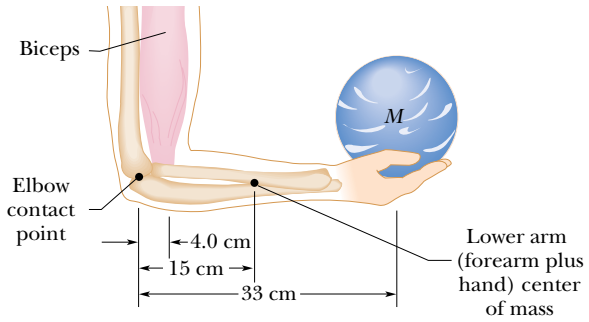

Figure 11-61 Problem 54.

PRACTICE QUESTIONS

1. A circular hoop rolls without slipping on a flat horizontal surface. Which one of the following is necessarily true?
- **(a)** All points on the rim of the hoop have the same speed.
- **(b)** All points on the rim of the hoop have the same velocity.
- **(c)** Every point on the rim of the wheel has a different velocity.
- **(d)** All points on the rim of the hoop have acceleration vectors that are tangent to the hoop.

2. A 1.0-kg wheel in the form of a solid disk rolls along a horizontal surface with a speed of 6.0 m/s. What is the total kinetic energy of the wheel?
- **(a)** 9.0 J **(b)** 18 J
- **(c)** 27 J **(d)** 36 J

3. A pulley of radius 2 m is rotated about its axis by a force $F = (20t - 5t^2)$ N (where t is measured in seconds) applied tangentially. If the moment if inertia of the pulley about its axis of rotation is 10 kg/m^2, the number of rotations make by the pulley before its direction of motion if reversed, is
- **(a)** more than 3 but less than 6.
- **(b)** more than 6 but less than 9.
- **(c)** more than 9.
- **(d)** less than 3.

4. Suppose you are riding a stationary exercise bicycle, and the electronic meter indicates that the wheel is rotating at 9.1 rad/s. The wheel has a radius of 0.45 m. If you ride the bike for 35 min, how far would you have gone if the bike could move?
- **(a)** 3900 m **(b)** 7800 m
- **(c)** 4300 m **(d)** 8600 m

5. Which statement concerning a wheel undergoing rolling motion is true?
- **(a)** The angular acceleration of the wheel must be zero m/s^2.
- **(b)** The tangential velocity is the same for all points on the wheel.
- **(c)** The linear velocity for all points on the rim of the wheel is non-zero.
- **(d)** There is no slipping at the point where the wheel touches the surface on which it is rolling.

6. A round uniform body of radius R, mass M, and moment of inertia I rolls down (without slipping) in an inclined plane making an angle θ with the horizontal. Then, its acceleration is

(a) $\dfrac{g\sin\theta}{1+(I/MR^2)}$ **(b)** $\dfrac{g\sin\theta}{1+(MR^2/I)}$

(c) $\dfrac{g\sin\theta}{1-(I/MR^2)}$ **(d)** $\dfrac{g\sin\theta}{1-(MR^2/I)}$

7. A solid sphere rolls without slipping along a horizontal surface. What percentage of its total kinetic energy is rotational kinetic energy?
- **(a)** 33% **(b)** 29%
- **(c)** 50% **(d)** 75%

8. An automobile tyre has a radius of 0.330 m, and its center moves forward with a linear speed of $v = 15.0$ m/s. Relative to the axle, what is the tangential speed of a point located 0.175 m from the axle?
- **(a)** 1.04 m/s **(b)** 1.50 m/s
- **(c)** 4.39 m/s **(d)** 7.96 m/s

9. Two motorcycles are riding around a circular track at the same angular velocity. One motorcycle is at a radius of 15 m;

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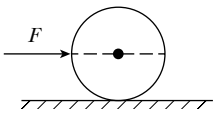
and the second is at a radius of 18 m. What is the ratio of their linear speeds, v_2/v_1?

(a) 1.0 **(b)** 1.2
(c) 0.83 **(d)** 0.71

10. A bicycle has tyres of radius 0.35 meters. If the bicycle is traveling at a constant speed of 12 m/s, at approximately what angular speed are the tyres rotating?

(a) 85 rev/min **(b)** 197 rev/min
(c) 214 rev/min **(d)** 327 rev/min

11. A ring of mass M and radius R is kept on horizontal surface as shown in the figure. The minimum coefficient of static friction such that ring does not slip is



(a) $\frac{4F}{Mg}$ **(b)** $\frac{F}{Mg}$
(c) $\frac{2F}{Mg}$ **(d)** $\frac{F}{2Mg}$

12. What happens when a spinning ice skater draws in her outstretched arms?

(a) Her angular momentum decreases.
(b) Her angular momentum increases.
(c) Her moment of inertia decreases causing her to speed up.
(d) Her moment of inertia decreases causing her to slow down.

13. A hollow cylinder of mass M and radius R rolls down an inclined plane. A block of mass M slides down an identical inclined plane. If both objects are released at the same time, then

(a) the cylinder reaches the bottom first.
(b) the block reaches the bottom first.
(c) the block reaches the bottom with the greater kinetic energy.
(d) the cylinder reaches the bottom with the greater kinetic energy.

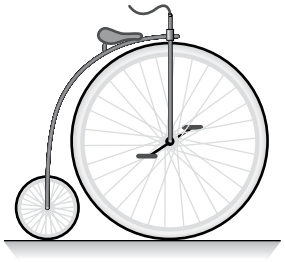
14. A child standing on the edge of a freely spinning merry-go-round moves quickly to the center. Which one of the following statements is necessarily true concerning this event and why?

(a) The angular speed of the system decreases because the moment of inertia of the system has increased.
(b) The angular speed of the system increases because the moment of inertia of the system has increased.
(c) The angular speed of the system decreases because the moment of inertia of the system has decreased.
(d) The angular speed of the system increases because the moment of inertia of the system has decreased.

15. Two disks are rotating about the same axis. Disk A has a moment of inertia of 3.4 kg·m^2 and an angular velocity of +7.2 rad/s. Disk B is rotating with an angular velocity of –9.8 rad/s. The two disks are then linked together without the aid of any external torques, so that they rotate as a single unit with an angular velocity of –2.4 rad/s. The axis of rotation for this unit is the same as that for the separate disks. What is the moment of inertia of disk B?

(a) 4.4 kg·m^2 **(b)** 2.4 kg·m^2
(c) 5.2 kg·m^2 **(d)** 3.6 kg·m^2

16. The penny-farthing is a bicycle that was popular between 1870 and 1890. As the figure shows, this type of bicycle has a large front wheel and a small rear wheel. On a Sunday ride in the park the front wheel (radius = 1.20 m) makes 276 rev. How many revolutions does the rear wheel (radius = 0.340 m) make?

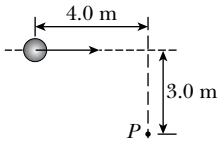


(a) 974 rev **(b)** 331 rev
(c) 78.2 rev **(d)** 812 rev

17. A 1500 kg satellite orbits a planet in a circular orbit of radius 6.2×10^6 m. What is the angular momentum of the satellite in its orbit around the planet if the satellite completes one orbit every 1.5×10^4 s?

(a) 3.9×10^6 kg · m^2/s **(b)** 2.4×10^{13} kg · m^2/s
(c) 1.4×10^{14} kg · m^2/s **(d)** 8.1×10^{11} kg · m^2/s

18. A 3.0 kg ball moves in a straight line at 10 m/s as shown in the figure. At the instant shown, what is its angular momentum about the point P?



(a) 30 kg·m^2/s **(b)** 90 kg·m^2/s
(c) 150 kg·m^2/s **(d)** 120 kg·m^2/s

19. A solid disk rotates in the horizontal plane at an angular velocity of 0.067 rad/s with respect to an axis perpendicular to the disk at its center. The moment of inertia of the disk is 0.10 kg·m^2. From above, sand is dropped straight down onto this rotating disk, so that a thin uniform ring of sand is formed at a distance of 0.40 m from the axis. The sand in the ring has a mass of 0.50 kg. After all the sand is in place, what is the angular velocity of the disk?

(a) 0.067 rad/s **(b)** 0.046 rad/s
(c) 0.059 rad/s **(d)** 0.037 rad/s

20. A flat uniform circular disk (radius = 2.00 m, mass = 1.00×10^2 kg) is initially stationary. The disk is free to rotate in the horizontal plane about a frictionless axis perpendicular to the center of the disk. A 40.0-kg person, standing 1.25 m from the axis, begins to run on the disk in a circular path and has a tangential speed of 2.00 m /s relative to the ground. Find the resulting angular speed of the disk (in rad/s) and describe the direction of the rotation.

(a) 0.500 rad/s **(b)** 0.250 rad/s
(c) 2.00 rad/s **(d)** 1.00 rad/s

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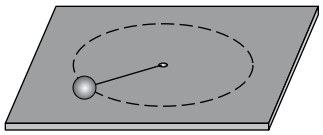
21. A thin uniform rod is rotating at an angular velocity of 7.0 rad/s about an axis that is perpendicular to the rod at its center. As the drawing indicates, the rod is hinged at two places, one-quarter of the length from each end. Without the aid of external torques, the rod suddenly assumes a u shape, with the arms of the u parallel to the rotation axis. What is the angular velocity of the rotating u?

(a) 3.5 rad/s **(b)** 7.0 rad/s
(c) 14 rad/s **(d)** 1.8 rad/s

22. A ball of mass M moves in a circular path on a horizontal, frictionless surface. It is attached to a light string that passes through a hole in the center of the table. If the string is pulled down, thereby reducing the radius of the path of the ball, the speed of the ball is observed to increase. This occurs because

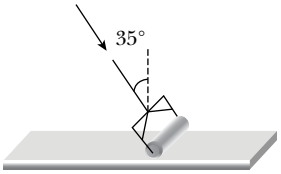


(a) the linear momentum of the ball is conserved.
(b) it is required by Newton's first law of motion.
(c) the angular momentum of the ball is conserved.
(d) the angular momentum of the ball must increase.

23. A small 0.500 kg object moves on a frictionless horizontal table in a circular path of radius 1.00 m. The angular speed is 6.28 rad/s. The object is attached to a string of negligible mass that passes through a small hole in the table at the center of the circle. Someone under the table begins to pull the string downward to make the circle smaller. If the string will tolerate a tension of no more than 105 N, what is the radius of the smallest possible circle on which the object can move?

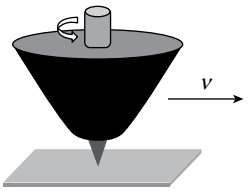
(a) 0.376 m **(b)** 0.434 m
(c) 0.573 m **(d)** 0.188 m

24. Joe is painting the floor of his basement using a paint roller. The roller has a mass of 2.4 kg and a radius of 3.8 cm. In rolling the roller across the floor, Joe applies a force $F = 16$ N directed at an angle of 35° as shown. Ignoring the mass of the roller handle, what is the magnitude of the angular acceleration of the roller?



(a) 7.2×10^1 rad/s^2 **(b)** 1.0×10^2 rad/s^2
(c) 2.3×10^2 rad/s^2 **(d)** 2.8×10^2 rad/s^2

25. A top is spinning counterclockwise as shown in the figure. It is also moving to the right with a linear speed v. What is the direction of the angular velocity?



(a) Upward **(b)** Right
(c) Downward **(d)** Into the paper

26. One end of a thin rod is attached to a pivot, about which it can rotate without friction. Air resistance is absent. The rod has a length of 0.80 m and is uniform. It is hanging vertically straight downward. The end of the rod nearest the floor is given a linear speed v_0, so that the rod begins to rotate upward about the pivot. What must be the value of v_0, such that the rod comes to a momentary halt in a straight-up orientation, exactly opposite to its initial orientation?

(a) 6.9 m/s **(b)** 5.6 m/s
(c) 4.7 m/s **(d)** 8.1 m/s

27. A marble and a cube are placed at the top of a ramp. Starting from rest at the same height, the marble rolls and the cube slides (no kinetic friction) down the ramp. Determine the ratio of the center-of-mass speed of the cube to the center-of-mass speed of the marble at the bottom of the ramp.

(a) 0.447 **(b)** 1.14
(c) 0.707 **(d)** 1.18

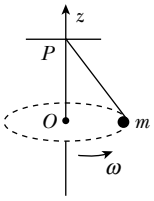
28. A particle of mass 2 kg is on a smooth horizontal table and moves in a circular path of radius 0.6 m. The height of the table from the ground is 0.8 m. If the angular speed of the particle is 12 and rad/s, the magnitude of its angular momentum about a point on the ground right under the center of the circle is

(a) 8.64 kg·m^2/s **(b)** 11.52 kg·m^2/s
(c) 14.4 kg·m^2/s **(d)** 20.16 kg·m^2/s

29. Two thin rectangular sheets (0.20 m × 0.40 m) are identical. In the first sheet the axis of rotation lies along the 0.20-m side, and in the second it lies along the 0.40-m side. The same torque is applied to each sheet. The first sheet, starting from rest, reaches its final angular velocity in 8.0 s. How long does it take for the second sheet, starting from rest, to reach the same angular velocity?

(a) 0.86 s **(b)** 1.4 s
(c) 2.0 s **(d)** 1.1 s

30. A small mass m is attached to a massless string whose other end is fixed at P as shown in the figure. The mass is undergoing circular motion in the xy plane with center at O and constant angular speed ω. If the angular momentum of the system, calculated about O and P are denoted by $\vec{L}_O$ and $\vec{L}_P$, respectively, then



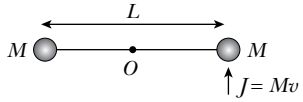
(a) $\vec{L}_O$ and $\vec{L}_P$ do not vary with time.

(b) $\vec{L}_O$ varies with time while $\vec{L}_P$ remains constant.

(c) $\vec{L}_O$ remains constant while $\vec{L}_P$ varies with time.

(d) $\vec{L}_O$ and $\vec{L}_P$ both vary with time.

31. Consider a body, as shown in the figure, consisting of two identical balls, each of mass M connected by a light rigid rod. If an impulse $J = Mv$ is imparted to the body at one of its ends, what would be its angular velocity?



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(a) $\frac{v}{L}$ **(b)** $\frac{2v}{L}$

(c) $\frac{v}{3L}$ **(d)** $\frac{v}{4L}$

32. A disk in xy plane, initially at rest, of mass 1 kg and radius 10 cm, with center at origin, can only rotate about its axis, that is, about z-axis due to a fixed axle through its center. A force $\vec{F} = (3\hat{i} - 4\hat{j} + \hat{k})$ N starts acting on it at the point $(3\hat{i} + 5\hat{j})$ cm. The work done by the force in the first one second is

(a) 2.39 J **(b)** 3.27 J

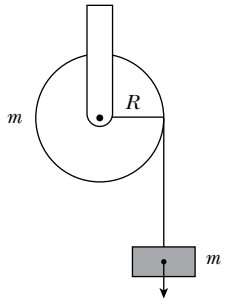
(c) 9.27 J **(d)** 7.29 J

33. A solid cylinder and a thin-walled hollow cylinder have the same mass and radius. They are rolling horizontally toward the bottom of an incline. The center of mass of each has the same translational speed. The cylinders roll up the incline and reach their highest points. Calculate the ratio of the distances (s_{solid}/s_{hollow}) along the incline through which each center of mass moves.

(a) $\frac{4}{3}$ **(b)** $\frac{3}{4}$

(c) $\frac{2}{3}$ **(d)** $\frac{3}{2}$

34. A mass m is supported by a massless string wound around a uniform hollow cylinder of mass m and radius R. If the string does not slip on the cylinder, with what acceleration will the mass fall on release?



(a) $\frac{2g}{3}$ **(b)** $\frac{g}{2}$

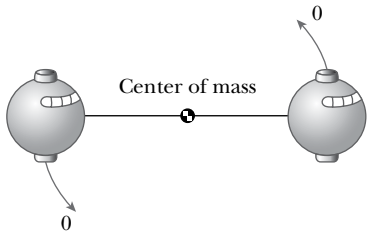
(c) $\frac{5g}{6}$ **(d)** g

35. A thin rod has a length of 0.25 m and rotates in a circle on a frictionless tabletop. The axis is perpendicular to the length of the rod at one of its ends. The rod has an angular velocity of 0.32 rad/s and a moment of inertia of 1.1×10^{-3} kg·m^2. A bug standing on the axis decides to crawl out to the other end of the rod. When the bug (mass = 4.2×10^{-3} kg) gets where it is going, what is the angular velocity of the rod?

(a) 1.32 rad/s **(b)** 0.26 rad/s

(c) 0.93 rad/s **(d)** 1.66 rad/s

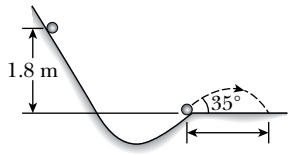
36. A playground merry-go-round is free to rotate about its center on frictionless bearings, and air resistance is negligible. The merry-go-round itself (without riders) has a moment of inertia of 125 kg·m^2. When one person is standing at a distance of 1.50 m from the center, the carousel has an angular velocity of 0.600 rad/s. However, as this person moves inward to a point located 0.750 m from the center, the angular velocity increases to 0.800 rad/s. What is the person's mass?



(a) 45 kg **(b)** 28 kg

(c) 41 kg **(d)** 33 kg

37. A tennis ball, starting from rest, rolls down the hill in the drawing. At the end of the hill the ball becomes airborne, leaving at an angle of 35° with respect to the ground. Treat the ball as a thin-walled spherical shell, and determine the range x.



(a) 4.0 m **(b)** 3.0 m

(c) 2.0 m **(d)** 1.0 m

38. A small 0.500 kg object moves on a frictionless horizontal table in a circular path of radius 1.00 m. The angular speed is 6.28 rad/s. The object is attached to a string of negligible mass that passes through a small hole in the table at the center of the circle. Someone under the table begins to pull the string downward to make the circle smaller. If the string will tolerate a tension of no more than 105 N, what is the radius of the smallest possible circle on which the object can move?

(a) 0.376 m **(b)** 0.434 m

(c) 0.573 m **(d)** 0.659 m

39. A ball of radius r is rolling (pure rolling) on a convex stationary circular surface of radius R with angular velocity ω and angular acceleration α. The magnitude of vertical acceleration of top point of the ball is

(a) $\omega^2 r\left(\frac{R+2r}{R+r}\right)$ **(b)** $\omega^2 r$

(c) $\frac{\omega^2 r^2}{R+r}$ **(d)** $\frac{\omega^2 rR}{R+r}$

More than One Option Correct Type

40. A ring (R), a disk (D), a solid sphere (S) and a hollow sphere with thin walls (H), which have the same mass but

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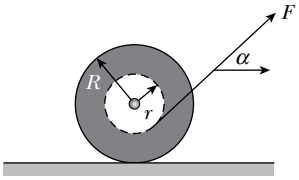
different radii, start together from rest at the top of an inclined plane and roll down without slipping. Then

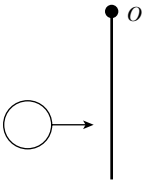
(a) all of them will reach the bottom of the incline together.
(b) the body with the maximum radius will reach the bottom first.
(c) they will reach the bottom in the order S, D, H and R.
(d) all of them will have the same kinetic energy at the bottom of the incline.

41. Inner and outer radii of a spool are r and R respectively. A thread is wound over its inner surface and spool is placed over a rough horizontal surface. Thread is pulled by a force F as shown in the figure. In case of pure rolling, which of the following statements are false?



(a) Thread unwinds, spool rotates anticlockwise and friction acts leftwards.
(b) Thread winds, spool rotates clockwise and friction acts leftwards.
(c) Thread winds, spool moves to the right and friction acts rightwards.
(d) Thread winds spool moves to the right and friction does not come into existence.

42. A ball moving horizontally collides with a uniform rod which is free to rotate about the horizontal axis passing through its upper end O as shown in the figure. Collision is instantaneous and inelastic. For this collision, which of the following statement(s) is/are not correct?



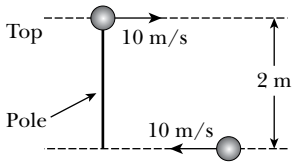
(a) Angular momentum about O does not remain constant.
(b) Angular moment about O remains constant but linear momentum must change.
(c) If angular momentum and linear momentum both remain constant, kinetic energy must remain constant.
(d) Linear momentum must change.

43. A body moves on a straight line with a uniform velocity. Its angular momentum

(a) is zero about a point on the straight line.
(b) remains constant about any given point.
(c) is always zero.
(d) is not zero about a point away from the straight line.

Linked Comprehension

Paragraph for Questions 44 and 45: Two skaters, each of mass 40 kg, approach each other along parallel paths that are separated by a distance of 2 m. Both skaters have a speed of 10 m/s. The first skater carries a 2-m pole that may be considered massless. As he passes the pole, the second skater catches hold of the end. The two skaters then go around in a circle about the center of the pole.



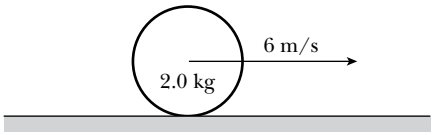
44. What is the angular speed of the skaters after they have linked together?

(a) 4 rad/s **(b)** 5 rad/s
(c) 10 rad/s **(d)** 20 rad/s

45. What is their combined angular momentum about the center of the pole?

(a) 2 kg·m²/s
(b) 800 kg·m²/s
(c) 40 kg·m²/s
(d) 400 kg·m²/s

Paragraph for Questions 46 and 47: A 2.0 kg hoop rolls without slipping on a horizontal surface so that its center proceeds to the right with a constant linear speed of 6.0 m/s.



46. Which one of the following statements is true concerning the angular momentum of this hoop?

(a) It points into the paper.
(b) It points to the right.
(c) It points out of the paper.
(d) It points to the left.

47. What is the total kinetic energy of the hoop?

(a) 36 J **(b)** 72 J
(c) 54 J **(d)** 96 J

Passage for Questions 48—50: Two disks A and B are mounted coaxially on a vertical axle. The disks have moments of inertia I and $2I$ respectively about the common axis. Disk A is imparted an initial angular velocity 2ω using the entire potential energy of a spring compressed by a distance x_1. Disk B is imparted an angular velocity ω by a spring having the same spring constant and compressed by a distance x_2. Both the disks rotate in the clockwise direction.

48. The ratio x_1/x_2 is

(a) 2 **(b)** $\frac{1}{2}$
(c) $\sqrt{2}$ **(d)** $\frac{1}{\sqrt{2}}$

49. When disk B is brought in contact with disk A, they acquire a common angular velocity in time t. The average

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frictional torque on one disk by the other during this period is

(a) $\frac{2I\omega}{3t}$ **(b)** $\frac{9I\omega}{2t}$

(c) $\frac{9I\omega}{4t}$ **(d)** $\frac{3I\omega}{2t}$

50. The loss of kinetic energy during the above process is

(a) $\frac{-I\omega^2}{2}$ **(b)** $\frac{-I\omega^2}{3}$

(c) $\frac{-I\omega^2}{4}$ **(d)** $\frac{-I\omega^2}{6}$

Matrix-Match

51. Match the statement in Column I with the statements in Column II. One or more than one choice from Column II can match with a statement from Column I.

Column I	Column II
(a) For a wheel of radius R rolling smoothly,	**(p)** $\vec{F}_{net} = 0$
(b) If a wheel is being accelerated but is still rolling smoothly,	**(q)** $a_{com,x} = -\frac{g\sin\theta}{1 + I_{com}/MR^2}$
(c) If the wheel rolls smoothly down a ramp, its acceleration is	**(r)** $a_{com} = \alpha R$
(d) For a body at static equilibrium, the vector sum of the external forces acting on it is	**(s)** $v_{com} = \omega R$

Directions for Questions 52 and 53: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

52. In the given table, Column I gives mass of a rotating cylinder, Column II gives its radius and Column III gives the angular speed.

Column I	Column II	Column III
(I) Mass of cylinder (m) rotating as pully is 23 kg	**(i)** The radius of cylinder (r) is 0.4 m	**(J)** The angular speed (ω) of the cylinder is 3.4 rad/s
(II) Mass of cylinder (m) rotating as pully is 20 kg	**(ii)** The radius of cylinder (r) is 0.5 m	**(K)** The angular speed (ω) of the cylinder is 4.1 rad/s
(III) Mass of cylinder (m) rotating as pully is 22.2 kg	**(iii)** The radius of cylinder (r) is 0.2 m	**(L)** The angular speed (ω) of the cylinder is 4 rad/s
(IV) Mass of cylinder (m) rotating as pully is 17 kg	**(iv)** The radius of cylinder (r) is 0.1 m	**(M)** The angular speed (ω) of the cylinder is 4.5 rad/s

(1) The combination whose rotational kinetic energy that equals 3.2 J is

(a) (I) (ii) (J) **(b)** (II) (iii) (L)
(c) (II) (i) (M) **(d)** (I) (ii) (M)

(2) The combination whose rotational kinetic energy that equals 16.6 J is

(a) (I) (iii) (K) **(b)** (IV) (iii) (L)
(c) (II) (iii) (L) **(d)** (I) (ii) (J)

(3) The combination whose rotational kinetic energy that equals 1.72 J is

(a) (III) (i) (K) **(b)** (I) (i) (L)
(c) (IV) (i) (M) **(d)** (II) (iii) (M)

53. In the given table, Column I denotes the mass of the cylindrical shell, Column II refers to the radius of the shell and Column III refers to its length.

Column I	Column II	Column III
(I) 2 kg	**(i)** 2 m	**(J)** 6 m
(II) 14 kg	**(ii)** 4 m	**(K)** 1 m
(III) 6 kg	**(iii)** 3 m	**(L)** 7 m
(IV) 12 kg	**(iv)** 5 m	**(M)** 10 m

(1) Find the combination that results in 348 kg·m^2 as rotational inertia.

(a) (I) (iii) (L) **(b)** (IV) (i) (M)
(c) (II) (iv) (K) **(d)** (III) (ii) (M)

(2) Find the combination that results in 270 kg·m^2 as rotational inertia.

(a) (III) (ii) (L) **(b)** (IV) (iii) (J)
(c) (II) (iii) (K) **(d)** (I) (i) (M)

(3) Find the combination that results in 371 kg·m^2 as rotational inertia.

(a) (II) (i) (L) **(b)** (I) (i) (K)
(c) (III) (iii) (L) **(d)** (I) (ii) (L)

Integer Type

54. A uniform circular disk of mass 50 kg and radius 0.4 m is rotating with an angular velocity of 10 rad/s about its own axis, which is vertical. Two uniform circular rings, each of mass 6.25 kg and radius 0.2 m, are gently placed symmetrically on the disk in such a manner that they are touching each other along the axis of the disk and are horizontal. Assume that the friction is large enough such that the rings are at rest relative to the disk and the system rotates

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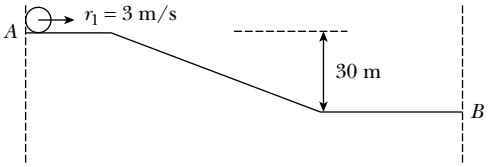
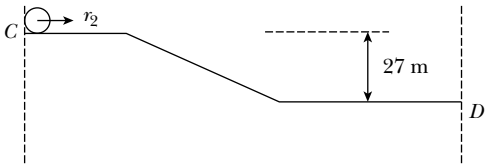
about the original axis. The new angular velocity (in rad/s) of the system is _____.

55. Two identical uniform disks all without slipping on two different surfaces AB and CD (see figure) starting at A and C with linear speeds r_1 and r_2, respectively, and always remain in contact with the surface. If they reach B and D with same linear speed and $r_1 = 3$ m/s then r_2 in m/s is _____. ($g = 10$ m/s^2)





56. A binary star consists of two stars A (mass $2.2M_s$) and B (mass $11M_s$), where M_s is the mass of the Sun. They are separated by distance d and are rotating about their center of mass, which is stationary. The ratio of the total angular momentum of the binary star to the angular momentum of star B about the center of mass is _____.

ANSWER KEY

Checkpoints

1. (a) same; (b) less **2.** Less (consider the transfer of energy from rotational kinetic energy to gravitational potential energy).

3. (a) $+z$; (b) $+y$; (c) $-x$ **4.** In situation depicted in (c), in situation depicted in (e), in situation depicted in (f).

5. (a) no; (b) at site of F_1 perpendicular to plane of figure; (c) 45 N.

Problems

1. (a) $v_{center} = 0$; (b) $\vec{v}_{top} = (+22\,\text{m/s})\hat{i}$; (c) $\vec{v}_{bottom} = (-22\,\text{m/s})\hat{i}$; (d) $a_{center} = 0$; (e) $a_{top} = 1.5\times10^3\,\text{m/s}^2$; (f) $a_{bottom} = 1.5 \times 10^3$ m/s^2; (g) $\vec{v} = (+22\,\text{m/s})\hat{i}$; (h) $2v = +44$ m/s; (i) zero; (j) it does not accelerate; (k) the answer is as it was in part (e), that is, $a = 1.5 \times 10^3$ m/s^2; (l) as explained in part (k), that is, $a = 1.5 \times 10^3$ m/s^2.

2. (a) $\omega_0 = 63.4$ rad/s; (b) $\omega^2 = 10.7$ rad/s; (c) 66.0 m **3.** ≈7.1m

4. (a) 12°; (b) The acceleration would be more. We can look at this in terms of forces or in terms of energy. In terms of forces, the uphill static friction would then be absent, so the downhill acceleration would be due only to the downhill gravitational pull. In terms of energy, the rotational term in Eq. 11-5 would be absent so that the potential energy it started with would simply become $1/2mv^2$ (without it being "shared" with another term) resulting in a greater speed (and, because of Eq. 2-16, greater acceleration).

5. $h' = \dfrac{5h}{7}$ **6.** 6.6×10^{-4} kg · m^2 **7.** (a) 63 rad/s; (b) 4.0 m

8. (a) We conclude that there is a turning point (and this is it). The ball, therefore, does not reach the origin; (b) 7.3 m/s.

9. 4.8 m **10.** (a) 8.0 J; (b) $v_{com} = 2.7$ m/s; (c) 4.3 J; (d) $v_{com}f = 1.3$ m/s

11. (a) $\vec{f}_s = (-4.0\ \text{N})\hat{i}$; (b) $I = 0.60\ \text{kg}\cdot\text{m}^2$

12. (a) 32.4 cm; (b) 2.24×10^{-2} N; (c) The direction is toward the center of the loop. **13.** 0.50

14. 1.56 m/s **15.** (a) $v_{com} = -R\omega = (-0.11\ \text{m})\omega$; (b) –2.1 m/s^2; (c) –47 rad/s^2; (d) 1.2 s; (e) 8.6 m; (f) 6.1 m/s

16. 0.25 **17.** (a) ≈13 cm/s^2; (b) ≈4.4 s; (c) 55 cm/s; (d) 1.8×10^{-2} J; (e) ≈1.4 J; (f) 27 rev/s

18. (a) 0.19 m/s^2; (b) Our result in part (a) applies to both the descending and the rising yo-yo motions; (c) 1.1×10^3 m; (d) Our result in part (c) indicates that the tension is well below the ultimate limit for the cord; (e) As we saw in our acceleration computation, all that mattered was the ratio R/R_0 (and, of course, g). So, if it is a scaled-up version, then such ratios are unchanged, and we obtain the same result; (f) Since the tension also depends on mass, then the larger yo-yo will involve a larger cord tension.

19. (a) 362 N (inward); (b) 1.14×10^3 N; (c) 71.4°

20. (a) 357 N; (b) 214 N; (c) the direction of the horizontal force is rightward; (d) 205 N; (e) the direction of the vertical force is upward.

21. (a) 43 N; (b) 25 N; (c) 51 N; (d) 29° **22.** (a) 1.44 m; (b) 433 N; (c) 235 N

23. (a) 40.4°; (b) 66.1 kg; (c) 13.2 kg

24. (a) 3.3×10^3 N; (b) The direction is upward since $F_{tricep} > 0$; (c) 3.5×10^3 N; (d) The negative sign implies that F_{humer} points downward.

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25. (a) As θ is increased from zero the crate slides before it tips; (b) It starts to slide when $\theta = 17°$; (c) The crate begins to slide when $\theta = 35.0°$ and begins to tip when $\theta = 33.7°$; thus, it tips first as the angle is increased; (d) Tipping begins at $\theta = 33.7° \approx 34°$.

26. (a) 1.62×10^3 N; (b) Since F_1 is negative, this force is downward; (c) 2.20×10^3 N (d) The result is positive, indicating that this force is upward; (e) The force of the diving board on the left pedestal is upward (opposite to the force of the pedestal on the diving board), so this pedestal is being stretched; (f) The force of the diving board on the right pedestal is downward, so this pedestal is being compressed.

27. (a) 8.8×10^2 N; (b) 5.4×10^2 N **28.** 2.38 m **29.** 0.218

30. 6.5×10^3 N **31.** $\approx 59\%$ **32.** 3.73×10^3 N

33. (a) 3.63 m/s^2; (b) 2.2×10^3 N; (c) 3.3×10^3 N; (d) 8.1×10^2 N; (e) 1.2×10^3 N **34.** 120°

35. 7.34 N **36.** (a) 68.0°; (b) 300 N **37.** 1.28 **38.** (a) 0.83; (b) 0.17; (c) 0.21

39. (a) 340 N; (b) 196 N; (c) 30.0° **40.** (a) 260 N; (b) 0.50; (c) 184 N

41. (a) 5.0 N; (b) 40 N; (c) 1.0 m

42. (a) 3.3×10^2 N; (b) 0.80 m; (c) Both intuitively and mathematically (since both coefficients are in the denominator) we see from part (a) that F_{N1} would increase in such a case; (d) h should decrease if the coefficients decrease.

43. (a) 17 N; (b) 1.8×10^2 N. **44.** 0.531 m **45.** (a) 8.63×10^3 N; (b) 10.7×10^3 N; (c) 45.8°

46. 0.64 **47.** 105 g **48.** (a) $F_{top} = (-104\text{ N})\hat{i} + (172\text{ N})\hat{j}$; (b) $F_{bottom} = (+104\text{ N})\hat{i} + (172\text{ N})\hat{j}$

49. (a) $\vec{F} = \vec{F}_x\hat{i} + \vec{F}_y\hat{j} = (-836\text{ N})\hat{i} + (284\text{ N})\hat{j}$; (b) $\vec{G} = \vec{G}_x\hat{i} + \vec{G}_y\hat{j} = (+836\text{ N})\hat{i} + (284\text{ N})\hat{j}$; (c) $\vec{H} = \vec{H}_x\hat{i} + \vec{H}_y\hat{j} = (+836\text{ N})\hat{i} + (970\text{ N})\hat{j}$; (d) $-\vec{G} = -\vec{G}_x\hat{i} + \vec{G}_y\hat{j} = (-836\text{ N})\hat{i} - (284\text{ N})\hat{j}$

50. (a) 254 N; (b) 660 N; (c) 386 N **51.** (a) 2.4×10^9 N; (b) 126

52. 451 N **53.** (a) Person 4; (b) Person 5 **54.** (a) 684 N $\approx 6.8 \times 10^2$ N; (b) 592 N $= 5.9 \times 10^2$ N

Practice Questions

Single Correct Choice Type

1. (c)	**2.** (c)	**3.** (a)	**4.** (d)	**5.** (d)
6. (a)	**7.** (b)	**8.** (d)	**9.** (b)	**10.** (d)
11. (d)	**12.** (c)	**13.** (b)	**14.** (d)	**15.** (a)
16. (a)	**17.** (b)	**18.** (b)	**19.** (d)	**20.** (a)
21. (c)	**22.** (c)	**23.** (c)	**24.** (b)	**25.** (a)
26. (a)	**27.** (d)	**28.** (b)	**29.** (c)	**30.** (c)
31. (a)	**32.** (d)	**33.** (b)	**34.** (b)	**35.** (b)
36. (b)	**37.** (c)	**38.** (c)	**39.** (a)	

More than One Correct Choice Type

40. (c), (d) **41.** (a), (c), (d) **42.** (a), (b), (c), (d) **43.** (a), (b), (c)

Linked Comprehension

44. (c)	**45.** (b)	**46.** (a)	**47.** (b)	**48.** (c)
49. (a)	**50.** (b)			

Matrix-Match

51. (a) → (s); (b) → (r); (c) → (q); (d) → (p) **52.** (1) → (a); (2) → (d); (3) → (c)

53. (1) → (d); (2) → (b); (3) → (a)

Integer Type

54. 8 **55.** 7 **56.** 6

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CHAPTER 12

Elasticity

12.1 | WHAT IS PHYSICS?

For the problems of Chapter 11 (Section 11.10), we have only three independent equations at our disposal, usually two balance of forces equations and one balance-of-torques equation about a given rotation axis. Thus, if a problem has more than three unknowns, we cannot solve it. It is easy to find such problems. In Sample Problem 11.10, for example, we could have assumed that there is friction between the wall and the top of the ladder. Then, there would have been a vertical frictional force acting where the ladder touches the wall, making a total of four unknown forces. With only three equations, we could not have solved this problem.

Consider an unsymmetrically loaded car. What are the forces—all different—on the four tires? Again, we cannot find them because we have only three independent equations. Similarly, we can solve an equilibrium problem for a table with three legs but not for one with four legs. Problems like these, in which there are more unknowns than equations, are called **indeterminate**.

Yet solutions to indeterminate problems exist in the real world. If you rest the tires of the car on four platform scales, each scale will register a definite reading, the sum of the readings being the weight of the car. What is eluding us in our efforts to find the individual forces by solving equations?

The problem is that we have assumed—without making a great point of it—that the bodies to which we apply the equations of static equilibrium are perfectly rigid. By this we mean that they do not deform when forces are applied to them. Strictly, there are no such bodies. The tires of the car, for example, deform easily under load until the car settles into a position of static equilibrium.

We have all had experience with a wobbly restaurant table, which we usually level by putting folded paper under one of the legs. If a big enough elephant sat on such a table, however, you may be sure that if the table did not collapse, it would deform just like the tires of a car. Its legs would all touch the floor, the forces acting upward on the table legs would all assume definite (and different) values as in Fig. 12-1, and the table would no longer wobble. Of course, we (and the elephant) would be thrown out onto the street but, in principle, how do we find the individual values of

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those forces acting on the legs in this or similar situations where there is deformation?

To solve such indeterminate equilibrium problems, we must supplement equilibrium equations with some knowledge of *elasticity*, the branch of physics and engineering that describes how real bodies deform when forces are applied to them.

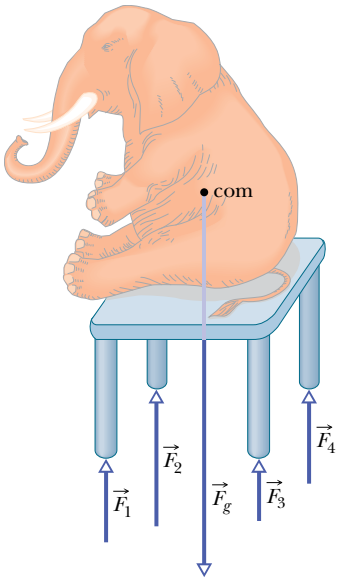
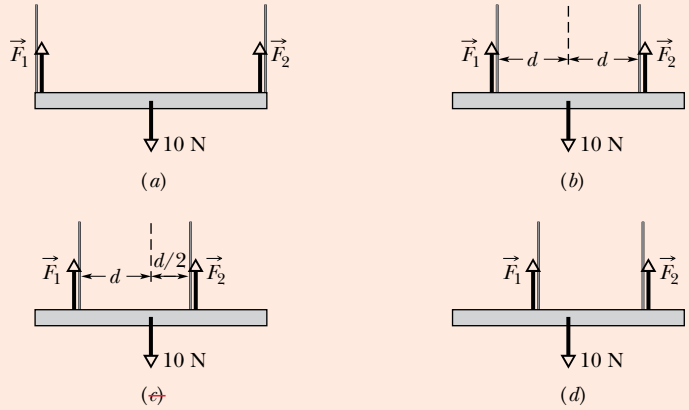


Figure 12-1 The table is an indeterminate structure. The four forces on the table legs differ from one another in magnitude and cannot be found from the laws of static equilibrium alone.

CHECKPOINT 1

A horizontal uniform bar of weight 10 N is to hang from a ceiling by two wires that exert upward forces $\vec{F}_1$ and $\vec{F}_2$ on the bar. The figure shows four arrangements for the wires. Which arrangements, if any, are indeterminate (so that we cannot solve for numerical values of $\vec{F}_1$ and $\vec{F}_2$)?



12.2 | ELASTICITY

When a large number of atoms come together to form a metallic solid, such as an iron nail, they settle into equilibrium positions in a three-dimensional *lattice*, a repetitive arrangement in which each atom is a well-defined equilibrium distance from its nearest neighbors. The atoms are held together by interatomic forces that are modeled as tiny springs in Fig. 12-2. The lattice is remarkably rigid, which is another way of saying that the "interatomic springs" are extremely stiff. It is for this reason that we perceive many ordinary objects, such as metal ladders, tables, and spoons, as perfectly rigid. Of course, some ordinary objects, such as garden hoses or rubber gloves, do not strike us as rigid at all. The atoms that make up these objects *do not* form a rigid lattice like that of Fig. 12-2 but are aligned in long, flexible molecular chains, each chain being only loosely bound to its neighbors.

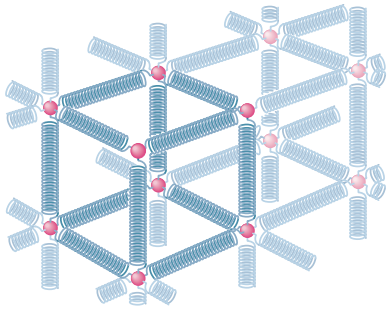


Figure 12-2 The atoms of a metallic solid are distributed on a repetitive three-dimensional lattice. The springs represent interatomic forces.

All real "rigid" bodies are to some extent **elastic**, which means that we can change their dimensions slightly by pulling, pushing, twisting, or compressing them. To get a feeling for the orders of magnitude involved, consider a vertical steel rod 1 m long and 1 cm in diameter attached to a factory ceiling. If you hang a subcompact car from the free end of such a rod, the rod will stretch but only by about 0.5 mm, or 0.05%. Furthermore, the rod will return to its original length when the car is removed.

If you hang two cars from the rod, the rod will be permanently stretched and will not recover its original length when you remove the load. If you hang three cars from the rod, the rod will break. Just before rupture, the elongation of the rod will be less than 0.2%. Although deformations of this size seem small, they are important in engineering practice. (Whether a wing under load will stay on an airplane is obviously important.)

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12.3 | STRESS AND STRAIN

Key Concepts

- Three elastic moduli are used to describe the elastic behavior (deformations) of objects as they respond to forces that act on them. The strain (fractional change in length) is linearly related to the applied stress (force per unit area) by the proper modulus, according to the general stress–strain relation

$$\text{stress} = \text{modulus} \times \text{strain}.$$

- When an object is under tension or compression, the stress–strain relation is written as

$$\frac{F}{A} = E\frac{\Delta L}{L},$$

where $\Delta L/L$ is the tensile or compressive strain of the object, F is the magnitude of the applied force $\vec{F}$ causing the strain, A is the cross-sectional area over which $\vec{F}$ is applied (perpendicular to A), and E is the Young's modulus for the object. The stress is F/A.

- When an object is under a shearing stress, the stress–strain relation is written as

$$\frac{F}{A} = G\frac{\Delta x}{L},$$

where $\Delta x/L$ is the shearing strain of the object, Δx is the displacement of one end of the object in the direction of the applied force $\vec{F}$, and G is the shear modulus of the object. The stress is F/A.

- When an object undergoes hydraulic compression due to a stress exerted by a surrounding fluid, the stress–strain relation is written as

$$p = B\frac{\Delta V}{V},$$

where p is the pressure (hydraulic stress) on the object due to the fluid, $\Delta V/V$ (the strain) is the absolute value of the fractional change in the object's volume due to that pressure, and B is the bulk modulus of the object.

Undeformed materials have the molecules in a state of equilibrium and all parts of the body are in mechanical equilibrium. When a deformation occurs, the arrangement of the molecules is changed, and the body ceases to be in its original state of equilibrium. Therefore, those forces arise which tend to bring the body back to equilibrium. These internal forces which occur when a body is deformed are called **internal stresses**. If no deformation occurs, there are no internal stresses.

The internal stresses are due to molecular forces, that is, the forces of interaction between the molecules. These molecular forces have a very short range of action. Their effect extends only to the neighboring molecules, or in other words the range of these forces is same as the distance between the molecules. On the other hand, the theory of elasticity is a macroscopic theory, and the only distances considered are those which are large compared with the distances between the molecules. The range of action of the molecular forces should therefore be taken as zero in the theory of elasticity. We can say that the forces that cause the internal stresses are, with regard to the theory of elasticity, "near-action" forces, which act from any point only to neighboring points. Hence, we conclude that the forces exerted on one part of the body by surrounding parts act only on the surface of that part.

Figure 12-3 shows three ways in which a solid might change its dimensions when forces act on it. In Fig. 12-3*a*, a cylinder is stretched. In Fig. 12-3*b*, a cylinder is deformed by a force perpendicular to its long axis, much as we might

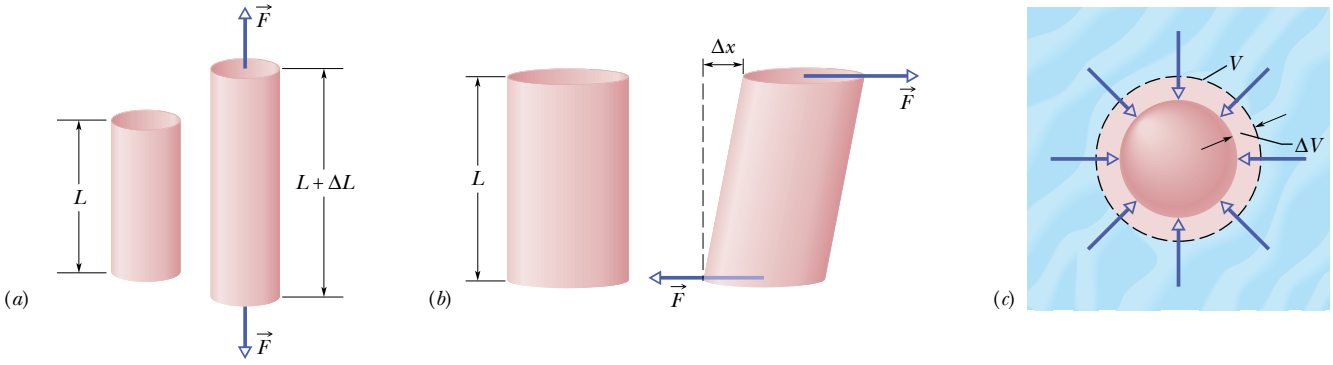

Figure 12-3 (*a*) A cylinder subject to *tensile stress* stretches by an amount ΔL. (*b*) A cylinder subject to *shearing stress* deforms by an amount Δx, somewhat like a pack of playing cards would. (*c*) A solid sphere subject to uniform *hydraulic stress* from a fluid shrinks in volume by an amount ΔV. All the deformations shown are greatly exaggerated.

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deform a pack of cards or a book. In Fig. 12-3*c*, a solid object placed in a fluid under high pressure is compressed uniformly on all sides. What the three deformation types have in common is that a **stress**, or deforming force per unit area, produces a **strain**, or unit deformation. In Fig. 12-3, *tensile stress* (associated with stretching) is illustrated in (*a*), *shearing stress* in (*b*), and *hydraulic stress* in (*c*).

The stresses and the strains take different forms in the three situations of Fig. 12-3, but—over the range of engineering usefulness—stress and strain are proportional to each other. The constant of proportionality is called a **modulus of elasticity**, so that

$$\text{stress} = \text{modulus} \times \text{strain}. \tag{12-1}$$

In a standard test of tensile properties, the tensile stress on a test cylinder (like that in Fig. 12-4) is slowly increased from zero to the point at which the cylinder fractures, and the strain is carefully measured and plotted. The result is a graph of stress versus strain like that in Fig. 12-5. For a substantial range of applied stresses, the stress–strain relation is linear, and the specimen recovers its original dimensions when the stress is removed; it is here that Eq. 12-1 applies. If the stress is increased beyond the **yield strength** S_y of the specimen, the specimen becomes permanently deformed. If the stress continues to increase, the specimen eventually ruptures, at a stress called the **ultimate strength** S_u.

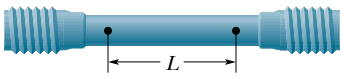


Figure 12-4 A test specimen used to determine a stress–strain curve such as that of Fig. 12-5. The change ΔL that occurs in a certain length L is measured in a tensile stress–strain test.

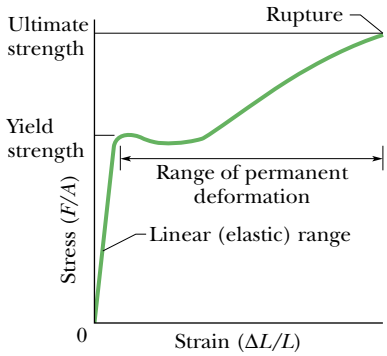


Figure 12-5 A stress–strain curve for a steel test specimen such as that of Fig. 12-4. The specimen deforms permanently when the stress is equal to the *yield strength* of the specimen's material. It ruptures when the stress is equal to the *ultimate strength* of the material.

Hooke's Law

The stress versus strain curve for many materials possess a linear region. The force required to stretch an elastic object such as a metal spring, within certain limits, is directly proportional to the extension of the spring, which is called **Hooke's law** and it is expressed as

$$F = -kx \tag{12-2}$$

where F is the force; x is the length of either extension or compression; and k is a constant of spring constant which is usually measured in N/m.

Tension and Compression

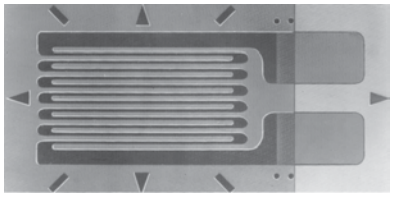
For simple tension or compression, the stress on the object is defined as F/A, where F is the magnitude of the force applied perpendicularly to an area A on the object. The strain, or unit deformation, is then the dimensionless quantity $\Delta L/L$, the fractional (or sometimes percentage) change in a length of the specimen. If the specimen is a long rod and the stress does not exceed the yield strength, then not only the entire rod but also every section of it experiences the same strain when a given stress is applied. Because the strain is dimensionless, the modulus in Eq. 12-1 has the same dimensions as the stress—namely, force per unit area.

The modulus for tensile and compressive stresses is called the **Young's modulus** and is represented in engineering practice by the symbol E. Equation 12-1 becomes

$$\frac{F}{A} = E\frac{\Delta L}{L}. \tag{12-3}$$

The strain $\Delta L/L$ in a specimen can often be measured conveniently with a *strain gage* (Fig. 12-6), which can be attached directly to operating machinery with an adhesive. Its electrical properties are dependent on the strain it undergoes.

Although the Young's modulus for an object may be almost the same for tension and compression, the object's ultimate strength may well be different for the two types of stress. Concrete, for example, is very strong in



Courtesy Micro Measurements, a Division of Vishay Precision Group, Raleigh, NC

Figure 12-6 A strain gage of overall dimensions 9.8 mm by 4.6 mm. The gage is fastened with adhesive to the object whose strain is to be measured; it experiences the same strain as the object. The electrical resistance of the gage varies with the strain, permitting strains up to 3% to be measured.

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Table 12-1 Some Elastic Properties of Selected Materials of Engineering Interest

Material	Density ρ (kg/m^3)	Young's Modulus E (10^9 N/m^2)	Ultimate Strength S_u (10^6 N/m^2)	Yield Strength S_y (10^6 N/m^2)
Steel[a]	7860	200	400	250
Aluminum	2710	70	110	95
Glass	2190	65	50[b]	–
Concrete[c]	2320	30	40[b]	–
Wood[d]	525	13	50[b]	–
Bone	1900	9[b]	170[b]	–
Polystyrene	1050	3	48	–

[a]Structural steel (ASTM-A36).
[b]In compression.
[c]High strength.
[d]Douglas fir.

compression but is so weak in tension that it is almost never used in that manner. Table 12-1 shows the Young's modulus and other elastic properties for some materials of engineering interest.

Shearing

In the case of shearing, the stress is also a force per unit area, but the force vector lies in the plane of the area rather than perpendicular to it. The strain is the dimensionless ratio $\Delta x/L$, with the quantities defined as shown in Fig. 12-3*b*. The corresponding modulus, which is given the symbol G in engineering practice, is called the **shear modulus**. For shearing, Eq. 12-1 is written as

$$\frac{F}{A} = G\frac{\Delta x}{L}. \tag{12-4}$$

Shearing occurs in rotating shafts under load and in bone fractures due to bending.

Hydraulic Stress

In Fig. 12-3*c*, the stress is the fluid pressure p on the object, and, as you will see in Chapter 14, pressure is a force per unit area. The strain is $\Delta V/V$, where V is the original volume of the specimen and ΔV is the absolute value of the change in volume. The corresponding modulus, with symbol B, is called the **bulk modulus** of the material. The object is said to be under *hydraulic compression*, and the pressure can be called the *hydraulic stress*. For this situation, we write Eq. 12-1 as

$$p = B\frac{\Delta V}{V}. \tag{12-5}$$

The bulk modulus is 2.2×10^9 N/m^2 for water and 1.6×10^{11} N/m^2 for steel. The pressure at the bottom of the Pacific Ocean, at its average depth of about 4000 m, is 4.0×10^7 N/m^2. The fractional compression $\Delta V/V$ of a volume of water due to this pressure is 1.8%; that for a steel object is only about 0.025%. In general, solids—with their rigid atomic lattices—are less compressible than liquids, in which the atoms or molecules are less tightly coupled to their neighbors.

SAMPLE PROBLEM 12.01

Stress and strain of elongated rod

One end of a steel rod of radius $R = 9.5$ mm and length $L = 81$ cm is held in a vise. A force of magnitude $F = 62$ kN is then applied perpendicularly to the end face (uniformly across the area) at the other end, pulling directly away from the vise. What are the stress on the rod and the elongation ΔL and strain of the rod?

KEY IDEAS

(1) Because the force is perpendicular to the end face and uniform, the stress is the ratio of the magnitude F of the force to the area A. The ratio is the left side of Eq. 12-3.
(2) The elongation ΔL is related to the stress and Young's

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modulus E by Eq. 12-3 ($F/A = E\,\Delta L/L$). (3) Strain is the ratio of the elongation to the initial length L.

Calculations: To find the stress, we write

$$\text{stress} = \frac{F}{A} = \frac{F}{\pi R^2} = \frac{6.2\times10^4\text{ N}}{(\pi)(9.5\times10^{-3}\text{ m})^2}$$
$$= 2.2\times10^8\text{ N/m}^2. \qquad \text{(Answer)}$$

The yield strength for structural steel is 2.5×10^8 N/m^2, so this rod is dangerously close to its yield strength.

We find the value of Young's modulus for steel in Table 12-1. Then from Eq. 12-3 we find the elongation:

$$\Delta L = \frac{(F/A)\,L}{E} = \frac{(2.2\times10^8\text{ N/m}^2)(0.81\text{ m})}{2.0\times10^{11}\text{ N/m}^2}$$
$$= 8.9\times10^{-4}\text{ m} = 0.89\text{ mm}. \qquad \text{(Answer)}$$

For the strain, we have

$$\frac{\Delta L}{L} = \frac{8.9\times10^{-4}\text{ m}}{0.81\text{ m}}$$
$$= 1.1\times10^{-3} = 0.11\%. \qquad \text{(Answer)}$$

SAMPLE PROBLEM 12.02

Balancing a wobbly table

A table has three legs that are 1.00 m in length and a fourth leg that is longer by $d = 0.50$ mm, so that the table wobbles slightly. A steel cylinder with mass $M = 290$ kg is placed on the table (which has a mass much less than M) so that all four legs are compressed but unbuckled and the table is level but no longer wobbles. The legs are wooden cylinders with cross-sectional area $A = 1.0$ cm^2; Young's modulus is $E = 1.3 \times 10^{10}$ N/m^2. What are the magnitudes of the forces on the legs from the floor?

KEY IDEAS

We take the table plus steel cylinder as our system. The situation is like that in Fig. 12-1, except now we have a steel cylinder on the table. If the tabletop remains level, the legs must be compressed in the following ways: Each of the short legs must be compressed by the same amount (call it ΔL_3) and thus by the same force of magnitude F_3. The single long leg must be compressed by a larger amount ΔL_4 and thus by a force with a larger magnitude F_4. In other words, for a level tabletop, we must have

$$\Delta L_4 = \Delta L_3 + d. \qquad (12\text{-}6)$$

From Eq. 12-3, we can relate a change in length to the force causing the change with $\Delta L = FL/AE$, where L is the original length of a leg. We can use this relation to replace ΔL_4 and ΔL_3 in Eq. 12-6. However, note that we can approximate the original length L as being the same for all four legs.

Calculations: Making those replacements and that approximation gives us

$$\frac{F_4 L}{AE} = \frac{F_3 L}{AE} + d. \qquad (12\text{-}7)$$

We cannot solve this equation because it has two unknowns, F_4 and F_3.

To get a second equation containing F_4 and F_3, we can use a vertical y axis and then write the balance of vertical forces ($F_{\text{net},y} = 0$) as

$$3F_3 + F_4 - Mg = 0, \qquad (12\text{-}8)$$

where Mg is equal to the magnitude of the gravitational force on the system. (*Three* legs have force $\vec{F}_3$ on them.) To solve the simultaneous Eqs. 12-7 and 12-8 for, say, F_3, we first use Eq. 12-8 to find that $F_4 = Mg - 3F_3$. Substituting that into Eq. 12-7 then yields, after some algebra,

$$F_3 = \frac{Mg}{4} - \frac{dAE}{4L}$$
$$= \frac{(290\text{ kg})(9.8\text{ m/s}^2)}{4}$$
$$- \frac{(5.0\times10^{-4}\text{ m})(10^{-4}\text{ m}^2)(1.3\times10^{10}\text{ N/m}^2)}{(4)(1.00\text{ m})}$$
$$= 548\text{ N} \approx 5.5\times10^2\text{ N}. \qquad \text{(Answer)}$$

From Eq. 12-8, we then find

$$F_4 = Mg - 3F_3 = (290\text{ kg})(9.8\text{ m/s}^2) - 3(548\text{ N})$$
$$\approx 1.2\text{ kN}. \qquad \text{(Answer)}$$

You can show that the three short legs are each compressed by 0.42 mm and the single long leg by 0.92 mm.

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12.4 | ELASTIC POTENTIAL ENERGY

Key Concept

- Total work done by the external force to extend the rod by x is

$$W = U = \frac{1}{2} \times \sigma \times S \times \text{Volume}$$

This is also the energy stored in the rod in form of potential energy as the internal forces are conservative in nature till the elastic region.

Let us consider a rod of relaxed length l_0 fixed at one end with the wall as shown in Fig. 12-7. Now, on the free end, we apply a force varying slowly such that the rod is always in equilibrium.

When the stress in the rod is s, the force we will be applying is F, where $F = A\sigma$. At this instant, the rod has extended by x, so the strain in the rod is $S = x/l$. Young's Modulus is given by

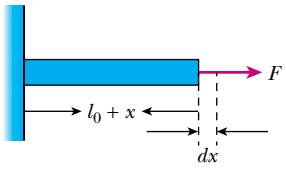

Figure 12-7 A rod attached to a wall is being pulled by force F.

$$Y = \frac{\sigma}{x/l} \Rightarrow \sigma = \frac{Yx}{l}.$$

Therefore,

$$F = \frac{YAx}{l}.$$

Thus, work done by the external force to extend the rod further by dx is Fdx. Therefore, the total work done is

$$W = \int Fdx = \int_0^x \frac{YAx}{l}\, dx$$

$$= \frac{1}{2}\frac{YA}{l}x^2 = \frac{1}{2}\left(\frac{Yx}{l}\right)\left(\frac{x}{l}\right)(lA) = \frac{1}{2} \times \sigma \times S \times \text{Volume}.$$

This is also the energy stored in the rod in form of potential energy as the internal forces are conservative in nature till the elastic region. Thus, elastic potential energy stored in the rod is

$$U = \frac{1}{2} \times \sigma \times S \times \text{Volume}. \tag{12-9}$$

SAMPLE PROBLEM 12.03

Strain and potential energy in the rod

In a rod of mass m, cross-sectional area A and Young's modulus Y:

(a) Find the strain in the rod under its own weight.

KEY IDEA

As gravity pulls the rod downward, there is tension developed in the rod. However, the lower points have lesser tension than the upper points, as they should support lesser weight.

Calculations: Tension in the rod at distance x from the lower end (Fig. 12-8) is T, where

$$T = \left(\frac{mx}{l}\right)g.$$

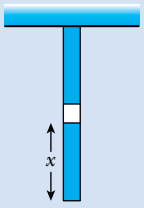

Figure 12-8 Tension in rod of mass m.

Thus, stress at distance x from the lower end is $\sigma = T/A$ and strain at this point is

$$S = \frac{\omega}{Y} = \frac{mxg/lA}{Y}.$$

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We can write strain at distance x as ds/dx; here, ds is the change in length in an element of length dx.

$$\frac{ds}{dx} = \frac{mxg/lA}{Y}. \qquad (12\text{-}10)$$

Therefore, the total change in the length of the rod,

$$\Delta L = \int_0^{\Delta L} ds = \int_0^{l} \left(\frac{mgx}{lAY}\right) dx = \frac{mg}{lAY}\left(\frac{l^2}{2}\right).$$

Hence, total strain in the rod is $mg/2AY$.

(b) Also find the elastic potential energy stored in the rod.

Calculations: The energy stored in the element at distance x from the lower end is

$$dU = \frac{1}{2} \times \frac{mgx}{lA} \times \frac{ds}{dx} \times A\,dx.$$

As we know from Eq. 12-10,

$$\frac{ds}{dx} = \frac{mxg/lA}{Y}.$$

Therefore, the total elastic potential energy stored in the rod is

$$U_0 = \int_0^{U_0} dU = \frac{m^2g^2}{2l^2AY}\int_0^{l} x^2\,dx$$

$$= \frac{m^2g^2}{2l^2AY} \times \frac{l^3}{3}$$

$$= \frac{m^2g^2l}{6AY}.$$

The answers are dimensionally correct.

SAMPLE PROBLEM 12.04

Change in the volume of submerged soild sphere

A solid sphere is initially kept in open air, and the pressure exerted on it by air is 1.0×10^5 N/m^2 (atmospheric pressure). The sphere is lowered into the ocean to a depth where the pressure is 200 times the atmospheric pressure. The volume of the sphere in air is 0.5 m^3. What is the change in the volume once the sphere is submerged? Given that bulk modulus is 6.1×10^{10} N/m^2.

KEY IDEA

What happens when you squeeze a tennis ball in your hand? Its volume reduces as it shrinks. We can expect same thing to happen to the sphere when it is subjected to high pressure from the ocean water. We need to perform a simple calculation using Eq. 12-5.

Calculation: $\Delta V = -\frac{V_i \Delta p}{B}$.

Substituting the numerical values:

$$\Delta V = \frac{(0.50\ \text{m}^3)(2.0\times10^7\ \text{N/m}^2 - 1.0\times10^5\ \text{N/m}^2)}{6.1\times10^{10}\ \text{N/m}^2}$$

$$= -1.6\times10^{-4}/\text{m}^3. \qquad \text{(Answer)}$$

The negative sign indicates that the volume of the sphere decreases when submerged.

12.5 | POISSON'S RATIO

Key Concept

- The ratio of axial strain ($\Delta l/l$) along the length of the wire and a lateral strain ($\Delta r/r$) in the directions perpendicular to the length is called Poisson's ratio, κ.

$$\kappa = \left(\frac{\Delta r}{r}\right) \times \left(\frac{l}{\Delta l}\right) = \frac{\text{Lateral strain}}{\text{Axial strain}} \quad \text{or} \quad \kappa = \frac{\Delta r/r}{\Delta l/l}.$$

When a length of wire is stretched, its cross-sectional area becomes smaller. Now, there will be axial strain $\Delta l/l$ along the length of the wire and a lateral strain in the directions perpendicular to the length. For a wire of circular cross-section, if the radius was r before stretching and becomes $r - \Delta r$ after stretching, the strain perpendicular to the length is $\Delta r/r$. The ratio of these two strains is called Poisson's ratio, κ.

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$$\kappa = \left(\frac{\Delta r}{r}\right) \times \left(\frac{l}{\Delta l}\right) = \frac{\text{Lateral strain}}{\text{Axial strain}} \quad \text{or} \quad \kappa = \frac{(\Delta r/r)}{(\Delta l/l)}. \tag{12-11}$$

Poisson's ratio, like Young's modulus and shear modulus, is a property of the solid and does not depend on its shape.

Relation between Poisson's Ratio and Young's Modulus

Imagine we have submerged a small solid object deep into a liquid so that it experiences uniform pressure p from all directions. This pressure will apply compressive force on the object reducing its size but keeping its shape unaltered.

For example, if the cube of side a in Fig. 12-9 experiences an equal inward force $\Delta Pl2$ on each of its six faces, there is an equal compression in the x, y and z directions, resulting in a cube with slightly shorter sides.

If the force was acting only in one dimension, then the length of a side reduces to becomes $l - \Delta l$ from l. The volume of the cube becomes $(l - \Delta l)^3$ changing from l^3. Since Δl is much smaller than l for practical values of pressures, the terms $(\Delta l)^2$ and $(\Delta l)^3$ may be neglected; so,

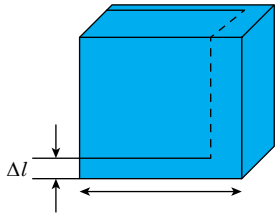

Figure 12-9 A cube immersed in a liquid, being compressed from all sides.

$$(l - \Delta l)^3 < l^3 - 3l^2\,\Delta l.$$

The fractional change in volume $\Delta V/V = 3\Delta l/l$.

As usual, stress/strain is constant for small deformations, and we can define the bulk modulus B of the material of the cube as

$$B = -\frac{\Delta p V}{\Delta V}.$$

Since the change in volume has occurred as a result of the application of three separate linear forces in the x, y and z directions, the bulk modulus can be related to Young's modulus. If the forces ΔPl^2 in the x, y and z direction are each applied in turn, each causes an extension in the two perpendicular directions by an amount given by Poisson's ratio. When the forces ΔPl^2 in the $\pm x$ directions are applied to the cube, the length of the sides parallel to the x axis becomes $l - \Delta l$, and the length of the other sides becomes $l + \kappa\Delta l$. The volume of the cube is now

$$(l - \Delta l) \times (l + \kappa\Delta l)^2 < l^3 - l^2\Delta l\,(1 - 2\kappa).$$

and the fractional change in volume is

$$\frac{\Delta V}{V} = \frac{\Delta l(1-2\kappa)}{l}.$$

As $\Delta l/l = p/Y$, we can write

$$\frac{\Delta l(1-2\kappa)}{l} = \frac{p(1-2\kappa)}{Y}.$$

The fractional change in volume, when forces Pl^2 are applied to all six faces of the cube, is simply three times this amount. The fractional change in volume is also given by the bulk modulus, and it follows that

$$B = \frac{pV}{\Delta V}.$$

Substituting ΔV, we get

$$B = \frac{Y}{3(1-2\kappa)}. \tag{12-12}$$

Steel has all four properties, putty has none of them. Glass is strong and stiff but not tough or ductile. Which properties would you ascribe to rubber, nylon and diamond?

12.6 | MECHANICAL PROPERTIES OF MATERIALS

The mechanical properties of a material govern its behavior under the action of external forces. When engineers select a material for a particular application they give special attention to these properties. Four important mechanical properties are strength, rigidity, ductility and toughness.

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Strength tells us the maximum magnitude of applied force a material can withstand before breaking. Rigidity tells us about the resistance material offers against distortion to its shape or size, or both. A stiff material is not very flexible. There is no such thing as a perfectly stiff or rigid (unyielding) material. All materials deform when subjected to force although the deformation may be very small.

Ductility means the ease with which the material can be hammered, pressed, bent, rolled, cut or stretched into useful shapes. A tough material is one that is not brittle, that is, it does not crack readily.

Steel has all four properties, putty has none of them. Glass is strong and stiff but not tough or ductile. Which properties would you ascribe to rubber, nylon and diamond?

REVIEW AND SUMMARY

Elastic Moduli Three **elastic moduli** are used to describe the elastic behavior (deformations) of objects as they respond to forces that act on them. The **strain** (fractional change in length) is linearly related to the applied **stress** (force per unit area) by the proper modulus, according to the general relation

$$\text{stress} = \text{modulus} \times \text{strain}. \tag{12-1}$$

The force required to stretch an elastic object such as a metal spring, within certain limits, is directly proportional to the extension of the spring, which is called **Hooke's law** and it is expressed as

$$F = -kx \tag{12-2}$$

where F is the force; x is the length of either extension or compression; and k is a constant of spring constant which is usually measured in N/m.

Tension and Compression When an object is under tension or compression, Eq. 12-1 is written as

$$\frac{F}{A} = E\frac{\Delta L}{L}, \tag{12-3}$$

where $\Delta L/L$ is the tensile or compressive strain of the object, F is the magnitude of the applied force $\vec{F}$ causing the strain, A is the cross-sectional area over which $\vec{F}$ is applied (perpendicular to A, as in Fig. 12-3*a*), and E is the **Young's modulus** for the object. The stress is F/A.

Shearing When an object is under a shearing stress, Eq. 12-1 is written as

$$\frac{F}{A} = G\frac{\Delta x}{L}, \tag{12-4}$$

where $\Delta x/L$ is the shearing strain of the object, Δx is the displacement of one end of the object in the direction of the applied force $\vec{F}$ (as in Fig. 12-3*b*), and G is the **shear modulus** of the object. The stress is F/A.

Hydraulic Stress When an object undergoes *hydraulic compression* due to a stress exerted by a surrounding fluid, Eq. 12-1 is written as

$$p = B\frac{\Delta V}{V}, \tag{12-5}$$

where p is the pressure (*hydraulic stress*) on the object due to the fluid, $\Delta V/V$ (the strain) is the absolute value of the fractional change in the object's volume due to that pressure, and B is the **bulk modulus** of the object.

Elastic Potential Energy The elastic potential energy stored in the rod is

$$U = \frac{1}{2} \times \sigma \times S \times \text{Volume}. \tag{12-9}$$

Poisson's Ratio The ratio of two strains (lateral strain and axial strain) is called Poisson's ratio, κ, which is expressed as

$$\kappa = \left(\frac{\Delta r}{r}\right) \times \left(\frac{l}{\Delta l}\right) = \frac{\text{Lateral strain}}{\text{Axial strain}} \quad \text{or} \quad \kappa = \frac{\Delta r/r}{\Delta l/l}. \tag{12-11}$$

Relation between Poisson's Ratio and Young's Modulus It is given by

$$B = \frac{Y}{3(1-2\kappa)}. \tag{12-12}$$

PROBLEMS

1. Figure 12-10 shows the stress–strain curve for a material. The scale of the stress axis is set by $s = 400$, in units of 10^6 N/m^2. What are (a) the Young's modulus and (b) the approximate yield strength for this material?

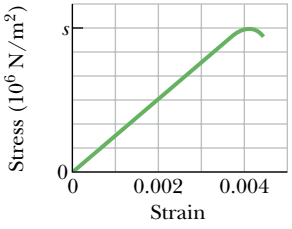

Figure 12-10 Problem 1.

2. In Fig. 12-11, a horizontal scaffold, of length 2.00 m and uniform mass 60.0 kg, is suspended from a building by two cables. The scaffold has dozens of paint cans stacked on it at various points. The total mass of the paint cans is 70.0 kg. The tension in the cable at the right is 722 N. How far

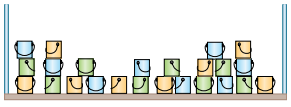

Figure 12-11 Problem 2.

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horizontally from *that* cable is the center of mass of the system of paint cans?

3. Figure 12-12 represents an insect caught at the midpoint of a spider-web thread. The thread breaks under a stress of 8.60×10^8 N/m^2 and a strain of 2.00. Initially, it was horizontal and had a length of 2.00 cm and a cross-sectional area of 8.00×10^{-12} m^2. As the thread was stretched under the weight of the insect, its volume remained constant. If the weight of the insect puts the thread on the verge of breaking, what is the insect's mass? (A spider's web is built to break if a potentially harmful insect, such as a bumble bee, becomes snared in the web.)

Figure 12-12 Problem 3.

4. In Fig. 12-13, a lead brick rests horizontally on cylinders A and B. The areas of the top faces of the cylinders are related by $A_A = 2.2A_B$; the Young's moduli of the cylinders are related by $E_A = 2.2E_B$. The cylinders had identical lengths before the brick was placed on them. What fraction of the brick's mass is supported (a) by cylinder A and (b) by cylinder B? The horizontal distances between the center of mass of the brick and the centerlines of the cylinders are d_A for cylinder A and d_B for cylinder B. (c) What is the ratio d_A/d_B?

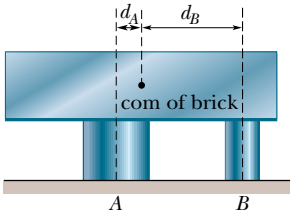

Figure 12-13 Problem 4.

5. A 1500 kg load is hung from the free end of a horizontal aluminum rod of length 7.0 cm, diameter 9.6 cm, and negligible mass. The other end of the rod is fixed in place. The shear modulus of aluminum is 3.0×10^{10} N/m^2. Find (a) the shear stress on the rod and (b) the vertical deflection of the rod's free end.

6. Figure 12-14 shows an approximate plot of stress versus strain for a spider-web thread, out to the point of breaking at a strain of 2.00. The vertical axis scale is set by values a = 0.10 GN/m^2, b = 0.30 GN/m^2, and c = 0.85 GN/m^2. Assume that the thread has an initial length of 0.80 cm, an initial cross-sectional area of 8.0×10^{-12} m^2, and (during stretching) a constant volume. The strain on the thread is the ratio of the change in the thread's length to that initial length, and the stress on the thread is the ratio of the collision force to that initial cross-sectional area. Assume that the work done on the thread by the collision force is given by the area under the curve on the graph. Assume also that when the single thread snares a flying insect, the insect's kinetic energy is transferred to the stretching of the thread. (a) How much kinetic energy would put the thread on the verge of breaking? What is the kinetic energy of (b) a fruit fly of mass 6.00 mg and speed 1.70 m/s and (c) a bumble bee of mass 0.388 g and speed 0.420 m/s? Would (d) the fruit fly and (e) the bumble bee break the thread?

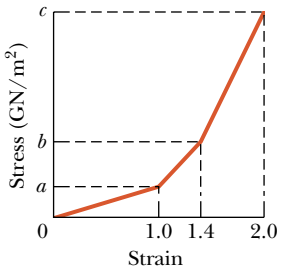

Figure 12-14 Problem 6.

7. Figure 12-15 shows the stress versus strain plot for an aluminum wire that is stretched by a machine pulling in opposite directions at the two ends of the wire. The scale of the stress axis is set by s = 7.0, in units of 10^7 N/m^2. The wire has an initial length of 0.800 m and an initial cross-sectional area of 2.30×10^{-6} m^2. How much work does the force from the machine do on the wire to produce a strain of 1.00×10^{-3}?

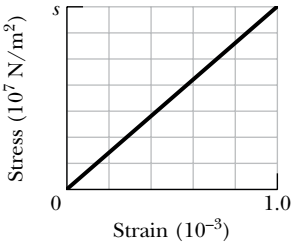

Figure 12-15 Problem 7.

8. In Fig. 12-16, a 103 kg uniform log hangs by two steel wires, A and B, both of radius 0.600 mm. Initially, wire A was 2.50 m long and 2.00 mm shorter than wire B. The log is now horizontal. What are the magnitudes of the forces on it from (a) wire A and (b) wire B? (c) What is the ratio d_A/d_B?

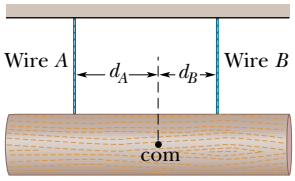

Figure 12-16 Problem 8.

9. Figure 12-17 is an overhead view of a rigid rod that turns about a vertical axle until the identical rubber stoppers A and B are forced against rigid walls at distances r_A = 5.0 cm and r_B = 2.0 cm from the axle. Initially the stoppers touch the walls without being compressed. Then force $\vec{F}$ of magnitude 150 N is applied perpendicular to the rod at a distance R = 5.0 cm from the axle. Find the magnitude of the force compressing (a) stopper A and (b) stopper B.

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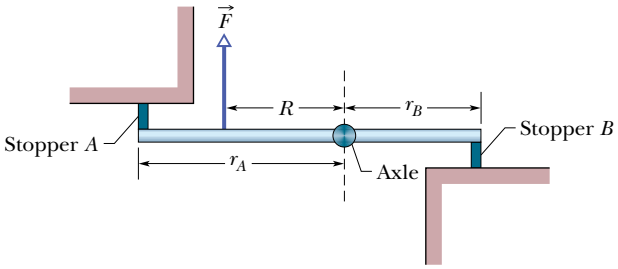


Figure 12-17 Problem 9.

10. A tunnel of length $L = 200$ m, height $H = 7.2$ m, and width 6.3 m (with a flat roof) is to be constructed at distance $d = 70$ m beneath the ground (Fig. 12-18). The tunnel roof is to be supported entirely by square steel columns, each with a cross-sectional area of 960 cm^2. The mass of 1.0 cm^3 of the ground material is 2.8 g. (a) What is the total weight of the ground material the columns must support? (b) How many columns are needed to keep the compressive stress on each column at one-half its ultimate strength?

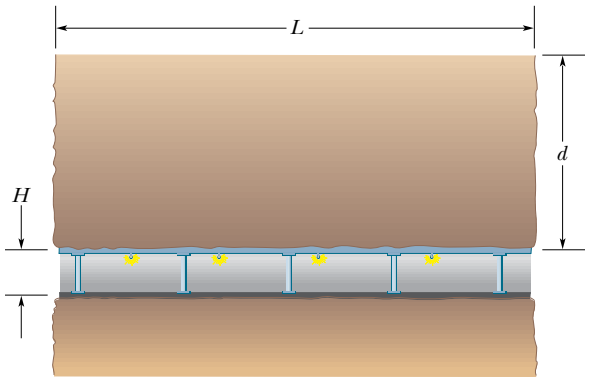
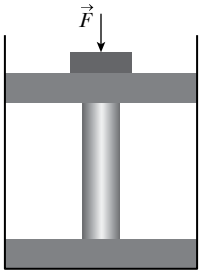


Figure 12-18 Problem 10.

PRACTICE QUESTIONS

1. If a metal wire is stretched a little beyond its elastic limit (or yield point), and released, it will
- **(a)** lose its elastic property completely.
- **(b)** not contract.
- **(c)** contract, but its final length will be greater than its initial length.
- **(d)** contract only up to its length at the elastic limit.

2. A cylinder with a diameter of 0.240 m is placed between the plates of a hydraulic press as illustrated in the figure. A force equal to 4.45×10^5 N is applied to the cylinder. What is the pressure on the end of the cylinder due to the applied force?

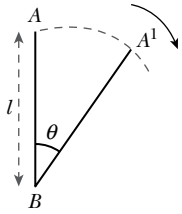


- **(a)** 9.84×10^6 Pa
- **(b)** 3.13×10^5 Pa
- **(c)** 6.18×10^6 Pa
- **(d)** 5.34×10^5 Pa

3. A wire elongates by l mm when a load W is hanged from it. If the wire goes over a pulley and two weights W each are hung at the two ends, the elongation of the wire (in mm) will be
- **(a)** $\frac{l}{2}$
- **(b)** l
- **(c)** $2l$
- **(d)** zero

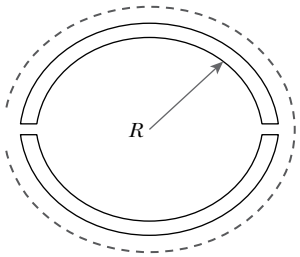
4. A cable stretches by an amount d when it supports a crate of mass M. The cable is replaced by another cable of the same material having the same length and twice the diameter. If the same crate is supported by the thicker cable, by how much will the cable stretch?
- **(a)** $\frac{d}{4}$
- **(b)** $\frac{d}{2}$
- **(c)** d
- **(d)** $2d$

5. A uniform rod of length l is free to rotate in a vertical plane about a fixed horizontal axis through B (see figure). The rod begins rotating from rest from its unstable equilibrium position. When it has turned through an angle θ, its angular velocity is given by



- **(a)** $\sqrt{\left(\frac{6g}{l}\right)}\sin\frac{\theta}{2}$
- **(b)** $\sqrt{\left(\frac{6g}{l}\right)}\cos\frac{\theta}{2}$
- **(c)** $\sqrt{\left(\frac{6g}{l}\right)}\sin\theta$
- **(d)** $\sqrt{\left(\frac{6g}{l}\right)}\cos\theta$

6. A wooden wheel of radius R is made of two semi-circular parts as shown in the figure. The two parts are held together by a ring made of a metal strip of cross-sectional area S and length L. L is slightly less than $2\pi R$. To fit the ring on the wheel, it is heated so that its temperature rises by ΔT and it just steps over the wheel. As it cools down to surrounding temperature, it presses the semi-circular parts together. If the coefficient of linear expansion of the metal is α, and its Young's modulus is Y, the force that one part of the wheel applies on the other part is



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(a) $2\pi SY\alpha\Delta T$ **(b)** $SY\alpha\Delta T$
(c) $\pi SY\alpha\Delta T$ **(d)** $2SY\alpha\Delta T$

7. A tow truck is pulling a car out of a ditch by means of a steel cable that is 9.1 m long and has a radius of 0.50 cm. When the car just begins to move, the tension in the cable is 890 N. How much has the cable stretched?
(a) 2.6×10^{-6} m **(b)** 2.3×10^{-2} m
(c) 1.7×10^{-4} m **(d)** 5.2×10^{-4} m

8. A rubber cube of side 5 cm has one side fixed while a tangential force equal to 180 kg wt. is applied to opposite face. Find the shearing strain produced and the lateral displacement of the strained face of the cube. Modulus of rigidity (or shear modulus) for rubber is 2.4×10^6 N/m^2.
(a) 0.2160 m **(b)** 0.0802 m
(c) 0.1093 m **(d)** 0.0147 m

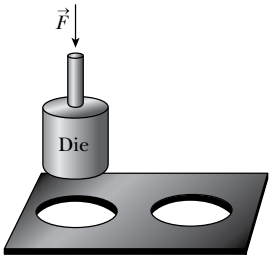
9. A steel wire of length 20 cm and cross-section 1 mm^2 is tied rigidly at both ends at room temperature 30 °C. If the temperature falls to 10 °C, what will be the change in tension? Given that $Y = 2 \times 10^{11}$ N/m^2 and $\alpha = 1.1 \times 10^{-5}$/°C
(a) 2.2 N **(b)** 4.4 N
(c) 22 N **(d)** 44 N

10. When subjected to a force of compression, the length of a bone decreases by 2.7×10^{-5} m. When this same bone is subjected to a tensile force of the same magnitude, by how much does it stretch?
(a) 4.6×10^{-5} m **(b)** 1.6×10^{-5} m
(c) 3.1×10^{-5} m **(d)** 2.0×10^{-5} m

11. A uniform rod of mass m, length L, and area of cross-section A is rotated about an axis passing through one of its ends and perpendicular to its length with a constant angular velocity ω in a horizontal plane. If Y is the Young's modulus of the material of rod, then the increase in its length due to rotation of rod is
(a) $\dfrac{m\omega^2L^2}{AY}$ **(b)** $\dfrac{m\omega^2L^2}{2AY}$
(c) $\dfrac{m\omega^2L^2}{3AY}$ **(d)** $\dfrac{2m\omega^2L^2}{AY}$

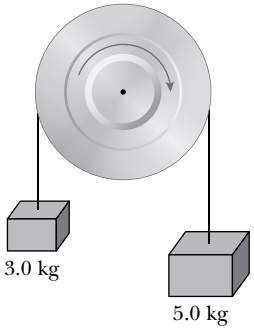
12. Between each pair of vertebrae in the spinal column is a cylindrical disc of cartilage. Typically, this disc has a radius of about 3.0×10^{-2} m and a thickness of about 7.0×10^{-3} m. The shear modulus of cartilage is 1.2×10^7 N/m^2. Suppose a shearing force of magnitude 11 N is applied parallel to the top surface of the disc while the bottom surface remains fixed in place. How far does the top surface move relative to the bottom surface?
(a) 2.0×10^{-5} m **(b)** 4.1×10^{-5} m
(c) 2.3×10^{-6} m **(d)** 3.5×10^{-6} m

13. A die is designed to punch holes with a radius of 1.00×10^{-2} m in a metal sheet that is 3.0×10^{-3} m thick, as shown in the figure. To punch through the sheet, the die must exert a shearing stress of 3.5×10^8 Pa. What force $\vec{F}$ must be applied to the die?



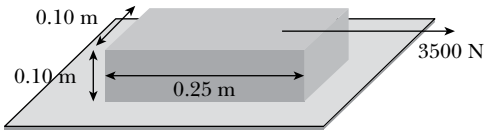
(a) 2.1×10^4 N **(b)** 4.2×10^4 N
(c) 3.3×10^4 N **(d)** 6.6×10^4 N

14. The figure shows two crates that are connected by a steel wire that passes over a pulley. The unstretched length of the wire is 1.5 m, and its cross-sectional area is 1.3×10^{-5} m^2. The pulley is frictionless and massless. When the crates are accelerating, determine the change in length of the wire. Ignore the mass of the wire.



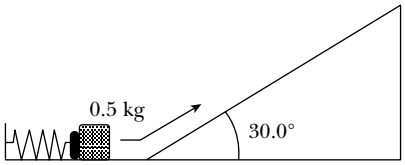
(a) 4.3×10^{-5} m **(b)** 3.2×10^{-5} m
(c) 2.1×10^{-5} m **(d)** 2.5×10^{-6} m

15. The brick shown in the figure is glued to the floor. A 3500 N force is applied to the top surface of the brick as shown. If the brick has a shear modulus of 5.4×10^9 N/m^2, how far to the right does the top face move relative to the stationary bottom face?



(a) 5.8×10^{-6} m **(b)** 2.6×10^{-6} m
(c) 1.1×10^{-6} m **(d)** 6.5×10^{-7} m

16. A spring with constant $k = 78$ N/m is at the base of a frictionless, 30.0° inclined plane. A 0.50 kg block is pressed against the spring, compressing it 0.20 m from its equilibrium position. The block is then released. If the block is not attached to the spring, how far up the incline will it travel before it stops?



(a) 0.080 m **(b)** 0.16 m
(c) 0.32 m **(d)** 0.64 m

17. The maximum compressional stress that a bone can withstand is 1.6×10^8 N/m^2 before it breaks. A thighbone (femur), which is the largest and longest bone in the human body, has a cross-sectional area of 7.7×10^{-4} m^2. What is the maximum compressional force that can be applied to the thighbone?
(a) 2.1×10^{11} N **(b)** 1.2×10^5 N
(c) 4.8×10^{12} N **(d)** 3.0×10^3 N

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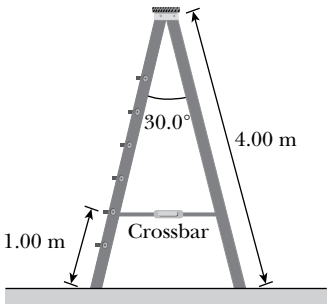
18. The shear modulus of steel is 8.1×10^{10} N/m^2. A steel nail of radius 7.5×10^{-4} m projects 0.040 m horizontally outward from a wall. A man hangs a wet raincoat of weight 28.5 N from the end of the nail. Assuming the wall holds its end of the nail, what is the vertical deflection of the other end of the nail?

(a) 1.8×10^{-3} m **(b)** 3.3×10^{-2} m
(c) 7.9×10^{-6} m **(d)** 4.2×10^{-4} m

19. A gymnast does a one-arm handstand. The humerus, which is the upper arm bone between the elbow and the shoulder joint, may be approximated as a 0.30-m-long cylinder with an outer radius of 1.0×10^{-2} m and a hollow inner core with a radius of 4.0×10^{-3} m. Excluding the arm, the mass of the gymnast is 63 kg. What is the compressional strain of the humerus?

(a) 4.9×10^{-4} **(b)** 3.5×10^{-4}
(c) 2.5×10^{-4} **(d)** 6.4×10^{-4}

20. The figure shows an A-shaped ladder. Both sides of the ladder are equal in length. This ladder is standing on a frictionless horizontal surface, and only the crossbar (which has a negligible mass) of the "A" keeps the ladder from collapsing. The ladder is uniform and has a mass of 20.0 kg. Determine the tension in the crossbar of the ladder.



(a) 16.9 N **(b)** 31.5 N
(c) 26.1 N **(d)** 17.5 N

21. A solid brass sphere is subjected to a pressure of 1.0×10^5 Pa due to the Earth's atmosphere. On Venus the pressure due to the atmosphere is 9.0×10^6 Pa. By what fraction $\Delta r/r_0$ (including the algebraic sign) does the radius of the sphere change when it is exposed to the atmosphere on Venus? Assume that the change in radius is very small relative to the initial radius.

(a) -4.4×10^{-5} **(b)** -4.4×10^{-6}
(c) $+5.4 \times 10^{-5}$ **(d)** -6.2×10^{-4}

22. A cylindrically shaped piece of collagen (a substance found in the body in connective tissue) is being stretched by a force that increases from 0 to 3.0×10^{-2} N. The length and radius of the collagen are, respectively, 2.5 and 0.091 cm, and Young's modulus is 3.1×10^6 N/m^2. If the stretching obeys Hooke's law, what is the spring constant k for collagen?

(a) 3.5×10^3 N/m **(b)** 1.0×10^3 N/m
(c) 3.2×10^2 N/m **(d)** 6.4×10^2 N/m

23. One end of a piano wire is wrapped around a cylindrical tuning peg and the other end is fixed in place. The tuning peg is turned so as to stretch the wire. The piano wire is made from steel ($Y = 2.0 \times 10^{11}$ N/m^2). It has a radius of 0.80 mm, and an unstrained length of 0.76 m. The radius of the tuning peg is 1.8 mm. Initially, there is no tension in the wire. Find the tension in the wire when the tuning peg is turned through two revolutions.

(a) 15,000 N **(b)** 12,000 N
(c) 9200 N **(d)** 6100 N

More than One Correct Choice Type

24. If σ is the breaking stress of a material of density ρ, then the length of the wire of that material that can hang freely without breaking is

(a) $\frac{\sigma}{\rho g}$ **(b)** $\frac{\sigma}{2\rho g}$
(c) $\frac{2\sigma}{\rho g}$ **(d)** $\frac{\rho}{2\sigma g}$

25. A uniform wire of radius r and length L hanging vertically and fixed at its upper end, is attached to a weight at its lower end. If Young's modulus for the material of the wire is Y, the extension is

(a) inversely proportional to Y.
(b) inversely proportional to r^2.
(c) directly proportional to L.
(d) inversely proportional to r.

26. A block is falling in a lake at a depth of 100 m shows 0.2% decrease in its volume at the bottom. What is the bulk modulus of the block?

(a) 4.9×10^7 dyn/mm^2 **(b)** 4.9×10^8 N/m^2
(c) 4.9×10^8 dyn/cm^2 **(d)** 4.9×10^{12} N/cm^2

27. The pressure increases by 1.0×10^4 N/m^2 for every meter of depth beneath the surface of the ocean. At what depth does the volume of a Pyrex glass cube of edge length 1.0×10^{-2} m at the ocean's surface decrease by 1.0×10^{-10} m^3? Bulk modulus of Pyrex glass is 2.6×10^{10} N/m^3

(a) 10.24×10^3 inch **(b)** 260 m
(c) 853 feet **(d)** 0.161 mile

28. For two different materials, it is given that $Y_1 > Y_2$ and $B_1 < B_2$. Here, Y is Young's modulus of elasticity and B is the Bulk modulus of elasticity. Then, we can conclude that

(a) 1 is more ductile.
(b) 2 is more ductile.
(c) 1 is more malleable.
(d) 2 is more malleable.

29. Choose the correct statement from the following:

(a) Elastic forces are always conservative.
(b) Elastic forces are not always conservative.
(c) Elastic forces are conservative only when Hooke's is obeyed.
(d) Elastic forces may be conservative even when Hooke's law is not obeyed.

30. For an ideal liquid:

(a) the shear modulus is infinite.
(b) the shear modulus is zero.
(c) the bulk modulus is zero.
(d) the bulk modulus is infinite.

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31. A copper and a steel wire of the same diameter are connected end to end. A deforming force F is applied to this composite wire, which causes a total elongation of 1 cm. The two wires will have

(a) the same stress. **(b)** different stress.
(c) same strain. **(d)** different strain.

Linked Comprehension

***Paragraph for Questions 32—34*:** A 5.0×10^2 N object is hung from the end of a wire of cross-sectional area 0.010 cm^2. The wire stretches from its original length of 200.00 cm to 200.50 cm.

32. What is the stress on the wire?

(a) 5.0×10^2 N/m^2 **(b)** 1.0×10^6 N/m^2
(c) 1.0×10^8 N/m^2 **(d)** 5.0×10^8 N/m^2

33. What is the elongation strain on the wire?

(a) 1.0×10^2 **(b)** 1.0×10^{-2}
(c) 5.0×10^2 **(d)** 2.5×10^{-3}

34. Determine the Young's modulus of the wire:

(a) 1.0×10^{11} N/m^2 **(b)** 1.0×10^9 N/m^2
(c) 2.0×10^{11} N/m^2 **(d)** 2.0×10^9 N/m^2

***Paragraph for Questions 35—37*:** Using Searle's apparatus while determining the Young's modulus of a wire, a girl is taking measurements of different parameters: Length of the wire, L, is 30 cm, which is measured with a millimeter scale; diameter of the wire, d, is 1 mm, which is measured with a screw gauge. (Least count of the screw gauge is 0.01 mm; least count of the spherometer attached to the apparatus frame is 0.005 mm; spherometer measured the extension e of the wire; W is the weight of the loads.)

35. The parameter that should be measured more carefully is

(a) load on the hanger. **(b)** length of the wire.
(c) diameter of the wire. **(d)** extension of the wire.

36. The load versus extension graph should be (if the extension is in elastic limit)

(a) straight line. **(b)** curved line.
(c) broken line. **(d)** any of the above.

37. The maximum fractional error in the measurement of Y is

(a) $\frac{\Delta W}{W}+\frac{\Delta L}{L}+\frac{\Delta d}{d}+\frac{\Delta l}{l}$

(b) $\frac{\Delta W}{W}+\frac{\Delta L}{L}-\frac{2\Delta d}{d}+\frac{\Delta l}{l}$

(c) $\frac{\Delta W}{W}+\frac{\Delta L}{L}+\frac{2\Delta d}{d}+\frac{\Delta l}{l}$

(d) $\frac{\Delta W}{W}-\frac{\Delta L}{L}+\frac{\Delta d}{d}+\frac{\Delta l}{l}$

Matrix-Match

Directions for Questions 38—39: Match the statement in Column I with the statements in Column II. One or more than one choice from Column II can match with a statement from Column I.

38.

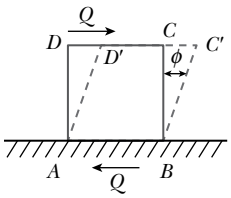
Column I	Column II
(a) If the wire is pulled at its ends by equal and opposite forces of magnitude F so that it undergoes an elongation x, according to Hooke's law, $F = -kx$, where k is the force constant. Force constant (k) of the wire will depend on	**(p)** Young's modulus (Y)
(b) Let us suspend the wire vertically from a rigid support and attach a mass m at its lower end. If the mass is slightly pulled down and released, it executes SHM of a time period that will depend on	**(q)** elongation (x)
(c) If the given wire is fixed between two rigid supports and its temperature is decreased, then the thermal stress that develops in the wire will depend on	**(r)** length (l)
(d) Work done in stretching the wire to a length $l + x$ will depend on	**(s)** area of cross-section (a)

39.

Column I	Column II
(a) Young's modulus	**(p)** $\frac{-\Delta V}{\rho V}$
(b) Bulk modulus	**(q)** $\frac{Fl}{A\Delta l}$
(c) Compressibility	**(r)** $\frac{-\Delta d/d}{\Delta Dl/l}$
(d) Poisson's ratio	**(s)** $\frac{-\rho V}{\Delta V}$

Directions for Question 40: There is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

40. In the given table, Young's modulus, bulk modulus, and shear modulus of some materials are listed in Columns I, II, and III, respectively.

Column I	Column II	Column III
(I) The ratio of the change in length of the body due to the deformation to its original length in the direction of the force.	**(i)** 	**(J)** Distortion in action

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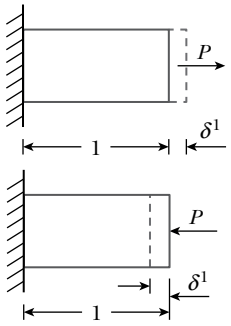
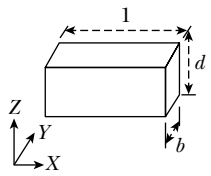
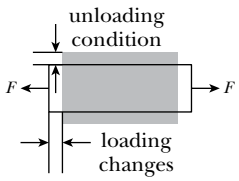
Column I	Column II	Column III
(II) The ratio of the change in volume of the body to the deformation to its original volume.	**(ii)** 	**(K)** Dilatation
(III) The ratio of the change in length (breadth of a rectangular bar or diameter of a circular bar) of the body due to the deformation to its original length (breadth of a rectangular bar or diameter of a circular bar) in the direction perpendicular to the force.	**(iii)** 	**(L)** Tensile or compressive strain
(IV) Strain accompanying a shearing action.	**(iv)** 	**(M)** Axial strain

(1) The correct combination that defines the characteristics of linear strain is

(a) (I) (iii) (L) **(b)** (IV) (i) (M)
(c) (II) (iv) (K) **(d)** (I) (ii) (L)

(2) The correct combination that defines the characteristics of lateral strain is

(a) (III) (ii) (L) **(b)** (III) (iv) (M)
(c) (II) (iii) (K) **(d)** (I) (i) (M)

(3) The correct combination that defines the characteristics volumetric strain is

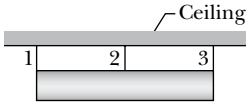
(a) (II) (iii) (K) **(b)** (I) (i) (K)
(c) (III) (iii) (L) **(d)** (I) (ii) (L)

Integer Type

41. The Leaning Tower of Pisa is 59.1 m high and 7.44 m in diameter. The top of the tower is displaced 4.01 m from the vertical. Treat the tower as a uniform, circular cylinder. What additional displacement (in m), measured at the top, would bring the tower to the verge of toppling?

42. A cylindrical aluminum rod, with an initial length of 0.8000 m and radius 1000.0 μm, is clamped in place at one end and then stretched by a machine pulling parallel to its length at its other end. Assuming that the rod's density (mass per unit volume) does not change, find the force magnitude that is required of the machine to decrease the radius to 999.9 μm. (The yield strength is not exceeded; $E_{\text{aluminium}} = 70 \times 10^9\ \text{N/m}^2$)

43. Figure shows a 300 kg cylinder that is horizontal. Three steel wires support the cylinder from a ceiling. Wires 1 and 3 are attached at the ends of the cylinder, and wire 2 is attached at the center. The wires each have a cross-sectional area of $2.00 \times 10^{-6}\ \text{m}^2$. Initially (before the cylinder was put in place), wires 1 and 3 were 2.0000 m long and wire 2 was 6.00 mm longer than that. Now (with the cylinder in place), all three wires have been stretched. What is the tension in wire 1 (kN)? ($E_{\text{steel}} = 200 \times 10^9\ \text{N/m}^2$)



ANSWER KEY

Checkpoint

1. (d)

Problems

1. (a) $7.5 \times 10^{10}\ \text{N/m}^2$; (b) $2.9 \times 10^8\ \text{N/m}^2$ **2.** 0.752 m **3.** 0.441 g

4. (a) 0.83; (b) 0.17; (c) 0.21 **5.** (a) 2.0×10^6 N/m; (b) $4.7\ 10^{-6}$ m.

6. (a) 3.04×10^{-5} J; (b) 8.67×10^6 J; (c) 3.42×10^{-5} J; (d) The fruit fly will not be able to break the thread; (e) The bumble bee will be able to break the thread.

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7. 0.0644 J
8. (a) 595 N; (b) 414 N; (c) 0.696
9. (a) $\approx 1.3 \times 10^2$ N; (b) 52 N
10. (a) 2.4×10^9 N; (b) 126

Practice Questions

Single Correct Choice Type

1. (c)
2. (a)
3. (b)
4. (a)
5. (c)
6. (d)
7. (d)
8. (d)
9. (c)
10. (b)
11. (b)
12. (c)
13. (d)
14. (c)
15. (b)
16. (d)
17. (b)
18. (c)
19. (c)
20. (d)
21. (a)
22. (c)
23. (b)

More than One Correct Choice Type

24. (a), (c)
25. (a), (b), (c)
26. (a), (b), (d)
27. (a), (b), (c), (d)
28. (b), (c)
29. (b), (d)
30. (b), (d)
31. (a), (d)

Linked Comprehension

32. (d)
33. (d)
34. (c)
35. (b)
36. (d)
37. (c)

Matrix-Match

38. (a) $\rightarrow$ (p), (r), (s); (b) $\rightarrow$ (p), (r), (s); (c) $\rightarrow$ (p); (d) $\rightarrow$ (p), (q), (r), (s)
39. (1) $\rightarrow$ (b); (2) $\rightarrow$ (d); (3) $\rightarrow$ (c)
40. (1) $\rightarrow$ (d); (2) $\rightarrow$ (b); (3) $\rightarrow$ (a)

Integer Type

41. 3
42. 44
43. 1

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CHAPTER 13

Gravitation

13.1 | WHAT IS PHYSICS?

One of the long-standing goals of physics is to understand the gravitational force—the force that holds you to Earth, holds the Moon in orbit around Earth, and holds Earth in orbit around the Sun. It also reaches out through the whole of our Milky Way galaxy, holding together the billions and billions of stars in the Galaxy and the countless molecules and dust particles between stars. We are located somewhat near the edge of this disk-shaped collection of stars and other matter, 2.6×10^4 light-years (2.5×10^{20} m) from the galactic center, around which we slowly revolve.

The gravitational force also reaches across intergalactic space, holding together the Local Group of galaxies, which includes, in addition to the Milky Way, the Andromeda Galaxy (Fig. 13-1) at a distance of 2.3×10^6 light-years away from Earth, plus several closer dwarf galaxies, such as the Large Magellanic Cloud. The Local Group is part of the Local Supercluster of galaxies that is being drawn by the gravitational force toward an exceptionally massive region of space called the Great Attractor. This region appears to be about 3.0×10^8 light-years from Earth, on the opposite side of the Milky Way. And the gravitational force is even more far-reaching because it attempts to hold together the entire universe, which is expanding.

This force is also responsible for some of the most mysterious structures in the universe: *black holes*. When a star considerably larger than

Courtesy NASA

Figure 13-1 The Andromeda Galaxy. Located 2.3×10^6 light-years from us, and faintly visible to the naked eye, it is very similar to our home galaxy, the Milky Way.

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13.1 What is Physics?
13.2 Newton's Law of Gravitation
13.3 Gravitation and the Principle of Superposition
13.4 Gravitation Near Earth's Surface
13.5 Gravitation Inside Earth
13.6 Gravitational Potential Energy
13.7 Planets and Satellites: Kepler's Laws
13.8 Satellites: Orbits and Energy
13.9 Einstein and Gravitation

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our Sun burns out, the gravitational force between all its particles can cause the star to collapse in on itself and thereby to form a black hole. The gravitational force at the surface of such a collapsed star is so strong that neither particles nor light can escape from the surface (thus the term "black hole"). Any star coming too near a black hole can be ripped apart by the strong gravitational force and pulled into the hole. Enough captures like this yields a *supermassive black hole*. Such mysterious monsters appear to be common in the universe. Indeed, such a monster lurks at the center of our Milky Way galaxy—the black hole there, called Sagittarius A*, has a mass of about 3.7×10^6 solar masses. The gravitational force near this black hole is so strong that it causes orbiting stars to whip around the black hole, completing an orbit in as little as 15.2 y.

Although the gravitational force is still not fully understood, the starting point in our understanding of it lies in the *law of gravitation* of Isaac Newton.

13.2 | NEWTON'S LAW OF GRAVITATION

Key Concepts

- Any particle in the universe attracts any other particle with a gravitational force whose magnitude is

  $$F = G\frac{m_1 m_2}{r^2} \quad \text{(Newton's law of gravitation)},$$

  where m_1 and m_2 are the masses of the particles, r is their separation, and G ($= 6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2$) is the gravitational constant.
- The gravitational force between extended bodies is found by adding (integrating) the individual forces on individual particles within the bodies. However, if either of the bodies is a uniform spherical shell or a spherically symmetric solid, the net gravitational force it exerts on an *external* object may be computed as if all the mass of the shell or body were located at its center.

Physicists like to study seemingly unrelated phenomena to show that a relationship can be found if the phenomena are examined closely enough. This search for unification has been going on for centuries. In 1665, the 23-year-old Isaac Newton made a basic contribution to physics when he showed that the force that holds the Moon in its orbit is the same force that makes an apple fall. We take this knowledge so much for granted now that it is not easy for us to comprehend the ancient belief that the motions of earthbound bodies and heavenly bodies were different in kind and were governed by different laws. Newton concluded not only that Earth attracts both apples and the Moon but also that every body in the universe attracts every other body; this tendency of bodies to move toward each other is called **gravitation**. Newton's conclusion takes a little getting used to, because the familiar attraction of Earth for earthbound bodies is so great that it overwhelms the attraction that earthbound bodies have for each other. For example, Earth attracts an apple with a force magnitude of about 0.8 N. You also attract a nearby apple (and it attracts you), but the force of attraction has less magnitude than the weight of a speck of dust. Newton proposed a *force law* that we call **newton's law of gravitation**: Every particle attracts any other particle with a **gravitational force** of magnitude

$$F = G\frac{m_1 m_2}{r^2} \quad \text{(Newton's law of gravitation)}. \tag{13-1}$$

Here m_1 and m_2 are the masses of the particles, r is the distance between them, and G is the **gravitational constant**, with a value that is now known to be

$$\begin{aligned} G &= 6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2 \\ &= 6.67 \times 10^{-11}\ \text{m}^3/\text{kg}\cdot\text{s}^2. \end{aligned} \tag{13-2}$$

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In Fig. 13-2*a*, $\vec{F}$ is the gravitational force acting on particle 1 (mass m_1) due to particle 2 (mass m_2). The force is directed toward particle 2 and is said to be an *attractive force* because particle 1 is attracted toward particle 2. The magnitude of the force is given by Eq. 13-1. We can describe $\vec{F}$ as being in the positive direction of an r axis extending radially from particle 1 through particle 2 (Fig. 13-2*b*). We can also describe $\vec{F}$ by using a radial unit vector $\hat{r}$ (a dimensionless vector of magnitude 1) that is directed away from particle 1 along the r axis (Fig. 13-2*c*). From Eq. 13-1, the force on particle 1 is then

$$\vec{F} = G\frac{m_1 m_2}{r^2}\hat{r}. \tag{13-3}$$

The gravitational force on particle 2 due to particle 1 has the same magnitude as the force on particle 1 but the opposite direction. These two forces form a third-law force pair, and we can speak of the gravitational force *between* the two particles as having a magnitude given by Eq. 13-1. This force between two particles is not altered by other objects, even if they are located between the particles. Put another way, no object can shield either particle from the gravitational force due to the other particle.

The strength of the gravitational force—that is, how strongly two particles with given masses at a given separation attract each other—depends on the value of the gravitational constant G. If G—by some miracle—were suddenly multiplied by a factor of 10, you would be crushed to the floor by Earth's attraction. If G were divided by this factor, Earth's attraction would be so weak that you could jump over a building.

Nonparticles. Although Newton's law of gravitation applies strictly to particles, we can also apply it to real objects as long as the sizes of the objects are small relative to the distance between them. The Moon and Earth are far enough apart so that, to a good approximation, we can treat them both as particles—but what about an apple and Earth? From the point of view of the apple, the broad and level Earth, stretching out to the horizon beneath the apple, certainly does not look like a particle.

Newton solved the apple–Earth problem with the *shell theorem:*

A uniform spherical shell of matter attracts a particle that is outside the shell as if all the shell's mass were concentrated at its center.

Earth can be thought of as a nest of such shells, one within another and each shell attracting a particle outside Earth's surface as if the mass of that shell were at the center of the shell. Thus, from the apple's point of view, Earth *does* behave like a particle. one that is located at the center of Earth and has a mass equal to that of Earth.

Third-Law Force Pair. Suppose that, as in Fig. 13-3, Earth pulls down on an apple with a force of magnitude 0.80 N. The apple must then pull up on Earth with a force of magnitude 0.80 N, which we take to act at the center of Earth. In the language of Chapter 5, these forces form a force pair in Newton's third law. Although they are matched in magnitude, they produce different accelerations when the apple is released. The acceleration of the apple is about 9.8 m/s^2, the familiar acceleration of a falling body near Earth's surface. The acceleration of Earth, however, measured in a reference frame attached to the center of mass of the apple–Earth system, is only about 1×10^{-25} m/s^2.

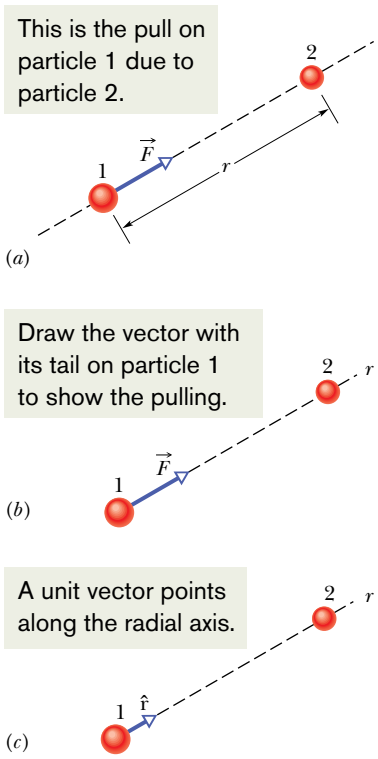

Figure 13-2 (*a*) The gravitational force $\vec{F}$ on particle 1 due to particle 2 is an attractive force because particle 1 is attracted to particle 2. (*b*) Force $\vec{F}$ is directed along a radial coordinate axis r extending from particle 1 through particle 2. (*c*) $\vec{F}$ is in the direction of a unit vector $\hat{r}$ along the r axis.

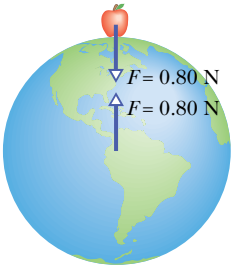

Figure 13-3 The apple pulls up on Earth just as hard as Earth pulls down on the apple.

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CHECKPOINT 1

A particle is to be placed, in turn, outside four objects, each of mass m: (1) a large uniform solid sphere, (2) a large uniform spherical shell, (3) a small uniform solid sphere, and (4) a small uniform shell. In each situation, the distance between the particle and the center of the object is d. Rank the objects according to the magnitude of the gravitational force they exert on the particle, greatest first.

13.3 | GRAVITATION AND THE PRINCIPLE OF SUPERPOSITION

Key Concepts

- Gravitational forces obey the principle of superposition; that is, if n particles interact, the net force $\vec{F}_{1,\text{net}}$ on a particle labeled particle 1 is the sum of the forces on it from all the other particles taken one at a time:

$$\vec{F}_{1,\text{net}} = \sum_{i=2}^{n} \vec{F}_{1i},$$

in which the sum is a vector sum of the forces $\vec{F}_{1i}$ on particle 1 from particles $2, 3, \ldots, n$.

- The gravitational force $\vec{F}_1$ on a particle from an extended body is found by first dividing the body into units of differential mass dm, each of which produces a differential force $d\vec{F}$ on the particle, and then integrating over all those units to find the sum of those forces:

$$\vec{F}_1 = \int d\vec{F}.$$

Given a group of particles, we find the net (or resultant) gravitational force on any one of them from the others by using the **principle of superposition**. This is a general principle that says a net effect is the sum of the individual effects. Here, the principle means that we first compute the individual gravitational forces that act on our selected particle due to each of the other particles. We then find the net force by adding these forces vectorially, just as we have done when adding forces in earlier chapters.

Let's look at two important points in that last (probably quickly read) sentence. (1) Forces are vectors and can be in different directions, and thus we must *add them as vectors*, taking into account their directions. (If two people pull on you in the opposite direction, their net force on you is clearly different than if they pull in the same direction.) (2) We *add* the individual forces. Think how impossible the world would be if the net force depended on some multiplying factor that varied from force to force depending on the situation, or if the presence of one force somehow amplified the magnitude of another force. No, thankfully, the world requires only simple vector addition of the forces.

For n interacting particles, we can write the principle of superposition for the gravitational forces on particle 1 as

$$\vec{F}_{1,\text{net}} = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \vec{F}_{15} + \cdots + \vec{F}_{1n}. \tag{13-4}$$

Here $\vec{F}_{1,\text{net}}$ is the net force on particle 1 due to the other particles and, for example, $\vec{F}_{13}$ is the force on particle 1 from particle 3. We can express this equation more compactly as a vector sum:

$$\vec{F}_{1,\text{net}} = \sum_{i=2}^{n} \vec{F}_{1i}. \tag{13-5}$$

Real Objects. What about the gravitational force on a particle from a real (extended) object? This force is found by dividing the object into parts small enough to treat as particles and then using Eq. 13-5 to find the vector sum of the forces on the particle from all the parts. In the limiting case, we can divide the extended object into differential parts each of mass dm and each producting a differential force $d\vec{F}$ on the particle. In this limit, the sum of Eq. 13-5 becomes an integral and we have

$$\vec{F}_1 = \int d\vec{F}, \tag{13-6}$$

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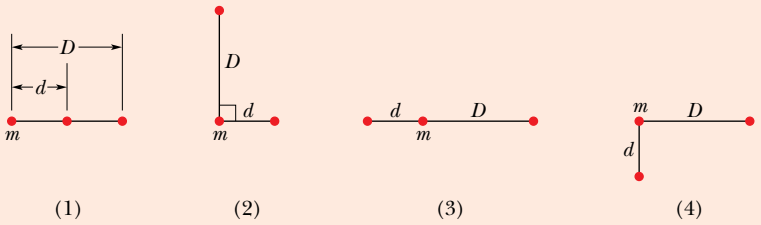
in which the integral is taken over the entire extended object and we drop the subscript "net." If the extended object is a uniform sphere or a spherical shell, we can avoid the integration of Eq. 13-6 by assuming that the object's mass is concentrated at the object's center and using Eq. 13-1.

CHECKPOINT 2

The figure shows four arrangements of three particles of equal masses. (a) Rank the arrangements according to the magnitude of the net gravitational force on the particle labeled m, greatest first. (b) In arrangement 2, is the direction of the net force closer to the line of length d or to the line of length D?



SAMPLE PROBLEM 13.01

Net gravitational force, 2D, three particles

Figure 13-4*a* shows an arrangement of three particles, particle 1 of mass $m_1 = 6.0$ kg and particles 2 and 3 of mass $m_2 = m_3 = 4.0$ kg, and distance $a = 2.0$ cm. What is the net gravitational force $\vec{F}_{1,\text{net}}$ on particle 1 due to the other particles?

KEY IDEAS

(1) Because we have particles, the magnitude of the gravitational force on particle 1 due to either of the other particles is given by Eq. 13-1 ($F = Gm_1m_2/r^2$). (2) The direction of either gravitational force on particle 1 is toward the particle responsible for it. (3) Because the forces are not along a single axis, we *cannot* simply add or subtract their magnitudes or their components to get the net force. Instead, we must add them as vectors.

Calculations: From Eq. 13-1, the magnitude of the force $\vec{F}_{12}$ on particle 1 from particle 2 is

$$F_{12} = \frac{Gm_1m_2}{a^2} = \frac{(6.67\times10^{-11}\ \text{m}^3/\text{kg}\cdot\text{s}^2)(6.0\ \text{kg})(4.0\ \text{kg})}{(0.020\ \text{m})^2} = 4.00\times10^{-6}\ \text{N}. \quad (13\text{-}7)$$

Similarly, the magnitude of force $\vec{F}_{13}$ on particle 1 from particle 3 is

$$F_{13} = \frac{Gm_1m_3}{(2a)^2} = \frac{(6.67\times10^{-11}\ \text{m}^3/\text{kg}\cdot\text{s}^2)(6.0\ \text{kg})(4.0\ \text{kg})}{(0.040\ \text{m})^2} = 1.00\times10^{-6}\ \text{N}. \quad (13\text{-}8)$$

Force $\vec{F}_{12}$ is directed in the positive direction of the y axis (Fig. 13-4*b*) and has only the y component F_{12}. Similarly, $\vec{F}_{13}$ is directed in the negative direction of the x axis and has only the x component $-F_{13}$ (Fig. 13-4*c*). (Note something important: We draw the force diagrams with the tail of a force vector anchored on the particle experiencing the force. Drawing them in other ways invites errors, especially on exams.)

To find the net force $\vec{F}_{1,\text{net}}$ on particle 1, we must add the two forces as vectors (Figs. 13-4*d* and *e*). We can do so on a vector-capable calculator. However, here we note that $-F_{13}$ and F_{12} are actually the x and y components of $\vec{F}_{1,\text{net}}$. Therefore, we can use Eq. 3-6 to find first the magnitude and then the direction of $\vec{F}_{1,\text{net}}$. The magnitude is

$$F_{1,\text{net}} = \sqrt{(F_{12})^2 + (-F_{13})^2} = \sqrt{(4.00\times10^{-6}\ \text{N})^2 + (-1.00\times10^{-6}\ \text{N})^2} = 4.1\times10^{-6}\ \text{N}. \quad \text{(Answer)}$$

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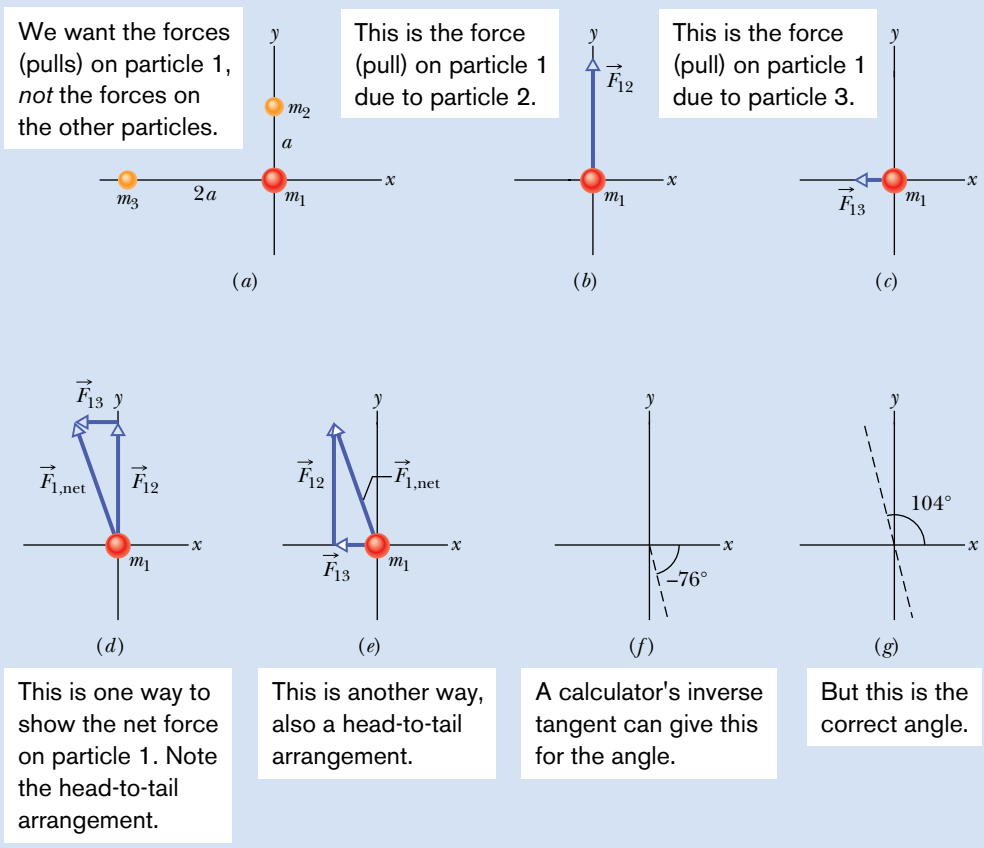
We want the forces (pulls) on particle 1, *not* the forces on the other particles.
This is the force (pull) on particle 1 due to particle 2.
This is the force (pull) on particle 1 due to particle 3.
(a)
(b)
(c)
(d)
(e)
(f)
(g)
This is one way to show the net force on particle 1. Note the head-to-tail arrangement.
This is another way, also a head-to-tail arrangement.
A calculator's inverse tangent can give this for the angle.
But this is the correct angle.
104°
−76°

Figure 13-4 (*a*) An arrangement of three particles. The force on particle 1 due to (*b*) particle 2 and (*c*) particle 3. (*d*)–(*g*) Ways to combine the forces to get the net force magnitude and orientation. In *WileyPLUS*, this figure is available as an animation with voiceover.

Relative to the positive direction of the x axis, Eq. 3-6 gives the direction of $\vec{F}_{1,\text{net}}$ as

$$\theta = \tan^{-1}\frac{F_{12}}{-F_{13}} = \tan^{-1}\frac{4.00\times10^{-6}\text{ N}}{-1.00\times10^{-6}\text{ N}} = -76°.$$

Is this a reasonable direction (Fig. 13-4*f*)? No, because the direction of $\vec{F}_{1,\text{net}}$ must be between the directions of $\vec{F}_{12}$ and $\vec{F}_{13}$. Recall from Chapter 2 that a calculator displays only one of the two possible answers to a $\tan^{-1}$ function. We find the other answer by adding 180°:

$$-76° + 180° = 104°, \qquad \text{(Answer)}$$

which *is* a reasonable direction for $\vec{F}_{1,\text{net}}$ (Fig. 13-4*g*).

SAMPLE PROBLEM 13.02

Gravitational force on one particle due to other particles

Figure 13-5*a* shows an arrangement of five particles, with masses $m_1 = 8.0$ kg, $m_2 = m_3 = m_4 = m_5 = 2.0$ kg, and with $a = 2.0$ cm and $\theta = 30°$. What is the net gravitational force $\vec{F}_{1,\text{net}}$ on particle 1 due to the other particles?

KEY IDEA

Because we have particles, the magnitude of the gravitational force on particle 1 due to either of the other

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particles is given by Eq. 13-1 ($F = Gm_1m_2/r^2$). The direction of a gravitational force on particle 1 is toward the particle responsible for the force. We can use symmetry to eliminate unneeded calculations.

Calculations: For the magnitudes of the forces on particle 1, first note that particles 2 and 4 have equal masses and equal distances of $r = 2a$ from particle 1. Thus, from Eq. 13-1, we find that

$$F_{12} = F_{14} = \frac{Gm_1m_3}{(2a)^2}. \qquad (13\text{-}9)$$

Similarly, since particles 3 and 5 have equal masses and are both at distance $r = a$ from particle 1, we find

$$F_{13} = F_{15} = \frac{Gm_1m_3}{a^2}. \qquad (13\text{-}10)$$

We could now substitute known data into these two equations to evaluate the magnitudes of the forces, indicate the directions of the forces on the free-body diagram of Fig. 13-5*b*, and then find the net force either (1) by resolving the vectors into x and y components, finding the net x and net y components, and then vectorially combining them or (2) by adding the vectors directly on a vector-capable calculator.

Instead, however, we shall make further use of the symmetry of the problem. First, we note that $\vec{F}_{12}$ and $\vec{F}_{14}$ are equal in magnitude but opposite in direction; thus, those forces *cancel*. Inspection of Fig. 13-5*b* and Eq. 13-10 reveals that the x components of $\vec{F}_{13}$ and $\vec{F}_{15}$ also *cancel*, and that their y components are identical in magnitude and both act in the positive direction of the y axis.

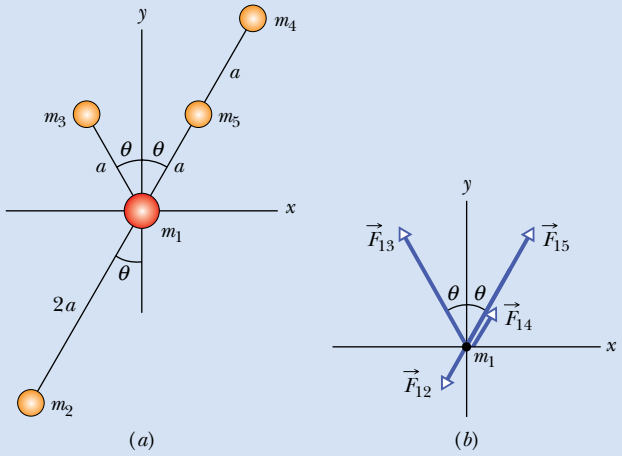

Figure 13-5 (*a*) An arrangement of five particles. (*b*) The forces acting on the particle of mass m_1 due to the other four particles.

Thus, $\vec{F}_{1,\text{net}}$ acts in that same direction, and its magnitude is twice the y component of $\vec{F}_{13}$:

$$F_{1,\text{net}} = 2F_{13}\cos\theta = 2\frac{Gm_1m_3}{a^2}\cos\theta$$
$$= \frac{2(6.67\times10^{-11}\ \text{m}^3/\text{kg}\cdot\text{s}^2)(8.0\ \text{kg})(2.0\ \text{kg})}{(0.020\ \text{m})^2}\cos 30°$$
$$= 4.6\times10^{-6}\ \text{N}. \qquad \text{(Answer)}$$

Note: The presence of particle 5 along the line between particles 1 and 4 does not alter the gravitational force on particle 1 from particle 4.

PROBLEM-SOLVING TACTICS

Tactic 1: Drawing Gravitational Force Vectors When you are given a diagram of particles, such as Fig. 13-4*a*, and asked to find the net gravitational force on one of them, you should usually draw a free-body diagram showing only the particle of concern and the forces on *it alone*, as in Fig. 13-4*b*. If, instead, you choose to superimpose the force vectors on the given diagram, be sure to draw the vectors with either all tails (preferably) or all heads on the particle experiencing those forces. If you draw the vectors elsewhere, you invite confusion—and confusion is guaranteed if you draw the vectors on the particles *causing* the forces.

Tactic 2: Simplifying a Sum of Forces with Symmetry In Sample Problem 13.02, we used the symmetry of the situation: By realizing that particles 2 and 4 are positioned symmetrically about particle 1, and thus that $\vec{F}_{12}$ and $\vec{F}_{14}$ cancel, we avoided calculating either force. By realizing that the x components of $\vec{F}_{13}$ and $\vec{F}_{15}$ cancel and that their y components are identical and add, we saved even more effort. In problems with symmetry, you can save much effort and reduce the chance of error by identifying which calculations are not needed because of the symmetry. Such identification is a skill acquired only by doing many problems.

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13.4 | GRAVITATION NEAR EARTH'S SURFACE

Key Concepts

- The gravitational acceleration a_g of a particle (of mass m) is due solely to the gravitational force acting on it. When the particle is at distance r from the center of a uniform, spherical body of mass M, the magnitude F of the gravitational force on the particle is given by Eq. 13-1. Thus, by Newton's second law,

$$F = ma_g,$$

which gives

$$a_g = \frac{GM}{r^2}.$$

- Because Earth's mass is not distributed uniformly, because the planet is not perfectly spherical, and because it rotates, the actual free-fall acceleration $\vec{g}$ of a particle near Earth differs slightly from the gravitational acceleration $\vec{a}_g$, and the particle's weight (equal to mg) differs from the magnitude of the gravitational force on it.

Let us assume that Earth is a uniform sphere of mass M. The magnitude of the gravitational force from Earth on a particle of mass m, located outside Earth a distance r from Earth's center, is then given by Eq. 13-1 as

$$F = G\frac{Mm}{r^2}. \tag{13-11}$$

If the particle is released, it will fall toward the center of Earth, as a result of the gravitational force $\vec{F}$, with an acceleration we shall call the **gravitational acceleration** $\vec{a}_g$. Newton's second law tells us that magnitudes F and a_g are related by

$$F = ma_g. \tag{13-12}$$

Now, substituting F from Eq. 13-11 into Eq. 13-12 and solving for a_g, we find

$$a_g = \frac{GM}{r^2}. \tag{13-13}$$

Table 13-1 Variation of a_g with Altitude

Altitude (km)	a_g (m/s²)	Altitude Example
0	9.83	Mean Earth surface
8.8	9.80	Mt. Everest
36.6	9.71	Highest crewed balloon
400	8.70	Space shuttle orbit
35 700	0.225	Communications satellite

Table 13-1 shows values of a_g computed for various altitudes above Earth's surface. Notice that a_g is significant even at 400 km.

Since Sections 5.1 to 5.6, we have assumed that Earth is an inertial frame by neglecting its rotation. This simplification has allowed us to assume that the free-fall acceleration g of a particle is the same as the particle's gravitational acceleration (which we now call a_g). Furthermore, we assumed that g has the constant value 9.8 m/s² any place on Earth's surface. However, any g value measured at a given location will differ from the a_g value calculated with Eq. 13-13 for that location for three reasons: (1) Earth's mass is not distributed uniformly, (2) Earth is not a perfect sphere, and (3) Earth rotates. Moreover, because g differs from a_g, the same three reasons mean that the measured weight mg of a particle differs from the magnitude of the gravitational force on the particle as given by Eq. 13-11. Let us now examine those reasons.

1. ***Earth's mass is not uniformly distributed.*** The density (mass per unit volume) of Earth varies radially as shown in Fig. 13-6, and the density of the crust (outer section) varies from region to region over Earth's surface. Thus, g varies from region to region over the surface.
2. ***Earth is not a sphere.*** Earth is approximately an ellipsoid, flattened at the poles and bulging at the equator. Its equatorial radius (from its center point out to the equator) is greater than its polar radius (from its center point out to either north or south pole) by 21 km. Thus, a point at the poles is closer to the dense core of Earth than is a point on the equator. This is one reason the free-fall acceleration g increases if you were to measure it while

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moving at sea level from the equator toward the north or south pole. As you move, you are actually getting closer to the center of Earth and thus, by Newton's law of gravitation, g increases.

3. ***Earth is rotating.*** The rotation axis runs through the north and south poles of Earth. An object located on Earth's surface anywhere except at those poles must rotate in a circle about the rotation axis and thus must have a centripetal acceleration directed toward the center of the circle. This centripetal acceleration requires a centripetal net force that is also directed toward that center.

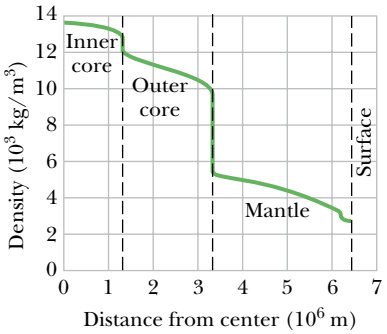

Figure 13-6 The density of Earth as a function of distance from the center. The limits of the solid inner core, the largely liquid outer core, and the solid mantle are shown, but the crust of Earth is too thin to show clearly on this plot.

To see how Earth's rotation causes g to differ from a_g, let us analyze a simple situation in which a crate of mass m is on a scale at the equator. Figure 13-7*a* shows this situation as viewed from a point in space above the north pole.

Figure 13-7*b*, a free-body diagram for the crate, shows the two forces on the crate, both acting along a radial r axis that extends from Earth's center. The normal force $\vec{F}_N$ on the crate from the scale is directed outward, in the positive direction of the r axis. The gravitational force, represented with its equivalent $m\vec{a}_g$, is directed inward. Because it travels in a circle about the center of Earth as Earth turns, the crate has a centripetal acceleration $\vec{a}$ directed toward Earth's center. From Eq. 10-23 ($a_r = \omega^2 r$), we know this acceleration is equal to $\omega^2 R$, where ω is Earth's angular speed and R is the circle's radius (approximately Earth's radius). Thus, we can write Newton's second law for forces along the r axis ($F_{\text{net},r} = ma_r$) as

$$F_N - ma_g = m(-\omega^2 R). \tag{13-14}$$

The magnitude F_N of the normal force is equal to the weight mg read on the scale. With mg substituted for F_N, Eq. 13-14 gives us

$$mg = ma_g - m(\omega^2 R), \tag{13-15}$$

which says

$$\begin{pmatrix}\text{measured}\\ \text{weight}\end{pmatrix} = \begin{pmatrix}\text{magnitude of}\\ \text{gravitational force}\end{pmatrix} - \begin{pmatrix}\text{mass times}\\ \text{centripetal acceleration}\end{pmatrix}.$$

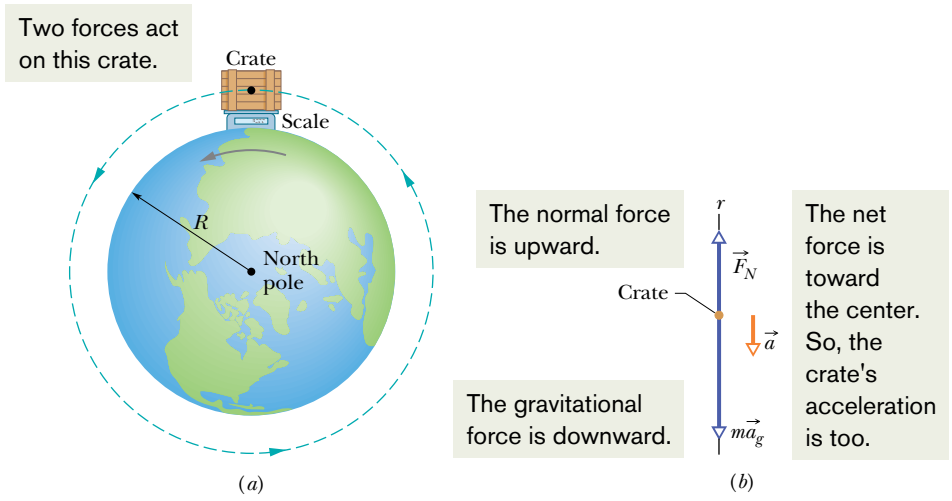

Figure 13-7 (*a*) A crate sitting on a scale at Earth's equator, as seen by an observer positioned on Earth's rotation axis at some point above the north pole. (*b*) A free-body diagram for the crate, with a radial r axis extending from Earth's center. The gravitational force on the crate is represented with its equivalent $m\vec{a}_g$. The normal force on the crate from the scale is $\vec{F}_N$. Because of Earth's rotation, the crate has a centripetal acceleration $\vec{a}$ that is directed toward Earth's center.

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Thus, the measured weight is less than the magnitude of the gravitational force on the crate, because of Earth's rotation.

Acceleration Difference. To find a corresponding expression for g and a_g, we cancel m from Eq. 13-15 to write

$$g = a_g - \omega^2 R, \tag{13-16}$$

which says

$$\begin{pmatrix}\text{free-fall}\\ \text{acceleration}\end{pmatrix} = \begin{pmatrix}\text{gravitational}\\ \text{acceleration}\end{pmatrix} - \begin{pmatrix}\text{centripetal}\\ \text{acceleration}\end{pmatrix}.$$

Thus, the measured free-fall acceleration is less than the gravitational acceleration because of Earth's rotation.

Equator. The difference between accelerations g and a_g is equal to $\omega^2 R$ and is greatest on the equator (for one reason, the radius of the circle traveled by the crate is greatest there). To find the difference, we can use Eq. 10-5 ($\omega = \Delta\theta/\Delta t$) and Earth's radius $R = 6.37 \times 10^6$ m. For one rotation of Earth, θ is 2π rad and the time period Δt is about 24 h. Using these values (and converting hours to seconds), we find that g is less than a_g by only about 0.034 m/s^2 (small compared to 9.8 m/s^2). Therefore, neglecting the difference in accelerations g and a_g is often justified. Similarly, neglecting the difference between weight and the magnitude of the gravitational force is also often justified.

SAMPLE PROBLEM 13.03

Difference in acceleration at head and feet

(a) An astronaut whose height h is 1.70 m floats "feet down" in an orbiting space shuttle at distance $r = 6.77 \times 10^6$ m away from the center of Earth. What is the difference between the gravitational acceleration at her feet and at her head?

KEY IDEAS

We can approximate Earth as a uniform sphere of mass M_E. Then, from Eq. 13-13, the gravitational acceleration at any distance r from the center of Earth is

$$a_g = \frac{GM_E}{r^2}. \tag{13-17}$$

We might simply apply this equation twice, first with $r = 6.77 \times 10^6$ m for the location of the feet and then with $r = 6.77 \times 10^6$ m + 1.70 m for the location of the head. However, a calculator may give us the same value for a_g twice, and thus a difference of zero, because h is so much smaller than r. Here's a more promising approach: Because we have a differential change dr in r between the astronaut's feet and head, we should differentiate Eq. 13-17 with respect to r.

Calculations: The differentiation gives us

$$da_g = -2\frac{GM_E}{r^3}dr, \tag{13-18}$$

where da_g is the differential change in the gravitational acceleration due to the differential change dr in r. For the astronaut, $dr = h$ and $r = 6.77 \times 10^6$ m. Substituting data into Eq. 13-18, we find

$$da_g = -2\frac{(6.67\times10^{-11}\,\text{m}^3/\text{kg}\cdot\text{s}^2)(5.98\times10^{24}\ \text{kg})}{(6.77\times10^6\ \text{m})^3}(1.70\ \text{m})$$
$$= -4.37\times10^{-6}\ \text{m/s}^2, \qquad \text{(Answer)}$$

where the M_E value is taken from Appendix C. This result means that the gravitational acceleration of the astronaut's feet toward Earth is slightly greater than the gravitational acceleration of her head toward Earth. This difference in acceleration (often called a *tidal effect*) tends to stretch her body, but the difference is so small that she would never even sense the stretching, much less suffer pain from it.

(b) If the astronaut is now "feet down" at the same orbital radius $r = 6.77 \times 10^6$ m about a black hole of mass $M_h = 1.99 \times 10^{31}$ kg (10 times our Sun's mass), what is the difference between the gravitational acceleration at her feet and at her head? The black hole has a mathematical surface (*event horizon*) of radius $R_h = 2.95 \times 10^4$ m. Nothing, not even light, can escape from that surface or anywhere inside it. Note that the astronaut is well outside the surface (at $r = 229R_h$).

Calculations: We again have a differential change dr in r between the astronaut's feet and head, so we

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can again use Eq. 13-18. However, now we substitute $M_h = 1.99 \times 10^{31}$ kg for M_E. We find

$$da_g = -2\frac{(6.67\times10^{-11}\,\text{m}^3/\text{kg}\cdot\text{s}^2)(1.99\times10^{31}\ \text{kg})}{(6.77\times10^{6}\ \text{m})^3}(1.70\ \text{m})$$
$$= -14.5\ \text{m/s}^2. \qquad \text{(Answer)}$$

This means that the gravitational acceleration of the astronaut's feet toward the black hole is noticeably larger than that of her head. The resulting tendency to stretch her body would be bearable but quite painful. If she drifted closer to the black hole, the stretching tendency would increase drastically.

13.5 | GRAVITATION INSIDE EARTH

Key Concepts

- A uniform shell of matter exerts no *net* gravitational force on a particle located inside it.
- The gravitational force $\vec{F}$ on a particle inside a uniform solid sphere, at a distance r from the center, is due only to mass M_{ins} in an "inside sphere" with that radius r:

$$M_{\text{ins}} = \frac{4}{3}\pi r^3\rho = \frac{M}{R^3}r^3,$$

where ρ is the solid sphere's density, R is its radius, and M is its mass. We can assign this inside mass to be that of a particle at the center of the solid sphere and then apply Newton's law of gravitation for particles. We find that the magnitude of the force acting on mass m is

$$F = \frac{GmM}{R^3}r.$$

Newton's shell theorem can also be applied to a situation in which a particle is located *inside* a uniform shell, to show the following:

A uniform shell of matter exerts no net gravitational force on a particle located inside it.

Caution: This statement does *not* mean that the gravitational forces on the particle from the various elements of the shell magically disappear. Rather, it means that the *sum* of the force vectors on the particle from all the elements is zero.

If Earth's mass were uniformly distributed, the gravitational force acting on a particle would be a maximum at Earth's surface and would decrease as the particle moved outward, away from the planet. If the particle were to move inward, perhaps down a deep mine shaft, the gravitational force would change for two reasons. (1) It would tend to increase because the particle would be moving closer to the center of Earth. (2) It would tend to decrease because the thickening shell of material lying outside the particle's radial position would not exert any net force on the particle.

To find an expression for the gravitational force inside a uniform Earth, let's use the plot in *Pole to Pole*, an early science fiction story by George Griffith. Three explorers attempt to travel by capsule through a naturally formed (and, of course, fictional) tunnel directly from the south pole to the north pole. Figure 13-8 shows the capsule (mass m) when it has fallen to a distance r from Earth's center. At that moment, the *net* gravitational force on the capsule is due to the mass M_{ins} inside the sphere with radius r (the mass enclosed by the dashed outline), not the mass in the outer spherical shell (outside the dashed outline). Moreover, we can assume that the inside mass M_{ins} is concentrated as a particle at Earth's center. Thus, we can write Eq. 13-1, for the magnitude of the gravitational force on the capsule, as

$$F = \frac{GmM_{\text{ins}}}{r^2}. \qquad (13\text{-}19)$$

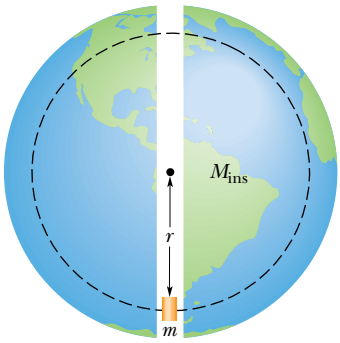

Figure 13-8 A capsule of mass m falls from rest through a tunnel that connects Earth's south and north poles. When the capsule is at distance r from Earth's center, the portion of Earth's mass that is contained in a sphere of that radius is M_{ins}.

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Because we assume a uniform density ρ, we can write this inside mass in terms of Earth's total mass M and its radius R:

$$\text{density} = \frac{\text{inside mass}}{\text{inside volume}} = \frac{\text{total mass}}{\text{total volume}},$$

$$\rho = \frac{M_{\text{ins}}}{\frac{4}{3}\pi r^3} = \frac{M}{\frac{4}{3}\pi R^3}.$$

Solving for M_{ins} we find

$$M_{\text{ins}} = \frac{4}{3}\pi r^3 \rho = \frac{M}{R^3} r^3. \tag{13-20}$$

Substituting the second expression for M_{ins} into Eq. 13-19 gives us the magnitude of the gravitational force on the capsule as a function of the capsule's distance r from Earth's center:

$$F = \frac{GmM}{R^3} r. \tag{13-21}$$

According to Griffith's story, as the capsule approaches Earth's center, the gravitational force on the explorers becomes alarmingly large and, exactly at the center, it suddenly but only momentarily disappears. From Eq. 13-21 we see that, in fact, the force magnitude decreases linearly as the capsule approaches the center, until it is zero at the center. At least Griffith got the zero-at-the-center detail correct.

Equation 13-21 can also be written in terms of the force vector $\vec{F}$ and the capsule's position vector $\vec{r}$ along a radial axis extending from Earth's center. Letting K represent the collection of constants in Eq. 13-21, we can rewrite the force in vector form as

$$\vec{F} = -K\vec{r}, \tag{13-22}$$

in which we have inserted a minus sign to indicate that $\vec{F}$ and $\vec{r}$ have opposite directions. Equation 13-22 has the form of Hooke's law (Eq. 8-20, $\vec{F} = -k\vec{d}$). Thus, under the idealized conditions of the story, the capsule would oscillate like a block on a spring, with the center of the oscillation at Earth's center. After the capsule had fallen from the south pole to Earth's center, it would travel from the center to the north pole (as Griffith said) and then back again, repeating the cycle forever.

For the real Earth, which certainly has a nonuniform distribution of mass (Fig. 13-6), the force on the capsule would initially *increase* as the capsule descends. The force would then reach a maximum at a certain depth, and only then would it begin to decrease as the capsule further descends.

13.6 | GRAVITATIONAL POTENTIAL ENERGY

Key Concepts

- The gravitational potential energy $U(r)$ of a system of two particles, with masses M and m and separated by a distance r, is the negative of the work that would be done by the gravitational force of either particle acting on the other if the separation between the particles were changed from infinite (very large) to r. This energy is

$$U = -\frac{GMm}{r} \quad \text{(gravitational potential energy)}.$$

- If a system contains more than two particles, its total gravitational potential energy U is the sum of the terms representing the potential energies of all the pairs. As an example, for three particles, of masses m_1, m_2 and m_3,

$$U = -\left(\frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}}\right).$$

- An object will escape the gravitational pull of an astronomical body of mass M and radius R (that is, it will reach an infinite distance) if the object's speed near the body's surface is at least equal to the escape speed, given by

$$v = \sqrt{\frac{2GM}{R}}.$$

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In Section 8.9, we discussed the gravitational potential energy of a particle–Earth system. We were careful to keep the particle near Earth's surface, so that we could regard the gravitational force as constant. We then chose some reference configuration of the system as having a gravitational potential energy of zero. Often, in this configuration the particle was on Earth's surface. For particles not on Earth's surface, the gravitational potential energy decreased when the separation between the particle and Earth decreased.

Here, we broaden our view and consider the gravitational potential energy U of two particles, of masses m and M, separated by a distance r. We again choose a reference configuration with U equal to zero. However, to simplify the equations, the separation distance r in the reference configuration is now large enough to be approximated as *infinite*. As before, the gravitational potential energy decreases when the separation decreases. Since $U = 0$ for $r = \infty$, the potential energy is negative for any finite separation and becomes progressively more negative as the particles move closer together.

With these facts in mind and as we shall justify next, we take the gravitational potential energy of the two-particle system to be

$$U = -\frac{GMm}{r} \quad \text{(gravitational potential energy)}. \tag{13-23}$$

Note that $U(r)$ approaches zero as r approaches infinity and that for any finite value of r, the value of $U(r)$ is negative.

Language. The potential energy given by Eq. 13-23 is a property of the system of two particles rather than of either particle alone. There is no way to divide this energy and say that so much belongs to one particle and so much to the other. However, if $M \gg m$, as is true for Earth (mass M) and a baseball (mass m), we often speak of "the potential energy of the baseball." We can get away with this because, when a baseball moves in the vicinity of Earth, changes in the potential energy of the baseball–Earth system appear almost entirely as changes in the kinetic energy of the baseball, since changes in the kinetic energy of Earth are too small to be measured. Similarly, in Section 13.8 we shall speak of "the potential energy of an artificial satellite" orbiting Earth, because the satellite's mass is so much smaller than Earth's mass. When we speak of the potential energy of bodies of comparable mass, however, we have to be careful to treat them as a system.

Multiple Particles. If our system contains more than two particles, we consider each pair of particles in turn, calculate the gravitational potential energy of that pair with Eq. 13-23 as if the other particles were not there, and then algebraically sum the results. Applying Eq. 13-23 to each of the three pairs of Fig. 13-9, for example, gives the potential energy of the system as

$$U = -\left(\frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}}\right). \tag{13-24}$$

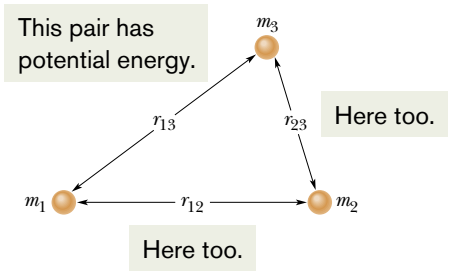

Figure 13-9 A system consisting of three particles. The gravitational potential energy *of the system* is the sum of the gravitational potential energies of all three pairs of particles.

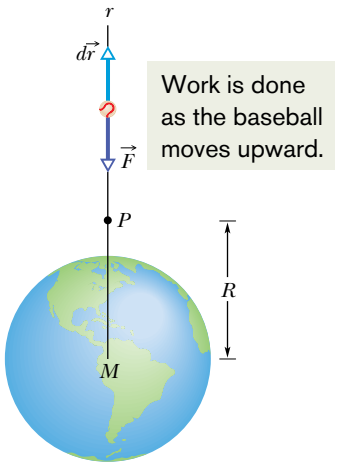

Figure 13-10 A baseball is shot directly away from Earth, through point P at radial distance R from Earth's center. The gravitational force $\vec{F}$ on the ball and a differential displacement vector $d\vec{r}$ are shown, both directed along a radial r axis.

Proof of Eq. 13-23

Let us shoot a baseball directly away from Earth along the path in Fig. 13-10. We want to find an expression for the gravitational potential energy U of the ball at point P along its path, at radial distance R from Earth's center. To do so, we first find the work W done on the ball by the gravitational force as the ball travels

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from point P to a great (infinite) distance from Earth. Because the gravitational force $\vec{F}(r)$ is a variable force (its magnitude depends on r), we must use the techniques of Section 8.7 to find the work. In vector notation, we can write

$$W = \int_R^\infty \vec{F}(r) \cdot d\vec{r}. \tag{13-25}$$

The integral contains the scalar (or dot) product of the force $\vec{F}(r)$ and the differential displacement vector $d\vec{r}$ along the ball's path. We can expand that product as

$$\vec{F}(r) \cdot d\vec{r} = F(r)dr\cos\phi, \tag{13-26}$$

where ϕ is the angle between the directions of $\vec{F}(r)$ and $d\vec{r}$. When we substitute 180° for ϕ and Eq. 13-1 for $F(r)$. Eq. 13-26 becomes

$$\vec{F}(r) \cdot d\vec{r} = -\frac{GMm}{r^2}dr,$$

where M is Earth's mass and m is the mass of the ball.

Substituting this into Eq. 13-25 and integrating give us

$$W = -GMm\int_R^\infty \frac{1}{r^2}dr = \left[\frac{GMm}{r}\right]_R^\infty$$
$$= 0 - \frac{GMm}{R} = -\frac{GMm}{R}, \tag{13-27}$$

where W is the work required to move the ball from point P (at distance R) to infinity. Equation 8-42 ($\Delta U = -W$) tells us that we can also write that work in terms of potential energies as

$$U_\infty - U = -W.$$

Because the potential energy U_∞ at infinity is zero, U is the potential energy at P, and W is given by Eq. 13-27, this equation becomes

$$U = W = -\frac{GMm}{R}.$$

Switching R to r gives us Eq. 13-23, which we set out to prove.

Path Independence

In Fig. 13-11, we move a baseball from point A to point G along a path consisting of three radial lengths and three circular arcs (centered on Earth). We are interested in the total work W done by Earth's gravitational force $\vec{F}$ on the ball as it moves from A to G. The work done along each circular arc is zero, because the direction of $\vec{F}$ is perpendicular to the arc at every point. Thus, W is the sum of only the works done by $\vec{F}$ along the three radial lengths.

Now, suppose we mentally shrink the arcs to zero. We would then be moving the ball directly from A to G along a single radial length. Does that change W? No. Because no work was done along the arcs, eliminating them does not change the work. The path taken from A to G now is clearly different, but the work done by $\vec{F}$ is the same.

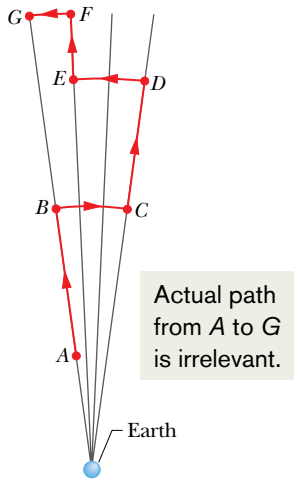

Figure 13-11 Near Earth, a baseball is moved from point A to point G along a path consisting of radial lengths and circular arcs.

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We discussed such a result in a general way in Section 8.9. Here is the point: The gravitational force is a conservative force. Thus, the work done by the gravitational force on a particle moving from an initial point i to a final point f is independent of the path taken between the points. From Eq. 8-42, the change ΔU in the gravitational potential energy from point i to point f is given by

$$\Delta U = U_f - U_i = -W. \tag{13-28}$$

Since the work W done by a conservative force is independent of the actual path taken, the change ΔU in gravitational potential energy is *also independent* of the path taken.

Potential Energy and Force

In the proof of Eq. 13-23, we derived the potential energy function $U(r)$ from the force function $\vec{F}(r)$. We should be able to go the other way—that is, to start from the potential energy function and derive the force function. Guided by Eq. 8-90 ($F(x) = -dU(x)/dx$), we can write

$$\begin{aligned} F &= -\frac{dU}{dr} = -\frac{d}{dr}\left(-\frac{GMm}{r}\right) \\ &= -\frac{GMm}{r^2}. \end{aligned} \tag{13-29}$$

This is Newton's law of gravitation (Eq. 13-1). The minus sign indicates that the force on mass m points radially inward, toward mass M.

Escape Speed

If you fire a projectile upward, usually it will slow, stop momentarily, and return to Earth. There is, however, a certain minimum initial speed that will cause it to move upward forever, theoretically coming to rest only at infinity. This minimum initial speed is called the (Earth) **escape speed**.

Consider a projectile of mass m, leaving the surface of a planet (or some other astronomical body or system) with escape speed v. The projectile has a kinetic energy K given by $1/2mv^2$ and a potential energy U given by Eq. 13-23:

$$U = -\frac{GMm}{R},$$

in which M is the mass of the planet and R is its radius.

When the projectile reaches infinity, it stops and thus has no kinetic energy. It also has no potential energy because an infinite separation between two bodies is our zero-potential-energy configuration. Its total energy at infinity is therefore zero. From the principle of conservation of energy, its total energy at the planet's surface must also have been zero, and so

$$K + U = \frac{1}{2}mv^2 + \left(-\frac{GMm}{R}\right) = 0.$$

This yields

$$v = \sqrt{\frac{2GM}{R}}. \tag{13-30}$$

Note that v does not depend on the direction in which a projectile is fired from a planet. However, attaining that speed is easier if the projectile is fired in the direction the launch site is moving as the planet rotates about its axis. For example, rockets are launched eastward at Cape Canaveral to take advantage of the Cape's eastward speed of 1500 km/h due to Earth's rotation.

Equation 13-30 can be applied to find the escape speed of a projectile from any astronomical body, provided we substitute the mass of the body for M and the radius of the body for R. Table 13-2 shows some escape speeds.

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Table 13-2 Some Escape Speeds

Body	Mass (kg)	Radius (m)	Escape Speed (km/s)
Ceres[a]	1.17×10^{21}	3.8×10^{5}	0.64
Earth's moon[a]	7.36×10^{22}	1.74×10^{6}	2.38
Earth	5.98×10^{24}	6.37×10^{6}	11.2
Jupiter	1.90×10^{27}	7.15×10^{7}	59.5
Sun	1.99×10^{30}	6.96×10^{8}	618
Sirius B[b]	2×10^{30}	1×10^{7}	5200
Neutron star[c]	2×10^{30}	1×10^{4}	2×10^{5}

[a]The most massive of the asteroids.
[b]A *white dwarf* (a star in a final stage of evolution) that is a companion of the bright star Sirius.
[c]The collapsed core of a star that remains after that star has exploded in a *supernova* event.

CHECKPOINT 3

You move a ball of mass m away from a sphere of mass M. (a) Does the gravitational potential energy of the system of ball and sphere increase or decrease? (b) Is positive work or negative work done by the gravitational force between the ball and the sphere?

SAMPLE PROBLEM 13.04

Asteroid falling from space, mechanical energy

An asteroid, headed directly toward Earth, has a speed of 12 km/s relative to the planet when the asteroid is 10 Earth radii from Earth's center. Neglecting the effects of Earth's atmosphere on the asteroid, find the asteroid's speed v_f when it reaches Earth's surface.

KEY IDEAS

Because we are to neglect the effects of the atmosphere on the asteroid, the mechanical energy of the asteroid–Earth system is conserved during the fall. Thus, the final mechanical energy (when the asteroid reaches Earth's surface) is equal to the initial mechanical energy. With kinetic energy K and gravitational potential energy U, we can write this as

$$K_f + U_f = K_i + U_i. \tag{13-31}$$

Also, if we assume the system is isolated, the system's linear momentum must be conserved during the fall. Therefore, the momentum change of the asteroid and that of Earth must be equal in magnitude and opposite in sign. However, because Earth's mass is so much greater than the asteroid's mass, the change in Earth's speed is negligible relative to the change in the asteroid's speed. So, the change in Earth's kinetic energy is also negligible. Thus, we can assume that the kinetic energies in Eq. 13-31 are those of the asteroid alone.

Calculations: Let m represent the asteroid's mass and M represent Earth's mass (5.98×10^{24} kg). The asteroid is initially at distance $10R_E$ and finally at distance R_E, where R_E, is Earth's radius (6.37×10^{6} m). Substituting Eq. 13-23 for U and $1/2mv^2$ for K, we rewrite Eq. 13-31 as

$$\frac{1}{2}mv_f^2 - \frac{GMm}{R_E} = \frac{1}{2}mv_i^2 - \frac{GMm}{10R_E}.$$

Rearranging and substituting known values, we find

$$\begin{aligned} v_f^2 &= v_i^2 + \frac{2GM}{R_E}\left(1 - \frac{1}{10}\right) \\ &= (12 \times 10^3 \text{ m/s})^2 \\ &\quad + \frac{2(6.67 \times 10^{-11} \text{ m}^3/\text{kg} \cdot \text{s}^2)(5.98 \times 10^{24} \text{ kg})}{6.37 \times 10^6 \text{ m}} 0.9 \\ &= 2.567 \times 10^8 \text{ m}^2/\text{s}^2, \end{aligned}$$

and $$v_f = 1.60 \times 10^4 \text{ m/s} = 16 \text{ km/s}. \qquad \text{(Answer)}$$

At this speed, the asteroid would not have to be particularly large to do considerable damage at impact. If it were only 5 m across, the impact could release about as much energy as the nuclear explosion at Hiroshima.

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13.7 | PLANETS AND SATELLITES: KEPLER'S LAWS

Key Concepts

- The motion of satellites, both natural and artificial, is governed by Kepler's laws:

1. *The law of orbits*. All planets move in elliptical orbits with the Sun at one focus.
2. *The law of areas*. A line joining any planet to the Sun sweeps out equal areas in equal time intervals. (This statement is equivalent to conservation of angular momentum.)
3. *The law of periods*. The square of the period T of any planet is proportional to the cube of the semimajor axis a of its orbit. For circular orbits with radius r,

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3 \quad \text{(law of periods)},$$

where M is the mass of the attracting body—the Sun in the case of the solar system. For elliptical planetary orbits, the semimajor axis a is substituted for r.

Planets and Satellites: Kepler's Laws

The motions of the planets, as they seemingly wander against the background of the stars, have been a puzzle since the dawn of history. The "loop-the-loop" motion of Mars, shown in Fig. 13-12, was particularly baffling. Johannes Kepler (1571–1630), after a lifetime of study, worked out the empirical laws that govern these motions. Tycho Brahe (1546–1601), the last of the great astronomers to make observations without the help of a telescope, compiled the extensive data from which Kepler was able to derive the three laws of planetary motion that now bear Kepler's name. Later, Newton (1642–1727) showed that his law of gravitation leads to Kepler's laws.

In this section we discuss each of Kepler's three laws. Although here we apply the laws to planets orbiting the Sun, they hold equally well for satellites, either natural or artificial, orbiting Earth or any other massive central body.

1. THE LAW OF ORBITS: All planets move in elliptical orbits, with the Sun at one focus.

Figure 13-13 shows a planet of mass m moving in such an orbit around the Sun, whose mass is M. We assume that $M \gg m$, so that the center of mass of the planet–Sun system is approximately at the center of the Sun.

The orbit in Fig. 13-13 is described by giving its **semimajor axis** a and its **eccentricity** e, the latter defined so that ea is the distance from the center of the ellipse to either focus F or F'. *An eccentricity of zero corresponds to a circle*, in which the two foci merge to a single central point. The eccentricities of the planetary orbits are not large; so if the orbits are drawn to scale, they look circular. The eccentricity of the ellipse of Fig. 13-13, which has been exaggerated for clarity, is 0.74. The eccentricity of Earth's orbit is only 0.0167.

2. THE LAW OF AREAS: A line that connects a planet to the Sun sweeps out equal areas in the plane of the planet's orbit in equal time intervals; that is, the rate dA/dt at which it sweeps out area A is constant.

Qualitatively, this second law tells us that the planet will move most slowly when it is farthest from the Sun and most rapidly when it is nearest to the Sun. As it turns out, Kepler's second law is totally equivalent to the law of conservation of angular momentum. Let us prove it.

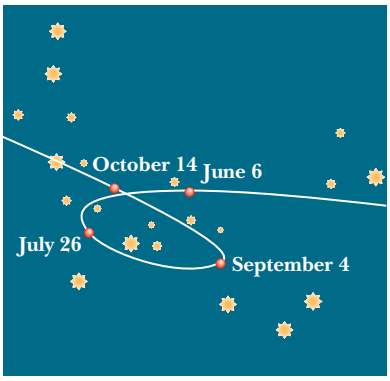

Figure 13-12 The path seen from Earth for the planet Mars as it moved against a background of the constellation Capricorn during 1971. The planet's position on four days is marked. Both Mars and Earth are moving in orbits around the Sun so that we see the position of Mars relative to us; this relative motion sometimes results in an apparent loop in the path of Mars.

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The area of the shaded wedge in Fig. 13-14*a* closely approximates the area swept out in time Δt by a line connecting the Sun and the planet, which are separated by distance r. The area ΔA of the wedge is approximately the area of a triangle with base $r\Delta\theta$ and height r. Since the area of a triangle is one-half of the base times the height, $\Delta A \approx 1/2 r^2 \Delta\theta$. This expression for ΔA becomes more exact as Δt (hence $\Delta\theta$) approaches zero. The instantaneous rate at which area is being swept out is then

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \omega, \tag{13-32}$$

in which ω is the angular speed of the line connecting Sun and planet, as the line rotates around the Sun.

Figure 13-14*b* shows the linear momentum $\vec{p}$ of the planet, along with the radial and perpendicular components of $\vec{p}$. From Eq. 10-89 ($L = rp_\perp$), the magnitude of the angular momentum $\vec{L}$ of the planet about the Sun is given by the product of r and $p_\perp$, the component of $\vec{p}$ perpendicular to r. Here, for a planet of mass m,

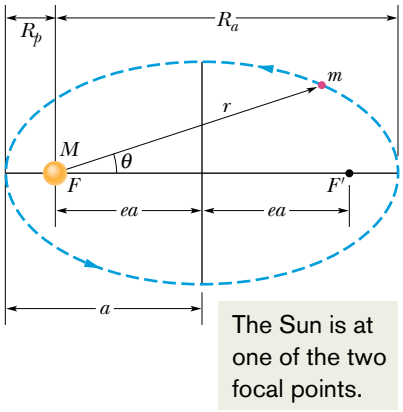

Figure 13-13 A planet of mass m moving in an elliptical orbit around the Sun. The Sun, of mass M, is at one focus F of the ellipse. The other focus is F', which is located in empty space. The semimajor axis a of the ellipse, the perihelion (nearest the Sun) distance R_p, and the aphelion (farthest from the Sun) distance R_a are also shown.

$$\begin{aligned} L &= rp_\perp = (r)(mv_\perp) = (r)(m\omega r) \\ &= mr^2\omega, \end{aligned} \tag{13-33}$$

where we have replaced $v_\perp$ with its equivalent ωr (Eq. 10-18). Eliminating $r^2\omega$ between Eqs. 13-32 and 13-33 leads to

$$\frac{dA}{dt} = \frac{L}{2m}. \tag{13-34}$$

If dA/dt is constant, as Kepler said it is, then Eq. 13-34 means that L must also be constant—angular momentum is conserved. Kepler's second law is indeed equivalent to the law of conservation of angular momentum.

3. THE LAW OF PERIODS: The square of the period of any planet is proportional to the cube of the semimajor axis of its orbit.

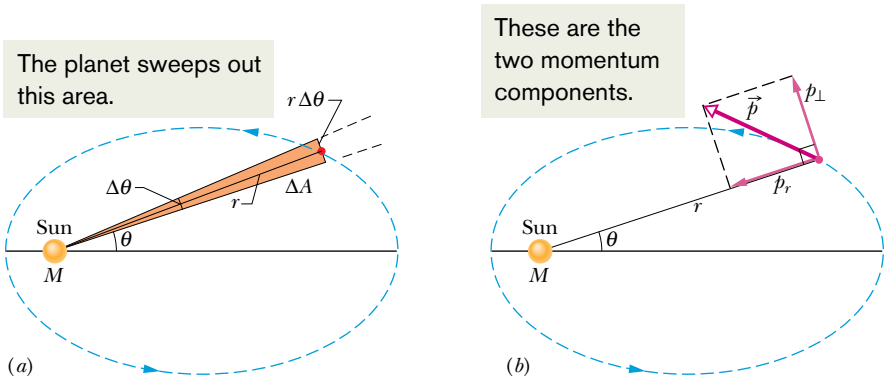

Figure 13-14 (*a*) In time Δt, the line r connecting the planet to the Sun moves through an angle $\Delta\theta$, sweeping out an area ΔA (shaded). (*b*) The linear momentum $\vec{p}$ of the planet and the components of $\vec{p}$.

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To see this, consider the circular orbit of Fig. 13-15, with radius r (the radius of a circle is equivalent to the semimajor axis of an ellipse). Applying Newton's second law ($F = ma$) to the orbiting planet in Fig. 13-15 yields

$$\frac{GMm}{r^2} = (m)(\omega^2 r). \qquad (13\text{-}35)$$

Here we have substituted from Eq. 13-1 for the force magnitude F and used Eq. 10-23 to substitute $\omega^2 r$ for the centripetal acceleration. If we now use Eq. 10-20 to replace ω with $2\pi/T$, where T is the period of the motion, we obtain Kepler's third law:

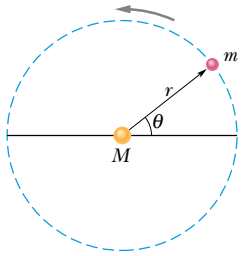

Figure 13-15 A planet of mass m moving around the Sun in a circular orbit of radius r.

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3 \quad \text{(law of periods)}. \qquad (13\text{-}36)$$

The quantity in parentheses is a constant that depends only on the mass M of the central body about which the planet orbits.

Equation 13-36 holds also for elliptical orbits, provided we replace r with a, the semimajor axis of the ellipse. This law predicts that the ratio T^2/a^3 has essentially the same value for every planetary orbit around a given massive body. Table 13-3 shows how well it holds for the orbits of the planets of the solar system.

Table 13-3 Kepler's Law of Periods for the Solar System

Planet	Semimajor Axis a (10^{10} m)	Period T (y)	T^2/a^3 (10^{-34} y^2/m^3)
Mercury	5.79	0.241	2.99
Venus	10.8	0.615	3.00
Earth	15.0	1.00	2.96
Mars	22.8	1.88	2.98
Jupiter	77.8	11.9	3.01
Saturn	143	29.5	2.98
Uranus	287	84.0	2.98
Neptune	450	165	2.99
Pluto	590	248	2.99

CHECKPOINT 4

Satellite 1 is in a certain circular orbit around a planet, while satellite 2 is in a larger circular orbit. Which satellite has (a) the longer period and (b) the greater speed?

SAMPLE PROBLEM 13.05

Kepler's law of periods, Comet Halley

Comet Halley orbits the Sun with a period of 76 years and, in 1986, had a distance of closest approach to the Sun, its *perihelion distance* R_p, of 8.9×10^{10} m. Table 13-3 shows that this is between the orbits of Mercury and Venus.

(a) What is the comet's farthest distance from the Sun, which is called its *aphelion distance* R_a?

KEY IDEAS

From Fig. 13-13 we see that $R_a + R_p = 2a$, where a is the semimajor axis of the orbit. Thus, we can find R_a if we first find a. We can relate a to the given period via the law of periods (Eq. 13-36) if we simply substitute the semimajor axis a for r.

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Calculations: Making that substitution and then solving for a, we have

$$a = \left(\frac{GMT^2}{4\pi^2}\right)^{1/3}. \tag{13-37}$$

If we substitute the mass M of the Sun, 1.99×10^{30} kg, and the period T of the comet, 76 years or 2.4×10^9 s, into Eq. 13-37, we find that $a = 2.7 \times 10^{12}$ m. Now we have

$$\begin{aligned} R_a &= 2a - R_p \\ &= (2)(2.7 \times 10^{12}\ \text{m}) - 8.9 \times 10^{10}\ \text{m} \\ &= 5.3 \times 10^{12}\ \text{m}. \end{aligned} \qquad \text{(Answer)}$$

Table 13-3 shows that this is a little less than the semimajor axis of the orbit of Pluto. Thus, the comet does not get farther from the Sun than Pluto.

(b) What is the eccentricity e of the orbit of comet Halley?

KEY IDEA

We can relate e, a, and R_p via Fig. 13-13, in which we see that $ea = a - R_p$.

Calculation: We have

$$e = \frac{a - R_p}{a} = 1 - \frac{R_p}{a} \tag{13-38}$$

$$= 1 - \frac{8.9 \times 10^{10}\ \text{m}}{2.7 \times 10^{12}\ \text{m}} = 0.97. \qquad \text{(Answer)}$$

This tells us that, with an eccentricity approaching unity, this orbit must be a long thin ellipse.

13.8 | SATELLITES: ORBITS AND ENERGY

Key Concept

- When a planet or satellite with mass m moves in a circular orbit with radius r, its potential energy U and kinetic energy K are given by

$$U = -\frac{GMm}{r} \quad \text{and} \quad K = \frac{GMm}{2r}.$$

The mechanical energy $E = K + U$ is then

$$E = -\frac{GMm}{2r}.$$

For an elliptical orbit of semimajor axis a,

$$E = -\frac{GMm}{2a}.$$

As a satellite orbits Earth in an elliptical path, both its speed, which fixes its kinetic energy K, and its distance from the center of Earth, which fixes its gravitational potential energy U, fluctuate with fixed periods. However, the mechanical energy E of the satellite remains constant. (Since the satellite's mass is so much smaller than Earth's mass, we assign U and E for the Earth–satellite system to the satellite alone.)

The potential energy of the system is given by Eq. 13-23:

$$U = -\frac{GMm}{r}$$

(with $U = 0$ for infinite separation). Here r is the radius of the satellite's orbit, assumed for the time being to be circular, and M and m are the masses of Earth and the satellite, respectively.

To find the kinetic energy of a satellite in a circular orbit, we write Newton's second law ($F = ma$) as

$$\frac{GMm}{r^2} = m\frac{v^2}{r}, \tag{13-39}$$

where v^2/r is the centripetal acceleration of the satellite. Then, from Eq. 13-39, the kinetic energy is

$$K = \frac{1}{2}mv^2 = \frac{GMm}{2r}, \tag{13-40}$$

which shows us that for a satellite in a circular orbit,

$$K = -\frac{U}{2} \quad \text{(circular orbit)}. \tag{13-41}$$

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The total mechanical energy of the orbiting satellite is

$$E = K + U = \frac{GMm}{2r} - \frac{GMm}{r}$$

or

$$E = -\frac{GMm}{2r} \quad \text{(circular orbit)}. \tag{13-42}$$

This tells us that for a satellite in a circular orbit, the total energy E is the negative of the kinetic energy K:

$$E = -K \quad \text{(circular orbit)}. \tag{13-43}$$

For a satellite in an elliptical orbit of semimajor axis a, we can substitute a for r in Eq. 13-42 to find the mechanical energy:

$$E = -\frac{GMm}{2a} \quad \text{(elliptical orbit)}. \tag{13-44}$$

Equation 13-44 tells us that the total energy of an orbiting satellite depends only on the semimajor axis of its orbit and not on its eccentricity e. For example, four orbits with the same semimajor axis are shown in Fig. 13-16; the same satellite would have the same total mechanical energy E in all four orbits. Figure 13-17 shows the variation of K, U, and E with r for a satellite moving in a circular orbit about a massive central body. Note that as r is increased, the kinetic energy (and thus also the orbital speed) decreases.

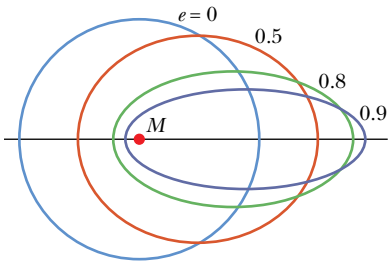


Figure 13-16 Four orbits with different eccentricities e about an object of mass M. All four orbits have the same semimajor axis a and thus correspond to the same total mechanical energy E.

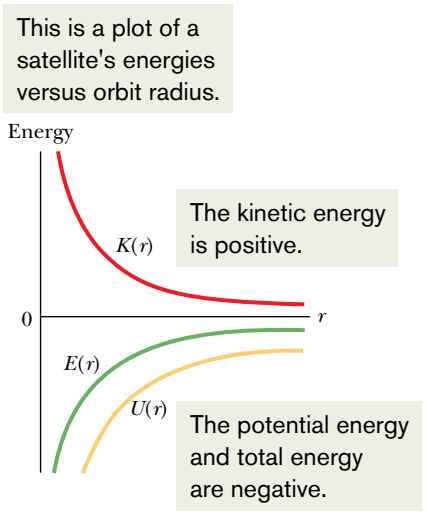
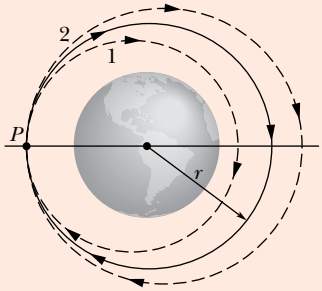


Figure 13-17 The variation of kinetic energy K, potential energy U, and total energy E with radius r for a satellite in a circular orbit. For any value of r, the values of U and E are negative, the value of K is positive, and $E = -K$. As $r \to \infty$, all three energy curves approach a value of zero.

CHECKPOINT 5

In the figure here, a space shuttle is initially in a circular orbit of radius r about Earth. At point P, the pilot briefly fires a forward-pointing thruster to decrease the shuttle's kinetic energy K and mechanical energy E. (a) Which of the dashed elliptical orbits shown in the figure will the shuttle then take? (b) Is the orbital period T of the shuttle (the time to return to P) then greater than, less than, or the same as in the circular orbit?



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SAMPLE PROBLEM 13.06

Mechanical energy of orbiting bowling ball

A playful astronaut releases a bowling ball, of mass $m = 7.20$ kg, into circular orbit about Earth at an altitude h of 350 km.

(a) What is the mechanical energy E of the ball in its orbit?

KEY IDEA

We can get E from the orbital energy, given by Eq. 13-42 ($E = -GMm/2r$), if we first find the orbital radius r. (It is *not* simply the given altitude.)

Calculations: The orbital radius must be

$$r = R + h = 6370 \text{ km} + 350 \text{ km} = 6.72 \times 10^6 \text{ m},$$

in which R is the radius of Earth. Then, from Eq. 13-42 with Earth mass $M = 5.98 \times 10^{24}$ kg, the mechanical energy is

$$E = -\frac{GMm}{2r}$$
$$= -\frac{(6.67\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98\times10^{24}\ \text{kg})(7.20\ \text{kg})}{(2)(6.72\times10^{6}\ \text{m})}$$
$$= -2.14\times10^{8}\ \text{J} = -214\ \text{MJ}. \quad \text{(Answer)}$$

(b) What is the mechanical energy E_0 of the ball on the launchpad at the Kennedy Space Center (before launch)? From there to the orbit, what is the change ΔE in the ball's mechanical energy?

KEY IDEA

On the launchpad, the ball is *not* in orbit and thus Eq. 13-42 does *not* apply. Instead, we must find $E_0 = K_0 + U_0$, where K_0 is the ball's kinetic energy and U_0 is the gravitational potential energy of the ball–Earth system.

Calculations: To find U_0, we use Eq. 13-23 to write

$$U_0 = -\frac{GMm}{R}$$
$$= -\frac{(6.67\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98\times10^{24}\ \text{kg})(7.20\ \text{kg})}{6.37\times10^{6}\ \text{m}}$$
$$= -4.51\times10^{8}\ \text{J} = -451\ \text{MJ}.$$

The kinetic energy K_0 of the ball is due to the ball's motion with Earth's rotation. You can show that K_0 is less than 1 MJ, which is negligible relative to U_0. Thus, the mechanical energy of the ball on the launchpad is

$$E_0 = K_0 + U_0 \approx 0 - 451\ \text{MJ} = -451\ \text{MJ}. \quad \text{(Answer)}$$

The *increase* in the mechanical energy of the ball from launchpad to orbit is

$$\Delta E = E - E_0 = (-214\ \text{MJ}) - (-451\ \text{MJ})$$
$$= 237\ \text{MJ}. \quad \text{(Answer)}$$

This is worth a few dollars at your utility company. Obviously the high cost of placing objects into orbit is not due to their required mechanical energy.

SAMPLE PROBLEM 13.07

Transforming a circular orbit into an elliptical orbit

A spaceship of mass $m = 4.50 \times 10^3$ kg is in a circular Earth orbit of radius $r = 8.00 \times 10^6$ m and period $T_0 = 118.6$ min $= 7.119 \times 10^3$ s when a thruster is fired in the forward direction to decrease the speed to 96.0% of the original speed. What is the period T of the resulting elliptical orbit (Fig. 13-18)?

KEY IDEAS

(1) An elliptical orbit period is related to the semimajor axis a by Kepler's third law, written as Eq. 13-36 ($T^2 = 4\pi^2 r^3/GM$) but with a replacing r. (2) The semimajor axis a is related to the total mechanical energy E of the ship by Eq. 13-44 ($E = -GMm/2a$), in which Earth's

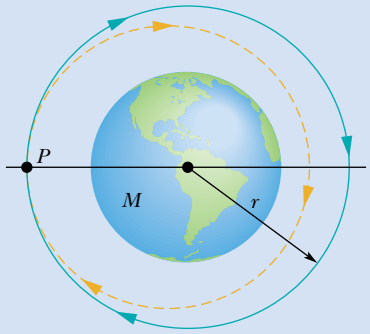

Figure 13-18 At point P a thruster is fired, changing a ship's orbit from circular to elliptical.

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mass is $M = 5.98 \times 10^{24}$ kg. (3) The potential energy of the ship at radius r from Earth's center is given by Eq. 13-23 ($U = -GMm/r$).

Calculations: Looking over the Key Ideas, we see that we need to calculate the total energy E to find the semimajor axis a, so that we can then determine the period of the elliptical orbit. Let's start with the kinetic energy, calculating it just after the thruster is fired. The speed v just then is 96% of the initial speed v_0, which was equal to the ratio of the circumference of the initial circular orbit to the initial period of the orbit. Thus, just after the thruster is fired, the kinetic energy is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(0.96v_0)^2 = \frac{1}{2}m(0.96)^2\left(\frac{2\pi r}{T_0}\right)^2$$
$$= \frac{1}{2}(4.50\times10^3\ \text{kg})(0.96)^2\left(\frac{2\pi(8.00\times10^6\ \text{m})}{7.119\times10^3\ \text{s}}\right)^2$$
$$= 1.0338\times10^{11}\ \text{J}.$$

Just after the thruster is fired, the ship is still at orbital radius r, and thus its gravitational potential energy is

$$U = -\frac{GMm}{r}$$
$$= -\frac{(6.67\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98\times10^{24}\ \text{kg})(4.50\times10^3\ \text{kg})}{8.00\times10^6\ \text{m}}$$
$$= -2.2436\times10^{11}\ \text{J}.$$

We can now find the semimajor axis by rearranging Eq. 13-44, substituting a for r, and then substituting in our energy results:

$$a = -\frac{GMm}{2E} = -\frac{GMm}{2(K+U)}$$
$$= -\frac{(6.67\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98\times10^{24}\ \text{kg})(4.50\times10^3\ \text{kg})}{2(1.0338\times10^{11}\ \text{J} - 2.2436\times10^{11}\ \text{J})}$$
$$= 7.418\times10^6\ \text{m}.$$

Now, one more step to go. We substitute a for r in Eq. 13-36 and then solve for the period T, substituting our result for a:

$$T = \left(\frac{4\pi^2a^3}{GM}\right)^{1/2}$$
$$= \left(\frac{4\pi^2(7.418\times10^6\ \text{m})^3}{(6.67\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98\times10^{24}\ \text{kg})}\right)^{1/2}$$
$$= 6.356\times10^3\ \text{s} = 106\ \text{min}. \qquad \text{(Answer)}$$

This is the period of the elliptical orbit that the ship takes after the thruster is fired. It is less than the period T_0 for the circular orbit for two reasons. (1) The orbital path length is now less. (2) The elliptical path takes the ship closer to Earth everywhere except at the point of firing (Fig. 13-18). The resulting decrease in gravitational potential energy increases the kinetic energy and thus also the speed of the ship.

13.9 | EINSTEIN AND GRAVITATION

Key Concept

- Einstein pointed out that gravitation and acceleration are equivalent. This principle of equivalence led him to a theory of gravitation (the general theory of relativity) that explains gravitational effects in terms of a curvature of space.

Principle of Equivalence

Albert Einstein once said: "I was … in the patent office at Bern when all of a sudden a thought occurred to me: 'If a person falls freely, he will not feel his own weight.' I was startled. This simple thought made a deep impression on me. It impelled me toward a theory of gravitation."

Thus Einstein tells us how he began to form his **general theory of relativity**. The fundamental postulate of this theory about gravitation (the gravitating of objects toward each other) is called the **principle of equivalence**, which says that gravitation and acceleration are equivalent. If a physicist were locked up in a small box as in Fig. 13-19, he would not be able to tell whether the box was at rest on Earth (and subject only to Earth's gravitational force), as in Fig. 13-19*a*, or accelerating through interstellar space at 9.8 m/s^2 (and subject only to the force producing that acceleration), as in Fig. 13-19*b*. In both situations he would feel the same and would read the same value for his weight on a scale. Moreover, if he watched an object fall past him, the object would have the same acceleration relative to him in both situations.

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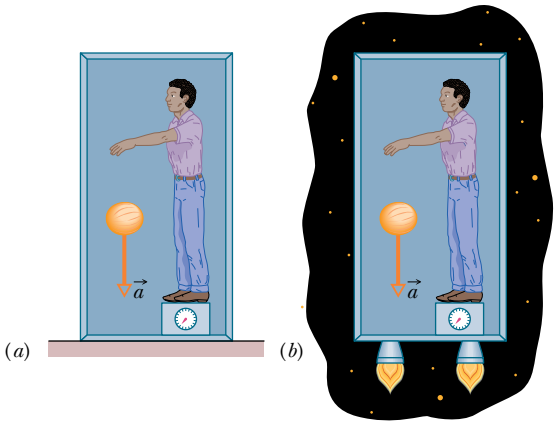

Figure 13-19 (*a*) A physicist in a box resting on Earth sees a cantaloupe (melon) falling with acceleration $a = 9.8\ \text{m/s}^2$. (*b*) If he and the box accelerate in deep space at $9.8\ \text{m/s}^2$, the cantaloupe has the same acceleration relative to him. It is not possible, by doing experiments within the box, for the physicist to tell which situation he is in. For example, the platform scale on which he stands reads the same weight in both situations.

Curvature of Space

We have thus far explained gravitation as due to a force between masses. Einstein showed that, instead, gravitation is due to a curvature of space that is caused by the masses. (As is discussed later in this book, space and time are entangled, so the curvature of which Einstein spoke is really a curvature of *spacetime*, the combined four dimensions of our universe.)

Picturing how space (such as vacuum) can have curvature is difficult. An analogy might help: Suppose that from orbit we watch a race in which two boats begin on Earth's equator with a separation of 20 km and head due south (Fig. 13-20*a*). To the sailors, the boats travel along flat, parallel paths. However, with time the boats draw together until, nearer the south pole, they touch. The sailors in the boats can interpret this drawing together in terms of a force acting on the boats. Looking on from space, however, we can see that the boats draw together simply because of the curvature of Earth's surface. We can see this because we are viewing the race from "outside" that surface.

Figure 13-20*b* shows a similar race: Two horizontally separated apples are dropped from the same height above Earth. Although the apples may appear to travel along parallel paths, they actually move toward each other because they both fall toward Earth's center. We can interpret the motion of the apples in terms of the gravitational force on the apples from Earth. We can also interpret the motion in terms of a curvature of the space near Earth, a curvature due to the presence of Earth's mass. This time we cannot see the curvature because we cannot get "outside"

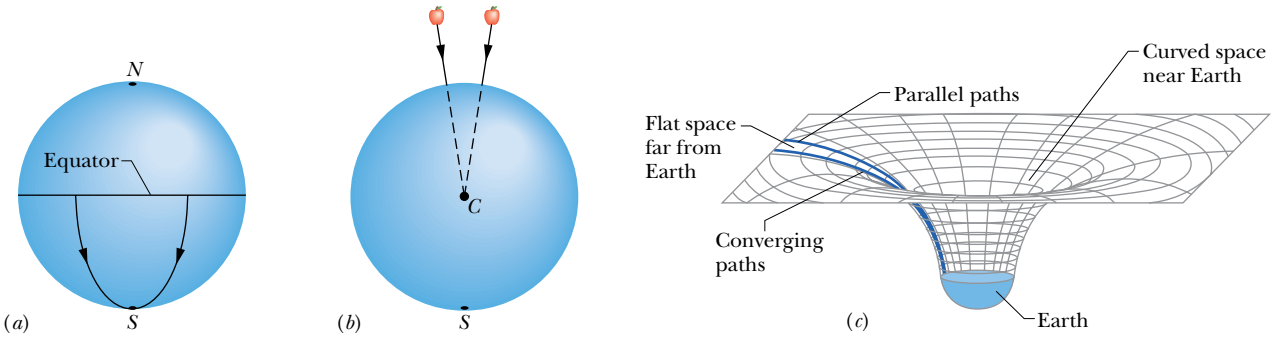

Figure 13-20 (*a*) Two objects moving along lines of longitude toward the south pole converge because of the curvature of Earth's surface. (*b*) Two objects falling freely near Earth move along lines that converge toward the center of Earth because of the curvature of space near Earth. (*c*) Far from Earth (and other masses), space is flat and parallel paths remain parallel. Close to Earth, the parallel paths begin to converge because space is curved by Earth's mass.

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the curved space, as we got "outside" the curved Earth in the boat example. However, we can depict the curvature with a drawing like Fig. 13-20*c*; there the apples would move along a surface that curves toward Earth because of Earth's mass.

When light passes near Earth, the path of the light bends slightly because of the curvature of space there, an effect called *gravitational tensing*. When light passes a more massive structure, like a galaxy or a black hole having large mass, its path can be bent more. If such a massive structure is between us and a quasar (an extremely bright, extremely distant source of light), the light from the quasar can bend around the massive structure and toward us (Fig. 13-21*a*), Then, because the light seems to be coming to us from a number of slightly different directions in the sky, we see the same quasar in all those different directions. In some situations, the quasars we see blend together to form a giant luminous arc, which is called an *Einstein ring* (Fig. 13-21*b*).

Should we attribute gravitation to the curvature of spacetime due to the presence of masses or to a force between masses? Or should we attribute it to the actions of a type of fundamental particle called a *graviton*, as conjectured in some modern physics theories? Although our theories about gravitation have been enormously successful in describing everything from falling apples to planetary and stellar motions, we still do not fully understand it on either the cosmological scale or the quantum physics scale.

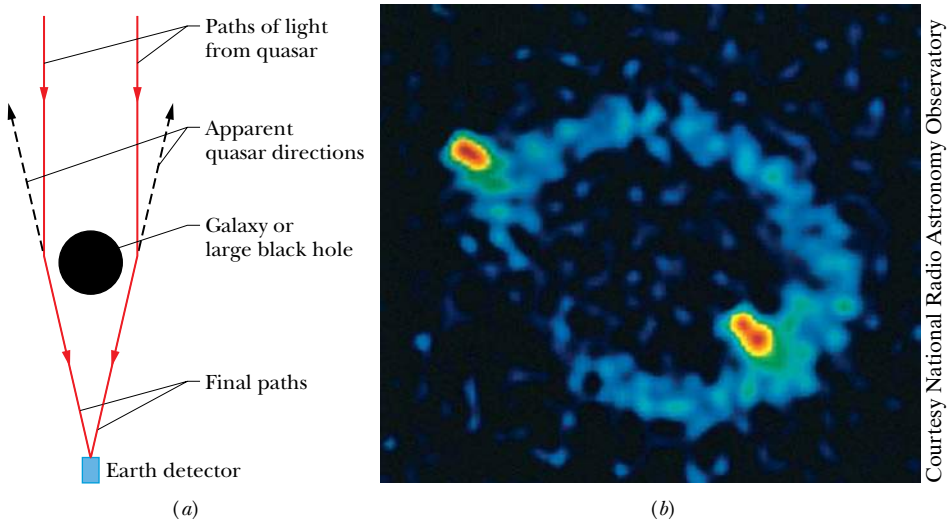

Figure 13-21 (*a*) Light from a distant quasar follows curved paths around a galaxy or a large black hole because the mass of the galaxy or black hole has curved the adjacent space. If the light is detected, it appears to have originated along the backward extensions of the final paths (dashed lines). (*b*) The Einstein ring known as MG1131 + 0456 on the computer screen of a telescope. The source of the light (actually, radio waves, which are a form of invisible light) is far behind the large, unseen galaxy that produces the ring; a portion of the source appears as the two bright spots seen along the ring.

REVIEW AND SUMMARY

The Law of Gravitation Any particle in the universe attracts any other particle with a **gravitational force** whose magnitude is

$$F = G\frac{m_1 m_2}{r^2} \quad \text{(Newton's law of gravitation)}, \qquad (13\text{-}1)$$

where m_1 and m_2 are the masses of the particles, r is their separation, and G ($=6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$) is the *gravitational constant*.

Gravitational Behavior of Uniform Spherical Shells The gravitational force between extended bodies is found by adding (integrating) the individual forces on individual particles within the bodies. However, if either of the bodies is a uniform spherical shell or a spherically symmetric solid, the net gravitational force it exerts on an *external* object may be computed as if all the mass of the shell or body were located at its center.

Superposition Gravitational forces obey the **principle of superposition**; that is, if n particles interact, the net force $\vec{F}_{1,\mathrm{net}}$ on a particle labeled particle 1 is the sum of the forces on it from all the other particles taken one at a time:

$$\vec{F}_{1,\mathrm{net}} = \sum_{i=2}^{n} \vec{F}_{1i}, \qquad (13\text{-}5)$$

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in which the sum is a vector sum of the forces $\vec{F}_{1i}$ on particle 1 from particles 2, 3, …, n. The gravitational force $\vec{F}_1$ on a particle from an extended body is found by dividing the body into units of differential mass dm, each of which produces a differential force $d\vec{F}$ on the particle, and then integrating to find the sum of those forces:

$$\vec{F}_1 = \int d\vec{F}. \tag{13-6}$$

Gravitational Acceleration The *gravitational acceleration* a_g of a particle (of mass m) is due solely to the gravitational force acting on it. When the particle is at distance r from the center of a uniform, spherical body of mass M, the magnitude F of the gravitational force on the particle is given by Eq. 13-1. Thus, by Newton's second law,

$$F = ma_g, \tag{13-12}$$

which gives

$$a_g = \frac{GM}{r^2}. \tag{13-13}$$

Free-Fall Acceleration and Weight Because Earth's mass is not distributed uniformly, because the planet is not perfectly spherical, and because it rotates, the actual free-fall acceleration $\vec{g}$ of a particle near Earth differs slightly from the gravitational acceleration $\vec{a}_g$, and the particle's weight (equal to mg) differs from the magnitude of the gravitational force on it as calculated by Newton's law of gravitation (Eq. 13-1).

Gravitation Within a Spherical Shell A uniform shell of matter exerts no net gravitational force on a particle located inside it. This means that if a particle is located inside a uniform solid sphere at distance r from its center, the gravitational force exerted on the particle is due only to the mass that lies inside a sphere of radius r (the *inside sphere*). The force magnitude is given by

$$F = \frac{GmM}{R^3}r, \tag{13-21}$$

where M is the sphere's mass and R is its radius.

Gravitational Potential Energy The gravitational potential energy $U(r)$ of a system of two particles, with masses M and m and separated by a distance r, is the negative of the work that would be done by the gravitational force of either particle acting on the other if the separation between the particles were changed from infinite (very large) to r. This energy is

$$U = -\frac{GMm}{r} \quad \text{(gravitational potential energy)}. \tag{13-23}$$

Potential Energy of a System If a system contains more than two particles, its total gravitational potential energy U is the sum of the terms representing the potential energies of all the pairs. As an example, for three particles, of masses m_1, m_2, and m_3,

$$U = -\left(\frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}}\right). \tag{13-24}$$

Escape Speed An object will escape the gravitational pull of an astronomical body of mass M and radius R (that is, it will reach an infinite distance) if the object's speed near the body's surface is at least equal to the **escape speed**, given by

$$v = \sqrt{\frac{2GM}{R}}. \tag{13-30}$$

Kepler's Laws The motion of satellites, both natural and artificial, is governed by these laws:

1. ***The law of orbits***. All planets move in elliptical orbits with the Sun at one focus.
2. ***The law of areas***. A line joining any planet to the Sun sweeps out equal areas in equal time intervals. (This statement is equivalent to conservation of angular momentum.)
3. ***The law of periods***. The square of the period T of any planet is proportional to the cube of the semimajor axis a of its orbit. For circular orbits with radius r,

$$T^2 = \left(\frac{4\pi^2}{GM}\right)r^3 \quad \text{(law of periods)}, \tag{13-36}$$

where M is the mass of the attracting body—the Sun in the case of the solar system. For elliptical planetary orbits, the semimajor axis a is substituted for r.

Energy in Planetary Motion When a planet or satellite with mass m moves in a circular orbit with radius r, its potential energy U and kinetic energy K are given by

$$U = -\frac{GMm}{r} \quad \text{and} \quad K = \frac{GMm}{2r}. \tag{13-23, 13-40}$$

The mechanical energy $E = K + U$ is then

$$E = -\frac{GMm}{2r}. \tag{13-42}$$

For an elliptical orbit of semimajor axis a,

$$E = -\frac{GMm}{2a}. \tag{13-44}$$

Einstein's View of Gravitation Einstein pointed out that gravitation and acceleration are equivalent. This **principle of equivalence** led him to a theory of gravitation (the **general theory of relativity**) that explains gravitational effects in terms of a curvature of space.

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PROBLEMS

1. In Fig. 13-22, a particle of mass $m_1 = 0.67$ kg is a distance $d = 23$ cm from one end of a uniform rod with length $L = 2.5$ m and mass $M = 4.0$ kg. What is the magnitude of the gravitational force $\vec{F}$ the particle from the rod?

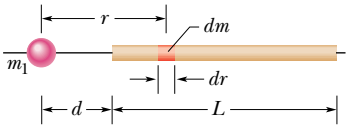

Figure 13-22 Problem 1.

2. Two neutron stars are separated by a distance of 1.0×10^{11} m. They each have a mass of 1.0×10^{30} kg and a radius of 2.0×10^5 m. They are initially at rest with respect to each other. As measured from that rest frame, how fast are they moving when (a) their separation has decreased to one-half its initial value and (b) they are about to collide?

3. In a certain binary-star system, each star has the same mass as our Sun, and they revolve about their center of mass. The distance between them is the same as the distance between Earth and the Sun. What is their period of revolution in years?

4. *Miniature black holes*. Left over from the big-bang beginning of the universe, tiny black holes might still wander through the universe. If one with a mass of 5.0×10^{11} kg (and a radius of only 5.0×10^{-16} m) reached Earth, at what distance from your head would its gravitational pull on you match that of Earth's?

5. A rocket is to be shot radially outward from Earth's surface. Neglecting Earth's rotation, find the radial distance from Earth's center that the rocket reaches if it is launched with (a) 0.400 times the escape speed from Earth and (b) 0.400 times the kinetic energy that is required to escape Earth. (c) At launch, what is the least mechanical energy required for it to escape Earth?

6. Two concentric spherical shells with uniformly distributed masses $M_1 = 5.00 \times 10^3$ kg and $M_2 = 9.00 \times 10^3$ kg are situated as shown in Fig. 13-23. Find the magnitude of the net gravitational force on a particle of mass $m = 2.00$ kg due to the shells, when the particle is located at radial distance (a) $a = 11.0$ m, (b) $b = 6.00$ m, and (c) $c = 2.00$ m.

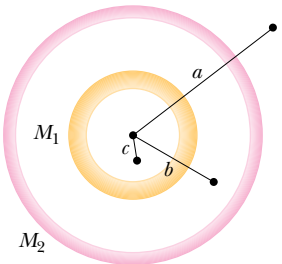

Figure 13-23 Problem 6.

7. (a) At what height above Earth's surface is the energy required to lift a satellite to that height equal to the kinetic energy required for the satellite to be in orbit at that height? (b) For greater heights, which is greater, the energy for lifting or the kinetic energy for orbiting?

8. We want to position a space probe along a line that extends directly toward the Sun in order to monitor solar flares. How far from Earth's center is the point on the line where the Sun's gravitational pull on the probe balances Earth's pull?

9. In Fig. 13-24, two satellites, A and B, both of mass $m = 150$ kg, move in the same circular orbit of radius $r = 7.87 \times 10^6$ m around Earth but in opposite senses of rotation and therefore on a collision course. (a) Find the total mechanical energy $E_A + E_B$ of the *two satellites + Earth* system before the collision. (b) If the collision is completely inelastic so that the wreckage remains as one piece of tangled material (mass = $2m$), find the total mechanical energy immediately after the collision. (c) Just after the collision, is the wreckage falling directly toward Earth's center or orbiting around Earth?

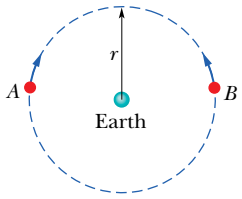

Figure 13-24 Problem 9.

10. A satellite is in a circular Earth orbit of radius r. The area A enclosed by the orbit depends on r^2 because $A = \pi r^2$. Determine how the following properties of the satellite depend on r: (a) period, (b) kinetic energy, (c) angular momentum, and (d) speed.

11. *Three dimensions.* Three point particles are fixed in place in an xyz coordinate system. Particle A, at the origin, has mass m_A. Particle B, at xyz coordinates $(2.00d, 1.00d, 2.00d)$, has mass $2.00m_A$, and particle C, at coordinates $(-1.00d, 2.00d, -3.00d)$, has mass $3.00m_A$. A fourth particle D, with mass $4.00m_A$, is to be placed near the other particles. In terms of distance d, at what (a) x, (b) y, and (c) z coordinate should D be placed so that the net gravitational force on A from B, C, and D is zero?

12. A comet that was seen in April 574 by Chinese astronomers on a day known by them as the Woo Woo day was spotted again in May 1994. Assume the time between observations is the period of the Woo Woo day comet and its eccentricity is 0.9932. What are (a) the semimajor axis of the comet's orbit and (b) its greatest distance from the Sun in terms of the mean orbital radius R_P of Pluto?

13. One model for a certain planet has a core of radius R and mass M surrounded by an outer shell of inner radius R, outer radius $2R$, and mass $4M$. If $M = 3.00 \times 10^{24}$ kg and $R = 12 \times 10^6$ m, what is the gravitational acceleration of a particle at points (a) R and (b) $3R$ from the center of the planet?

14. The three spheres in Fig. 13-25, with masses $m_A = 80$ g, $m_B = 10$ g, and $m_C = 20$ g, have their centers on a common

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line, with $L = 14$ cm and $d = 2.0$ cm. You move sphere B along the line until its center-to-center separation from C is $d = 2.0$ cm. How much work is done on sphere B (a) by you and (b) by the net gravitational force on B due to spheres A and C?

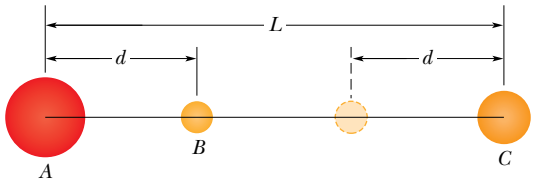

Figure 13-25 Problem 14.

15. The Sun and Earth each exert a gravitational force on the Moon. What is the ratio F_{Sun}/F_{Earth} of these two forces? (The average Sun–Moon distance is equal to the Sun–Earth distance.)

16. The first known collision between space debris and a functioning satellite occurred in 1996: At an altitude of 700 km, a year-old French spy satellite was hit by a piece of an Ariane rocket. A stabilizing boom on the satellite was demolished, and the satellite was sent spinning out of control. Just before the collision and in kilometers per hour, what was the speed of the rocket piece relative to the satellite if both were in circular orbits and the collision was (a) head-on and (b) along perpendicular paths?

17. The Sun's center is at one focus of Earth's orbit. How far from this focus is the other focus, (a) in meters and (b) in terms of the solar radius, 6.96×10^8 m? The eccentricity is 0.0167, and the semimajor axis is 1.50×10^{11} m.

18. As seen in Fig. 13-26, two spheres of mass m and a third sphere of mass M form an equilateral triangle, and a fourth sphere of mass m_4 is at the center of the triangle. The net gravitational force on that central sphere from the three other spheres is zero. (a) What is M in terms of m? (b) If we double the value of m_4, what then is the magnitude of the net gravitational force on the central sphere?

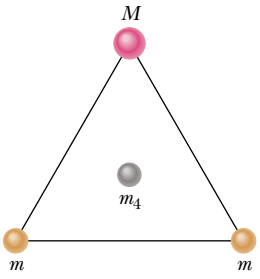

Figure 13-26 Problem 18.

19. A satellite is in circular orbit about Earth at an altitude of 300 km. What are its (a) orbital period, (b) orbital frequency, and (c) linear speed?

20. An orbiting satellite stays over a certain spot on the equator of (rotating) Earth. What is the altitude of the orbit (called a *geosynchronous orbit*)?

21. What multiple of the energy needed to escape from Earth gives the energy needed to escape from (a) a moon of mass 1.472×10^{23} kg and radius 3.480×10^6 m and (b) Jupiter?

22. Two particles are separated by 19 m when their gravitational attraction has a magnitude of 4.0×10^{-12} N. If the mass of particle 1 is 5.2 kg, what is the mass of particle 2?

23. Figure 13-27 gives the potential energy function $U(r)$ of a projectile, plotted outward from the surface of a planet of radius R_s. What least kinetic energy is required of a projectile launched at the surface if the projectile is to "escape" the planet?

24. Figure 13-27 gives the potential energy function $U(r)$ of a projectile, plotted outward from the surface of a planet of radius R_s. If the projectile is launched radially outward from the surface with a mechanical energy of -2.0×10^9 J, what are (a) its kinetic energy at radius $r = 1.15R_s$, and (b) its *turning point* (see Section 8.4) in terms of R_s?

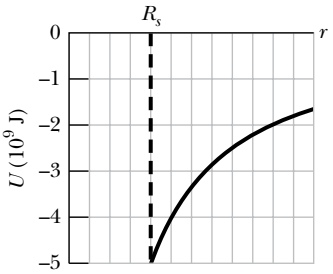

Figure 13-27 Problems 23 and 24.

25. In Fig. 13-28, three 8.00 kg spheres are located at distances $d_1 = 0.300$ m and $d_2 = 0.400$ m. What are the (a) magnitude and (b) direction (relative to the positive direction of the x axis) of the net gravitational force on sphere B due to spheres A and C?

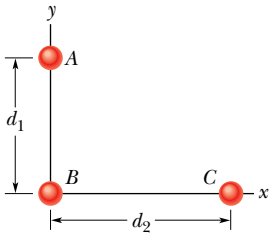

Figure 13-28 Problem 25.

26. In Fig. 13-29, a square of edge length 20.0 cm is formed by four spheres of masses $m_1 = 5.00$ g, $m_2 = 3.00$ g, $m_3 = 1.00$ g, and $m_4 = 5.00$ g. In unit-vector notation, what is the net gravitational force from them on a central sphere with mass $m_5 = 2.00$ g?

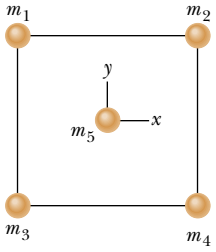

Figure 13-29 Problem 26.

27. In deep space, sphere A of mass 20 kg is located at the origin of an x axis and sphere B of mass 10 kg is located on the axis at $x = 0.95$ m. Sphere B is released from rest while sphere A is held at the origin. (a) What is the gravitational potential energy of the two-sphere system just as B is released? (b) What is the kinetic energy of B when it has moved 0.15 m toward A?

28. Consider an object of mass M that is small enough to be treated as a particle but large enough that a section of mass m can be removed and positioned 3.0 mm from the remaining part. To maximize the magnitude of the gravitational force between the removed section and the remaining part, what should the ratio of m/M be?

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29. The Martian satellite Phobos travels in an approximately circular orbit of radius 9.4×10^6 m with a period of 7 h 39 min. Calculate the mass of Mars from this information.

30. In 1610, Galileo used his telescope to discover four moons around Jupiter, with these mean orbital radii a and periods T:

Name	$a(10^8$ m)	T (days)
Io	4.22	1.77
Europa	6.71	3.55
Ganymede	10.7	7.16
Callisto	18.8	16.7

(a) Plot log a (y axis) against log T (x axis) and show that you get a straight line. (b) Measure the slope of the line and compare it with the value that you expect from Kepler's third law. (c) Find the mass of Jupiter from the intercept of this line with the y axis.

31. Three identical stars of mass M form an equilateral triangle that rotates around the triangle's center as the stars move in a common circle about that center. The triangle has edge length L. What is the speed of the stars?

32. Three point particles are fixed in position in an xy plane. Two of them, particle A of mass 6.00 g and particle B of mass 12.0 g, are shown in Fig. 13-30, with a separation of $d_{AB} = 0.500$ m at angle $\theta = 30°$. Particle C, with mass 10.0 g, is not shown. The net gravitational force acting on particle A due to particles B and C is 2.77×10^{-14} N at an angle of $-163.8°$ from the positive direction of the x axis. What are (a) the x coordinate and (b) the y coordinate of particle C?

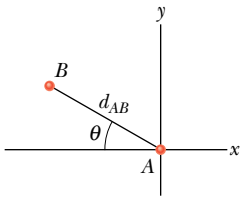

Figure 13-30 Problem 32.

33. A satellite, moving in an elliptical orbit, is 1440 km above Earth's surface at its farthest point and 180 km above at its closest point. Calculate (a) the semimajor axis and (b) the eccentricity of the orbit.

34. *Hunting a black hole.* Observations of the light from a certain star indicate that it is part of a binary (two-star) system. This visible star has orbital speed $v = 270$ km/s, orbital period $T = 1.70$ days, and approximate mass $m_1 = 6M_s$, where M_s is the Sun's mass, 1.99×10^{30} kg. Assume that the visible star and its companion star, which is dark and unseen, are both in circular orbits (Fig. 13-31). What integer multiple of M_s gives the *approximate* mass m_2 of the dark star?

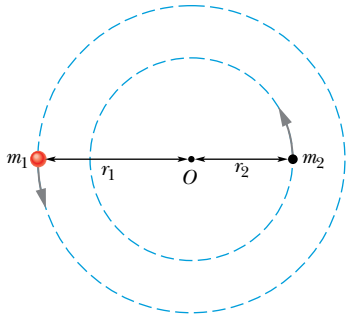

Figure 13-31 Problem 34.

35. (a) What is the escape speed on a spherical asteroid whose radius is 700 km and whose gravitational acceleration at the surface is 4.5 m/s^2? (b) How far from the surface will a particle go if it leaves the asteroid's surface with a radial speed of 1000 m/s? (c) With what speed will an object hit the asteroid if it is dropped from 1000 km above the surface?

36. At what altitude above Earth's surface would the gravitational acceleration be 2.0 m/s^2?

37. An asteroid, whose mass is 2.0×10^{-4} times the mass of Earth, revolves in a circular orbit around the Sun at a distance that is 3.0 times Earth's distance from the Sun. (a) Calculate the period of revolution of the asteroid in years. (b) What is the ratio of the kinetic energy of the asteroid to the kinetic energy of Earth?

38. The radius R_h and mass M_h of a black hole are related by $R_h = 2GM_h/c^2$, where c is the speed of light. Assume that the gravitational acceleration a_g of an object at a distance $r_o = 1.001R_h$ from the center of a black hole is given by Eq. 13-13 (it is, for large black holes). (a) In terms of M_h, find a_g at r_o. (b) Does a_g at r_o increase or decrease as M_h increases? (c) What is a_g at r_o for a very large black hole whose mass is 1.55×10^{12} times the solar mass of 1.99×10^{30} kg? (d) If an astronaut of height 1.66 m is at r_o with her feet down, what is the difference in gravitational acceleration between her head and feet? (e) Is the tendency to stretch the astronaut severe?

39. One way to attack a satellite in Earth orbit is to launch a swarm of pellets in the same orbit as the satellite but in the opposite direction. Suppose a satellite in a circular orbit 500 km above Earth's surface collides with a pellet having mass 5.0 g. (a) What is the kinetic energy of the pellet in the reference frame of the satellite just before the collision? (b) What is the ratio of this kinetic energy to the kinetic energy of a 5.0 g bullet from a modern army rifle with a muzzle speed of 950 m/s?

40. Figure 13-32 shows, not to scale, a cross section through the interior of Earth. Rather than being uniform throughout, Earth is divided into three zones: an outer *crust*, a *mantle*, and an inner *core*. The dimensions of these zones and the masses contained within them are shown on the figure. Earth has a total mass of 5.98×10^{24} kg and a radius of 6370 km. Ignore rotation and assume that Earth is spherical. (a) Calculate a_g at the surface. (b) Suppose that a bore hole (the *Mohole*) is driven to the crust–mantle interface at a depth of 25.0 km; what would be the value of a_g at the bottom of the hole? (c) Suppose that Earth were a uniform sphere with the

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same total mass and size. What would be the value of a_g at a depth of 25.0 km? (Precise measurements of a_g are sensitive probes of the interior structure of Earth, although results can be clouded by local variations in mass distribution.)

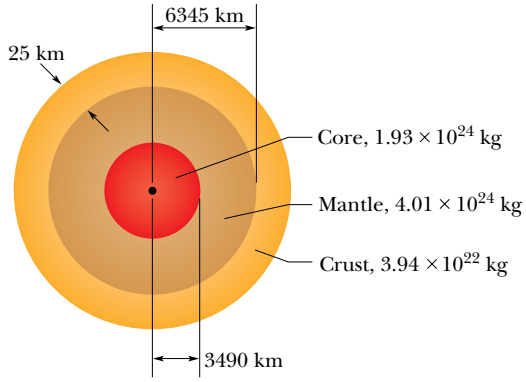

Figure 13-32 Problem 40.

41. In Fig. 13-19*b*, the scale on which the 80 kg physicist stands reads 205 N. How long will the cantaloupe (melon) take to reach the floor if the physicist drops it (from rest relative to himself) at a height of 3.0 m above the floor?

42. *Moon effect.* Some people believe that the Moon controls their activities. If the Moon moves from being directly on the opposite side of Earth from you to being directly overhead, by what percent does (a) the Moon's gravitational pull on you increase and (b) your weight (as measured on a scale) decrease? Assume that the Earth–Moon (center-to-center) distance is 3.82×10^8 m and Earth's radius is 6.37×10^6 m.

43. In Problem 29, what ratio m/M gives the least gravitational potential energy for the system?

44. A solid sphere has a uniformly distributed mass of 1.0×10^4 kg and a radius of 1.0 m. What is the magnitude of the gravitational force due to the sphere on a particle of mass $m = 2.0$ kg when the particle is located at a distance of (a) 1.5 m and (b) 0.50 m from the center of the sphere? (c) Write a general expression for the magnitude of the gravitational force on the particle at a distance $r \leq 1.0$ m from the center of the sphere.

45. Two Earth satellites, A and B, each of mass m, are to be launched into circular orbits about Earth's center. Satellite A is to orbit at an altitude of 6370 km. Satellite B is to orbit at an altitude of 19 850 km. The radius of Earth R_E is 6370 km. (a) What is the ratio of the potential energy of satellite B to that of satellite A, in orbit? (b) What is the ratio of the kinetic energy of satellite B to that of satellite A, in orbit? (c) Which satellite has the greater total energy if each has a mass of 14.6 kg? (d) By how much?

46. Figure 13-33 shows four particles, each of mass 20.0 g, that form a square with an edge length of $d = 0.600$ m. If d is increased to 1.20 m, what is the change in the gravitational potential energy of the four-particle system?

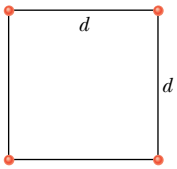

Figure 13-33 Problem 46.

47. *Two dimensions.* In Fig. 13-34, three point particles are fixed in place in an xy plane. Particle A has mass m_A, particle B has mass $2.00m_A$, and particle C has mass $3.00m_A$. A fourth particle D, with mass $12.0m_A$, is to be placed near the other three particles. In order for the net force on particle A due to the other three particles to be zero, where should particle D be located? Give (a) the distance from A in terms of d and (b) the angle relative to the positive direction of the x axis?

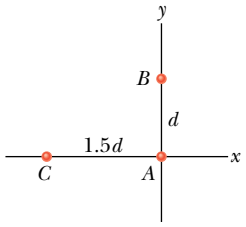

Figure 13-34 Problem 47.

48. What are (a) the speed and (b) the period of a 400 kg satellite in an approximately circular orbit 900 km above the surface of Earth? Suppose the satellite loses mechanical energy at the average rate of 1.4×10^5 J per orbital revolution. Adopting the reasonable approximation that the satellite's orbit becomes a "circle of slowly diminishing radius," determine the satellite's (c) altitude, (d) speed, and (e) period at the end of its 1500th revolution. (f) What is the magnitude of the average retarding force on the satellite? Is angular momentum around Earth's center conserved for (g) the satellite and (h) the satellite–Earth system (assuming that system is isolated)?

49. Zero, a hypothetical planet, has a mass of 5.0×10^{23} kg, a radius of 3.5×10^6 m, and no atmosphere. A 10 kg space probe is to be launched vertically from its surface. (a) If the probe is launched with an initial energy of 5.0×10^7 J, what will be its kinetic energy when it is 4.0×10^6 m from the center of Zero? (b) If the probe is to achieve a maximum distance of 8.0×10^6 m from the center of Zero, with what initial kinetic energy must it be launched from the surface of Zero?

50. Two small spaceships, each with mass $m = 1500$ kg, are in the circular Earth orbit of Fig. 13-35, at an altitude h of 400 km. Igor, the commander of one of the ships, arrives at any fixed point in the orbit 90 s ahead of Picard, the commander of the other ship. What are the (a) period T_0 and (b) speed v_0 of the ships? At point P in Fig. 13-35, Picard fires an instantaneous burst in the forward direction, *reducing* his ship's speed by 1.00%. After this burst, he follows the elliptical orbit shown dashed in the figure. What are the (c) kinetic energy and (d) potential energy of his ship immediately after the burst? In Picard's new elliptical orbit, what are (e) the total energy E, (f) the semimajor axis a, and (g) the orbital period T? (h) How much earlier than Igor will Picard return to P?

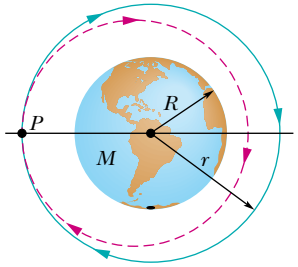

Figure 13-35 Problem 50.

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51. *One dimension.* In Fig. 13-36, two point particles are fixed on an x axis separated by distance d. Particle A has mass m_A and particle B has mass $25.0m_A$. A third particle C, of mass $75.0m_A$, is to be placed on the x axis and near particles A and B. In terms of distance d, at what x coordinate should C be placed so that the net gravitational force on particle A from particles B and C is zero?

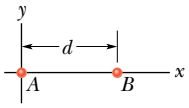

Figure 13-36 Problem 51.

52. The mean distance of Mars from the Sun is 1.52 times that of Earth from the Sun. From Kepler's law of periods, calculate the number of years required for Mars to make one revolution around the Sun; compare your answer with the value given in Appendix C.

53. (a) What will an object weigh on the Earth's surface if it weighs 160 N on the Moon's surface? (b) How many Earth radii must this same object be from the center of Earth if it is to weigh the same as it does on the Moon?

54. In Fig. 13-37*a*, particle A is fixed in place at $x = -0.20$ m on the x axis and particle B, with a mass of 1.5 kg, is fixed in place at the origin. Particle C (not shown) can be moved along the x axis, between particle B and $x = \infty$. Figure 13-37*b* shows the x component $F_{\text{net},x}$ of the net gravitational force on particle B due to particles A and C, as a function of position x of particle C. The plot actually extends to the right, approaching an asymptote of -4.17×10^{-10} N as $x \to \infty$. What are the masses of (a) particle A and (b) particle C?

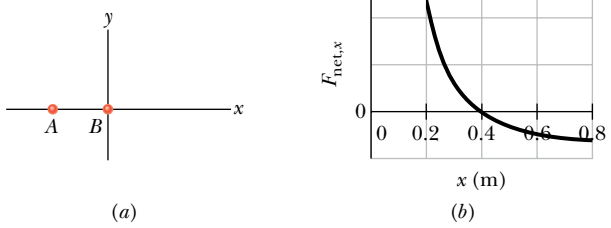

Figure 13-37 Problem 54.

55. A uniform solid sphere of radius R produces a gravitational acceleration of a_g on its surface. At what distance from the sphere's center are there points (a) inside and (b) outside the sphere where the gravitational acceleration is $a_g/2$?

56. A satellite is put in a circular orbit about Earth with a radius equal to two-thirds the radius of the Moon's orbit. What is its period of revolution in lunar months? (A lunar month is the period of revolution of the Moon.)

57. The mean diameters of Mars and Earth are 6.9×10^3 km and 1.3×10^4 km, respectively. The mass of Mars is 0.11 times Earth's mass. (a) What is the ratio of the mean density (mass per unit volume) of Mars to that of Earth? (b) What is the value of the gravitational acceleration on Mars? (c) What is the escape speed on Mars?

58. Figure 13-38 shows a spherical hollow inside a lead sphere of radius $R = 2.00$ cm; the surface of the hollow passes through the center of the sphere and "touches" the right side of the sphere. The mass of the sphere before hollowing was $M = 2.95$ kg. With what gravitational force does the hollowed-out lead sphere attract a small sphere of mass $m = 0.950$ kg that lies at a distance $d = 9.00$ cm from the center of the lead sphere, on the straight line connecting the centers of the spheres and of the hollow?

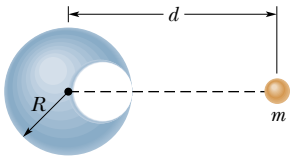

Figure 13-38 Problem 58.

59. At what angular speed of rotation is the surface material on the equator of a neutron star on the verge of flying off the star if the star is spherical with a radius of 18.0 km and a mass of 7.72×10^{24} kg?

60. *Mountain pull.* A large mountain can slightly affect the direction of "down" as determined by a plumb line. Assume that we can model a mountain as a sphere of radius $R = 1.80$ km and density (mass per unit volume) 2.6×10^3 kg/m^3. Assume also that we hang a 0.50 m plumb line at a distance of $3R$ from the sphere's center and such that the sphere pulls horizontally on the lower end. How far would the lower end move toward the sphere?

61. The Sun, which is 2.2×10^{20} m from the center of the Milky Way galaxy, revolves around that center once every 2.5×10^8 years. Assuming each star in the Galaxy has a mass equal to the Sun's mass of 2.0×10^{30} kg, the stars are distributed uniformly in a sphere about the galactic center, and the Sun is at the edge of that sphere, estimate the number of stars in the Galaxy.

62. Assume a planet is a uniform sphere of radius R that (somehow) has a narrow radial tunnel through its center (Fig. 13-8). Also assume we can position an apple anywhere along the tunnel or outside the sphere. Let F_R be the magnitude of the gravitational force on the apple when it is located at the planet's surface. How far from the surface is there a point where the magnitude is $1/3F_R$ if we move the apple (a) away from the planet and (b) into the tunnel?

63. (a) What is the gravitational potential energy of the two-particle system in Problem 22? If you triple the separation between the particles, how much work is done (b) by the gravitational force between the particles and (c) by you?

64. A satellite orbits a planet of unknown mass in a circle of radius 3.0×10^7 m. The magnitude of the gravitational force on the satellite from the planet is $F = 80$ N. (a) What is the kinetic energy of the satellite in this orbit? (b) What would F be if the orbit radius were increased to 4.0×10^7 m?

65. A 30 kg satellite has a circular orbit with a period of 2.0 h and a radius of 9.0×10^6 m around a planet of unknown mass. If the magnitude of the gravitational acceleration on the surface of the planet is 6.0 m/s^2, what is the radius of the planet?

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PRACTICE QUESTIONS

Single Correct Choice Type

1. A spring scale, calibrated in newtons, is used to weigh sugar. If it were possible to weigh sugar at the following locations, where will the buyer get the most sugar to a newton?
(a) At the North Pole.
(b) At the Equator.
(c) Near the center of Earth.
(d) On the Moon.

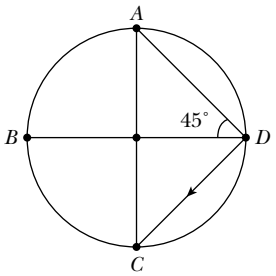
2. A spaceship is launched into a circular orbit close to the Earth's surface. What additional velocity has now to be imparted to the spaceship in the orbit to overcome the gravitational pull? (Radius of the Earth = 6400 km; $g = 9.8\ \text{m/s}^2$)
(a) 2.458 km/s **(b)** 5.343 km/s
(c) 3.278 km/s **(d)** 4.811 km/s

3. A mass m is located at the origin; a second mass m is at $x = d$. A third mass m is above the first two so that the three masses form an equilateral triangle. What is the net gravitational force on the third mass?
(a) $\dfrac{2Gm^2}{d^2}$ **(b)** $\dfrac{Gm^2}{d^2}$
(c) $\dfrac{\sqrt{3}Gm^2}{d^2}$ **(d)** $\dfrac{\sqrt{3}Gm^2}{2d^2}$

4. A satellite is launched into a circular orbit of radius R around the Earth. A second satellite is launched into an orbit of radius $(1.01)R$. The period of the second satellite is larger than the first one by approximately
(a) 0.7% **(b)** 1.5%
(c) 1.0% **(d)** 3.0%

5. A very long (length L) cylindrical galaxy is made of uniformly distributed mass and has radius $R(R \ll L)$. A star outside the galaxy is orbiting the galaxy in a plane perpendicular to the galaxy and passing through its center. If the time period of star is T and its distance from the galaxy's axis is r, then
(a) $T^2 \propto r^3$ **(b)** $T \propto r^2$
(c) $T \propto r$ **(d)** $T \propto \sqrt{r}$

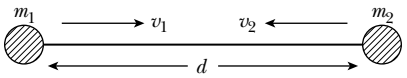
6. Four particles of equal mass m are placed at the corners of square and moving along a circle of radius r under the influence of mutual gravitational attraction. The speed of each particle will be



(a) $\sqrt{\dfrac{GM}{R}(2\sqrt{2}+1)}$ **(b)** $\sqrt{\dfrac{Gm}{r}\left(\dfrac{2\sqrt{2}+1}{4}\right)}$
(c) $\sqrt{\dfrac{Gm}{r}\left(\dfrac{\sqrt{2}+1}{4}\right)}$ **(d)** $\sqrt{\dfrac{Gm}{r}\left(\dfrac{2\sqrt{2}-1}{4}\right)}$

7. The orbit of a certain a satellite has a semi-major axis of 1.5×10^7 m and an eccentricity of 0.20. Its perigee (minimum distance) and apogee (maximum distance), respectively, are
(a) 6.0×10^6 m; 9.0×10^6 m
(b) 3.0×10^6 m; 1.2×10^7 m
(c) 1.2×10^7 m; 1.8×10^7 m
(d) 1.0×10^7 m; 1.2×10^7 m

8. Two hypothetical planets of masses m_1 and m_2 are at rest when they are infinite distance apart. Because of the gravitational force, they move toward each other along the line joining their centers. What is their speed when their separation is d?
(Speed of m_1 is v_1 and that of m_2 is v_2.)



(a) $v_1 = v_2$
(b) $v_1 = m_2\sqrt{\dfrac{2G}{d(m_1+m_2)}};\ v_2 = m_1\sqrt{\dfrac{2G}{d(m_1+m_2)}}$
(c) $v_1 = m_1\sqrt{\dfrac{2G}{d(m_1+m_2)}};\ v_2 = m_2\sqrt{\dfrac{2G}{d(m_1+m_2)}}$
(d) $v_1 = m_2\sqrt{\dfrac{2G}{m_1)}};\ v_2 = m_1\sqrt{\dfrac{2G}{m_1)}}$

9. Let the acceleration due to gravity be g_1 at a height h above the Earth's surface, and g_2 at a depth d below the Earth's surface. If $g_1 = g_2$, $h < R$, and $d < R$, then
(a) $h = d$
(b) $h = 2d$
(c) $2h = d$
(d) It is not possible for g_1 to be equal to g_2.

10. Consider a planet in some other solar system which has a mass double the mass of the Earth and density equal to the average density of the Earth. An object weighing W on the Earth will weigh
(a) W **(b)** $2W$
(c) $\dfrac{W}{2}$ **(d)** $2^{1/3}\,W$ at the planet

11. An astronaut finishes some work on the outside of his satellite, which is in circular orbit around the Earth. He leaves his wrench outside the satellite. If there is no air resistance, the wrench
(a) falls directly down to the Earth.
(b) continues in orbit at reduced speed.
(c) continues in orbit with the satellite.
(d) flies off tangentially into space.

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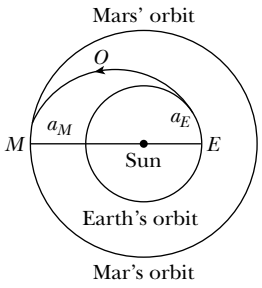
12. A particle of mass 1 kg is placed at a distance of 4 m from the center and on the axis of a uniform ring of mass 5 kg and radius 3 m. Find the work required to increase the distance of the particle from 4 m to $3\sqrt{3}$ m.

(a) 1.05×10^{-11} J **(b)** 1.11×10^{-11} J
(c) 1.43×10^{-11} J **(d)** 1.80×10^{-11} J

13. India's Mangalyan was sent to the Mars by launching it into a transfer orbit EOM around the Sun. It leaves the Earth at E and meets Mars at M. If the semi-major axis of Earth's orbit is $a_E = 1.5 \times 10^{11}$ m, that of Mar's orbit $a_M = 2.28 \times 10^{11}$ m, taken Kepler's laws give the estimate of time for Mangalyan to reach Mars from Earth to be close to

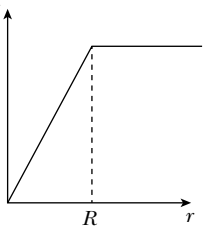


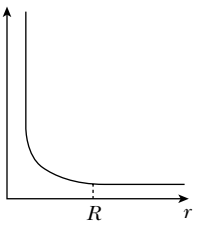
(a) 260 days **(b)** 320 days
(c) 500 days **(d)** 220 days

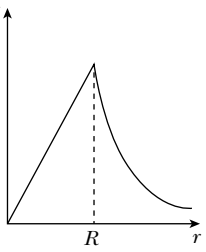
14. A spherically symmetric gravitational system of particles has a mass density

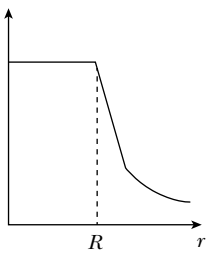
$$\rho = \begin{cases} \rho_0 & \text{for} \quad r \le R, \\ 0 & \text{for} \quad r > R, \end{cases}$$

where ρ_0 is a constant. A test mass can undergo circular motion under the influence of the gravitational field of particles. Its speed v as a function of distance r $(0 < r < \infty)$ from the center of the system is represented by

(a)


(b)


(c)


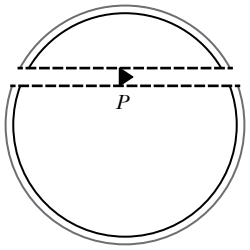
(d)


15. A planet of radius R = 1/10 × (radius of Earth) has the same mass density as Earth. Scientists dig a well of depth $R/5$ on it and lower a wire of the same length and of linear mass density 10^{-3} kg/m into it. If the wire is not touching anywhere, the force applied at the top of the wire by a person holding it in place is

(Radius of Earth = 6×10^6 m; Acceleration due to gravity of Earth = 10 m/s^2)

(a) 96 N **(b)** 108 N
(c) 120 N **(d)** 150 N

16. A spherical shell is cut into two pieces along a chord as shown in the figure. P is a point on the plane of the chord. The gravitational field at P due to the upper part is I_1 and that due to the lower part is I_2. What is the relation between them?



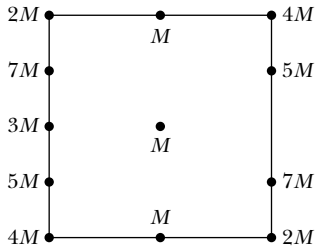
(a) $I_1 > I_2$ **(b)** $I_1 < I_2$
(c) $I_1 = I_2$ **(d)** No definite relation

17. A system of binary stars of masses m_A and m_B are moving in circular orbits of radii r_A and r_B, respectively. If T_A and T_B are the time periods of masses m_A and m_B, respectively, then

(a) $\dfrac{T_A}{T_B} = \left(\dfrac{r_A}{r_B}\right)^{3/2}$ **(b)** $T_A > T_B$ if $r_A > r_B$

(c) $T_A > T_B$ if $m_A > m_B$ **(d)** $T_A = T_B$

18. In the following figure, a central particle of mass M is surrounded by a square array of other particles, separated by either distance d or distance $d/2$ along the perimeter of the square. What are the values of the net gravitational force on the central particle due to the other particles?



(a) $\dfrac{40GM^2}{d^2}$ **(b)** $\dfrac{3GM^2}{d^2}$

(c) $\dfrac{21GM^2}{2d^2}$ **(d)** $\dfrac{2GM^2}{21d^2}$

19. How far from Earth must a space probe be along a line toward the Sun so that the Sun's gravitational pull on the probe balances Earth's pull?

(Mass of Sun = 1.99×10^{30} kg; Mass of Earth = 5.98×10^{24} kg; Distance between Earth and Sun = 150×10^9 m)

(a) 1.30×10^8 m **(b)** 2.60×10^8 m
(c) 3.90×10^8 m **(d)** 5.20×10^8 m

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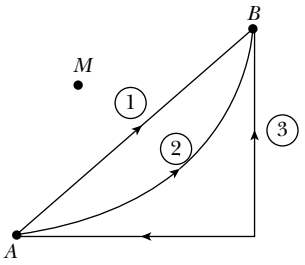
20. If W_1, W_2 and W_3 represent the work done in moving a particle of mass m from the point A to the point B along three different paths in the gravitational field of a point mass M as shown in the figure. The work done will satisfy the relation

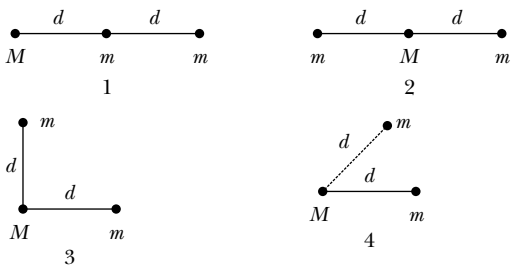


(a) $W_1 > W_2 > W_3$ **(b)** $W_1 < W_2 < W_3$
(c) $W_1 = W_2 = W_3$ **(d)** $W_1 > W_2 < W_3$

21. An elevator supported by a single cable descends a shaft at a constant speed. The only forces acting on the elevator are the tension in the cable and the gravitational force. Which one of the following statements is true?
(a) The net work done by the two forces is zero.
(b) The magnitude of work done by the gravitational force is greater than that done by tension.
(c) The magnitude of the work done by the tension is greater than that done by the gravitational force.
(d) The net work done by the gravitational force is negative.

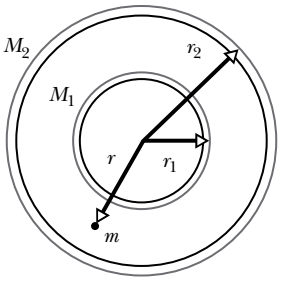
22. Inside a satellite orbiting very close to the Earth's surface, water does not fall out of a glass when it is inverted. Which of the following is the best explanation for this?
(a) The Earth does not exert any force on the water.
(b) The Earth's force of attraction on the water is exactly balanced by the force created by the satellite's motion.
(c) The water and the glass have the same acceleration, equal to g, toward the center of the Earth, and hence there is no relative motion between them.
(d) The gravitational attraction between the glass and the water balances the Earth's attraction on the water.

23. Three particles, two with mass m and one mass M, might be arranged in any of the four configurations shown below. Rank the configurations according to the magnitude of the gravitational force on M, least to greatest.



(a) 1, 2, 3, 4 **(b)** 2, 1, 3, 4
(c) 2, 1, 4, 3 **(d)** 2, 3, 4, 2

24. Two concentric shells of mass M_1 and M_2 are having radii r_1 and r_2. Which of the following is the correct expression for the gravitational field on a mass m?



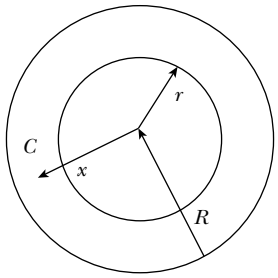
(a) $I = \dfrac{G(M_1 + M_2)}{r^2}$ for $r < r_1$

(b) $I = \dfrac{G(M_1 + M_2)}{r^2}$ for $r < r_2$

(c) $I = G\dfrac{M_2}{r^2}$ for $r_1 < r < r_2$

(d) $I = \dfrac{GM_1}{r^2}$ for $r_1 < r < r_2$

25. Gravitational force acting on a 1 kg particle placed at point C in a spherical shell of inner radius r and outer radius R and mass m and which is uniformly distributed is



(a) $\dfrac{Gm(x^3 - r^3)}{x^3(R^3 - r^3)}$ **(b)** $\dfrac{Gm(x^3 - r^3)}{x^2(R^2 - r^2)}$

(c) $\dfrac{Gm(x^3 - r^3)}{x^2(R^3 - r^3)}$ **(d)** $\dfrac{Gm(x^2 - r^2)}{x(R^2 - r^2)}$

More than One Correct Choice Type

26. For a satellite to be geostationary, which of the following are essential conditions?
(a) It must always be stationed above the Equator.
(b) It must rotate from West to East.
(c) It must be about 36,000 km above the Earth.
(d) Its orbit must be circular, and not elliptical.

27. Two bodies of masses m_1 and m_2 are initially at rest at infinite distance apart. They are then allowed to move toward each other under mutual gravitational attraction. Their relative velocity of approach at a separation distance r between them is

(a) $\left[2G\dfrac{(m_1 - m_2)}{r}\right]^{1/2}$ **(b)** $\left[\dfrac{2G}{r}(m_1 + m_2)\right]^{1/2}$

(c) $\left[\dfrac{r}{2G(m_1 m_2)}\right]^{1/2}$ **(d)** $\left[\dfrac{2G}{r}m_1 m_2\right]^{1/2}$

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28. For a satellite revolving in a circular orbit suppose v_0 is the orbital speed, T its time period, U its potential energy and K the kinetic energy. Now, the value of G is decreased. Then

(a) v_0 decreases
(b) T decreases
(c) $U = v/2$ decreases
(d) K decreases

29. Two spherical planets P and Q have the same uniform density ρ, masses M_P and M_Q and surface areas A and $4A$, respectively. A spherical planet R also has uniform density ρ and its mass is $M_P + M_Q$. The escape velocities from the planets P, Q and R are v_P, v_Q and v_R, respectively. Then

(a) $v_Q > v_R > v_P$ **(b)** $v_R > v_Q > v_P$
(c) $\frac{v_R}{v_P} = 3$ **(d)** $\frac{v_P}{v_Q} = \frac{1}{2}$

30. Two bodies, each of mass M, are kept fixed with a separation $2L$. A particle of mass m is projected from the midpoint of the line joining their centers, perpendicular to the line. The gravitational constant is G. The correct statement(s) is(are):

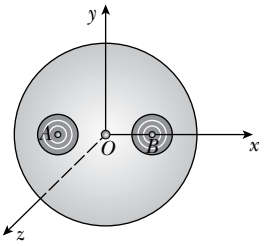
(a) The minimum initial velocity of the mass m to escape the gravitational field of the two bodies is $4\sqrt{\frac{GM}{L}}$.
(b) The minimum initial velocity of the mass m to escape the gravitational field of the two bodies is $2\sqrt{\frac{GM}{L}}$.
(c) The minimum initial velocity of the mass m to escape the gravitational field of the two bodies is $\sqrt{\frac{2GM}{L}}$.
(d) The energy of the mass m remains constant.

31. The magnitudes of gravitational field at distances r_1 and r_2 from the center of a uniform sphere of radius R and mass M are F_1 and F_2, respectively. Then

(a) $\frac{F_1}{F_2} = \frac{r_1}{r_2}$ if $r_1 < R$ and $r_2 < R$
(b) $\frac{F_1}{F_2} = \frac{r_1^2}{r_2^2}$ if $r_1 > R$ and $r_2 > R$
(c) $\frac{F_1}{F_2} = \frac{r_1}{r_2}$ if $r_1 > R$ and $r_2 > R$
(d) $\frac{F_1}{F_2} = \frac{r_1^2}{r_2^2}$ if $r_1 < R$ and $r_2 < R$

32. A solid sphere of uniform density and radius 4 unit is located with its center at the origin O of coordinates. Two spheres of equal radii 1 unit, with their centers at $A(-2, 0, 0)$ and $B(2, 0, 0)$, respectively, are taken out of the solid leaving behind spherical cavities as shown in the below figure, then

(a) the gravitational force due to this object at the origin is zero.
(b) the gravitational force at the point $B(2, 0, 0)$ is zero.
(c) the gravitational potential is the same at all points of the circle $y^2 + z^2 = 36$.
(d) the gravitational potential is the same at all points on the circle $y^2 + z^2 = 4$.



33. A small mass m is moved slowly from the surface of the Earth to a height h above the surface. The work done (by an external agency) in doing this is

(a) mgh, for all values of h **(b)** mgh, for $h \ll R$
(c) $\frac{1}{2}mgh$, for $h = R$ **(d)** $-\frac{1}{2}mgh$, for $h = R$

Linked Comprehension

Paragraph for Questions 34 to 36: While weight of a body depends directly on acceleration due to gravity g, the value of g depends upon many factors. It depends on the shape of the Earth, rotation of the Earth, etc. Weight of a body at a pole is more than that at a place on equator because g is maximum at poles and minimum on equator. Acceleration due to gravity g varies with latitude λ according to the equation $g_x = g - R\omega^2 \cos^2\lambda$, where R is radius of Earth and ω is angular velocity of Earth. A body of mass m weighs W_r in a train at rest. The train then begins to run with a velocity v around the equator from west to east. It conserved that weight W_m of the same body in the moving train is different from W_r. Let v_e be the velocity of a point on equator with respect to axis of rotation of Earth and R be the radius of the Earth. Clearly, the relative velocity between Earth and train will affect the weight of the body.

34. Difference between weight W_r and the gravitational attraction on the body can be given as

(a) $\frac{mv^2}{R}$ **(b)** $\frac{1}{2}\frac{mv^2}{R}$
(c) $\frac{mv_e^2}{R}$ **(d)** $\frac{mv_e^2}{R}$

35. Weight W_m of the body can be given as

(a) $mg - m\frac{(v_e + v)^2}{R}$ **(b)** $mg - m\frac{(v_e - v)^2}{R}$
(c) $\frac{m}{R}[v_e^2 - (v_e + v)^2]$ **(d)** $mg + m\frac{(v_e + v)^2}{R}$

36. Difference in the weight in the two given states

(a) $\frac{m}{R}v_e^2$ **(b)** $\frac{m}{R}(v_e + v)^2$
(c) $\frac{m}{R}[(v_e + v)^2 - v_e^2]$ **(d)** $\frac{m}{R}[(v_e - v)^2 - v_e^2]$

Paragraph for Questions 37–39: Two small particles of mass m lie on either side of Earth. The particles located along the line joining the centers of the Earth and Moon. The difference between

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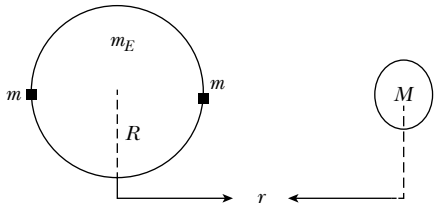
the force exerted by Moon on particles is called tidal force. The astronomical objects are held together by force of gravity. If the tidal force on such object is greater than the gravitational forces holding the objects together the object will fly off.

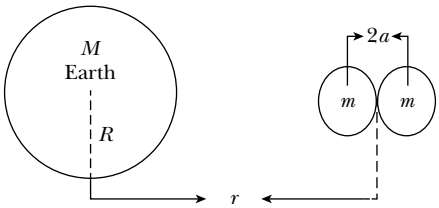
37. Tidal force if $r \gg R$, if M mass of Moon, R radius of Earth, r distance between Moon to Earth.



(a) $\frac{GMmR}{r^3}$ **(b)** $\frac{2GMmR}{r^3}$

(c) $\frac{3GMmR}{r^3}$ **(d)** $\frac{4GMmR}{r^3}$

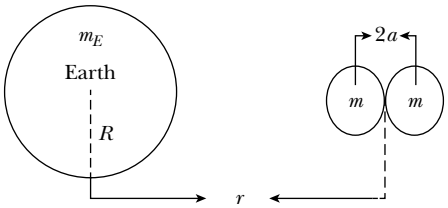
38. Two spheres of masses m and radius a are in contact in space minimum distance of this system from Earth is (this minimum distance is called Rochie limit) (M is mass of the Earth)



(a) $\left(\frac{2M}{m}\right)^{1/3} a$ **(b)** $\left(\frac{4M}{m}\right)^{1/3} a$

(c) $\left(\frac{8M}{m}\right)^{1/3} a$ **(d)** $\left(\frac{16M}{m}\right)^{1/3} a$

39. In the above case, two spheres replaced by a satellite of density ρ_s can exist near a planet of density ρ_p and radius R, Rochie limit is



(a) $\left(\frac{2\rho_p}{\rho_s}\right)^{1/3} R$ **(b)** $\left(\frac{16\rho_p}{\rho_s}\right)^{1/3} R$

(c) $\left(\frac{4\rho_p}{\rho_s}\right)^{1/3} R$ **(d)** $\left(\frac{8\rho_p}{\rho_s}\right)^{1/3} R$

Paragraph for Questions 40–42: A small body is dropped into a narrow channel drilled along the Earth axis. Considering the Earth to be a homogeneous sphere and disregarding air resistance answer the following questions. (Acceleration due gravity on the surface of the Earth is g; Radius of Earth is R)

40. The magnitude of body's acceleration, $a(r)$ as a function of the distance r from the Earth's center is

(a) $g\frac{r}{R}$ **(b)** $2g\frac{r}{R}$

(c) $g\frac{r}{2R}$ **(d)** None of these

41. The magnitude of body's speed as a function of r is

(a) $\sqrt{\frac{2g(R^2-r^2)}{R}}$ **(b)** $\sqrt{\frac{g(R^2-r^2)}{R}}$

(c) $\sqrt{\frac{g(R^2-r^2)}{2R}}$ **(d)** None of these

42. The ratio of speed of the body at the instant when it reaches the center of Earth to the escape speed on the surface of the Earth is

(a) $\frac{1}{2}$ **(b)** 1

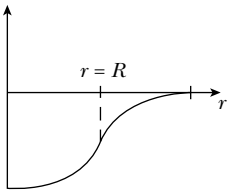
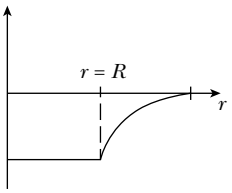
(c) $\frac{1}{\sqrt{2}}$ **(d)** None of these

Matrix-Match

43. A satellite of mass m is orbiting around the Earth of mass M and radius R. Match the correct expression in Column II with the statements given in Column I.

Column I	Column II
(a) Gravitational potential energy of satellite orbiting close to Earth is	**(p)** $\frac{GMm}{2R}$
(b) Kinetic energy of orbiting satellite close to Earth is	**(q)** $-\frac{GMm}{2R}$
(c) Kinetic energy of orbiting satellite at height R from the surface of Earth	**(r)** $\frac{GMm}{4R}$
(d) Total mechanical energy of satellite orbiting close to Earth	**(s)** $-\frac{GMm}{R}$

44. Map the correct graph(s) in Column II for each statement in Column I.

Column I	Column II
(a) g due to a spherical shell of mass M and radius R	**(p)** 
(b) g due to a solid sphere of mass M and radius R	**(q)** 

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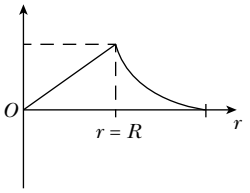
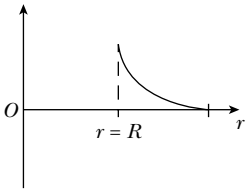
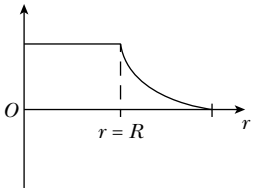
Column I	Column II
(c) Gravitational potential V for a solid sphere	**(r)** 
(d) Gravitational potential V for a spherical shell	**(s)** 
	(t) 

Directions for Question 45 and 46: There is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

45. Given are the two different masses M_1 and M_2 separated by a distance r in different combinations.

Column I	Column II	Column III
(I) $M_1 = 22$ kg	**(i)** $M_2 = 14$ kg	**(J)** $r = 6$ cm
(II) $M_1 = 21$ kg	**(ii)** $M_2 = 15$ kg	**(K)** $r = 4$ cm
(III) $M_1 = 12$ kg	**(iii)** $M_2 = 16$ kg	**(L)** $r = 5$ cm
(IV) $M_1 = 10$ kg	**(iv)** $M_2 = 18$ kg	**(M)** $r = 7$ cm

(1) The combination that results in the gravitational force 420.21×10^{-11} N is
- **(a)** (III) (ii) (J)
- **(b)** (II) (iv) (J)
- **(c)** (II) (i) (M)
- **(d)** (II) (i) (K)

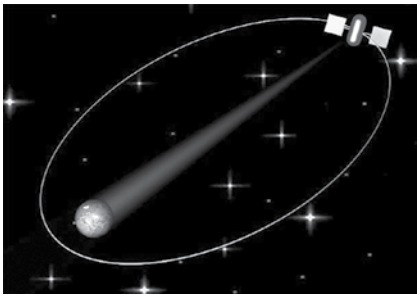
(2) The combination that results in the gravitational force 160.08×10^{-11} N is
- **(a)** (I) (iii) (K)
- **(b)** (IV) (iii) (L)
- **(c)** (II) (iii) (L)
- **(d)** (III) (i) (M)

(3) The combination that results in the gravitational force 200.1×10^{-11} N is
- **(a)** (III) (i) (K)
- **(b)** (I) (i) (L)
- **(c)** (IV) (ii) (L)
- **(d)** (I) (iv) (L)

46. In the three columns, information related satellites is given.

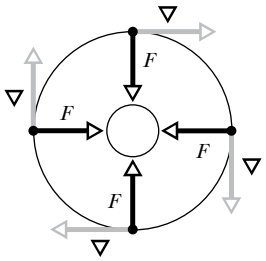
Column I	Column II	Column III
(I) $v_0 < v < v_e$	**(i)** The satellite follows an elliptical orbit	**(J)** along a parabolic trajectory.
(II) $v = v_e$	**(ii)** The satellite follows a circular orbit	**(K)** with center of the Earth as the further focus.
(III) $v = v_0$	**(iii)** The satellite follows an elliptical path	**(L)** with center of the Earth as the nearer focus.
(IV) $v < v_0$	**(iv)** The satellite escapes the gravitational field of the Earth.	**(M)** with center of Earth as the center of the orbit.

(1) What does the following figure signify?



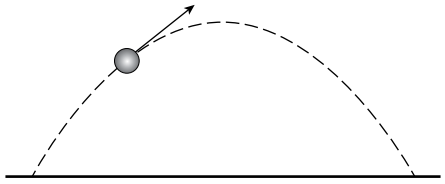
- **(a)** (I) (iii) (L)
- **(b)** (IV) (ii) (M)
- **(c)** (II) (iv) (K)
- **(d)** (IV) (iii) (K)

(2) What does the following figure signify?



- **(a)** (III) (ii) (L)
- **(b)** (III) (ii) (M)
- **(c)** (II) (iii) (K)
- **(d)** (I) (i) (M)

(3) What does the following figure signify?



- **(a)** (II) (iv) (J)
- **(b)** (I) (i) (K)
- **(c)** (III) (iii) (L)
- **(d)** (I) (ii) (L)

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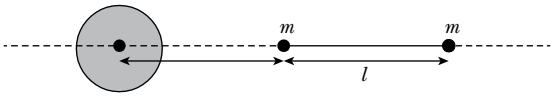
Integer Type

47. Gravitational acceleration on the surface of a planet is $\sqrt{6}/11\,g$, where g is the gravitational acceleration on the surface of the Earth. The average mass density of the planet is 2/3 times that of the Earth. If the escape speed on the surface of the Earth is taken to be 11 km/s, the escape speed on the surface of the planet (in km/s) is _____.

48. A large spherical mass M is fixed at one position and two identical point masses m are kept on a line passing through the center of M (see figure). The point masses are connected by a rigid massless rod of length l and this assembly is free to move along the line connecting them. All three masses interact only through their mutual gravitational interaction. When the point mass nearer to M is at a distance $r = 3l$ from M, the tension in the rod is zero for $m = k(M/288)$. The value of k is _____.



49. A body weighs 64 N on the surface of the Earth. What is the gravitational force on it due to the Earth at a height equal to one-third of the radius of the Earth?

50. What must the separation be between a 5.2 kg particle and a 2.4 kg particle for their gravitational attraction to have a magnitude of 2.3×10^{-12} N?

51. If the value of gravitational constant becomes double, what is the ratio of inertial mass to gravitational mass of Moon?

ANSWER KEY

Checkpoints

1. All tie
2. (a) 1, tie of 2 and 4, then 3; (b) line of length d
3. (a) Increase; (b) Negative
4. (a) 2; (b) 1
5. (a) Path 1 [decreased E (more negative) gives decreased a]; (b) Less (decreased a gives decreased T)

Problems

1. 2.8×10^{-10} N
2. (a) 2.58×10^4 m/s or about 26 km/s; (b) 1.29×10^7 m/s $\approx 1.3 \times 10^4$ km/s
3. 0.71 y
4. ≈1.8 m
5. (a) 7.58×10^6 m; (b) 1.06×10^7 m; (c) The mechanical energy is zero for the "escape condition."
6. (a) 1.54×10^{-8} N; (b) 1.85×10^{-8} N; (c) If the particle is at C, then no other mass is at smaller radius and the gravitational force on it is zero.
7. (a) 3.19×10^6 m; (b) The energy of lifting is greater.
8. 2.60×10^8 m
9. (a) -7.60×10^9 J; (b) -7.60×10^9 J; (c) Toward the center of the planet.
10. (a) $T \propto r^{3/2}$; (b) $K \propto \frac{1}{r}$; (c) and (d) (c) and (d) From the previous part, knowing that K is proportional to v^2, we find that v is proportional to $1/\sqrt{r}$. Thus, by Eq. 13-33, the angular momentum (which depends on the product rv) is proportional to $r/\sqrt{r} = \sqrt{r}$.
11. (a) $-1.88d$; (b) $-3.90d$; (c) $0.489d$. In this way, we are able to deduce that $(x, y, z) = (1.88\text{d}, 3.90\text{d}, 0.489\text{d})$.
12. (a) 1.89×10^{13} m; (b) $\approx 6.4R_P$
13. (a) 1.39 m/s; (b) 0.772 m/s
14. (a) +1.7 pJ; (b) –1.7 pJ
15. 2.16
16. (a) 5.4×10^4 km/h; (b) 3.8×10^4 km/h
17. (a) 5.01×10^9 m; (b) 7.20
18. (a) $m = M$; (b) The net force is still zero.
19. (a) 90.3 min; (b) 1.85×10^{-4} Hz; (c) 7.73 km/s
20. 3.58×10^7 m
21. (a) 0.0451; (b) 28.5
22. ≈4.2 kg
23. 5.0×10^9 J
24. (a) $\approx 2.3 \times 10^9$ J; (b) $r = 2.17(1.15R_s) = 2.5R_s$
25. (a) 5.45×10^{-8} N; (b) The direction of the force relative to the $+x$ axis is 60.6°.
26. $\vec{F}_{\text{net}} = F_{\text{net}}(\cos 45°\hat{\text{i}} + \sin 45°\hat{\text{j}}) = (9.43 \times 10^{-15}\ \text{N})\hat{\text{i}} + (9.43 \times 10^{-15}\ \text{N})\hat{\text{j}}$
27. (a) $\approx 1.4 \times 10^{-8}$ J; (b) 2.6×10^{-9} J
28. 1/2
29. 6.5×10^{23} kg

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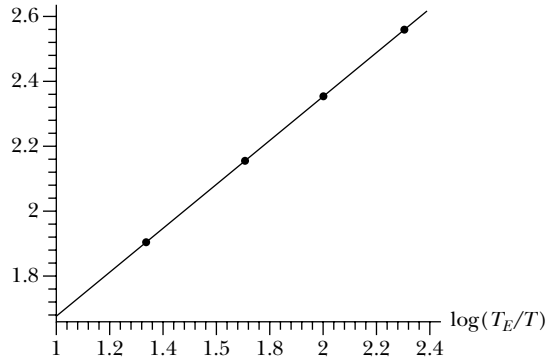
30. (a) $\log(r_E/a)$



(b) Slope = 2/3; (c) 1.86×10^{27} kg

31. $v = \sqrt{GM/L}$ **32.** (a) –0.22 m; (b) –0.39 m **33.** (a) 7.18×10^6 m; (b) 0.0877

34. $m_2/M_s \approx 9$ **35.** (a) 2.5×10^3 m/s; (b) 1.3×10^5 m; (c) 1.9×10^3 m/s

36. 7.8×10^6 m **37.** (a) $T = 5.2$ y; (b) 6.7×10^{-5}

38. (a) $(3.02 \times 10^{43}\ \text{kg} \cdot \text{m/s}^2)/M_h$; (b) a_g decreases as M_h increases; (c) $a_g = 9.82\ \text{m/s}^2$; (d) $7.10 \times 10^{-15}\ \text{m/s}^2$; (e) The miniscule result of the previous part implies that, in this case, any effects due to the differences of gravitational forces on the body are negligible.

39. (a) $\approx 5.8 \times 10^5$ J; (b) $\approx 2.6 \times 10^2$

40. (a) 9.83 m/s; (b) 9.84 m/s^2; (c) 9.79 m/s^2 **41.** 1.5 s **42.** (a) 6.9%; (b) $\approx (2.3 \times 10^{-5})\%$

43. $m/M = 1/2$ **44.** (a) 5.9×10^{-7} N; (b) 6.7×10^{-7} N; (c) $(1.3 \times 10^{-6}\ \text{N/m})r$

45. (a) 0.486; (b) 0.486; (c) Satellite B has the largest energy; (d) $\Delta E = 2.3 \times 10^8$ J. **46.** 1.20×10^{-13} J

47. (a) $r^2 = \left(\dfrac{12}{2.404}\right) d^2$; (b) –56.3°

48. (a) 7.4 km/s; (b) 6.2×10^3 s $\gg$ 103 min; (c) 7.6×10^5 m; (d) $\approx$7.5 km/s; (e) 6.0×10^3 s; (f) 3.1×10^{-3} N; (g) The angular momentum of the satellite is not conserved; (h) the angular momentum of the satellite–Earth system is very nearly conserved.

49. (a) 3.8×10^7 J; (b) 5.4×10^7 J

50. (a) $\approx$92.3 min; (b) $\approx 7.68 \times 10^3$ m/s^2; (c) 4.33×10^{10} J; (d) -8.84×10^{10} J; (e) -4.51×10^{10} J; (f) 6.64×10^6 m; (g) $\approx$89.7 min; (h) 72 s

51. $-1.73d$ **52.** 1.87 y; The value in Appendix C (1.88 y) is quite close.

53. (a) 939 N; (b) 1.5×10^7 m

54. (a) 0.17 kg; (b) 0.67 kg **55.** (a) $r = R/2$; (b) $r = \sqrt{3}R$

56. 0.54 lunar month for the period of the satellite.

57. (a) 0.74; (b) 3.8 m/s^2; (c) 5.0×10 m/s **58.** 1.94×10^{-8} N **59.** $\approx$1.50 rev/s **60.** 7.4×10^{-6} m

61. $5\ 10^{10}$ **62.** (a) $r = R\sqrt{3}$; (b) $r = R/3$ **63.** (a) -7.6×10^{-11} J; (b) -5.1×10^{-11} J; (c) 5.1×10^{-11} J

64. (a) 1.2×10^9 J; (b) 45 N **65.** 9.6×10^6 m

Practice Questions

Single Correct Choice Type

1. (c) **2.** (c) **3.** (c) **4.** (b) **5.** (c)
6. (b) **7.** (a) **8.** (b) **9.** (c) **10.** (d)
11. (c) **12.** (b) **13.** (a) **14.** (c) **15.** (d)
16. (c) **17.** (d) **18.** (b) **19.** (b) **20.** (c)
21. (a) **22.** (c) **23.** (b) **24.** (d) **25.** (c)

More than One Correct Choice Type

26. (a), (b), (c), (d) **27.** (a), (c), (d) **28.** (a), (d) **29.** (b), (d) **30.** (b), (d)
31. (a), (b) **32.** (a), (c), (d) **33.** (b), (c)

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Linked Comprehension

34. (d) 35. (c) 36. (c) 37. (d) 38. (d)
39. (b) 40. (d) 41. (b) 42. (c)

Matrix-Match

43. (a) → (r); (b) → (p); (c) → (s); (d) → (q)
44. (a) → (s); (b) → (r); (c) → (p); (d) → (q)
45. (1) → (b); (2) → (d); (3) → (c)
46. (1) → (d); (2) → (b); (3) → (a)

Integer Type

47. 3 48. 7 49. 36 50. 19 51. 1

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CHAPTER 14

Fluids

14.1 | WHAT IS PHYSICS?

The physics of fluids is the basis of hydraulic engineering, a branch of engineering that is applied in a great many fields. A nuclear engineer might study the fluid flow in the hydraulic system of an aging nuclear reactor, while a medical engineer might study the blood flow in the arteries of an aging patient. An environmental engineer might be concerned about the drainage from waste sites or the efficient irrigation of farmlands. A naval engineer might be concerned with the dangers faced by a deep-sea diver or with the possibility of a crew escaping from a downed submarine. An aeronautical engineer might design the hydraulic systems controlling the wing flaps that allow a jet airplane to land. Hydraulic engineering is also applied in many Broadway and Las Vegas shows, where huge sets are quickly put up and brought down by hydraulic systems.

Before we can study any such application of the physics of fluids, we must first answer the question "What is a fluid?"

What Is a Fluid?

A **fluid**, in contrast to a solid, is a substance that can flow. Fluids conform to the boundaries of any container in which we put them. They do so because a fluid cannot sustain a force that is tangential to its surface. (In the more formal language, a fluid is a substance that flows because it cannot withstand a shearing stress. It can, however, exert a force in the direction perpendicular to its surface.) Some materials, such as pitch, take a long time to conform to the boundaries of a container, but they do so eventually; thus, we classify even those materials as fluids.

You may wonder why we lump liquids and gases together and call them fluids. After all (you may say), liquid water is as different from steam as it is from ice. Actually, it is not. Ice, like other crystalline solids, has its constituent atoms organized in a fairly rigid three-dimensional array called a crystalline lattice. In neither steam nor liquid water, however, is there any such orderly long-range arrangement.

Contents

14.1 What is Physics?
14.2 Density and Pressure
14.3 Fluids at Rest
14.4 Measuring Pressure
14.5 Pascal's Principle
14.6 Archimedes' Principle
14.7 Hydrostatic Force on a Curved Surface
14.8 Linear Accelerated Motion of a Fluid
14.9 Motion of a Fluid in a Rotating Vessel
14.10 Buoyant Force
14.11 Fluid Dynamics
14.12 Bernoulli's Equation

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14.2 | DENSITY AND PRESSURE

Key Concepts

- The density ρ of any material is defined as the material's mass per unit volume:

$$\rho = \frac{\Delta m}{\Delta V}.$$

  Usually, where a material sample is much larger than atomic dimensions, we can write this as

$$\rho = \frac{m}{V}.$$

- A fluid is a substance that can flow; it conforms to the boundaries of its container because it cannot withstand shearing stress. It can, however, exert a force perpendicular to its surface. That force is described in terms of pressure p:

$$p = \frac{\Delta F}{\Delta A},$$

  in which ΔF is the force acting on a surface element of area ΔA. If the force is uniform over a flat area, this can be written as

$$p = \frac{F}{A}.$$

- The force resulting from fluid pressure at a particular point in a fluid has the same magnitude in all directions.

When we discuss rigid bodies, we are concerned with particular lumps of matter, such as wooden blocks, baseballs, or metal rods. Physical quantities that we find useful, and in whose terms we express Newton's laws, are mass and force. We might speak, for example, of a 3.6 kg block acted on by a 25 N force.

With fluids, we are more interested in the extended substance and in properties that can vary from point to point in that substance. It is more useful to speak of **density** and **pressure** than of mass and force.

Density

To find the density ρ of a fluid at any point, we isolate a small volume element ΔV around that point and measure the mass Δm of the fluid contained within that element. The **density** is then

$$\rho = \frac{\Delta m}{\Delta V}. \qquad (14\text{-}1)$$

In theory, the density at any point in a fluid is the limit of this ratio as the volume element ΔV at that point is made smaller and smaller. In practice, we assume that a fluid sample is large relative to atomic dimensions and thus is "smooth" (with uniform density), rather than "lumpy" with atoms. This assumption allows us to write the density in terms of the mass m and volume V of the sample:

$$\rho = \frac{m}{V} \quad \text{(uniform density)}. \qquad (14\text{-}2)$$

Density is a scalar property; its SI unit is the kilogram per cubic meter. Table 14-1 shows the densities of some substances and the average densities of some objects. Note that the density of a gas (see Air in the table) varies considerably with pressure, but the density of a liquid (see Water) does not; that is, gases are readily *compressible* but liquids are not.

Table 14-1 Some Densities

Material or Object	Density (kg/m^3)
Interstellar space	10^{-20}
Best laboratory vacuum	10^{-17}
Air: 20°C and 1 atm pressure	1.21
20°C and 50 atm	60.5
Styrofoam	1×10^2
Ice	0.917×10^3
Water: 20°C and 1 atm	0.998×10^3
20°C and 50 atm	1.000×10^3
Seawater: 20°C and 1 atm	1.024×10^3
Whole blood	1.060×10^3
Iron	7.9×10^3
Mercury (the metal, not the planet)	13.6×10^3
Earth: average	5.5×10^3
core	9.5×10^3
crust	2.8×10^3
Sun: average	1.4×10^3
core	1.6×10^5
White dwarf star (core)	10^{10}
Uranium nucleus	3×10^{17}
Neutron star (core)	10^{18}

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Pressure

Let a small pressure-sensing device be suspended inside a fluid-filled vessel, as in Fig. 14-1*a*. The sensor (Fig. 14-1*b*) consists of a piston of surface area ΔA riding in a close-fitting cylinder and resting against a spring. A readout arrangement allows us to record the amount by which the (calibrated) spring is compressed by the surrounding fluid, thus indicating the magnitude ΔF of the force that acts normal to the piston. We define the **pressure** on the piston as

$$p = \frac{\Delta F}{\Delta A}. \tag{14-3}$$

In theory, the pressure at any point in the fluid is the limit of this ratio as the surface area ΔA of the piston, centered on that point, is made smaller and smaller. However, if the force is uniform over a flat area A (it is evenly distributed over every point of the area), we can write Eq. 14-3 as

$$p = \frac{F}{A} \quad \text{(pressure of uniform force on flat area)}, \tag{14-4}$$

where F is the magnitude of the normal force on area A.

We find by experiment that at a given point in a fluid at rest, the pressure p defined by Eq. 14-4 has the same value no matter how the pressure sensor is oriented. Pressure is a scalar, having no directional properties. It is true that the force acting on the piston of our pressure sensor is a vector quantity, but Eq. 14-4 involves only the *magnitude* of that force, a scalar quantity.

The SI unit of pressure is the newton per square meter, which is given a special name, the **pascal** (Pa). In metric countries, tire pressure gauges are calibrated in kilopascals. The pascal is related to some other common (non-SI) pressure units as follows:

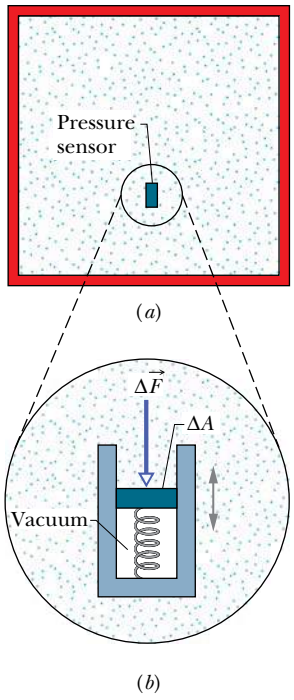

Figure 14-1 (*a*) A fluid-filled vessel containing a small pressure sensor, shown in (*b*). The pressure is measured by the relative position of the movable piston in the sensor.

$$1 \text{ atm} = 1.01 \times 10^5 \text{ Pa} = 760 \text{ torr} = 14.7 \text{ lb/in.}^2.$$

The *atmosphere* (atm) is, as the name suggests, the approximate average pressure of the atmosphere at sea level. The *torr* (named for Evangelista Torricelli, who invented the mercury barometer in 1674) was formerly called the *millimeter of mercury* (mm Hg). The pound per square inch is often abbreviated psi. Table 14-2 shows some pressures.

Table 14-2 Some Pressures

	Pressure (Pa)
Center of the Sun	2×10^{16}
Center of Earth	4×10^{11}
Highest sustained laboratory pressure	1.5×10^{10}
Deepest ocean trench (bottom)	1.1×10^{8}
Spike heels on a dance floor	10^{6}
Automobile tire[a]	2×10^{5}
Atmosphere at sea level	1.0×10^{5}
Normal blood systolic pressure[a,b]	1.6×10^{4}
Best laboratory vacuum	10^{-12}

[a]Pressure in excess of atmospheric pressure.
[b]Equivalent to 120 torr on the physician's pressure gauge.

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SAMPLE PROBLEM 14.01

Atmospheric pressure and force

A living room has floor dimensions of 3.5 m and 4.2 m and a height of 2.4 m.
(a) What does the air in the room weigh when the air pressure is 1.0 atm?

KEY IDEAS

(1) The air's weight is equal to mg, where m is its mass. (2) Mass m is related to the air density ρ and the air volume V by Eq. 14-2 ($\rho = m/V$).

Calculation: Putting the two ideas together and taking the density of air at 1.0 atm from Table 14-1, we find

$$\begin{aligned} mg &= (\rho V)g \\ &= (1.21\ \text{kg/m}^3)(3.5\ \text{m} \times 4.2\ \text{m} \times 2.4\ \text{m})(9.8\ \text{m/s}^2) \\ &= 418\ \text{N} \approx 420\ \text{N}. \end{aligned}$$ (Answer)

This is the weight of about 110 cans of Pepsi.

(b) What is the magnitude of the atmosphere's downward force on the top of your head, which we take to have an area of 0.040 m^2?

KEY IDEA

When the fluid pressure p on a surface of area A is uniform, the fluid force on the surface can be obtained from Eq. 14-4 ($p = F/A$).

Calculation: Although air pressure varies daily, we can approximate that $p = 1.0$ atm. Then Eq. 14-4 gives

$$\begin{aligned} F = pA &= (1.0\ \text{atm})\left(\frac{1.01\times10^5\ \text{N/m}^2}{1.0\ \text{atm}}\right)(0.040\,\text{m}^2) \\ &= 4.0\times10^3\ \text{N}. \end{aligned}$$ (Answer)

This large force is equal to the weight of the air column from the top of your head to the top of the atmosphere.

14.3 | FLUIDS AT REST

Key Concepts

- Pressure in a fluid at rest varies with vertical position y. For y measured positive upward,
$$p_2 = p_1 + \rho g(y_1 - y_2).$$
If h is the *depth* of a fluid sample *below* some reference level at which the pressure is p_0, this equation becomes
$$p = p_0 + \rho g h,$$
where p is the pressure in the sample.
- The pressure in a fluid is the same for all points at the same level.
- Gauge pressure is the difference between the actual pressure (or absolute pressure) at a point and the atmospheric pressure.

Figure 14-2*a* shows a tank of water—or other liquid—open to the atmosphere. As every diver knows, the pressure *increases* with depth below the air–water interface. The diver's depth gauge, in fact, is a pressure sensor much like that of Fig. 14-1*b*. As every mountaineer knows, the pressure *decreases* with altitude as one ascends into the atmosphere. The pressures encountered by the diver and the mountaineer are usually called *hydrostatic pressures*, because they are due to fluids that are static (at rest). Here we want to find an expression for hydrostatic pressure as a function of depth or altitude.

Let us look first at the increase in pressure with depth below the water's surface. We set up a vertical y axis in the tank, with its origin at the air–water interface and the positive direction upward. We next consider a water sample contained in an imaginary right circular cylinder of horizontal base (or face) area A, such that y_1 and y_2 (both of which are *negative* numbers) are the depths below the surface of the upper and lower cylinder faces, respectively.

Figure 14-2*e* is a free-body diagram for the water in the cylinder. The water is in *static equilibrium*; that is, it is stationary and the forces on it balance. Three forces act on it vertically: Force $\vec{F}_1$ acts at the top surface of the cylinder and is due to the water above the cylinder (Fig. 14-2*b*). Force $\vec{F}_2$ acts at the bottom surface of the cylinder and is due

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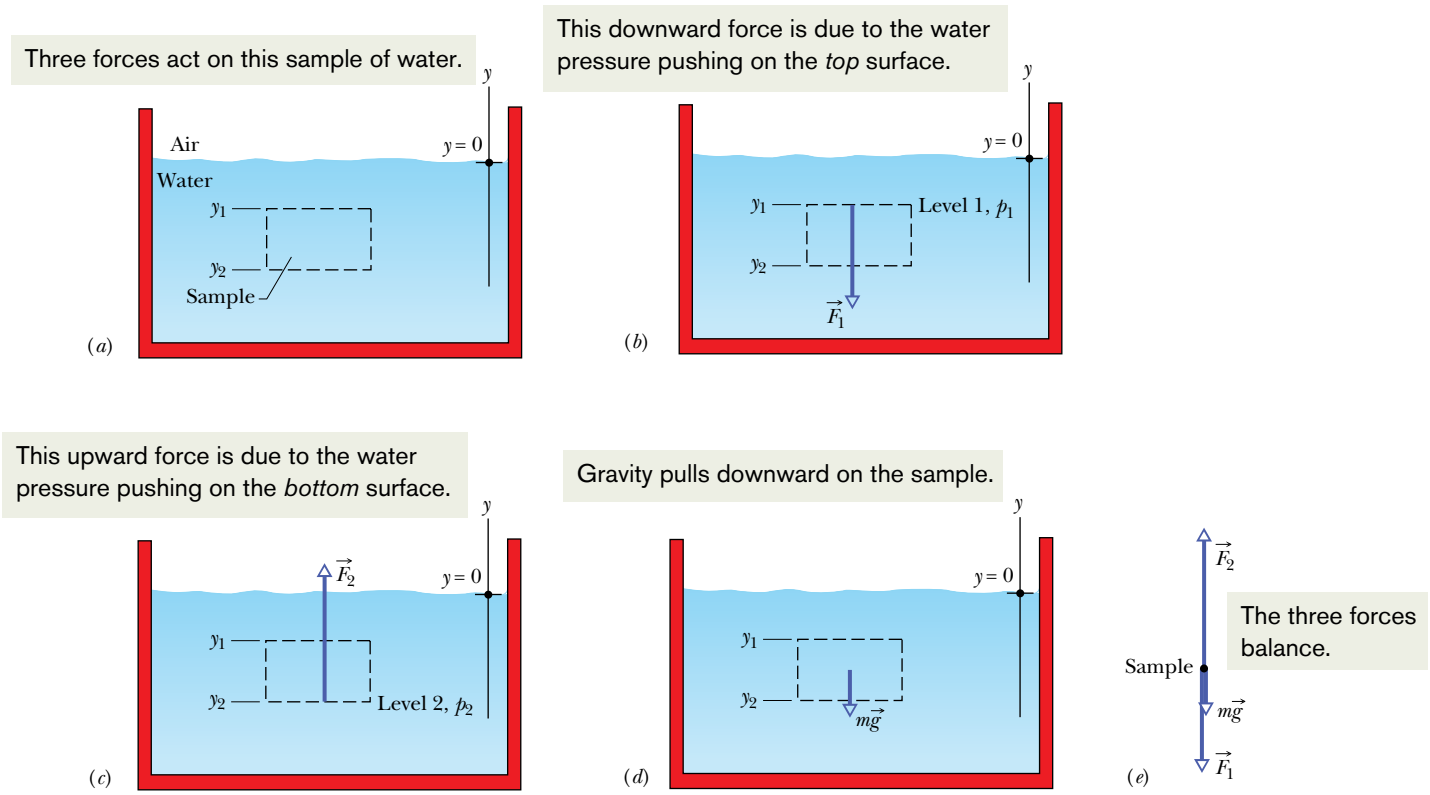

Figure 14-2 (*a*) A tank of water in which a sample of water is contained in an imaginary cylinder of horizontal base area A. (*b*)–(*d*) Force $\vec{F}_1$ acts at the top surface of the cylinder; force $\vec{F}_2$ acts at the bottom surface of the cylinder; the gravitational force on the water in the cylinder is represented by $m\vec{g}$. (*e*) A free-body diagram of the water sample. In *WileyPLUS*, this figure is available as an animation with voiceover.

to the water just below the cylinder (Fig. 14-2*c*). The gravitational force on the water is $m\vec{g}$, where m is the mass of the water in the cylinder (Fig. 14-2*d*). The balance of these forces is written as

$$F_2 = F_1 + mg. \tag{14-5}$$

To involve pressures, we use Eq. 14-4 to write

$$F_1 = p_1 A \quad \text{and} \quad F_2 = p_2 A. \tag{14-6}$$

The mass m of the water in the cylinder is, from Eq. 14-2, $m = \rho V$, where the cylinder's volume V is the product of its face area A and its height $y_1 - y_2$. Thus, m is equal to $\rho A(y_1 - y_2)$. Substituting this and Eq. 14-6 into Eq. 14-5, we find

$$p_2 A = p_1 A + \rho A g(y_1 - y_2)$$

or

$$p_2 = p_1 + \rho g(y_1 - y_2). \tag{14-7}$$

This equation can be used to find pressure both in a liquid (as a function of depth) and in the atmosphere (as a function of altitude or height). For the former, suppose we seek the pressure p at a depth h below the liquid surface. Then we choose level 1 to be the surface, level 2 to be a distance h below it (as in Fig. 14-3), and p_0 to represent the atmospheric pressure on the surface. We then substitute

$$y_1 = 0, p_1 = p_0 \quad \text{and} \quad y_2 = -h, p_2 = p$$

into Eq. 14-7, which becomes

$$p = p_0 + \rho g h \quad \text{(pressure at depth } h\text{)}. \tag{14-8}$$

Note that the pressure at a given depth in the liquid depends on that depth but not on any horizontal dimension.

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The pressure at a point in a fluid in static equilibrium depends on the depth of that point but not on any horizontal dimension of the fluid or its container.

Thus, Eq. 14-8 holds no matter what the shape of the container. If the bottom surface of the container is at depth h, then Eq. 14-8 gives the pressure p there.

In Eq. 14-8, p is said to be the total pressure, or **absolute pressure**, at level 2. To see why, note in Fig. 14-3 that the pressure p at level 2 consists of two contributions: (1) p_0, the pressure due to the atmosphere, which bears down on the liquid, and (2) ρgh, the pressure due to the liquid above level 2, which bears down on level 2. In general, the difference between an absolute pressure and an atmospheric pressure is called the **gauge pressure** (because we use a gauge to measure this pressure difference). For Fig. 14-3, the gauge pressure is ρgh.

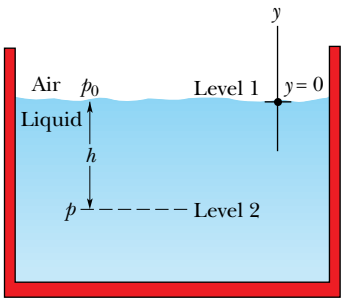


Figure 14-3 The pressure p increases with depth h below the liquid surface according to Eq. 14-8.

Equation 14-7 also holds above the liquid surface: It gives the atmospheric pressure at a given distance above level 1 in terms of the atmospheric pressure p_1 at level 1 (*assuming* that the atmospheric density is uniform over that distance). For example, to find the atmospheric pressure at a distance d above level 1 in Fig. 14-3, we substitute

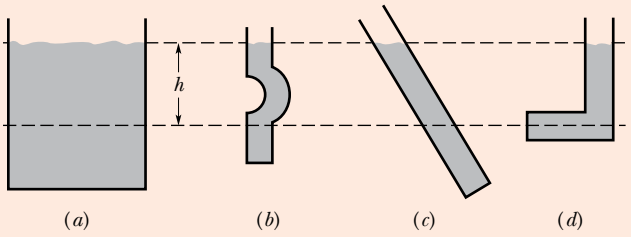
$$y_1 = 0, p_1 = p_0 \quad \text{and} \quad y_2 = d, p_2 = p.$$

Then with $\rho = \rho_{air}$, we obtain

$$p = p_0 - \rho_{air} g d.$$

CHECKPOINT 1

The figure shows four containers of olive oil. Rank them according to the pressure at depth h, greatest first.



SAMPLE PROBLEM 14.02

Gauge pressure on a scuba diver

A novice scuba diver practicing in a swimming pool takes enough air from his tank to fully expand his lungs before abandoning the tank at depth L and swimming to the surface, failing to exhale during his ascent. At the surface, the difference Δp between the external pressure on him and the air pressure in his lungs is 9.3 kPa. From what depth does he start? What potentially lethal danger does he face?

KEY IDEA

The pressure at depth h in a liquid of density ρ is given by Eq. 14-8 ($p = p_0 + \rho gh$), where the gauge pressure ρgh is added to the atmospheric pressure p_0.

Calculations: Here, when the diver fills his lungs at depth L, the external pressure on him (and thus the air pressure within his lungs) is greater than normal and given by Eq. 14-8 as

$$p = p_0 + \rho gL,$$

where ρ is the water's density (998 kg/m^3, Table 14-1). As he ascends, the external pressure on him decreases, until it is atmospheric pressure p_0 at the surface. His blood pressure also decreases, until it is normal. However, because he does not exhale, the air pressure in his lungs

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remains at the value it had at depth L. At the surface, the pressure difference Δp is

$$\Delta p = p - p_0 = \rho g L,$$

so

$$L = \frac{\Delta p}{\rho g} = \frac{9300\text{Pa}}{(998\ \text{kg/m}^3)(9.8\ \text{m/s}^2)}$$
$$= 0.95\ \text{m}. \qquad \text{(Answer)}$$

This is not deep! Yet, the pressure difference of 9.3 kPa (about 9% of atmospheric pressure) is sufficient to rupture the diver's lungs and force air from them into the depressurized blood, which then carries the air to the heart, killing the diver. If the diver follows instructions and gradually exhales as he ascends, he allows the pressure in his lungs to equalize with the external pressure, and then there is no danger.

SAMPLE PROBLEM 14.03

Balancing of pressure in a U-tube

The U-tube in Fig. 14-4 contains two liquids in static equilibrium: Water of density ρ_w (= 998 kg/m³) is in the right arm, and oil of unknown density ρ_x is in the left. Measurement gives $l = 135$ mm and $d = 12.3$ mm. What is the density of the oil?

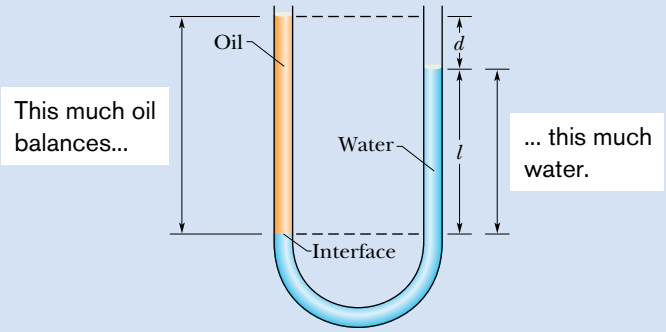

Figure 14-4 The oil in the left arm stands higher than the water.

KEY IDEAS

(1) The pressure p_{int} at the level of the oil–water interface in the left arm depends on the density ρ_x and height of the oil above the interface. (2) The water in the right arm *at the same level* must be at the same pressure p_{int}. The reason is that, because the water is in static equilibrium, pressures at points in the water at the same level must be the same.

Calculations: In the right arm, the interface is a distance l below the free surface of the *water*, and we have, from Eq. 14-8,

$$p_{int} = p_0 + \rho_w g l \quad \text{(right arm)}.$$

In the left arm, the interface is a distance $l + d$ below the free surface of the *oil*, and we have, again from Eq. 14-8,

$$p_{int} = p_0 + \rho_x g(l + d) \quad \text{(left arm)}.$$

Equating these two expressions and solving for the unknown density yield

$$\rho_x = \rho_w \frac{l}{l+d} = (998\ \text{kg/m}^3)\frac{135\ \text{mm}}{135\ \text{mm} + 12.3\ \text{mm}}$$
$$= 915\ \text{kg/m}^3. \qquad \text{(Answer)}$$

Note that the answer does not depend on the atmospheric pressure p_0 or the free-fall acceleration g.

14.4 | MEASURING PRESSURE

Key Concepts

- A mercury barometer can be used to measure atmospheric pressure.
- An open-tube manometer can be used to measure the gauge pressure of a confined gas.

The Mercury Barometer

Figure 14-5*a* shows a very basic *mercury barometer*, a device used to measure the pressure of the atmosphere. The long glass tube is filled with mercury and inverted with its open end in a dish of mercury, as the figure shows. The space above the mercury column contains only mercury vapor, whose pressure is so small at ordinary temperatures that it can be neglected.

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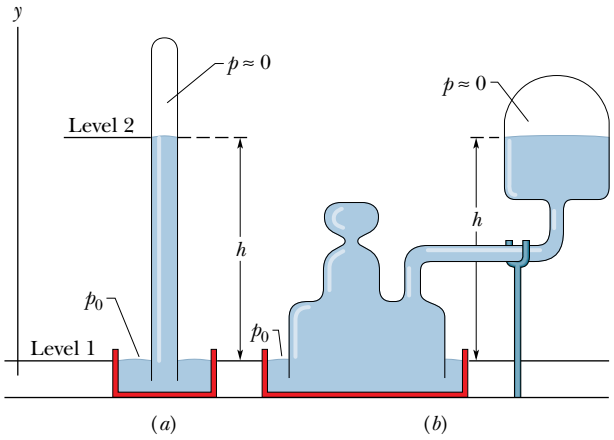

Figure 14-5 (*a*) A mercury barometer. (*b*) Another mercury barometer. The distance h is the same in both cases.

We can use Eq. 14-7 to find the atmospheric pressure p_0 in terms of the height h of the mercury column. We choose level 1 of Fig. 14-2 to be that of the air–mercury interface and level 2 to be that of the top of the mercury column, as labeled in Fig. 14-5*a*. We then substitute

$$y_1 = 0, p_1 = p_0 \quad \text{and} \quad y_2 = h, p_2 = 0$$

into Eq. 14-7, finding that

$$p_0 = \rho g h, \tag{14-9}$$

where ρ is the density of the mercury.

For a given pressure, the height h of the mercury column does not depend on the cross-sectional area of the vertical tube. The fanciful mercury barometer of Fig. 14-5*b* gives the same reading as that of Fig. 14-5*a*; all that counts is the vertical distance h between the mercury levels.

Equation 14-9 shows that, for a given pressure, the height of the column of mercury depends on the value of g at the location of the barometer and on the density of mercury, which varies with temperature. The height of the column (in millimeters) is numerically equal to the pressure (in torr) *only* if the barometer is at a place where g has its accepted standard value of 9.80665 m/s^2 *and* the temperature of the mercury is 0°C. If these conditions do not prevail (and they rarely do), small corrections must be made before the height of the mercury column can be transformed into a pressure.

The Open-Tube Manometer

An *open-tube manometer* (Fig. 14-6) measures the gauge pressure p_g of a gas. It consists of a **U**-tube containing a liquid, with one end of the tube connected to the vessel whose gauge pressure we wish to measure and the other end open to the atmosphere. We can use Eq. 14-7 to find the gauge pressure in terms of the height h shown in Fig. 14-6. Let us choose levels 1 and 2 as shown in Fig. 14-6. With

$$y_1 = 0, p_1 = p_0 \quad \text{and} \quad y_2 = -h, p_2 = p$$

substituted into Eq. 14-7, we find that

$$p_g = p - p_0 = \rho g h, \tag{14-10}$$

where ρ is the liquid's density. The gauge pressure p_g is directly proportional to h.

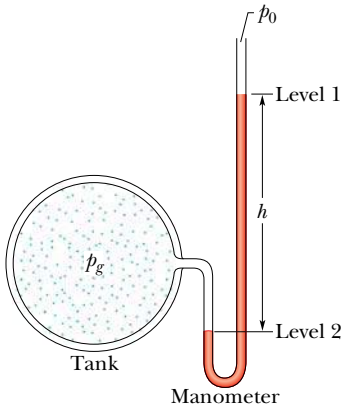

Figure 14-6 An open-tube manometer, connected to measure the gauge pressure of the gas in the tank on the left. The right arm of the **U**-tube is open to the atmosphere.

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The gauge pressure can be positive or negative, depending on whether $p > p_0$ or $p < p_0$. In inflated tires or the human circulatory system, the (absolute) pressure is greater than atmospheric pressure, so the gauge pressure is a positive quantity, sometimes called the *overpressure*. If you suck on a straw to pull fluid up the straw, the (absolute) pressure in your lungs is actually less than atmospheric pressure. The gauge pressure in your lungs is then a negative quantity.

14.5 | PASCAL'S PRINCIPLE

Key Concept

- Pascal's principle states that a change in the pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid and to the walls of the containing vessel.

When you squeeze one end of a tube to get toothpaste out the other end, you are watching **Pascal's principle** in action. This principle is also the basis for the Heimlich maneuver, in which a sharp pressure increase properly applied to the abdomen is transmitted to the throat, forcefully ejecting food lodged there. The principle was first stated clearly in 1652 by Blaise Pascal (for whom the unit of pressure is named):

A change in the pressure applied to an enclosed incompressible fluid is transmitted undiminished to every portion of the fluid and to the walls of its container.

Demonstrating Pascal's Principle

Consider the case in which the incompressible fluid is a liquid contained in a tall cylinder, as in Fig. 14-7. The cylinder is fitted with a piston on which a container of lead shot rests. The atmosphere, container, and shot exert pressure p_{ext} on the piston and thus on the liquid. The pressure p at any point P in the liquid is then

$$p = p_{ext} + \rho g h. \tag{14-11}$$

Let us add a little more lead shot to the container to increase p_{ext} by an amount Δp_{ext}. The quantities ρ, g, and h in Eq. 14-11 are unchanged, so the pressure change at P is

$$\Delta p = \Delta p_{ext}. \tag{14-12}$$

This pressure change is independent of h, so it must hold for all points within the liquid, as Pascal's principle states.

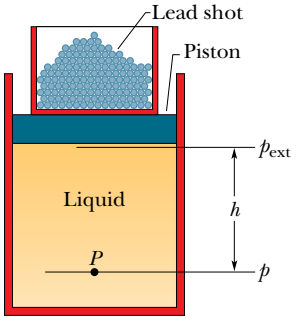

Figure 14-7 Lead shot (small balls of lead) loaded onto the piston create a pressure p_{ext} at the top of the enclosed (incompressible) liquid. If p_{ext} is increased, by adding more lead shot, the pressure increases by the same amount at all points within the liquid.

Pascal's Principle and the Hydraulic Lever

Figure 14-8 shows how Pascal's principle can be made the basis of a hydraulic lever. In operation, let an external force of magnitude F_i be directed downward on the left-hand (or input) piston, whose surface area is A_i. An incompressible liquid in the device then produces an upward force of magnitude F_o on the right-hand (or output) piston, whose surface area is A_o. To keep the system in equilibrium, there must be a downward force of magnitude F_o on the output piston from an external load (not shown). The force $\vec{F}_i$ applied on the left and the downward force $\vec{F}_o$ from the load on the right produce a change Δp in the pressure of the liquid that is given by

$$\Delta p = \frac{F_i}{A_i} = \frac{F_o}{A_o},$$

so

$$F_o = F_i \frac{A_o}{A_i}. \tag{14-13}$$

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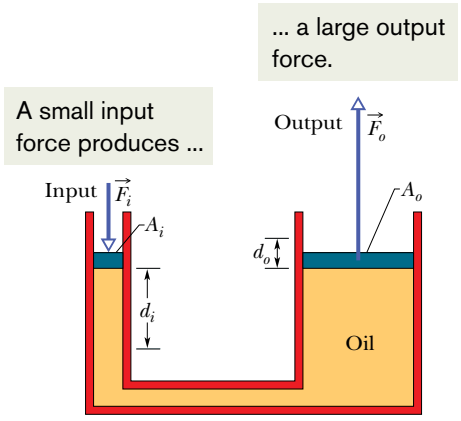

Figure 14-8 A hydraulic arrangement that can be used to magnify a force $\vec{F}_i$. The work done is, however, not magnified and is the same for both the input and output forces.

Equation 14-13 shows that the output force F_o on the load must be greater than the input force F_i if $A_o > A_i$, as is the case in Fig. 14-8.

If we move the input piston downward a distance d_i, the output piston moves upward a distance d_o, such that the same volume V of the incompressible liquid is displaced at both pistons. Then

$$V = A_i d_i = A_o d_o,$$

which we can write as

$$d_o = d_i \frac{A_i}{A_o}. \tag{14-14}$$

This shows that, if $A_o > A_i$ (as in Fig. 14-8), the output piston moves a smaller distance than the input piston moves.

From Eqs. 14-13 and 14-14 we can write the output work as

$$W = F_o d_o = \left(F_i \frac{A_o}{A_i}\right)\left(d_i \frac{A_i}{A_o}\right) = F_i d_i, \tag{14-15}$$

which shows that the work W done *on* the input piston by the applied force is equal to the work W done *by* the output piston in lifting the load placed on it.

The advantage of a hydraulic lever is this:

With a hydraulic lever, a given force applied over a given distance can be transformed to a greater force applied over a smaller distance.

The product of force and distance remains unchanged so that the same work is done. However, there is often tremendous advantage in being able to exert the larger force. Most of us, for example, cannot lift an automobile directly but can with a hydraulic jack, even though we have to pump the handle farther than the automobile rises and in a series of small strokes.

14.6 | ARCHIMEDES' PRINCIPLE

Key Concepts

- Archimedes' principle states that when a body is fully or partially submerged in a fluid, the fluid pushes upward with a buoyant force with magnitude

$$F_b = m_f g,$$

where m_f is the mass of the fluid that has been pushed out of the way by the body.

- When a body floats in a fluid, the magnitude F_b of the (upward) buoyant force on the body is equal to the magnitude F_g of the (downward) gravitational force on the body.
- The apparent weight of a body on which a buoyant force acts is related to its actual weight by

$$\text{weight}_{\text{app}} = \text{weight} - F_b.$$

Figure 14-9 shows a student in a swimming pool, manipulating a very thin plastic sack (of negligible mass) that is filled with water. She finds that the sack and its contained water are in static equilibrium, tending neither to rise nor to sink. The downward gravitational force $\vec{F}_g$ on the contained water must be balanced by a net upward force from the water surrounding the sack.

This net upward force is a **buoyant force** $\vec{F}_b$. It exists because the pressure in the surrounding water increases with depth below the surface. Thus, the pressure near the bottom of the sack is greater than the pressure near the

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top, which means the forces on the sack due to this pressure are greater in magnitude near the bottom of the sack than near the top. Some of the forces are represented in Fig. 14-10*a*, where the space occupied by the sack has been left empty. Note that the force vectors drawn near the bottom of that space (with upward components) have longer lengths than those drawn near the top of the sack (with downward components). If we vectorially add all the forces on the sack from the water, the horizontal components cancel and the vertical components add to yield the upward buoyant force $\vec{F}_b$ on the sack. (Force $\vec{F}_b$ is shown to the right of the pool in Fig. 14-10*a*.)

Because the sack of water is in static equilibrium, the magnitude of $\vec{F}_b$ is equal to the magnitude $m_f g$ of the gravitational force $\vec{F}_g$ on the sack of water: $F_b = m_f g$. (Subscript f refers to *fluid*, here the water.) In words, the magnitude of the buoyant force is equal to the weight of the water in the sack.

In Fig. 14-10*b*, we have replaced the sack of water with a stone that exactly fills the hole in Fig. 14-10*a*. The stone is said to *displace* the water, meaning that the stone occupies space that would otherwise be occupied by water. We have changed nothing about the shape of the hole, so the forces at the hole's surface must be the same as when the water-filled sack was in place. Thus, the same upward buoyant force that acted on the water-filled sack now acts on the stone; that is, the magnitude F_b of the buoyant force is equal to $m_f g$, the weight of the water displaced by the stone.

The upward buoyant force on this sack of water equals the weight of the water.

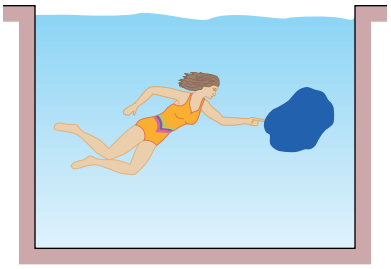

Figure 14-9 A thin-walled plastic sack of water is in static equilibrium in the pool. The gravitational force on the sack must be balanced by a net upward force on it from the surrounding water.

Unlike the water-filled sack, the stone is not in static equilibrium. The downward gravitational force $\vec{F}_g$ on the stone is greater in magnitude than the upward buoyant force (Fig. 14-10*b*). The stone thus accelerates downward, sinking.

Let us next exactly fill the hole in Fig. 14-10*a* with a block of lightweight wood, as in Fig. 14-10*c*. Again, nothing has changed about the forces at the hole's surface, so the magnitude F_b of the buoyant force is still equal to $m_f g$, the weight of the displaced water. Like the stone, the block is not in static equilibrium. However, this time the gravitational force $\vec{F}_g$ is lesser in magnitude than the buoyant force (as shown to the right of the pool), and so the block accelerates upward, rising to the top surface of the water.

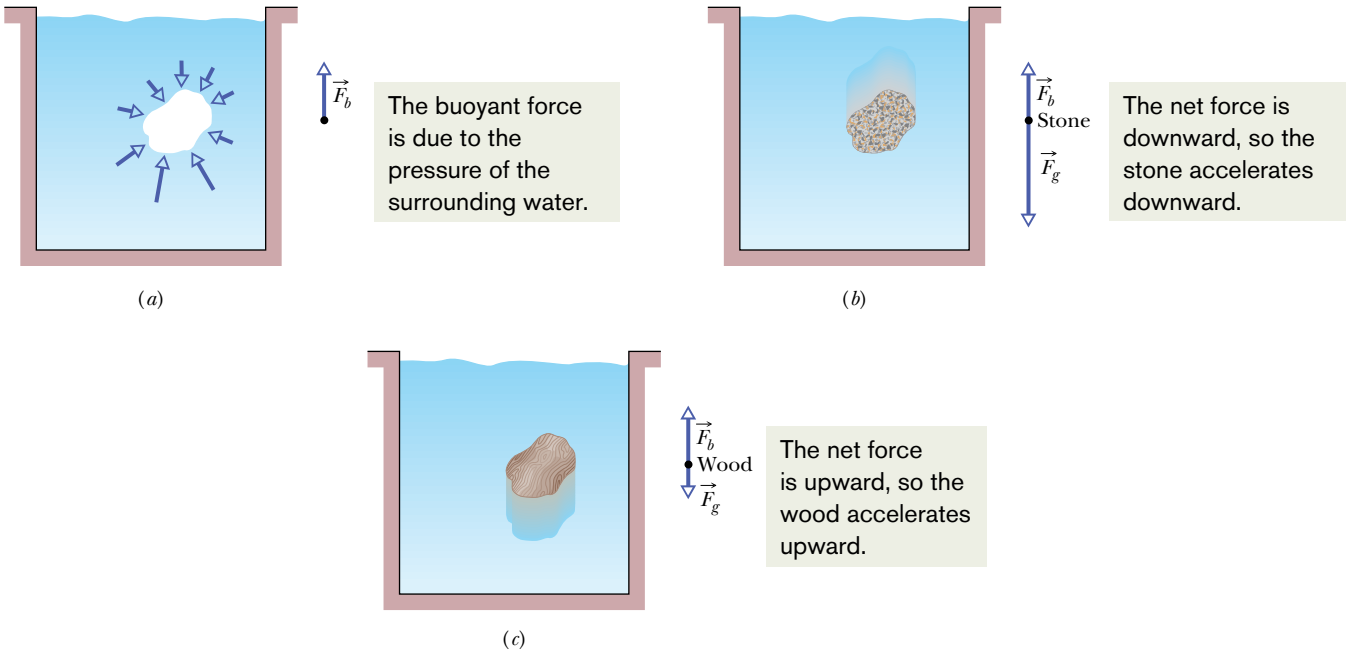

Figure 14-10 (*a*) The water surrounding the hole in the water produces a net upward buoyant force on whatever fills the hole. (*b*) For a stone of the same volume as the hole, the gravitational force exceeds the buoyant force in magnitude. (*c*) For a lump of wood of the same volume, the gravitational force is less than the buoyant force in magnitude.

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Our results with the sack, stone, and block apply to all fluids and are summarized in **Archimedes' principle**:

When a body is fully or partially submerged in a fluid, a buoyant force $\vec{F}_b$ from the surrounding fluid acts on the body. The force is directed upward and has a magnitude equal to the weight $m_f g$ of the fluid that has been displaced by the body.

The buoyant force on a body in a fluid has the magnitude

$$F_b = m_f g \quad \text{(buoyant force)}, \tag{14-16}$$

where m_f is the mass of the fluid that is displaced by the body.

Floating

When we release a block of lightweight wood just above the water in a pool, the block moves into the water because the gravitational force on it pulls it downward. As the block displaces more and more water, the magnitude F_b of the upward buoyant force acting on it increases. Eventually, F_b is large enough to equal the magnitude F_g of the downward gravitational force on the block, and the block comes to rest. The block is then in static equilibrium and is said to be *floating* in the water. In general,

When a body floats in a fluid, the magnitude F_b of the buoyant force on the body is equal to the magnitude F_g of the gravitational force on the body.

We can write this statement as

$$F_b = F_g \quad \text{(floating)}. \tag{14-17}$$

From Eq. 14-16, we know that $F_b = m_f g$. Thus,

When a body floats in a fluid, the magnitude F_g of the gravitational force on the body is equal to the weight $m_f g$ of the fluid that has been displaced by the body.

We can write this statement as

$$F_g = m_f g \quad \text{(floating)}. \tag{14-18}$$

In other words, a floating body displaces its own weight of fluid.

Apparent Weight in a Fluid

If we place a stone on a scale that is calibrated to measure weight, then the reading on the scale is the stone's weight. However, if we do this underwater, the upward buoyant force on the stone from the water decreases the reading. That reading is then an apparent weight. In general, an **apparent weight** is related to the actual weight of a body and the buoyant force on the body by

$$\begin{pmatrix}\text{apparent}\\ \text{weight}\end{pmatrix} = \begin{pmatrix}\text{actual}\\ \text{weight}\end{pmatrix} - \begin{pmatrix}\text{magnitude of}\\ \text{buoyant force}\end{pmatrix},$$

which we can write as

$$\text{weight}_{\text{app}} = \text{weight} - F_b \quad \text{(apparent weight)}. \tag{14-19}$$

If, in some test of strength, you had to lift a heavy stone, you could do it more easily with the stone underwater. Then your applied force would need to exceed only the stone's apparent weight, not its larger actual weight.

The magnitude of the buoyant force on a floating body is equal to the body's weight. Equation 14-19 thus tells us that a floating body has an apparent weight of zero—the body would produce a reading of zero on a scale.

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For example, when astronauts prepare to perform a complex task in space, they practice the task floating underwater, where their suits are adjusted to give them an apparent weight of zero.

CHECKPOINT 2

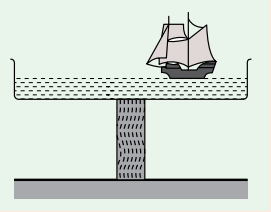
A penguin floats first in a fluid of density ρ_0, then in a fluid of density $0.95\rho_0$, and then in a fluid of density $1.1\rho_0$. (a) Rank the densities according to the magnitude of the buoyant force on the penguin, greatest first. (b) Rank the densities according to the amount of fluid displaced by the penguin, greatest first.

CHECKPOINT 3

A wooden block floats vertically in a glass filled with water. How will the level of the water in the glass change if the block is kept in a horizontal position?

CHECKPOINT 4

A vessel filled with water is placed exactly in middle of a thin wall as shown in figure. Will the system topple if a small wooden boat carrying some weight is floated in the vessel?



CHECKPOINT 5

A piece of ice is floating in a tub filled with water. How will the level of the water in the tub change when the ice melts? Consider the following cases:

(1) A stone is frozen in the ice.
(2) The ice contains an air bubble.

SAMPLE PROBLEM 14.04

Floating, buoyancy, and density

In Fig. 14-11, a block of density $\rho = 800\ \text{kg/m}^3$ floats face down in a fluid of density $\rho_f = 1200\ \text{kg/m}^3$. The block has height $H = 6.0$ cm.
(a) By what depth h is the block submerged?

Floating means that the buoyant force matches the gravitational force.

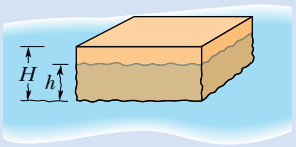


Figure 14-11 Block of height H floats in a fluid, to a depth of h.

KEY IDEAS

(1) Floating requires that the upward buoyant force on the block match the downward gravitational force on the block. (2) The buoyant force is equal to the weight $m_f g$ of the fluid displaced by the submerged portion of the block.

Calculations: From Eq. 14-16, we know that the buoyant force has the magnitude $F_b = m_f g$, where m_f is the mass of the fluid displaced by the block's submerged volume V_f. From Eq. 14-2 ($\rho = m/V$), we know that the mass of the displaced fluid is $m_f = \rho_f V_f$. We do not know V_f but if

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we symbolize the block's face length as L and its width as W, then from Fig. 14-11 we see that the submerged volume must be $V_f = LWh$. If we now combine our three expressions, we find that the upward buoyant force has magnitude

$$F_b = m_f g = \rho_f V_f g = \rho_f LWhg. \tag{14-20}$$

Similarly, we can write the magnitude F_g of the gravitational force on the block, first in terms of the block's mass m, then in terms of the block's density ρ and (full) volume V, and then in terms of the block's dimensions L, W, and H (the full height):

$$F_g = mg = \rho Vg = \rho LWHg. \tag{14-21}$$

The floating block is stationary. Thus, writing Newton's second law for components along a vertical y axis with the positive direction upward ($F_{\text{net},y} = ma_y$), we have

$$F_b - F_g = m(0),$$

or from Eqs. 14-20 and 14-21,

$$\rho_f LWhg - \rho LWHg = 0,$$

which gives us

$$h = \frac{\rho}{\rho_f} H = \frac{800 \text{ kg/m}^3}{1200 \text{ kg/m}^3}(6.0 \text{ cm})$$
$$= 4.0 \text{ cm}. \qquad \text{(Answer)}$$

(b) If the block is held fully submerged and then released, what is the magnitude of its acceleration?

Calculations: The gravitational force on the block is the same but now, with the block fully submerged, the volume of the displaced water is $V = LWH$. (The full height of the block is used.) This means that the value of F_b is now larger, and the block will no longer be stationary but will accelerate upward. Now, Newton's second law yields

$$F_b - F_g = ma,$$

or

$$\rho_f LWHg - \rho LWHg = \rho LWHa,$$

where we inserted ρLWH for the mass m of the block. Solving for a leads to

$$a = \left(\frac{\rho_f}{\rho} - 1\right) g = \left(\frac{1200 \text{ kg/m}^3}{800 \text{ kg/m}^3} - 1\right)(9.8 \text{ m/s}^2)$$
$$= 4.9 \text{ m/s}^2. \qquad \text{(Answer)}$$

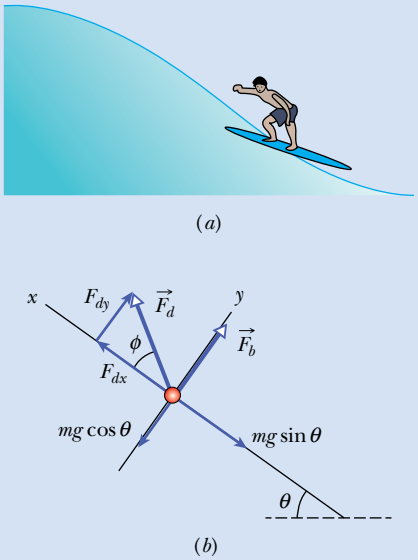
SAMPLE PROBLEM 14.05

Let's go surfing

In part *a* of Fig. 14-12, a surfer rides on the front side of a wave, at a point where a tangent to the wave has a slope of $\theta = 30.0°$. The combined mass of surfer and surfboard is $m = 83.0$ kg, and the board has a submerged volume of $V = 2.50 \times 10^{-2}$ m^3. The surfer maintains his position on the wave as the wave moves at constant speed toward shore. What is the magnitude and direction (relative to the positive direction of the x axis in part *b* of Fig. 14-12) of the drag force on the surfboard from the water?

KEY IDEAS

(1) The buoyancy force on the surfer has a magnitude F_b equal to the weight of the seawater displaced by the submerged volume of the surfboard. The direction of the force is perpendicular to the surface at the surfer's location.
(2) By Newton's second law, because the surfer moves at constant speed toward the shore, the (vector) sum of the buoyancy force $\vec{F}_b$, the gravitational force $\vec{F}_g$, and the drag force $\vec{F}_d$ must be 0.



(*a*) (*b*)

Figure 14-12 (*a*) Surfer. (*b*) Free-body diagram showing the forces on the surfer–surfboard system.

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Calculations: The forces and their components are shown in the free-body diagram of part *b* of Fig. 14-12. The gravitational force $\vec{F}_g$ is downward and (as we saw in Chapter 5) has a component of $mg \sin\theta$ down the slope and a component of $mg \cos\theta$ perpendicular to the slope. A drag force $\vec{F}_d$ from the water acts on the surfboard because water is continuously forced up into the wave as the wave continues to move toward the shore. This push on the surfboard is upward and to the rear, at angle ϕ to the x axis. The buoyancy force $\vec{F}_b$ is perpendicular to the water surface; its magnitude depends on the mass m_f the water displaced by the surfboard, as given by Eq. 14-16 ($F_b = m_f g$). From Eq. 14-2 ($\rho = m/V$), we can write the mass in terms of the seawater density ρ_w and the submerged volume V of the surfboard: $m_f = \rho_w V$. From Table 14-1, seawater density ρ_w is 1.024×10^3 kg/m^3. Thus, the magnitude of the buoyant force is

$$\begin{aligned} F_b &= m_f g = \rho_w V g \\ &= (1.024 \times 10^3 \text{ kg/m}^3)(2.50 \times 10^{-2} \text{ m}^3)(9.8 \text{ m/s}^2) \\ &= 2.509 \times 10^2 \text{ N}. \end{aligned}$$

So, Newton's second law for the y axis,

$$F_{dy} + F_b - mg\cos\theta = m(0),$$

becomes

$$F_{dy} + 2.509 \times 10^2 \text{ N} - (83 \text{ kg})(9.8 \text{ m/s}^2)\cos 30.0° = 0,$$

yielding

$$F_{dy} = 453.5 \text{ N}.$$

Similarly, Newton's second law $\vec{F}_{\text{net}} = m\vec{a}$ for the x axis,

$$F_{dx} - mg\sin\theta = m(0),$$

yields

$$F_{dx} = 406.7 \text{ N}.$$

Combining the two components of the drag force tells us that the force has magnitude

$$\begin{aligned} F_d &= \sqrt{(406.7 \text{ N})^2 + (453.5 \text{ N})^2} \\ &= 609 \text{ N} \end{aligned} \qquad \text{(Answer)}$$

and angle

$$\phi = \tan^{-1}\left(\frac{453.5 \text{ N}}{406.7 \text{ N}}\right) = 48.1°. \qquad \text{(Answer)}$$

Wipeout avoided: If the surfer tilts the board slightly forward, the magnitude of the drag force decreases and angle ϕ changes. The result is that the net force is no longer zero and the surfer moves down the face of the wave. The descent is somewhat self-adjusting because, as the surfer descends, the tilt angle θ of the wave surface decreases and thus so does the component of the gravitational force $mg \sin\theta$ pulling the surfer down the slope. So, the surfer can adjust the board to re-establish equilibrium, now lower on the wave. Similarly, by tilting the board slightly backward, the surfer increases the drag and moves up the face of the wave. If the surfer is still on the lower part of the wave, then both θ and $mg \sin\theta$ increase and again the surfer can control the forces and re-establish equilibrium.

SAMPLE PROBLEM 14.06

Floating, fraction of object that is submerged

In Fig. 14-13, a rectangular block floats in a liquid. A block of mass $m = 0.500$ kg and density $\rho = 700$ kg/m^3 floats face down in a liquid of density $\rho_f = 1200$ kg/m^3. What are the magnitude of the buoyant force on the block from the liquid, the weight of the liquid displaced by the block (the liquid that would be in the space now occupied by the block), and the mass of that displaced liquid? What fraction of the block's volume is submerged? What fraction of its height is submerged?

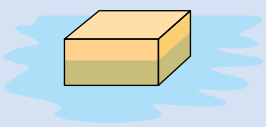

Figure 14-13 A rectangular block that is floating in a liquid.

KEY IDEAS

(1) The magnitude of the buoyant force is equal to the weight of the displaced liquid.

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(2) Because the block is floating, the magnitude of the upward buoyant force is equal to the magnitude of the downward gravitational force.

Calculations: Because the block is floating, the magnitude of the (upward) buoyant force on it must match the magnitude of the (downward) gravitational force on it:

$$F_b = mg \tag{14-22}$$
$$= (0.500\text{ kg})(9.8\text{ m/s}^2) = 4.9\text{ N}. \quad \text{(Answer)}$$

Regardless of whether the block is floating, the magnitude of the buoyant force must equal the weight $m_f g$ of the liquid displaced by the block. Thus, we have

$$m_f g = F_b \tag{14-23}$$
$$= 4.9\text{ N}. \quad \text{(Answer)}$$

The mass of the displaced liquid is then

$$m_f g = 0.500\text{ kg}. \quad \text{(Answer)}$$

the same as the block.

To find the fraction of the block's volume that is submerged, let us combine Eqs. 14-22 and 14-23 to write

$$m_f g = mg \tag{14-24}$$

and then relate the masses to the densities and volume using the general relation $\rho = m/V$. Here, we have the submerged volume V_f and the full block volume V. We also have the liquid's density ρ_f and the block's density ρ. Equation 14-24 then becomes

$$\rho_f V_f g = \rho V g.$$

or

$$\frac{V_f}{V} = \frac{\rho}{\rho_f}$$
$$= \frac{700\text{ kg/m}^3}{1200\text{ kg/m}^3} = 0.58. \tag{14-24a}$$

Thus, 58% of the block's volume is submerged.

To find the submerged height h, we note that the submerged volume V_f and the area of the bottom face are related by

$$V_f = Ah.$$

Similarly, the full volume and the full height H are related by

$$V = AH.$$

Substituting these expressions into Eq. 14-24a leads to

$$\frac{h}{H} = \frac{\rho}{\rho_f} = 0.58.$$

Thus, 58% of the block's height is submerged.

SAMPLE PROBLEM 14.07

Force on a submerged vertical wall

Figure 14-14*a* shows a vertical wall of horizontal length $L = 5.0$ m and depth $H = 3.0$ m that is fully submerged in water, with its top at the top surface of the water. What is the force on the wall due to the water pressure?

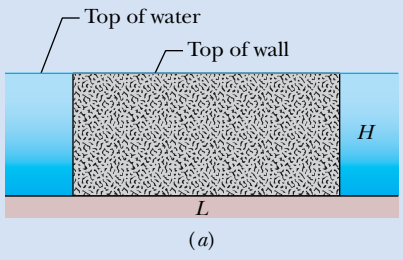

Figure 14-14 (*a*) A vertical wall that is fully submerged in water.

KEY IDEAS

(1) The water pressure increases with depth according to Eq. 14-8 ($p = p_0 + \rho gh$, where p_0 is the pressure at the top of the water, ρ is the water density, and h is the depth).

(2) The pressure p and force ΔF on a small area ΔA are related by Eq. 14-3 ($p = \Delta F/\Delta A$). We can rewrite that equation in terms of differential quantities: $p = dF/dA$.

(3) Thus, the magnitude of the force increases with depth.

Calculations: Let y represent the *depth* below the top water surface. Then the gauge pressure is

$$p_g = \rho g y \tag{14-25}$$

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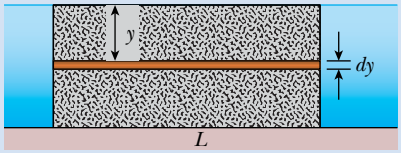
Note that because the expression contains a variable (namely, depth y), we cannot use $p = F/A$ to find the force on the wall because there is no one value of pressure. However, if we pick a small area dA on the wall (Fig. 14-14*b*), we can relate the pressure p and force dF on the wall by

$$p_g = \frac{dF}{dA}. \tag{14-26}$$



(*b*)

Figure 14-14 (*b*) Considering small area on the wall.

(We use the "d" notation to indicate that we are dealing with tiny portions and differential amounts.) Equation 14-26 tells us that the force on dA is

$$dF = p_g dA. \tag{14-27}$$

For area dA, we could choose a small square at some depth y. However, note that the variable in Eq. 14-25 is y and not x. Thus, the pressure at a given depth y is the same at any horizontal position along the wall. So, we can choose dA to be the area of a strip of length L and tiny height dy. That is,

$$dA = L\,dy \tag{14-28}$$

Then the force on that strip is

$$dF = p_g L\,dy,$$

or, with Eq. 14-25,

$$dF = \rho g y L\,dy \tag{14-29}$$

To find the total force on the wall, we need to add the forces on all the horizontal strips from top to bottom of the wall. Of course, because each strip has the tiny height dy, there are a huge number of strips along the wall and adding up their forces would take "for ever" (much longer than our patience). However, we can add them via integration rather quickly. Integrating both sides of Eq. 14-29, we find

$$\int dF = \int \rho g y L\,dy \tag{14-30}$$

We want to include all the strips between $y = 0$ (at the top of the wall) and $y = 3.0$ m (at the bottom of the wall). To do so, we use those values as the limits to the integral on the right side of Eq. 14-30. The integration on the left side yields the total force F we seek. We find

$$\begin{aligned} F &= \rho g L \int_0^{3.0\text{ m}} y\,dy \\ &= \rho g L \frac{1}{2} y^2 \Big|_0^{3.0\text{ m}} \\ &= (1000\text{ kg/m}^3)(9.8\text{ m/s}^2)(5.0\text{ m})\frac{1}{2}(3.0\text{ m})^2 \\ &= 2.21\times 10^5\text{ N}. \end{aligned} \tag{Answer}$$

SAMPLE PROBLEM 14.08

Buoyancy of a floating body in a descending lift

A vessel with a body floating in it is kept in an lift accelerating downward with acceleration a such that $a < g$. Will the body rise or sink further in the vessel?

Reasoning: The force of buoyancy on the body can be written as $F_b = \rho V_2(g - a)$, where V_2 is the volume of the submerged portion of the body in the lift. The pressure at a point h below the surface will become $\rho(g - a)h$ instead of ρgh. Applying the Newton's second law, remembering that the body was accelerating upward at with acceleration a, we have

$$Mg - \rho V_2(g - a) = Ma$$

or

$$V_2 = \frac{M}{\rho}$$

Thus, $V_2 = V$, as in a stationary vessel, $V = M/\rho$. So, the body does not rise to the surface.

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SAMPLE PROBLEM 14.09

Pressure due to fluid in vessels of different shapes but same base area

Three vessels having different shapes are shown in Fig. 14-15*a*, having the same base area and the same weight when empty. The vessels are filled with mercury to the same level. (Neglect the effect of the atmosphere.)

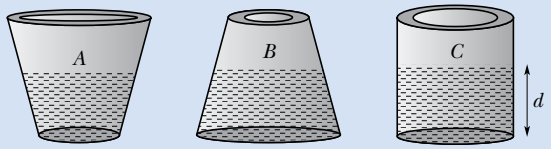

Figure 14-15 (*a*) Three differently shaped vessels filled with mercury to the same level.

(a) Find which vessel has the maximum and which vessel has the minimum pressure at the bottom, or is the pressure same?

Reasoning: The mercury at the bottom of each vessel is at the same depth d below the surface. Neglecting the pressure at the surface, the pressures at the bottom must be equal, hence

$$p = \rho g d$$

(b) Which vessel shows the maximum weight when weighed on a weighing scale or do they all weigh the same?

Reasoning: The weight of each filled vessel is equal to the weight of the vessel itself plus the weight of the mercury inside. The vessels themselves are of equal weights, but vessel A holds more mercury than C, while vessel B holds less mercury than C. Vessel A weighs the most and vessel B weighs the least.

(c) Even though mercury exerts the same downward force on the bottom of each vessel, then why do the vessels weigh differently?

Reasoning: In vessel C, forces due to fluid pressure on the sides of the container are horizontal. Forces on any two diametrically opposite points on the walls of the container are equal and opposite; thus, the net force on the container walls is zero. The force on the bottom is

$$F = pA = (\rho g d)(\pi r^2)$$

The volume of mercury in the cylinder is $V = \pi r^2 d$. So

$$F = \rho g V = (\rho V) g = mg$$

The force on the bottom of vessel C is equal to the weight of the mercury. The forces due to fluid pressure on the sides of the containers A and B have vertical components also. So, the force between the fluid and the base of container will not be equal to the weight of the fluid. These containers support the fluid by exerting an upward force equal in magnitude to the weight of the fluid but some force is being applied by the sidewalls and the remaining by the bottom. Fig. 14-15*b* shows the forces acting on each container due to mercury.

The force on the bottom of vessel A is less than the weight of the mercury in the container, while the force on the bottom of vessel B is greater than the weight of the mercury. In vessel A, the forces on the container walls have downward components as well as horizontal components. The sum of the downward components of the forces on the walls and the downward force on the bottom of the container is equal to the weight of the water. Similarly, the forces on the walls of vessel B have upward components. In each case, the total force on the bottom and sides of the container due to the mercury is equal to the weight of the mercury.

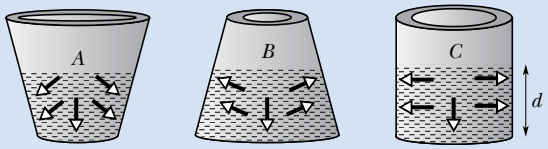

Figure 14-15 (*b*) Forces exerted on the containers by mercury.

14.7 | HYDROSTATIC FORCE ON A CURVED SURFACE

Vessels may have walls and even bottoms which are nonplanar. Although theoretically speaking, the resultant fluid force on a surface can be determined by integration, as was done for the plane surfaces. This is generally a tough process and we cannot offer simple, general formulas. Here we discuss an alternative approach where we will find the horizontal and vertical components of the forces by considering the fluid volume enclosed by the curved surface in equilibrium.

For example, consider the curved portion BC of the open tank, having a unit length perpendicular to the plane of the paper, as shown in Fig. 14-16*a*. To find the resultant fluid force acting on this section, we consider entire fluid that is enclosed by the surface BC, the horizontal plane surface AB, and the vertical plane surface AC as a separate body.

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Imagine that this body is separated by the remaining fluid mass by an imaginary surface. The free-body diagram for the body is shown in Fig. 14-16*b*. From the discussion of forces on planar surfaces, we know the magnitude and the effective point of application of forces F_1 and F_2. The weight, W of the mass of fluid enclosed by the imaginary surface is a product of density of the fluid and the enclosed volume and acts through the center of gravity (cog) of the mass of fluid contained within the imaginary surface. The forces F_H and F_V represent the horizontal and vertical components of the force that the tank exerts on the fluid.

For this force system to be in translational and rotational equilibrium, the horizontal component F_H must be equal in magnitude and collinear with F_2, and the vertical component F_V equal in magnitude and collinear with the resultant of the vertical forces F_1 and W. Thus,

$$F_H = F_2$$
$$F_V = F_1 + W$$

and the magnitude of the resultant force is obtained from the equation:

$$F_R = \sqrt{(F_H)^2 + (F_V)^2}$$

From the principles of equilibrium discussed earlier, it is known that when a body is held in equilibrium by three nonparallel forces their lines of action intersect at a common point, and it should be coplanar. Although our analysis is being done for the fluid, by Newton's third law, the force experienced by the curved surface is equal and opposite to F_R. The desired fluid force is shown in Fig. 14-16*c*.

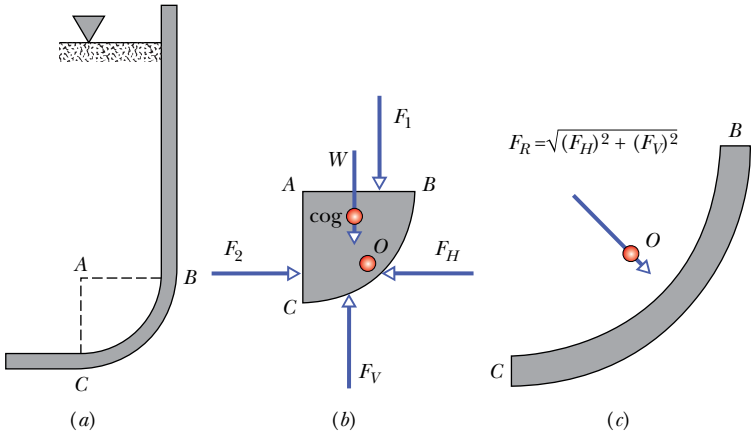

Figure 14.16 (*a*) Curved portion of an open tank. (*b*) Free-body diagram of the body of fluid enclosed in the space *ABC* as shown in part (*a*). (*c*) Force on the curved wall of the tank due to fluid.

SAMPLE PROBLEM 14.10

Dividing fluid in two parts for calculating its volume

A conical vessel without a bottom tightly stands on a table. A liquid is poured into the vessel and as soon as its level reaches the height h, the pressure of the liquid raises the vessel. The radius of the bottom greater base of the vessel is R (Fig. 14-17*a*), the semi-vertex angle of the cone is α, and the weight of the vessel is W. What is the density of the liquid?

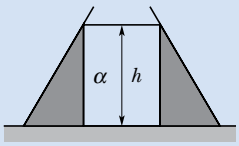

Figure 14-17 (*a*) Dividing fluid in two parts for calculating its volume.

KEY IDEAS

(1) The forces applied by the liquid on the side walls of cone will have vertical components. As the amount of liquid increases, this force will increase till it lifts the vessel and we need to calculate this force.

(2) The direct solution may involve taking elemental rings and integrating the force. But, if we consider the liquid as a system in equilibrium, then only unknown force acting on it will be the one applied by the sidewalls. According to Newton's third law, an identical force acts on the vessel.

Calculation: Let's assume that when the height of the liquid is h in the vessel, the vessel rises.

$$\tan\alpha = \frac{r}{x} = \frac{R}{x+h}$$

or
$$r = h - h\tan\alpha \qquad (14\text{-}31)$$

The pressure on the *bottom* of the vessel is ρgh and the force with which the marked portion (the truncated cone minus the cylinder volume) of the liquid (Fig. 14-17*b*) presses on the table is

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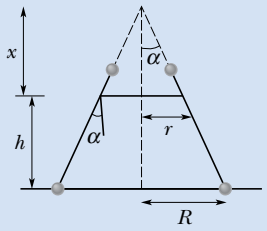

Figure 14-17 (*b*) A frustum without any bottom, filled with fluid.

According to Newton's third law, an identical force acts on the liquid. At the time when the vessel starts rising, for equilibrium of the liquid, we have

$$W + W_1 = \rho g h \pi (2Rh\tan\alpha - h^2\tan^2\alpha)$$

where W_1 is the weight of the shaded portion of the liquid

$$W_1 = \frac{\rho g h}{3}[\pi R^2 + \pi(R - h\tan\alpha)^2 + \pi R(R - h\tan\alpha)] - \rho g h \pi (R - h\tan\alpha)^2$$

Therefore, $$\rho = \frac{W}{\pi g h^2 \tan\alpha\left(R - \frac{1}{3}h\tan\alpha\right)}$$

Note: If the vessel were a cylinder it will be impossible to lift it up by putting liquid in the vessel. We can verify this by keeping value of a as zero. We get required density as infinite, which is meaningless.

14.8 | LINEAR ACCELERATED MOTION OF A FLUID

We consider an open container of a liquid that is moving along a straight line with a constant acceleration a as shown in Fig. 14-18.

Let us consider a small horizontal cylinder of length dx and cross-sectional area A located at y distance below the free surface of the fluid. This cylinder is accelerating in ground frame with an acceleration a; hence, the net horizontal force acting on it should be equal to the product of mass (dm) and acceleration.

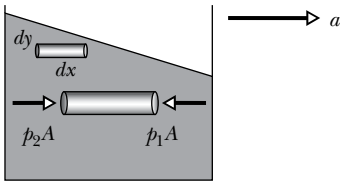

Figure 14-18 Linear accelerated motion of the vessel containing fluid.

$$dm = A\,dx\,\rho$$

$$p_2A - p_1A = (A\,dx\rho)\,a$$

If we say that the right face of the cylinder is at a distance y below the free surface of the fluid, then the left surface is $y + dy$ below the surface of liquid. Thus,

$$p_2 - p_1 = \rho g\,dy$$

Therefore, $$\frac{dy}{dx} = \frac{a}{g}$$

As the slope of the free surface is coming out to be constant, we can say that it must be a straight line.

Hence, $$\tan\theta = \frac{a}{g} \qquad (14\text{-}32)$$

If the container has acceleration along y axis also, then the slope of this line is given by the relationship.

$$\frac{dy}{dx} = \frac{a}{g + a_y}$$

Along a free surface, the pressure is constant, so that for the accelerating mass shown in Fig. 14-19, the free surface will be inclined if $a_x \neq 0$. In addition, all lines parallel to the free surface will have same pressure.

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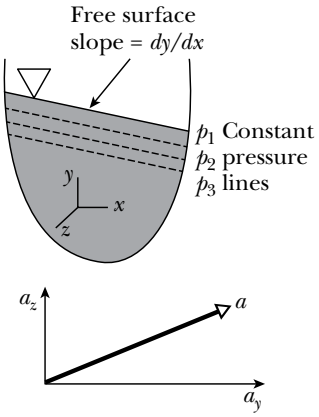

Figure 14-19 A vessel accelerating at some angle with the horizontal.

For a special circumstance, when $a_x = 0, a_y \neq 0$, which corresponds to the mass of fluid accelerating in the vertical direction, the fluid surface will be horizontal. However, we see that the pressure variation is not $\rho g\, dy$, but is given by the equation

$$dp = \rho(g + a_y)\, dy$$

Thus, the pressure on the bottom of a liquid-filled tank which is resting on the floor of an elevator and accelerating upward will be more than, if the, tank would have been at rest (or moving with a constant velocity). It is to be noted that for a freely falling fluid mass ($a_y = -g$), the pressure variation in all three coordinate directions is zero, which means that the pressure throughout will be the same. The pressure throughout a "blob" of a liquid floating in an orbiting space shuttle (a form of free fall) is zero. The only force holding the liquid together is surface tension.

SAMPLE PROBLEM 14.11

Linear accelerated motion of a fluid in a tank

The cross-section of a tank kept on a vehicle is shown in Fig 14-20*a*. The rectangular tank is open to the atmosphere. During motion of the vehicle, the tank is subjected to a constant linear acceleration, $a = 2.5$ m/s^2. How much fluid will be left inside the tank if initially the tank is half filled. The vessel is 5 m wide and 2 m high.

KEY IDEAS

(1) We know that the fluid will have an inclined free surface as shown in the figures, having angle θ with horizontal. You can imagine that if we accelerate it at a higher rate more fluid will get spilled in the process as it will rise higher.

(2) If the height of the liquid on the left wall is greater than 2 m, the fluid will spill out.

Calculations: Using Eq. 14-31, we can find the angle that the fluid will make with the horizontal

$$\tan\theta = \frac{2.5}{10} = \frac{1}{4}$$

Let's assume that the dimension of tank in the plane perpendicular to the page is d.

From geometry, it is easy to see that free surface on right-hand side will go down and the same will rise on left-hand side. Thus, if we assume that fluid on right-hand side has not touched the floor, we will have fluid taking a shape as shown in Fig 14-20*b*. The cuboid part will have volume $x \times 5 \times d$, where x is the height above the bottom.

The wedge part will have the volume $1/2 \times h \times 5 \times d$, where h can be found in the following manner:

$$\left(\frac{h}{5}\right) = \tan\theta = \left(\frac{1}{4}\right)$$

Thus, total volume will be

$$\frac{1}{2} \times \frac{5}{4} \times 5 \times d + x \times 5 \times d$$

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and if we assume there is no spilling then it must be equal to the final volume, so

$$\frac{1}{2}\times\frac{5}{4}\times 5\times d + x\times 5\times d = 1\times 5\times d$$

Solving we get, $x = 3/8$ m.

Therefore, total length

$$\frac{5}{4}+\frac{3}{8}=\frac{10+3}{8}=\frac{13}{8}<2\text{ m}$$

Thus, height is less than 2 m. Hence, water will not spill.

Learn: The answer may appear like an anti-climax as there is no spill, but at least we know that as acceleration increases, the chances of spilling will also increase.

In Case of: How much fluid will be left inside the tank if the vehicle accelerates with an acceleration $a = 10$ m/s^2?

Calculations: Using Eq. 14-31 we can find the angle that the fluid will make with the horizontal

$$\tan\theta=\frac{10}{10}=1 \quad \text{or} \quad \theta=\frac{\pi}{4}$$

In this case, fluid cannot remain inside. Fluid having an inclined free surface at 45° angle, and covering the bottom of length 5 m, will also be 5 m high. This will require the wall to be of 5 m height, which is just 2 m for the given vessel. Instead, if we think it other way round, to keep in contact with the left-hand side wall, bottom will have to be covered only 2 m with the fluid as shown in Fig. 14-20*c*.

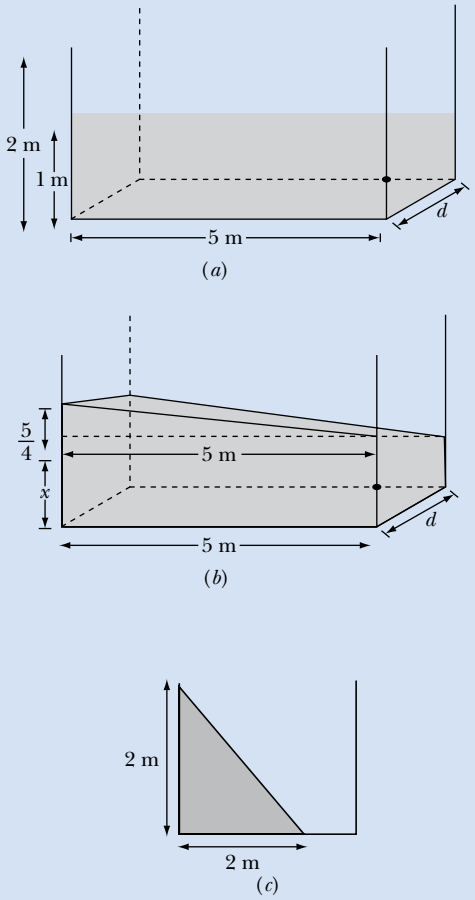

Figure 14-20 (*a*) A tank open to the atmosphere, filled with fluid, and kept on top of a car. (*b*) Fluid in the tank when car is in motion. (*c*) Fluid in the tank when acceleration of car is 10 m/s^2.

Volume of fluid inside = (1/2) × 2 × 2 × d m^3 = 2d m^3.

Thus, volume of fluid gone outside = 3d m^3. (Answer)

CHECKPOINT 6

If the vessel is closed from the top and now accelerated at 10 m/s^2, then what length of the floor will be uncovered?

14.9 | MOTION OF A FLUID IN A ROTATING VESSEL

Consider a cylindrical vessel, rotating at constant angular velocity about its axis. If it contains fluid then after an initial irregular shape, it will rotate with the tank as a rigid body. The acceleration of fluid particles located at a distance r from the axis of rotation will be equal to $\omega^2 r$, and the direction of the acceleration is toward the axis of rotation as shown in Fig. 14-21*a*. The fluid particles will undergo circular motion.

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Let us consider a small horizontal cylinder of length dr and cross-sectional area A located at a distance y below the free surface of the fluid and at a distance r from the axis (Fig. 14-21a). This cylinder is accelerating in ground frame with acceleration $\omega^2 r$ toward the axis; hence, the net horizontal force acting on it should be equal to the product of mass (dm) and acceleration.

$$dm = A dr \rho$$

$$p_2 A - p_1 A = (A dr \rho)\omega^2 r$$

If we say that the left face of the cylinder is at a distance y below the free surface of the fluid, then the right surface is $y + dy$ below the surface of liquid. Thus,

$$p_2 - p_1 = \rho g\, dy$$

Solving, we get

$$\frac{dy}{dr} = \frac{r\omega^2}{g}$$

and, therefore, the equation for surfaces of constant pressure is

$$y = \frac{\omega^2 r^2}{2g} + \text{constant}$$

This equation shows that these surfaces of constant pressure are parabolic as shown in Fig. 14-21b. The pressure varies with the distance from the axis of rotation, but at a fixed radius, the pressure varies hydrostatically in the vertical direction as shown in Fig. 14-21b.

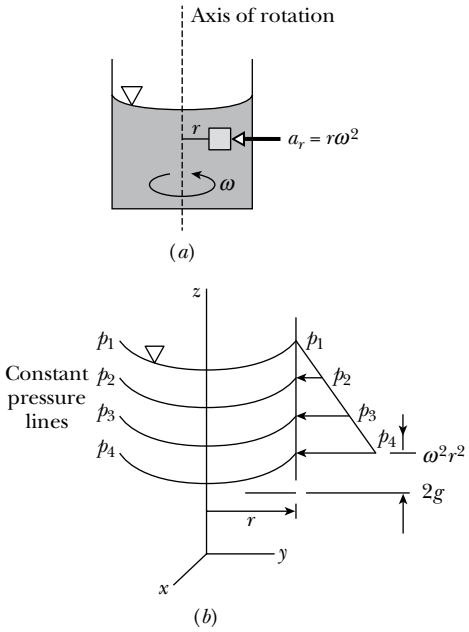

Figure 14-21 (a) A cylinder containing fluid is rotating about its axis. (b) Constant pressure lines in the rotating cylinder.

SAMPLE PROBLEM 14.12

Shape of free surface liquid in a rotating tank

The angular velocity, ω, of a rotating body or shaft can be measured by attaching an open cylinder of liquid, as shown in Fig. 14-22a, and measuring the change in the fluid level, $H - h_0$, caused by the rotation of the fluid. Determine the relationship between this change in fluid level and the angular velocity. Assume no fluid is spilling out of the vessel.

KEY IDEA

We know that the free surface of fluid has parabolic shape. Thus, fluid at the axis will be lower than the fluid around it.

Calculations: The height, h, of the free surface above the tank bottom can be determined as follows

$$h = \frac{\omega^2 r^2}{2g} + h_0$$

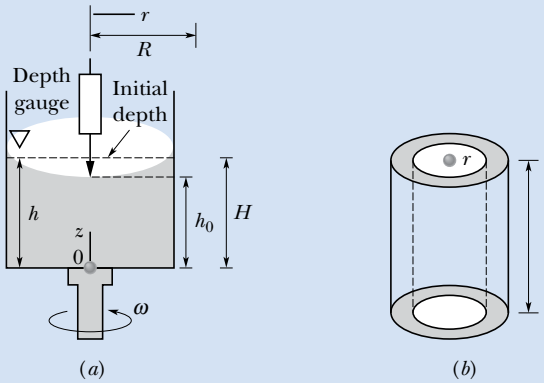

Figure 14-22 (a) An open cylinder of liquid attached to a rotating shaft. (b) Element of cylindrical shape.

The initial volume of fluid in the tank,

$$V_i = \pi R^2 H$$

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The volume of the fluid with the rotating tank can be found by considering a differential element of cylindrical shape as shown in Fig. 14-22*b*. This cylindrical shell is taken at some arbitrary radius, *r*, and thickness *dr*. Thus, its volume is

$$dV = 2\pi rh\,dr$$

The total volume is, therefore,

$$V = 2\pi\int_0^R r\left(\frac{\omega^2 r^2}{2g} + h_0\right)dr = \frac{\pi\omega^2 R^4}{4g} + \pi R^2 h_0$$

Since the volume of the fluid in the tank must remain constant (given that none spills over the top), it follows that

$$\pi R^2 H = \frac{\pi\omega^2 R^4}{4g} + \pi R^2 h_0$$

or

$$H - h_0 = \frac{\omega^2 R^2}{4g} \qquad \text{(Answer)}$$

Learn: This is the relationship we were looking for. It shows that the change in depth could indeed be used to determine the rotational speed.

14.10 | BUOYANT FORCE

The location of the line of action of the buoyant force can be determined by adding torques of the forces due to pressure forces, with respect to some convenient axis. The buoyant force must pass through the center of mass of the displaced volume, as shown in Fig. 14-23*a*, as it was in translational and rotational equilibrium. The point through which the buoyant force acts is called the center of buoyancy.

The same results apply to floating bodies which are only partially submerged, as shown in Fig. 14-23*b*, if the density of the fluid above the liquid surface is very small compared with the liquid in which the body floats. As the fluid above the surface is usually air, for practical purposes, this condition is satisfied.

In the above discussion, the fluid is assumed to have a constant density. If a body is immersed in a fluid in which density varies with depth, such as having multiple layers of fluid, the magnitude of the buoyant force remains equal to the weight of the displaced fluid and the buoyant force passes through the center of mass of the displaced volume.

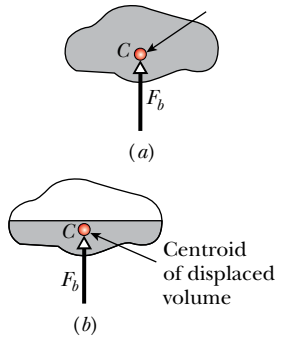

Figure 14-23 Buoyant force passing through center of mass of the displaced volume when a body is (*a*) totally submerged and (*b*) partially submerged. The fluid is assumed to have constant density.

SAMPLE PROBLEM 14.13

Buoyant force on a geometric body kept in a vessel into which fluid is poured

A square with side *b* and rectangle with sides *b* and 2*b* are cut out from two plates of equal thickness with densities of 3.5 g/cm^3 and 2 g/cm^3, the square being cut out of the heavier material. The square and the rectangle are fastened together in the form of the letter *L* and placed upside down on the bottom of an empty vessel (Fig. 14-24*a*). Will the object be stable if the vessel is filled with water?

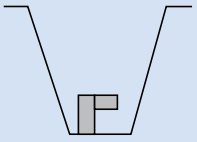

Figure 14-24 (*a*) A geometric body kept in vessel in which fluid is going to be poured.

KEY IDEAS

(1) The inverted *L* will be stable on the bottom of the empty vessel, since a perpendicular dropped from the center of gravity *b* (Fig. 14-24*b*) is within the limits of the supporting area. As water is poured into the vessel, the buoyant force acting on the rectangle will grow in magnitude (it is assumed that the water can flow between the base of the block and the bottom of the vessel).

(2) When the depth of the water in the vessel is 0.5*r*, the sum of the torques of the forces which tend to turn the body clockwise, will be equal to the sum of those acting in the opposite direction. As the vessel is filled further, the block will fall.

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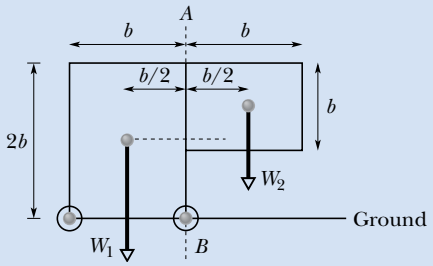


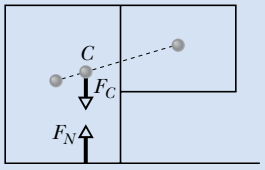
Figure 14-24 (*b*) Weight acting through the center of mass of the square and rectangle.

Calculation: Initially,

$$W_1 = 2b^2(2) = 4b^2$$
$$W_2 = 3.5b^2$$

First we need to find exact location of normal. Normal must develop from ground exactly at the point where center of gravity of whole system (rectangle and square) is located. In this case it is somewhere on left side of line *AB* as shown in Fig. 14-24*c*.

Now we can say that whole system is in rotational equilibrium.



C is center of gravity of whole system

Figure 14-24 (*c*) For equilibrium, normal must act through the center of mass when no liquid is poured.

When we start pouring liquid gradually, normal force decreases (Fig. 14-24*d*) according to the following equation

$$F_b + F_N = F_C = (W_1 + W_2)$$

where F_b is buoyant force on rectangular body.

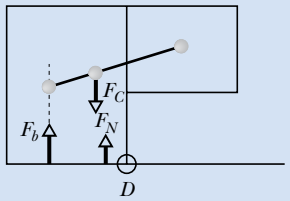


Figure 14-24 (*d*) For equilibrium, normal must shift when liquid is poured.

SAMPLE PROBLEM 14.14

Buoyant force on a hinged wooden board partially submerged in water

One end of a board of length *l* is hinged on top of a stone protruding from water. Length *a* of the board is above the point of support (Fig. 14-25*a*). What part of the board is below the surface of the water in equilibrium state, if the density of wood is γ?

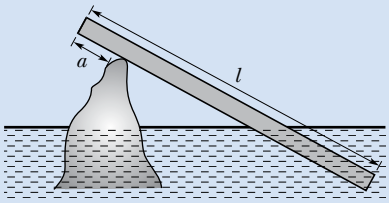


Figure 14-25 (*a*) A board partially submerged inside the fluid.

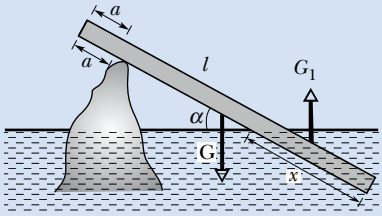


Figure 14-25 (*b*) Free-body diagram of the board.

KEY IDEAS

(1) The rod is experiencing no net force and torque.

(2) We cannot solve this problem by equating net force to zero, as the force applied by the stone is unknown. Thus, we will balance the torques of the forces acting on the board with respect to point *C* (Fig. 14-25*b*). This will exclude the force applied by the stone, as the moment arm for this force will be zero.

Calculations: Here F_{b_1} is the buoyant force applied by the water on the submerged portion and is equal to the

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volume of submerged portion multiplied by the density of water.

$$F_{b_1} = Ax\gamma_0$$

The length of moment arm is $\left(l - a - \frac{x}{2}\right)\cos\alpha$.

Weight of the board is $W = Al\gamma$. Here A is the cross-sectional area of the board and γ the density of the wood. Equating torque, we get

$$F_{b_1} = \left(1 - a - \frac{x}{2}\right)\cos\alpha = W\left(\frac{l}{2} - a\right)\cos\alpha$$

Hence, $$x = (l-a) \pm \sqrt{(l-a)^2 - \frac{\gamma}{\gamma_0} l(l-2a)}$$

Since $x < l - a$, only one solution is valid, so

$$x = (l-a) - \sqrt{(l-a)^2 - \frac{\gamma}{\gamma_0} l(l-2a)}$$

Learn: The values of weight and buoyant force are not equal as there must be some force acting on the board due to the stone.

Stability

The center of buoyancy and center of gravity do not, necessarily, coincide so the floating or submerged body may not be in stable equilibrium. A small rotation can cause the buoyant force to produce either a restoring or overturning torque. For example, for the completely submerged body as shown in Fig. 14-26*a*, which has a center of gravity below the center of buoyancy, a rotation from its equilibrium position will create a restoring torque by the buoyant force, F_b, which causes the body to rotate back to its original position. Thus, if the center of gravity falls below the center of buoyancy, the body is stable.

However, as shown in Fig. 14-26*b*, if the center of gravity of the completely submerged body is above the center of buoyancy, the resulting torque formed by the weight and the buoyant force will cause the body to overturn and move to a new equilibrium position. Thus, a completely submerged body with its center of gravity above its center of buoyancy is in an unstable equilibrium position.

For floating bodies, the stability problem is more complicated, as the body rotates the location of the center of buoyancy may change.

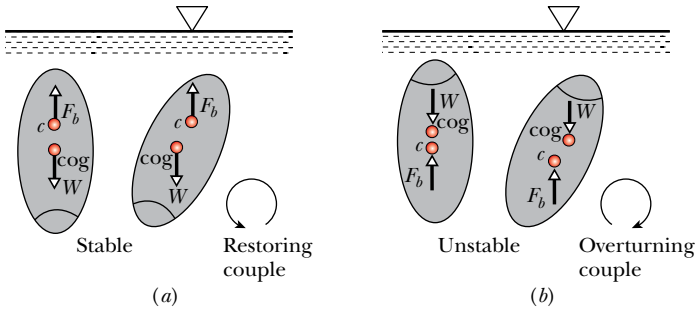

Figure 14-26 Different situations where rotational equilibrium of body is discussed.

14.11 | FLUID DYNAMICS

Key Concepts

- An ideal fluid is incompressible and lacks viscosity, and its flow is steady and irrotational.
- A *streamline* is the path followed by an individual fluid particle.
- A *tube of flow* is a bundle of streamlines.
- The flow within any tube of flow obeys the equation of continuity:

$$R_V = Av = \text{a constant},$$

in which R_V is the volume flow rate, A is the cross-sectional area of the tube of flow at any point, and v is the speed of the fluid at that point.

- The mass flow rate R_m is

$$R_m = \rho R_V = \rho A v = \text{a constant}.$$

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Ideal Fluids in Motion

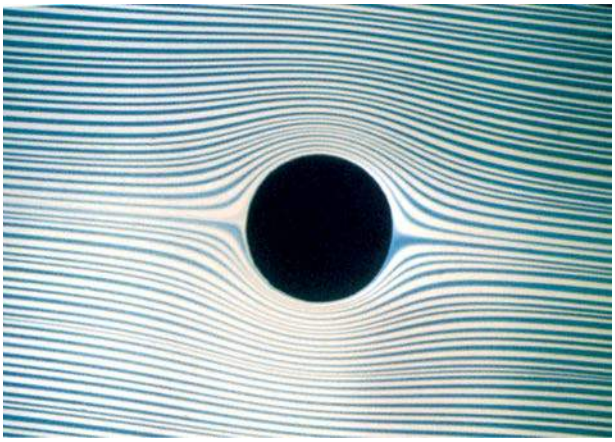
The motion of *real fluids* is very complicated and not yet fully understood. Instead, we shall discuss the motion of an **ideal fluid**, which is simpler to handle mathematically and yet provides useful results. Here are four assumptions that we make about our ideal fluid; they all are concerned with *flow:*

Will McIntyre/Photo Researchers, Inc.

Figure 14-27 At a certain point, the rising flow of smoke and heated gas changes from steady to turbulent.

1. ***Steady flow.*** In *steady* (or *laminar*) *flow*, the velocity of the moving fluid at any fixed point does not change with time. The gentle flow of water near the center of a quiet stream is steady; the flow in a chain of rapids is not. Figure 14-27 shows a transition from steady flow to *nonsteady* (or *nonlaminar* or *turbulent*) *flow* for a rising stream of smoke. The speed of the smoke particles increases as they rise and, at a certain critical speed, the flow changes from steady to nonsteady.
2. ***Incompressible flow.*** We assume, as for fluids at rest, that our ideal fluid is incompressible; that is, its density has a constant, uniform value.
3. ***Nonviscous flow.*** Roughly speaking, the viscosity of a fluid is a measure of how resistive the fluid is to flow. For example, thick honey is more resistive to flow than water, and so honey is said to be more viscous than water. Viscosity is the fluid analog of friction between solids; both are mechanisms by which the kinetic energy of moving objects can be transferred to thermal energy. In the absence of friction, a block could glide at constant speed along a horizontal surface. In the same way, an object moving through a nonviscous fluid would experience no *viscous drag force*—that is, no resistive force due to viscosity; it could move at constant speed through the fluid. The British scientist Lord Rayleigh noted that in an ideal fluid a ship's propeller would not work, but, on the other hand, in an ideal fluid a ship (once set into motion) would not need a propeller!
4. ***Irrotational flow.*** Although it need not concern us further, we also assume that the flow is *irrotational*. To test for this property, let a tiny grain of dust move with the fluid. Although this test body may (or may not) move in a circular path, in irrotational flow the test body will not rotate about an axis through its own center of mass. For a loose analogy, the motion of a Ferris wheel is rotational; that of its passengers is irrotational.

We can make the flow of a fluid visible by adding a *tracer*. This might be a dye injected into many points across a liquid stream (Fig. 14-28) or smoke particles added to a gas flow (Fig. 14-27). Each bit of a tracer follows a *streamline*, which is the path that a tiny element of the fluid would take as the fluid flows. We know that the velocity of a particle is always tangent to the path taken by the particle. Here the particle is the fluid element, and its velocity



Courtesy D. H. Peregrine, University of Bristol

Figure 14-28 The steady flow of a fluid around a cylinder, as revealed by a dye tracer that was injected into the fluid upstream of the cylinder.

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$\vec{v}$ is always tangent to a streamline (Fig. 14-29). For this reason, two streamlines can never intersect; if they did, then an element arriving at their intersection would have two different velocities simultaneously—an impossibility.

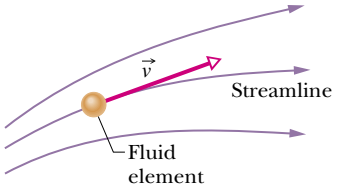

Figure 14-29 A fluid element traces out a streamline as it moves. The velocity vector of the element is tangent to the streamline at every point.

The Equation of Continuity

You may have noticed that you can increase the speed of the water emerging from a garden hose by partially closing the hose opening with your thumb. Apparently the speed v of the water depends on the cross-sectional area A through which the water flows.

Here we wish to derive an expression that relates v and A for the steady flow of an ideal fluid through a tube with varying cross section, like that in Fig. 14-30. The flow there is toward the right, and the tube segment shown (part of a longer tube) has length L. The fluid has speeds v_1 at the left end of the segment and v_2 at the right end. The tube has cross-sectional areas A_1 at the left end and A_2 at the right end. Suppose that in a time interval Δt a volume ΔV of fluid enters the tube segment at its left end (that volume is colored purple in Fig. 14-30). Then, because the fluid is incompressible, an identical volume ΔV must emerge from the right end of the segment (it is colored green in Fig. 14-30).

We can use this common volume ΔV to relate the speeds and areas. To do so, we first consider Fig. 14-31, which shows a side view of a tube of *uniform* cross-sectional area A. In Fig. 14-31*a*, a fluid element e is about to pass through the dashed line drawn across the tube width. The element's speed is v, so during a time interval Δt, the element moves along the tube a distance $\Delta x = v\Delta t$. The volume ΔV of fluid that has passed through the dashed line in that time interval Δt is

$$\Delta V = A\,\Delta x = Av\,\Delta t. \tag{14-33}$$

Applying Eq. 14-33 to both the left and right ends of the tube segment in Fig. 14-30, we have

$$\Delta V = A_1 v_1\,\Delta t = A_2 v_2\,\Delta t$$

or

$$A_1 v_1 = A_2 v_2 \quad \text{(equation of continuity)}. \tag{14-34}$$

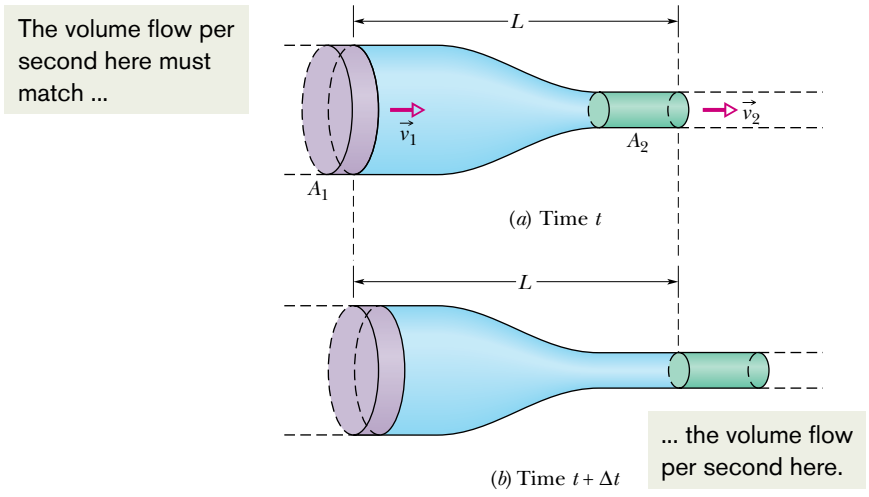

Figure 14-30 Fluid flows from left to right at a steady rate through a tube segment of length L. The fluid's speed is v_1 at the left side and v_2 at the right side. The tube's cross-sectional area is A_1 at the left side and A_2 at the right side. From time t in (*a*) to time $t + \Delta t$ in (*b*), the amount of fluid shown in purple enters at the left side and the equal amount of fluid shown in green emerges at the right side.

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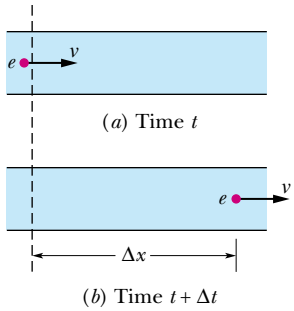


Figure 14-31 Fluid flows at a constant speed v through a tube. (*a*) At time t, fluid element e is about to pass the dashed line. (*b*) At time $t + \Delta t$, element e is a distance $\Delta x = v\,\Delta t$ from the dashed line.

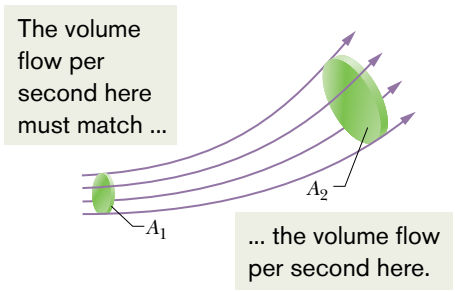


Figure 14-32 A tube of flow is defined by the streamlines that form the boundary of the tube. The volume flow rate must be the same for all cross sections of the tube of flow.

This relation between speed and cross-sectional area is called the **equation of continuity** for the flow of an ideal fluid. It tells us that the flow speed increases when we decrease the cross-sectional area through which the fluid flows.

Equation 14-34 applies not only to an actual tube but also to any so-called *tube of flow*, or imaginary tube whose boundary consists of streamlines. Such a tube acts like a real tube because no fluid element can cross a streamline; thus, all the fluid within a tube of flow must remain within its boundary. Figure 14-32 shows a tube of flow in which the cross-sectional area increases from area A_1 to area A_2 along the flow direction. From Eq. 14-34 we know that, with the increase in area, the speed must decrease, as is indicated by the greater spacing between streamlines at the right in Fig. 14-32. Similarly, you can see that in Fig. 14-28 the speed of the flow is greatest just above and just below the cylinder.

We can rewrite Eq. 14-34 as

$$R_V = Av = \text{a constant} \quad \text{(volume flow rate, equation of continuity)}, \tag{14-35}$$

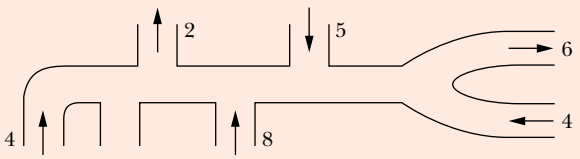
in which R_V is the **volume flow rate** of the fluid (volume past a given point per unit time). Its SI unit is the cubic meter per second (m^3/s). If the density ρ of the fluid is uniform, we can multiply Eq. 14-35 by that density to get the **mass flow rate** R_m (mass per unit time):

$$R_m = \rho R_V = \rho Av = \text{a constant} \quad \text{(mass flow rate)}. \tag{14-36}$$

The SI unit of mass flow rate is the kilogram per second (kg/s). Equation 14-36 says that the mass that flows into the tube segment of Fig. 14-30 each second must be equal to the mass that flows out of that segment each second.

CHECKPOINT 7

The figure shows a pipe and gives the volume flow rate (in cm^3/s) and the direction of flow for all but one section. What are the volume flow rate and the direction of flow for that section?



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SAMPLE PROBLEM 14.15

A water stream narrows as it falls

Figure 14-33 shows how the stream of water emerging from a faucet "necks down" as it falls. This change in the horizontal cross-sectional area is characteristic of any laminar (nonturbulant) falling stream because the gravitational force increases the speed of the stream. Here the indicated cross-sectional areas are $A_0 = 1.2\ \text{cm}^2$ and $A = 0.35\ \text{cm}^2$. The two levels are separated by a vertical distance $h = 45$ mm. What is the volume flow rate from the tap?

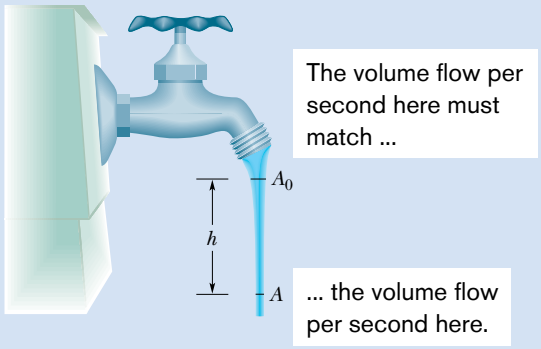

Figure 14-33 As water falls from a tap, its speed increases. Because the volume flow rate must be the same at all horizontal cross sections of the stream, the stream must "neck down" (narrow).

KEY IDEA

The volume flow rate through the higher cross section must be the same as that through the lower cross section.

Calculations: From Eq. 14-35, we have

$$A_0 v_0 = Av, \tag{14-37}$$

where v_0 and v are the water speeds at the levels corresponding to A_0 and A. We can also write, because the water is falling freely with acceleration g,

$$v^2 = v_0^2 + 2gh. \tag{14-38}$$

Eliminating v between Eqs. 14-37 and 14-38 and solving for v_0, we obtain

$$v_0 = \sqrt{\frac{2ghA^2}{A_0^2 - A^2}}$$

$$= \sqrt{\frac{(2)(9.8\ \text{m/s}^2)(0.045\ \text{m})(0.35\ \text{cm}^2)^2}{(1.2\ \text{cm}^2)^2 - (0.35\ \text{cm}^2)^2}}$$

$$= 0.286\ \text{m/s} = 28.6\ \text{cm/s}.$$

From Eq. 14-35, the volume flow rate R_V is then

$$R_V = A_0 v_0 = (1.2\ \text{cm}^2)(28.6\ \text{cm/s})$$

$$= 34\ \text{cm}^3/\text{s}. \qquad \text{(Answer)}$$

14.12 | BERNOULLI'S EQUATION

Key Concept

- Applying the principle of conservation of mechanical energy to the flow of an ideal fluid leads to Bernoulli's equation:

$$p + \frac{1}{2}\rho v^2 + \rho g y = \text{a constant}$$

along any tube of flow.

Figure 14-34 represents a tube through which an ideal fluid is flowing at a steady rate. In a time interval Δt, suppose that a volume of fluid ΔV, colored purple in Fig. 14-34, enters the tube at the left (or input) end and an identical volume, colored green in Fig. 14-34, emerges at the right (or output) end. The emerging volume must be the same as the entering volume because the fluid is incompressible, with an assumed constant density ρ.

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Let y_1, v_1, and p_1 be the elevation, speed, and pressure of the fluid entering at the left, and y_2, v_2, and p_2 be the corresponding quantities for the fluid emerging at the right. By applying the principle of conservation of energy to the fluid, we shall show that these quantities are related by

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2. \qquad (14\text{-}39)$$

In general, the term $1/2\,\rho v^2$ is called the fluid's **kinetic energy density** (kinetic energy per unit volume). We can also write Eq. 14-39 as

$$p + \frac{1}{2}\rho v^2 + \rho g y = \text{a constant} \qquad \text{(Bernoulli's equation)}. \qquad (14\text{-}40)$$

Equations 14-39 and 14-40 are equivalent forms of **Bernoulli's equation**, after Daniel Bernoulli, who studied fluid flow in the 1700s.* Like the equation of continuity (Eq. 14-35), Bernoulli's equation is not a new principle but simply the reformulation of a familiar principle in a form more suitable to fluid mechanics. As a check, let us apply Bernoulli's equation to fluids at rest, by putting $v_1 = v_2 = 0$ in Eq. 14-39. The result is Eq. 14-7:

$$p_2 = p_1 + \rho g(y_1 - y_2).$$

A major prediction of Bernoulli's equation emerges if we take y to be a constant ($y = 0$, say) so that the fluid does not change elevation as it flows. Equation 14-39 then becomes

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2, \qquad (14\text{-}41)$$

which tells us that:

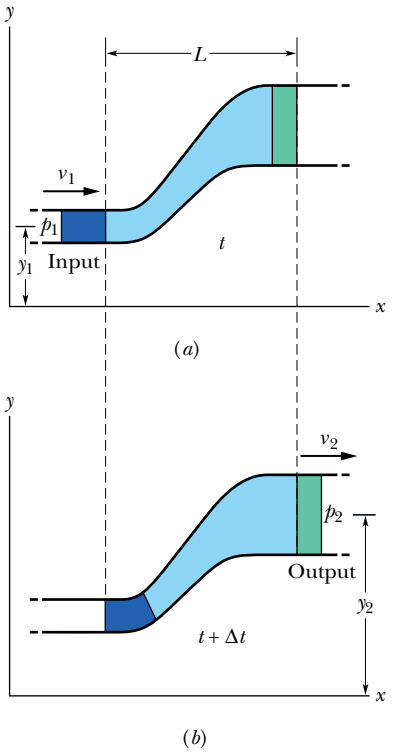

Figure 14-34 Fluid flows at a steady rate through a length L of a tube, from the input end at the left to the output end at the right. From time t in (a) to time $t + \Delta t$ in (b), the amount of fluid shown in purple enters the input end and the equal amount shown in green emerges from the output end.

If the speed of a fluid element increases as the element travels along a horizontal streamline, the pressure of the fluid must decrease, and conversely.

Put another way, where the streamlines are relatively close together (where the velocity is relatively great), the pressure is relatively low, and conversely.

The link between a change in speed and a change in pressure makes sense if you consider a fluid element that travels through a tube of various widths. Recall that the element's speed in the narrower regions is fast and its speed in the wider regions is slow. By Newton's second law, forces (or pressures) must cause the changes in speed (the accelerations). When the element nears a narrow region, the higher pressure behind it accelerates it so that it then has a greater speed in the narrow region. When it nears a wide region, the higher pressure ahead of it decelerates it so that it then has a lesser speed in the wide region.

Bernoulli's equation is strictly valid only to the extent that the fluid is ideal. If viscous forces are present, thermal energy will be involved, which here we neglect.

Proof of Bernoulli's Equation

Let us take as our system the entire volume of the (ideal) fluid shown in Fig. 14-34. We shall apply the principle of conservation of energy to this system as it moves from its initial state (Fig. 14-34a) to its final state (Fig. 14-34b).

* For irrotational flow (which we assume), the constant in Eq. 14-40 has the same value for all points within the tube of flow; the points do not have to lie along the same streamline. Similarly, the points 1 and 2 in Eq. 14-39 can lie anywhere within the tube of flow.

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The fluid lying between the two vertical planes separated by a distance L in Fig. 14-34 does not change its properties during this process; we need be concerned only with changes that take place at the input and output ends.

First, we apply energy conservation in the form of the work–kinetic energy theorem,

$$W = \Delta K, \tag{14-42}$$

which tells us that the change in the kinetic energy of our system must equal the net work done on the system. The change in kinetic energy results from the change in speed between the ends of the tube and is

$$\begin{aligned}\Delta K &= \frac{1}{2}\Delta m v_2^2 - \frac{1}{2}\Delta m v_1^2 \\ &= \frac{1}{2}\rho \Delta V(v_2^2 - v_1^2),\end{aligned} \tag{14-43}$$

in which Δm $(= \rho\, \Delta V)$ is the mass of the fluid that enters at the input end and leaves at the output end during a small time interval Δt.

The work done on the system arises from two sources. The work W_g done by the gravitational force $(\Delta m\vec{g})$ on the fluid of mass Δm during the vertical lift of the mass from the input level to the output level is

$$\begin{aligned}W_g &= -\Delta m g(y_2 - y_1) \\ &= -\rho g\, \Delta V(y_2 - y_1).\end{aligned} \tag{14-44}$$

This work is negative because the upward displacement and the downward gravitational force have opposite directions.

Work must also be done *on* the system (at the input end) to push the entering fluid into the tube and *by* the system (at the output end) to push forward the fluid that is located ahead of the emerging fluid. In general, the work done by a force of magnitude F, acting on a fluid sample contained in a tube of area A to move the fluid through a distance Δx, is

$$F\,\Delta x = (pA)(\Delta x) = p(A\,\Delta x) = p\,\Delta V.$$

The work done on the system is then $p_1\,\Delta V$, and the work done by the system is $-p_2\,\Delta V$. Their sum W_p is

$$\begin{aligned}W_p &= -p_2\,\Delta V + p_1\,\Delta V \\ &= -(p_2 - p_1)\,\Delta V.\end{aligned} \tag{14-45}$$

The work–kinetic energy theorem of Eq. 14-42 now becomes

$$W = W_g + W_p = \Delta K.$$

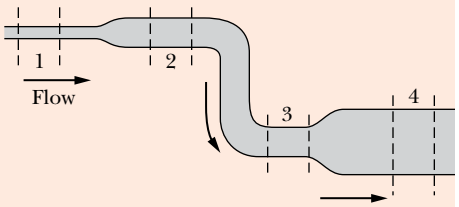
Substituting from Eqs. 14-43, 14-44, and 14-45 yields

$$-\rho g \Delta V(y_2 - y_1) - \Delta V(p_2 - p_1) = \frac{1}{2}\rho \Delta V(v_2^2 - v_1^2).$$

This, after a slight rearrangement, matches Eq. 14-39, which we set out to prove.

CHECKPOINT 8

Water flows smoothly through the pipe shown in the figure, descending in the process. Rank the four numbered sections of pipe according to (a) the volume flow rate R_V through them, (b) the flow speed v through them, and (c) the water pressure p within them, greatest first.



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SAMPLE PROBLEM 14.16

Bernoulli principle of fluid through a narrowing pipe

Ethanol of density $\rho = 791$ kg/m^3 flows smoothly through a horizontal pipe that tapers (as in Fig. 14-30) in cross-sectional area from $A_1 = 1.20 \times 10^{-3}$ m^2 to $A_2 = A_1/2$. The pressure difference between the wide and narrow sections of pipe is 4120 Pa. What is the volume flow rate R_V of the ethanol?

KEY IDEAS

(1) Because the fluid flowing through the wide section of pipe must entirely pass through the narrow section, the volume flow rate R_V must be the same in the two sections. Thus, from Eq. 14-35,

$$R_V = v_1 A_1 = v_2 A_2. \tag{14-46}$$

However, with two unknown speeds, we cannot evaluate this equation for R_V. (2) Because the flow is smooth, we can apply Bernoulli's equation. From Eq. 14-39, we can write

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g y = p_2 + \frac{1}{2}\rho v_2^2 + \rho g y, \tag{14-47}$$

where subscripts 1 and 2 refer to the wide and narrow sections of pipe, respectively, and y is their common elevation. This equation hardly seems to help because it does not contain the desired R_V and it contains the unknown speeds v_1 and v_2.

Calculations: There is a neat way to make Eq. 14-47 work for us: First, we can use Eq. 14-46 and the fact that $A_2 = A_1/2$ to write

$$v_1 = \frac{R_V}{A_1} \quad \text{and} \quad v_2 = \frac{R_V}{A_2} = \frac{2R_V}{A_1}. \tag{14-48}$$

Then we can substitute these expressions into Eq. 14-47 to eliminate the unknown speeds and introduce the desired volume flow rate. Doing this and solving for R_V yield

$$R_V = A_1\sqrt{\frac{2(p_1 - p_2)}{3\rho}}. \tag{14-49}$$

We still have a decision to make: We know that the pressure difference between the two sections is 4120 Pa, but does that mean that $p_1 - p_2$ is 4120 Pa or –4120 Pa? We could guess the former is true, or otherwise the square root in Eq. 14-49 would give us an imaginary number. However, let's try some reasoning. From Eq. 14-46 we see that speed v_2 in the narrow section (small A_2) must be greater than speed v_1 in the wider section (larger A_1). Recall that if the speed of a fluid increases as the fluid travels along a horizontal path (as here), the pressure of the fluid must decrease. Thus, p_1 is greater than p_2, and $p_1 - p_2 = 4120$ Pa. Inserting this and known data into Eq. 14-49 gives

$$R_V = 1.20 \times 10^{-3}\,\text{m}^2\sqrt{\frac{(2)(4120\ \text{Pa})}{(3)(791\ \text{kg/m}^3)}}$$
$$= 2.24 \times 10^{-3}\,\text{m}^3/\text{s}. \qquad \text{(Answer)}$$

SAMPLE PROBLEM 14.17

Bernoulli principle for a leaky water tank

In the old West, a desperado fires a bullet into an open water tank (Fig. 14-35), creating a hole a distance h below the water surface. What is the speed v of the water exiting the tank?

KEY IDEAS

(1) This situation is essentially that of water moving (downward) with speed v_0 through a wide pipe (the tank) of cross-sectional area A and then moving (horizontally) with speed v through a narrow pipe (the hole) of cross-sectional area a. (2) Because the water flowing through the wide pipe must entirely pass through the narrow pipe, the volume flow rate R_V must be the same in the two "pipes." (3) We can also relate v to v_0 (and to h) through Bernoulli's equation (Eq. 14-39).

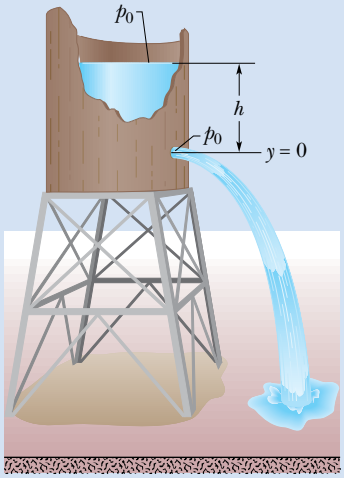

Figure 14-35 Water pours through a hole in a water tank, at a distance h below the water surface. The pressure at the water surface and at the hole is atmospheric pressure p_0.

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Calculations: From Eq. 14-35,

$$R_V = av = Av_0$$

and thus

$$v_0 = \frac{a}{A}v.$$

Because $a \ll A$, we see that $v_0 \ll v$. To apply Bernoulli's equation, we take the level of the hole as our reference level for measuring elevations (and thus gravitational potential energy). Noting that the pressure at the top of the tank and at the bullet hole is the atmospheric pressure p_0 (because both places are exposed to the atmosphere), we write Eq. 14-39 as

$$p_0 + \frac{1}{2}\rho v_0^2 + \rho gh = p_0 + \frac{1}{2}\rho v^2 + \rho g(0). \qquad (14\text{-}50)$$

(Here the top of the tank is represented by the left side of the equation and the hole by the right side. The zero on the right indicates that the hole is at our reference level.) Before we solve Eq. 14-50 for v, we can use our result that $v_0 \ll v$ to simplify it: We assume that v_0^2, and thus the term $1/2\,\rho v_0^2$ in Eq. 14-50, is negligible relative to the other terms, and we drop it. Solving the remaining equation for v then yields

$$v = \sqrt{2gh}. \qquad \text{(Answer)}$$

This is the same speed that an object would have when falling a height h from rest.

SAMPLE PROBLEM 14.18

Race car, the air forced to flow under the car, Bernoulli's equation

Many types of race cars depend on *negative lift* (or *downforce*) to push them down against the track surface so they can take turns quickly without sliding out into the track wall. Part of the negative lift is the *ground force*, which is a force due to the airflow beneath the car. As the race car in Fig. 14-36 moves forward at 27.25 m/s, air is forced to flow over and under the car (Fig. 14-36*a*). The air forced to flow under the car enters through a vertical cross-sectional area $A_0 = 0.0330\ \text{m}^2$ at the front of the car (Fig. 14-36*b*) and then flows beneath the car where the vertical cross-sectional area is $A_1 = 0.0310\ \text{m}^2$. Treat this flow as steady flow through a stationary horizontal pipe that decreases in cross-sectional area from A_0 to A_1 (Fig. 14-36*c*).

(a) At the moment it passes through A_0, the air is at atmospheric pressure p_0. At what pressure p_1 is the air as it moves through A_1?

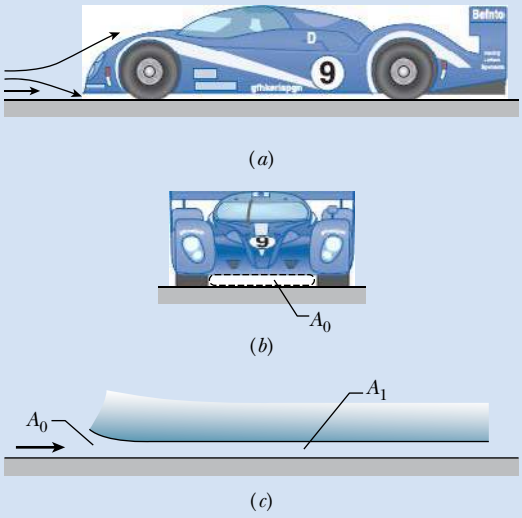

Figure 14-36 (*a*) Air flows above and below a race car. (*b*) The flow beneath the car enters through vertical cross-sectional area A_0. (*c*) The flow is then constrained as in a pipe that narrows to vertical cross-sectional area A_1.

KEY IDEA

Because the flow is steady, we can apply Bernoulli's equation (Eq. 14-39) to the flow. To be consistent with the given subscripts, we write the equation as

$$p_0 + \frac{1}{2}\rho v_0^2 + \rho gy = p_1 + \frac{1}{2}\rho v_1^2 + \rho gy \qquad (14\text{-}51)$$

where r is the air density and y is the distance above the ground of the flowing air. Because all the air entering through cross-sectional area A_0 flows through cross-sectional area A_1, the volume flow rate R_V through the two areas must be the same.

Calculations: From Eq. 14-35, we can write

$$A_0 v_0 = A_1 v_1,$$

or

$$v_1 = v_0 - \frac{A_0}{A_1} \qquad (14\text{-}52)$$

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Substituting Eq. 14-52 into Eq. 14-51 and rearranging give us

$$p_1 = p_0 - \frac{1}{2}\rho v_0^2\left(\frac{A_0^2}{A_1^2} - 1\right) \qquad (14\text{-}53)$$

The speed of the air as it enters A_0 at the front of the car is equal to 27.25 m/s, the speed of the car as it moves forward through the air. Substituting this speed, the air density $\rho = 1.21$ kg/m^3, and the values for A_0 and A_1 into Eq. 14-53, we find

$$p_1 = p_0 - \frac{1}{2}(1.21\ \text{kg/m}^3)(27.25\ \text{m/s})^2\left[\frac{(0.0330\ \text{m}^2)^2}{(0.0310\ \text{m}^2)^2} - 1\right]$$

$$p_1 = p_0 - 59.838\ \text{Pa} \approx p_0 - 59.8\ \text{Pa} \qquad \text{(Answer)}$$

Learn: Thus, the air pressure beneath the car is 59.8 Pa less than atmospheric pressure.

(b) If the horizontal cross-sectional area of the car is $A_h = 4.86$ m^2, what is the magnitude of the net vertical force $F_{\text{net},y}$ on the car due to the air pressures above and below the car?

KEY IDEA

The pressure on a surface is the force per unit area, as given by the equation $p = F/A$.

Calculations: Here, we are concerned with the top and bottom surfaces of the car, where we take both surfaces to have area A_h. Above the car, the air is at atmospheric pressure p_0 and presses down on the car with a vertical component

$$F_{y,\text{above}} = -p_0 A_h$$

Below the car, the air is at pressure $p_1 = p_0 - 59.838$ Pa and presses up on the car with a vertical component

$$F_{y,\text{below}} = (p_0 - 59.838\ \text{Pa})A_h$$

The net vertical force is then

$$\begin{aligned} F_{\text{net},y} &= F_{y,\text{below}} + F_{y,\text{above}} \\ &= (p_0 - 59.838\ \text{Pa})A_h - p_0 A_h \\ &= -(59.838\ \text{Pa})(4.86\ \text{m}^2) = -291\ \text{N} \qquad \text{(Answer)} \end{aligned}$$

Learn: Danger of drafting—This net downward force, which is due to the reduced air pressure beneath the car, is the ground effect acting on the car. It is about 30% of the total negative lift that helps hold the car on the track. Without negative lift, a car must greatly slow for turns or else it will slide outward into the track wall. In a race, a driver can reduce the air drag on his car by closely following another car, a procedure known as *drafting*. However, the leading car disrupts the steady flow of air under the trailing car, eliminating the ground effect on the trailing car. If the trailing driver does not anticipate that elimination and slow accordingly, sliding into the track wall may be unavoidable.

SAMPLE PROBLEM 14.19

A siphon and its water flow, Bernoulli's equation

Figure 14-37 shows a *siphon*, which is a device for removing liquid from a container. Tube ABC must initially be filled, but once this has been done, liquid will flow through the tube until the liquid surface in the container is level with the tube opening at A. The liquid has density 1000 kg/m^3 and negligible viscosity. The distances shown are $h_1 = 25$ cm, $d = 12$ cm and $h_2 = 40$ cm.

(a) With what speed does the liquid emerge from the tube at C?

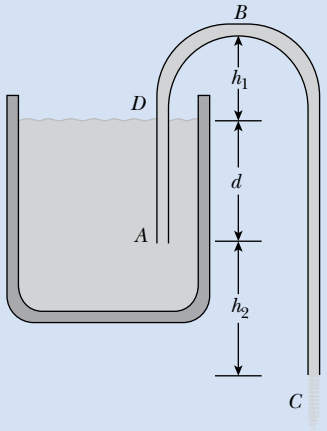

Figure 14-37 A standard siphon arrangement.

KEY IDEAS

You may have used siphon and you may recollect that lower the exit point of the fluid, faster is the fluid flows out.

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We consider a point D on the surface of the liquid in the container, in the same tube of flow with points A, B, and C. Applying Bernoulli's equation to points D and C, we obtain

$$p_D + \frac{1}{2}\rho v_D^2 + \rho g h_D = p_C + \frac{1}{2}\rho v_C^2 + \rho g h_C$$

Calculations: This leads to

$$v_C = \sqrt{\frac{2(p_D - p_C)}{\rho} + 2g(h_D - h_C) + v_D^2} \approx \sqrt{2g(d + h_2)}$$

where in the last step we set $p_D = p_C = p_{air}$ and $v_D/v_C \approx 0$. Plugging in the values, we obtain

$$v_C = \sqrt{2(9.8\ \text{m/s}^2)(0.40\ \text{m} + 0.12\ \text{m})} = 3.2\ \text{m/s}$$

Learn: The result confirms our experience.

(b) If the atmospheric pressure is 1.0×10^5 Pa, what is the pressure in the liquid at the topmost point B?

KEY IDEA

We now consider points B and C:

$$p_B + \frac{1}{2}\rho v_B^2 + \rho g h_B = p_C + \frac{1}{2}\rho v_C^2 + \rho g h_C$$

Calculations: Since $v_B = v_C$ by equation of continuity, and $p_C = p_{air}$, Bernoulli's equation becomes

$$\begin{aligned} p_B &= p_C + \rho g(h_C - h_B) = p_{air} - \rho g(h_1 + h_2 + d) \\ &= 10 \times 10^5\ \text{Pa} - (1.0 \times 10^3\ \text{kg/m}^3)(9.8\ \text{m/s}^2) \\ &\quad \times (0.25\ \text{m} + 0.40\ \text{m} + 0.12\ \text{m}) \\ &= 9.2 \times 10^4\ \text{Pa} \end{aligned}$$

(c) Theoretically, what is the greatest possible height h_1 that a siphon can lift water?

Calculations: Since $p_B \geq 0$, we must let

$$p_{air} - \rho g(h_1 + d + h_2) \geq 0$$

which yields

$$h_1 \leq h_{1,max} = \frac{p_{air}}{\rho g} - d - h_2 = 10.3\ \text{m} \qquad \text{(Answer)}$$

CHECKPOINT 9

A siphon of uniform diameter is used to drain water from a tank as illustrated in Fig. 14-37. Assuming steady flow without friction, consider a small hole in the hose at location D as indicated. When the siphon is used, will water leak out of the hose, or will air leak into the hose and thus causing the siphon to stop?

SAMPLE PROBLEM 14.20

Pitot tube, velocity of the fluid in terms of the density

Figure 14-38 shows a device called Pitot tube. It measures the velocity of moving fluids. Determine the velocity of the fluid in terms of the density ρ, the density of the fluid in manometer (U-tube) σ and the height h.

KEY IDEA

The difference in the two tubes is that liquid will flow into the tube B with full kinetic energy while it will just pass over the tube A without directly entering into it.

This problem is based on the use of Bernoulli's principle, in two different situations.

Calculations: The fluid inside the right tube must be at rest as the fluid exactly at the end is in contact with the fluid in Pitot tube, which is at rest.

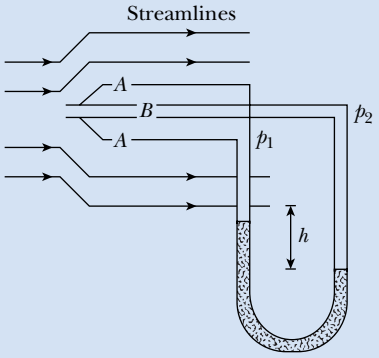

Figure 14-38 Pitot tube.

The velocity v_1 is the fluid velocity v_f. The velocity v_2 of the fluid at point B is zero and the pressure in the right arm is p_2 (called stagnation pressure).

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Thus, using Bernoulli's principle

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$

we get

$$p_2 = p_1 + \frac{1}{2}\rho v_f^2$$

On the other hand, the openings at point A is not along the flow lines, so we do not need to use Bernoulli's equation. We can simply say that the pressure just outside the opening is same as that within the Pitot tube.

Therefore, the pressure at the left arm of the manometer is same as the fluid pressure p_f, that is, $p_1 = p_f$.

Also, $$p_2 = p_1 + (\rho - \sigma)gh \qquad (14\text{-}54)$$

Generally, $\sigma \ll \rho$, so it is ignored.

Thus, $$p_2 = p_f + \rho gh \qquad (14\text{-}55)$$

From Eqs. 14-54 and 14-55

$$\frac{1}{2}\sigma v_f^2 = \rho gh$$

or $$v_f = \sqrt{\frac{2\rho gh}{\sigma}}$$

Learn: Thus, we can see that we have measured the fluid velocity as this was the only difference between the two tubes leading to the pressure difference between the tubes.

SAMPLE PROBLEM 14.21

Venturi meter, flow rate of fluid

A *venturi meter* is used to measure the flow speed of a fluid in a pipe. The meter is connected between two sections of the pipe (Fig. 14-39); the cross-sectional area A of the entrance and exit of the meter matches the pipe's cross-sectional area. Between the entrance and exit, the fluid flows from the pipe with speed V and then through a narrow "throat" of cross-sectional area a with speed v. A manometer connects the wider portion of the meter to the narrower portion. The change in the fluid's speed is accompanied by a change Δp in the fluid's pressure, which causes a height difference h of the liquid in the two arms of the manometer. (Here, Δp means pressure in the throat minus pressure in the pipe.)

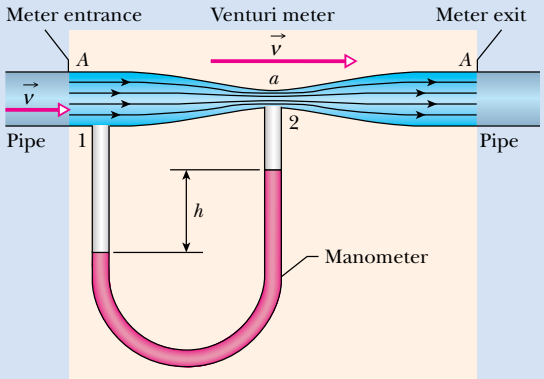

Figure 14-39 Venturi meter.

(a) By applying Bernoulli's equation and the equation of continuity to points 1 and 2 in Fig. 14-39, show that

$$V = \sqrt{\frac{2a^2\Delta p}{\rho(a^2 - A^2)}}$$

where r is the density of the fluid.

Reasoning: The continuity equation yields $AV = av$, and Bernoulli's equation yields $\Delta p + 1/2\ \rho V^2 = 1/2\ \rho V^2$, where $\Delta p = p_1 - p_2$. The first equation gives $v = (A/a)V$. We use this to substitute for v in the second equation and obtain $\Delta p + 1/2\ \rho V^2 = 1/2\ \rho(A/a)^2 V$. We solve this equation for V, to get

$$V = \sqrt{\frac{2\Delta p}{\rho(A/a)^2 - 1}} = \sqrt{\frac{2a^2\Delta p}{\rho(A^2 - a^2)}}$$

(b) Suppose that the fluid is fresh water, that the cross-sectional areas are 64 cm^2 in the pipe and 32 cm^2 in the throat, and that the pressure is 55 kPa in the pipe and 41 kPa in the throat. What is the rate of water flow in cubic meters per second?

Calculation: We substitute the values to obtain

$$v = \sqrt{\frac{2(32\times10^{-4}\ \text{m}^2)^2[(55\times10^3\ \text{Pa}) - (41\times10^3\ \text{Pa})]}{(1000\ \text{kg/m}^3)[(64\times10^{-4}\ \text{m}^2)^2 - (32\times10^{-4}\ \text{m}^2)^2]}}$$

Consequently, the flow rate is

$$Av = (64 \times 10^{-4}\ \text{m}^2)(3.06\ \text{m/s}) = 2.0 \times 10^{-2}\ \text{m}^3/\text{s}$$

(Answer)

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SAMPLE PROBLEM 14.22

Continuity equation

In Fig. 14-40, water flows through a pipe that tapers from radius $r_1 = 0.020$ m, where the water has a uniform speed $v_1 = 1.5$ m/s, to a radius $r_2 = 0.010$ m. What is the speed in the smaller pipe?

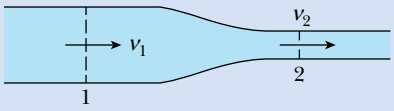

Figure 14-40 Water flows through a pipe that has different radius.

KEY IDEA

If a flowing fluid is incompressible (its volume cannot be changed by pressure), the volume flow rate (the rate at which a certain volume passes a point in the flow) is constant along the flow path. This means that at any two points in the flow path, the product of flow speed v and cross-sectional area A must be constant.

Calculation: Pick a cross-sectional area through point 1 in the wider pipe and another one through point 2 in the narrower pipe. Then we can write

$$v_2 A_2 = v_1 A_1$$

Solving for v_2 and substituting for the area of a circle lead to

$$v_2 = v_1 \frac{\pi r_1^2}{\pi r_2^2}$$

$$= (1.5 \text{ m/s}) \frac{(0.020 \text{ m})^2}{(0.010 \text{ m})^2} = 6.0 \text{ m/s} \qquad \text{(Answer)}$$

SAMPLE PROBLEM 14.23

Applying Bernoulli's equation to determine orifice discharge and velocity

A tank has two outlets (i) a rounded orifice A of diameter D and (ii) a pipe B with well-rounded entry and of length L, as shown in Fig. 14-41. For a height of water H in the tank, determine the (a) discharge from the outlets A and B, (b) velocities in the two outlets at levels 1 and 2 indicated in Fig. 14-41.

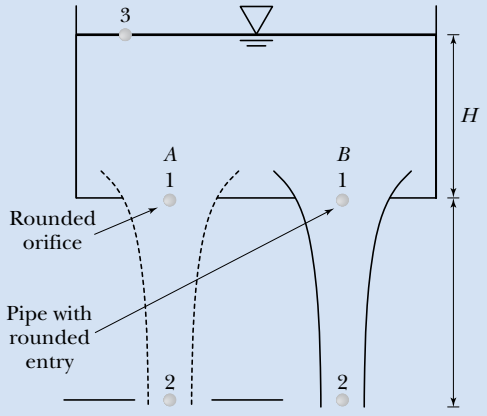

Figure 14-41 A tank having two outlets, a rounded orifice A and a pipe with round entry B.

KEY IDEA

(1) This problem is based on the use of Bernoulli's principle in two different situations.

(2) The difference in the two situations is that in part (i) pressure at the point 1 and all points below it will be atmospheric. On the other hand in part (ii) pressure will be atmospheric only at point 2.

Calculations:

(a) Rounded orifice A: Applying Bernoulli's equation to a point on the water surface 3 and point 1, we get

$$0 + 0 + H = \frac{p_1}{\gamma} + \frac{V_{1A}^2}{2g} + 0$$

As the orifice discharges to atmosphere,

$$\frac{p_1}{\gamma} = 0 \quad \text{and} \quad V_{1A} = \sqrt{2gH}$$

The discharge is,

$$Q_A = \frac{\pi}{4} D^2 \sqrt{2gH}$$

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At point 2, the pressure is atmospheric and hence by applying Bernoulli's equation between points 3 and 2, we get

$$0+0+(H+L)=0+\frac{V_{2A}^2}{2g}+0$$

or $$V_{2A}=\sqrt{2g(H+L)}$$

As the discharge is Q_A, the diameter at 2 will be smaller than D.

(b) Pipe B: By applying Bernoulli's equation between points 3 and 2.

$$0+0+(H+L)=0+\frac{V_{2B}^2}{2g}+0$$

or $$V_{2B}=\sqrt{2g(H+L)}$$

As the pipe size is uniform from points 1 to 2, by continuity equation

$$V_{1B}+V_{2B}=\sqrt{2g(H+L)}$$

thus, the results are:

	Orifice	Pipe
Velocity at 1	$\sqrt{2gH}$	$\sqrt{2g(H+L)}$
Velocity at 2	$\sqrt{2g(H+L)}$	$\sqrt{2g(H+L)}$
Discharge Q	$\frac{\pi}{4}D^2\sqrt{2gH}$	$\frac{\pi}{4}D^2\sqrt{2g(H+L)}$

Learn: We can see that velocity at point 2 is same in both the cases, this could have been directly concluded by applying Bernoulli's equation between points 3 and 2 in both parts.

SAMPLE PROBLEM 14.24

Hydraulic ram

Figure 14-42*a* shows a self-acting water lifting device called a hydraulic ram. Its working is based on the phenomenon of hydraulic impact — when a liquid flowing along a tube is suddenly stopped, then there is a sharp increase in the pressure of the liquid. Its flow can be suddenly stopped, for example, by the shutting of a valve that discharges the water from the tube. A tube with a length of $l = 2$ m and a diameter of $d = 20$ cm is lowered into a stream with a current velocity of $v = 4$ m/s.

First, let valve V_2 be open and valve V_1 be shut. A sharp increase in pressure will cause valve V_1 to open (valve V_2 will close at the same time) and the water will flow up into vessel A. The pressure drops, valve V_1 shuts and V_2 opens. The water in the tube assumes its course and the phenomenon is repeated in the previous sequence.

Find the amount of water raised by the ram in one hour to a height of $h = 30$ m if each valve opens thirty times a minute.

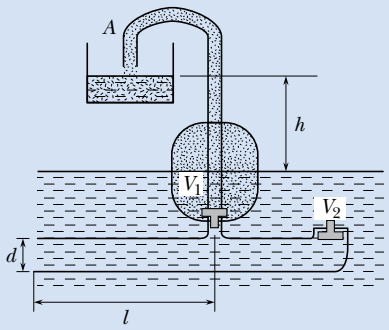

Figure 14-42 (*a*) A self-acting hydraulic ram in which, as valve 2 closes, water flows through valve 1 and reaches A.

KEY IDEAS

(1) Here we can imagine that as the water rushes along the tube it will be push both the valves but due to the design V_2 will close and V_1 will open. Thus, water will move along the vertical tube.

(2) Now this problem involves conservation of mechanical energy as in the ideal fluid there is no loss of mechanical energy.

Calculations: Thus, on the basis of the law of conservation of energy, we can write

$$\frac{Mv^2}{2}=mgh$$

where M is the mass of the water in the tube stopped by valve V_2 and m is the mass of the water raised to the height h.

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Therefore, $$\frac{\rho l \pi d^2}{4} \times \frac{v^2}{2} = \rho V_0 g h$$

where V_0 is the volume of water having mass m.

The average volume raised in two seconds is

$$V_0 = \frac{l \pi d^2 v^2}{8gh} = 1.7 \times 10^{-3}\,\text{m}^3$$

One hour of ram operation will raise

$$V = 1.7 \times 10^{-3} \times 30 \times 60 \cong 3\,\text{m}^3 \qquad \text{(Answer)}$$

Learn: Above device converts kinetic energy of water into potential energy. When water comes inside tube, it has some kinetic energy and exerts impulsive force on valve V_1 and this way valve opens and gains potential. This process repeats once in two seconds. The flow lines of fluid motion in the ram are shown in Fig. 14-42*b*.

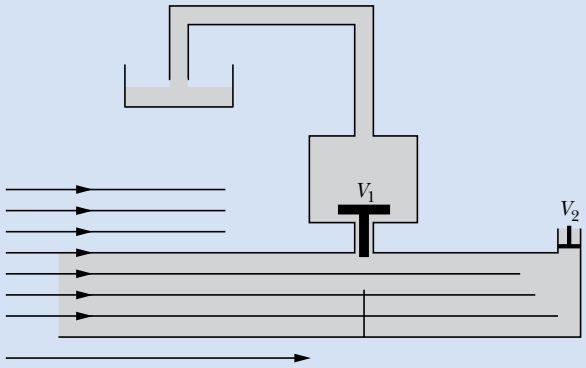

Figure. 14-42 (*b*) Flow lines in the ram.

REVIEW AND SUMMARY

Density The **density** ρ of any material is defined as the material's mass per unit volume:

$$\rho = \frac{\Delta m}{\Delta V}. \qquad (14\text{-}1)$$

Usually, where a material sample is much larger than atomic dimensions, we can write Eq. 14-1 as

$$\rho = \frac{m}{V}. \qquad (14\text{-}2)$$

Fluid Pressure A **fluid** is a substance that can flow; it conforms to the boundaries of its container because it cannot withstand shearing stress. It can, however, exert a force perpendicular to its surface. That force is described in terms of **pressure** p:

$$p = \frac{\Delta F}{\Delta A}, \qquad (14\text{-}3)$$

in which ΔF is the force acting on a surface element of area ΔA. If the force is uniform over a flat area, Eq. 14-3 can be written as

$$p = \frac{F}{A}. \qquad (14\text{-}4)$$

The force resulting from fluid pressure at a particular point in a fluid has the same magnitude in all directions. **Gauge pressure** is the difference between the actual pressure (or *absolute pressure*) at a point and the atmospheric pressure.

Pressure Variation with Height and Depth Pressure in a fluid at rest varies with vertical position y. For y measured positive upward,

$$p_2 = p_1 + \rho g(y_1 - y_2). \qquad (14\text{-}7)$$

The pressure in a fluid is the same for all points at the same level. If h is the *depth* of a fluid sample below some reference level at which the pressure is p_0, then the pressure in the sample is

$$p = p_0 + \rho g h. \qquad (14\text{-}8)$$

Pascal's Principle A change in the pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid and to the walls of the containing vessel.

Archimedes' Principle When a body is fully or partially submerged in a fluid, a buoyant force $\vec{F}_b$ from the surrounding fluid acts on the body. The force is directed upward and has a magnitude given by

$$F_b = m_f g, \qquad (14\text{-}16)$$

where m_f is the mass of the fluid that has been displaced by the body (that is, the fluid that has been pushed out of the way by the body).

When a body floats in a fluid, the magnitude F_b of the (upward) buoyant force on the body is equal to the magnitude F_g of the (downward) gravitational force on the body. The **apparent weight** of a body on which a buoyant force acts is related to its actual weight by

$$\text{weight}_{\text{app}} = \text{weight} - F_b. \qquad (14\text{-}19)$$

Flow of Ideal Fluids An **ideal fluid** is incompressible and lacks viscosity, and its flow is steady and irrotational. A *streamline* is the path followed by an individual fluid particle. A *tube of flow* is a bundle of streamlines. The flow within any tube of flow obeys the **equation of continuity**:

$$R_V = Av = \text{a constant}, \qquad (14\text{-}35)$$

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in which R_V is the **volume flow rate**, A is the cross-sectional area of the tube of flow at any point, and v is the speed of the fluid at that point. The **mass flow rate** R_m is

$$R_m = \rho R_V = \rho A v = \text{a constant}. \quad (14\text{-}36)$$

Bernoulli's Equation Applying the principle of conservation of mechanical energy to the flow of an ideal fluid leads to **Bernoulli's equation** along any tube of flow:

$$p + \frac{1}{2}\rho v^2 + \rho g y = \text{a constant}. \quad (14\text{-}40)$$

PROBLEMS

1. *Giraffe bending to drink*. In a giraffe with its head 1.8 m above its heart, and its heart 2.0 m above its feet, the (hydrostatic) gauge pressure in the blood at its heart is 250 torr. Assume that the giraffe stands upright and the blood density is 1.06×10^3 kg/m^3. In torr (or mm Hg), find the (gauge) blood pressure (a) at the brain (the pressure is enough to perfuse the brain with blood, to keep the giraffe from fainting) and (b) at the feet (the pressure must be countered by tight-fitting skin acting like a pressure stocking). (c) If the giraffe were to lower its head to drink from a pond without splaying its legs and moving slowly, what would be the increase in the blood pressure in the brain? (Such action would probably be lethal.)

2. In Fig. 14-43, the fresh water behind a reservoir dam has depth D = 15 m. A horizontal pipe 4.0 cm in diameter passes through the dam at depth d = 6.0 m. A plug secures the pipe opening. (a) Find the magnitude of the frictional force between plug and pipe wall. (b) The plug is removed. What water volume exits the pipe in 3.0 h?

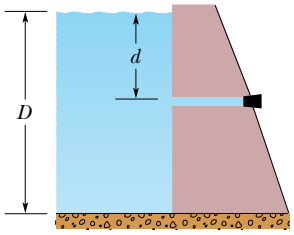

Figure 14-43 Problem 2.

3. In Fig. 14-44, water stands at depth D = 30.0 m behind the vertical upstream face of a dam of width W = 250 m. Find (a) the net horizontal force on the dam from the gauge pressure of the water and (b) the net torque due to that force about a horizontal line through O parallel to the (long) width of the dam. This torque tends to rotate the dam around that line, which would cause the dam to fail. (c) Find the moment arm of the torque.

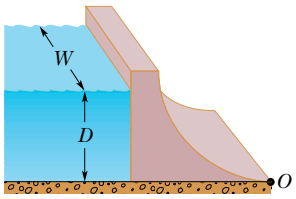

Figure 14-44 Problem 3.

4. The volume of air space in the passenger compartment of an 1800 kg car is 5.00 m^3. The volume of the motor and front wheels is 0.710 m^3, and the volume of the rear wheels, gas tank, and trunk is 0.800 m^3; water cannot enter these two regions. The car rolls into a lake. (a) At first, no water enters the passenger compartment. How much of the car, in cubic meters, is below the water surface with the car floating (Fig. 14-45)? (b) As water slowly enters, the car sinks. How many cubic meters of water are in the car as it disappears below the water surface? (The car, with a heavy load in the trunk, remains horizontal.)

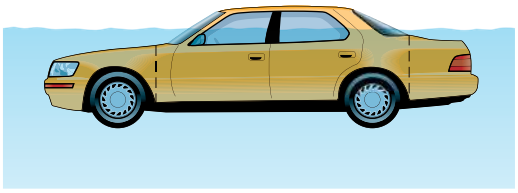

Figure 14-45 Problem 4.

5. The maximum depth d_{max} that a diver can snorkel is set by the density of the water and the fact that human lungs can function against a maximum pressure difference (between inside and outside the chest cavity) of 0.050 atm. What is the difference in d_{max} for fresh water and the water of the Dead Sea (the saltiest natural water in the world, with a density of 1.5×10^3 kg/m^3)?

6. A small solid ball is released from rest while fully submerged in a liquid and then its kinetic energy is measured when it has moved 2.0 cm in the liquid. Figure 14-46 gives the results after many liquids are used: The kinetic energy K is plotted versus the liquid density ρ_{liq}, and K_s = 2.4 J sets the scale on the vertical axis. What are (a) the density and (b) the volume of the ball?

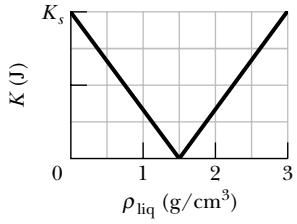

Figure 14-46 Problem 6.

7. Fresh water flows horizontally from pipe section 1 of cross-sectional area A_1 into pipe section 2 of cross-sectional

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area A_2. Figure 14-47 gives a plot of the pressure difference $p_2 - p_1$ versus the inverse area squared A_1^{-2} that would be expected for a volume flow rate of a certain value if the water flow were laminar under all circumstances. The scale on the vertical axis is set by $\Delta p_s = 600$ kN/m². For the conditions of the figure, what are the values of (a) A_2 and (b) the volume flow rate?

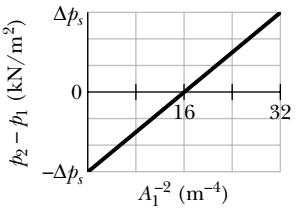

Figure 14-47 Problem 7.

8. In Fig. 14-48, water flows steadily from the left pipe section (radius $r_1 = 2.00R$), through the middle section (radius R), and into the right section (radius $r_3 = 3.00R$). The speed of the water in the middle section is 0.620 m/s. What is the net work done on 0.700 m³ of the water as it moves from the left section to the right section?

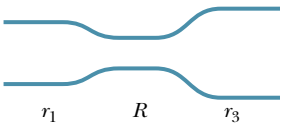

Figure 14-48 Problem 8.

9. A fluid is to be pushed through a narrow tube of radius 0.56 cm. What is the pressure increase in the fluid when the applied force is 120 N?

10. A liquid of density 900 kg/m³ flows through a horizontal pipe that has a cross-sectional area of 1.90×10^{-2} m² in region A and a cross-sectional area of 9.50×10^{-2} m² in region B. The pressure difference between the two regions is 7.20×10^3 Pa. What are (a) the volume flow rate and (b) the mass flow rate?

11. The plastic tube in Fig. 14-49 has a cross-sectional area of 5.00 cm². The tube is filled with water until the short arm (of length $d = 0.800$ m) is full. Then the short arm is sealed and more water is gradually poured into the long arm. When the total height of water in the long arm reaches 2.80 m, the seal is on the verging of popping. What force is then on the seal?

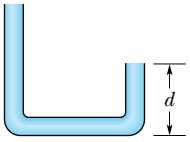

Figure 14-49 Problems 11.

12. When researchers find a reasonably complete fossil of a dinosaur, they can determine the mass and weight of the living dinosaur with a scale model sculpted from plastic and based on the dimensions of the fossil bones. The scale of the model is 1/20; that is, lengths are 1/20 actual length, areas are $(1/20)^2$ actual areas, and volumes are $(1/20)^3$ actual volumes. First, the model is suspended from one arm of a balance and weights are added to the other arm until equilibrium is reached. Then the model is fully submerged in water and enough weights are removed from the second arm to reestablish equilibrium (Fig. 14-50). For a model of a particular *T. rex* fossil, 791.10 g had to be removed to reestablish equilibrium. What was the volume of (a) the model and (b) the actual *T. rex*? (c) If the density of *T. rex* was approximately the density of water, what was its mass?

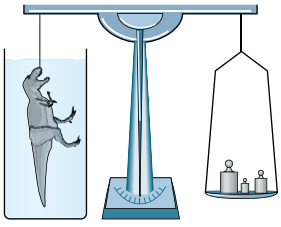

Figure 14-50 Problem 12.

13. In analyzing certain geological features, it is often appropriate to assume that the pressure at some horizontal *level of compensation*, deep inside Earth, is the same over a large region and is equal to the pressure due to the gravitational force on the overlying material. Thus, the pressure on the level of compensation is given by the fluid pressure formula. This model requires, for one thing, that mountains have *roots* of continental rock extending into the denser mantle (Fig. 14-51). Consider a mountain of height $H = 6.0$ km on a continent of thickness $T = 28$ km. The continental rock has a density of 2.9 g/cm³, and beneath this rock the mantle has a density of 3.3 g/cm³. Calculate the depth D of the root. (*Hint*: Set the pressure at points a and b equal; the depth y of the level of compensation will cancel out.)

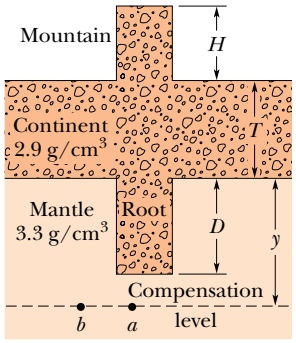

Figure 14-51 Problem 13.

14. Two streams merge to form a river. One stream has a width of 8.2 m, depth of 3.4 m, and current speed of 2.3 m/s. The other stream is 6.8 m wide and 3.2 m deep, and flows at 2.6 m/s. If the river has width 10.5 m and depth 4.5 m, what is its speed?

15. A partially evacuated airtight container has a tight-fitting lid of surface area 50 m² and negligible mass. If the force required to remove the lid is 480 N and the atmospheric pressure is 1.0×10^5 Pa, what is the internal air pressure?

16. A 3.00 kg object is released from rest while fully submerged in a liquid. The liquid displaced by the submerged object has a mass of 5.00 kg. How far and in what direction does the object move in 0.200 s, assuming that it moves freely and that the drag force on it from the liquid is negligible?

17. *Blood pressure in Argentinosaurus.* (a) If this long-necked, gigantic sauropod had a head height of 18 m and a heart height of 8.0 m, what (hydrostatic) gauge pressure in its blood was required at the heart such that the blood pressure at the brain was 80 torr (just enough to perfuse the brain with blood)? Assume the blood had a density of 1.06×10^3 kg/m³. (b) What was the blood pressure (in torr or mm Hg) at the feet?

18. A boat floating in fresh water displaces water weighing 50.1 kN. (a) What is the weight of the water this boat

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displaces when floating in salt water of density 1.10×10^3 kg/m^3? (b) What is the difference between the volume of fresh water displaced and the volume of salt water displaced?

19. *Snorkeling by humans and elephants.* When a person snorkels, the lungs are connected directly to the atmosphere through the snorkel tube and thus are at atmospheric pressure. In atmospheres, what is the difference Δp between this internal air pressure and the water pressure against the body if the length of the snorkel tube is (a) 20 cm (standard situation) and (b) 3.5 m (probably lethal situation)? In the latter, the pressure difference causes blood vessels on the walls of the lungs to rupture, releasing blood into the lungs. As depicted in Fig. 14-52, an elephant can safely snorkel through its trunk while swimming with its lungs 3.5 m below the water surface because the membrane around its lungs contains connective tissue that holds and protects the blood vessels, preventing rupturing.

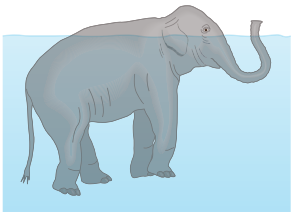

Figure 14-52 Problem 19.

20. Three children, each of weight 356 N, make a log raft by lashing together logs of diameter 0.30 m and length 2.00 m. How many logs will be needed to keep them afloat in fresh water? Take the density of the logs to be 800 kg/m^3.

21. What gauge pressure must a machine produce in order to suck mud of density 1800 kg/m^3 up a tube by a height of 2.0 m?

22. A garden hose with an internal diameter of 1.9 cm is connected to a (stationary) lawn sprinkler that consists merely of a container with 20 holes, each 0.15 cm in diameter. If the water in the hose has a speed of 0.91 m/s, at what speed does it leave the sprinkler holes?

23. *g-LOC in dogfights.* When a pilot takes a tight turn at high speed in a modern fighter airplane, the blood pressure at the brain level decreases, blood no longer perfuses the brain, and the blood in the brain drains. If the heart maintains the (hydrostatic) gauge pressure in the aorta at 120 torr (or mm Hg) when the pilot undergoes a horizontal centripetal acceleration of $4.5g$, what is the blood pressure (in torr) at the brain, 30 cm radially inward from the heart? The perfusion in the brain is small enough that the vision switches to black and white and narrows to "tunnel vision" and the pilot can undergo *g-LOC* ("*g*-induced loss of consciousness"). Blood density is 1.06×10^3 kg/m^3.

24. A piston of cross-sectional area a is used in a hydraulic press to exert a small force of magnitude f on the enclosed liquid. A connecting pipe leads to a larger piston of cross-sectional area A (Fig. 14-53). (a) What force magnitude F will the larger piston sustain without moving? (b) If the piston diameters are 3.50 cm and 60.0 cm, what force magnitude on the large piston will balance a 20.0 N force on the small piston?

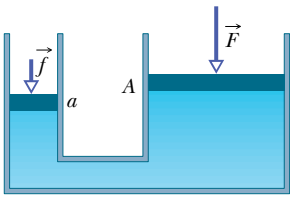

Figure 14-53 Problem 24.

25. How much work is done by pressure in forcing 2.0 m^3 of water through a pipe having an internal diameter of 13 mm if the difference in pressure at the two ends of the pipe is 1.0 atm?

26. Calculate the hydrostatic difference in blood pressure between the brain and the foot in a person of height 1.50 m. The density of blood is 1.06×10^3 kg/m^3.

27. Figure 14-54 shows an iron ball suspended by thread of negligible mass from an upright cylinder that floats partially submerged in water. The cylinder has a height of 6.00 cm, a face area of 12.0 cm^2 on the top and bottom, and a density of 0.25 g/cm^3, and 1.00 cm of its height is above the water surface. What is the radius of the iron ball?

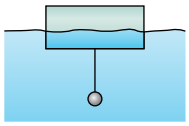

Figure 14-54 Problem 27.

28. A flotation device is in the shape of a right cylinder, with a height of 0.650 m and a face area of 4.00 m^2 on top and bottom, and its density is 0.300 times that of fresh water. It is initially held fully submerged in fresh water, with its top face at the water surface. Then it is allowed to ascend gradually until it begins to float. How much work does the buoyant force do on the device during the ascent?

29. Suppose that two tanks, 1 and 2, each with a large opening at the top, contain different liquids. A small hole is made in the side of each tank at the same depth h below the liquid surface, but the hole in tank 1 has half the cross-sectional area of the hole in tank 2. (a) What is the ratio ρ_1/ρ_2 of the densities of the liquids if the mass flow rate is the same for the two holes? (b) What is the ratio R_{V1}/R_{V2} of the volume flow rates from the two tanks? (c) At one instant, the liquid in tank 1 is 16.0 cm above the hole. If the tanks are to have *equal* volume flow rates, what height above the hole must the liquid in tank 2 be just then?

30. In Fig. 14-55, a spring of spring constant 3.75×10^4 N/m is between a rigid beam and the output piston of a hydraulic lever. An empty container with negligible mass sits on the input piston. The input piston has area A_i, and the output piston has area $18.0A_i$. Initially the spring is at its rest length. How many kilograms of sand must be (slowly) poured into the container to compress the spring by 5.00 cm?

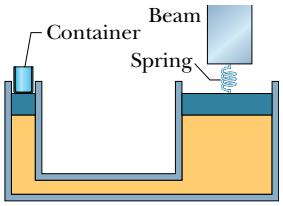

Figure 14-55 Problem 30.

31. Two identical cylindrical vessels with their bases at the same level each contain a liquid of density 1.30×10^3 kg/m^3. The area of each base is 4.25 cm^2, but in one vessel the liquid height is 0.854 m and in the other it is 1.560 m. Find the work done by the gravitational force in equalizing the levels when the two vessels are connected.

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32. Figure 14-56 shows two sections of an old pipe system that runs through a hill, with distances $d_A = d_B = 40$ m and $D = 110$ m. On each side of the hill, the pipe radius is 2.00 cm. However, the radius of the pipe inside the hill is no longer known. To determine it, hydraulic engineers first establish that water flows through the left and right sections at 2.50 m/s. Then they release a dye in the water at point A and find that it takes 88.8 s to reach point B. What is the average radius of the pipe within the hill?

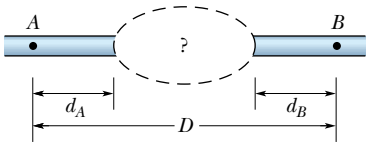

Figure 14-56 Problem 32.

33. A hollow spherical iron shell floats almost completely submerged in water. The outer diameter is 50.0 cm, and the density of iron is 7.87 g/cm^3. Find the inner diameter.

34. An office window has dimensions 3.4 m by 2.1 m. As a result of the passage of a storm, the outside air pressure drops to 0.93 atm, but inside the pressure is held at 1.0 atm. What net force pushes out on the window?

35. *Canal effect.* Figure 14-57 shows an anchored barge that extends across a canal by distance $d = 30$ m and into the water by distance $b = 12$ m. The canal has a width $D = 55$ m, a water depth $H = 14$ m, and a uniform water-flow speed $v_i = 1.2$ m/s. Assume that the flow around the barge is uniform. As the water passes the bow, the water level undergoes a dramatic dip known as the canal effect. If the dip has depth $h = 0.80$ m, what is the water speed alongside the boat through the vertical cross sections at (a) point a and (b) point b? The erosion due to the speed increase is a common concern to hydraulic engineers.

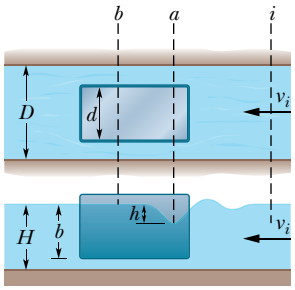

Figure 14-57 Problem 35.

36. What would be the height of the atmosphere if the air density (a) were uniform and (b) decreased linearly to zero with height? Assume that at sea level the air pressure is 1.0 atm and the air density is 1.3 kg/m^3.

37. The intake in Fig. 14-58 has cross-sectional area of 0.740 m^2 and water flow at 0.400 m/s. At the outlet the cross-sectional area is smaller than at the intake and the water flows out at 9.50 m/s. The pressure difference between intake and outlet is 3.00 MPa. What is vertical distance D?

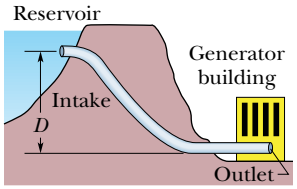

Figure 14-58 Problem 37.

38. A cylindrical tank with a large diameter is filled with water to a depth $D = 0.30$ m. A hole of cross-sectional area $A = 6.2$ cm^2 in the bottom of the tank allows water to drain out. (a) What is the drainage rate in cubic meters per second? (b) At what distance below the bottom of the tank is the cross-sectional area of the stream equal to one-half the area of the hole?

39. You inflate the front tires on your car to 32 psi. Later, you measure your blood pressure, obtaining a reading of 135/70, the readings being in mm Hg. In metric countries (which is to say, most of the world), these pressures are customarily reported in kilopascals (kPa). In kilopascals, what are (a) your tire pressure and (b) your blood pressure?

40. If you can produce a minimum gauge pressure of -2.5×10^{-3} atm in your lungs, to what maximum height can you suck tea (density 1000 kg/m^3) up a straw?

41. *Lurking alligators.* An alligator waits for prey by floating with only the top of its head exposed, so that the prey cannot easily see it. One way it can adjust the extent of sinking is by controlling the size of its lungs. Another way may be by swallowing stones (*gastrolithes*) that then reside in the stomach. Figure 14-59 shows a highly simplified model (a "rhombohedron gater") of mass 145 kg that roams with its head partially exposed. The top head surface has area 0.15 m^2. If the alligator were to swallow stones with a total mass of 1.0% of its body mass (a typical amount), how far would it sink?

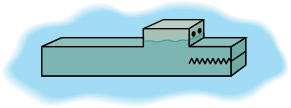

Figure 14-59 Problem 41.

42. The water flowing through a 1.9 cm (inside diameter) pipe flows out through three 1.5 cm pipes. (a) If the flow rates in the three smaller pipes are 26, 21, and 16 L/min, what is the flow rate in the 1.9 cm pipe? (b) What is the ratio of the speed in the 1.9 cm pipe to that in the pipe carrying 26 L/min?

43. An inflatable device with a density of 1.20 g/cm^3 is fully submerged in fresh water and then inflated enough for the device's density to match that of the water. What then is the ratio of the expanded volume to the full volume of the device?

44. Figure 14-60 shows a stream of water flowing through a hole at depth $h = 12$ cm in a tank holding water to height $H = 40$ cm. (a) At what distance x does the stream strike the floor? (b) At what depth should a second hole be made to give the same value of x? (c) At what depth should a hole be made to maximize x?

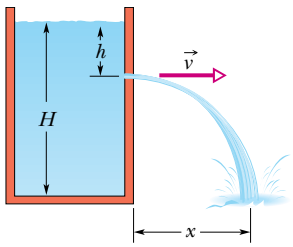

Figure 14-60 Problem 44.

45. A large aquarium of height 5.00 m is filled with fresh water to a depth of 2.00 m. One wall of the aquarium consists of thick plastic 9.00 m wide. By how much does the total force on that wall increase if the aquarium is next filled to a depth of 4.00 m?

46. A pitot tube (Fig. 14-61) is used to determine the airspeed of an airplane. It consists of an outer tube with a number

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of small holes B (four are shown) that allow air into the tube; that tube is connected to one arm of a U-tube. The other arm of the U-tube is connected to hole A at the front end of the device, which points in the direction the plane is headed. At A the air becomes stagnant so that $v_A = 0$. At B, however, the speed of the air presumably equals the airspeed v of the plane. (a) Use Bernoulli's equation to show that

$$v = \sqrt{\frac{2\rho gh}{\rho_{air}}},$$

where ρ is the density of the liquid in the U-tube and h is the difference in the liquid levels in that tube. (b) Suppose that the tube contains alcohol and the level difference h is 20.0 cm. What is the plane's speed relative to the air? The density of the air is 1.03 kg/m^3 and that of alcohol is 810 kg/m^3.

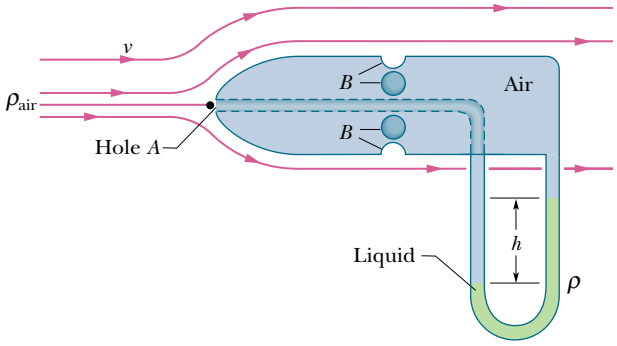

Figure 14-61 Problem 46 and 47.

47. A pitot tube (see Problem 46) on a high-altitude aircraft measures a differential pressure of 150 Pa. What is the aircraft's airspeed if the density of the air is 0.031 kg/m^3?

48. Models of torpedoes are sometimes tested in a horizontal pipe of flowing water, much as a wind tunnel is used to test model airplanes. Consider a circular pipe of internal diameter 25.0 cm and a torpedo model aligned along the long axis of the pipe. The model has a 6.00 cm diameter and is to be tested with water flowing past it at 2.00 m/s. (a) With what speed must the water flow in the part of the pipe that is unconstricted by the model? (b) What will the pressure difference be between the constricted and unconstricted parts of the pipe?

49. In one observation, the column in a mercury barometer (as is shown in Fig. 14-5*a*) has a measured height h of 740.35 mm. The temperature is –5.0°C, at which temperature the density of mercury ρ is 1.3608×10^4 kg/m^3. The free-fall acceleration g at the site of the barometer is 9.7828 m/s^2. What is the atmospheric pressure at that site in pascals and in torr (which is the common unit for barometer readings)?

50. What fraction of the volume of an iceberg (density 917 kg/m^3) would be visible if the iceberg floats (a) in the ocean (salt water, density 1024 kg/m^3) and (b) in a river (fresh water, density 1000 kg/m^3)? (When salt water freezes to form ice, the salt is excluded. So, an iceberg could provide fresh water to a community.)

51. Water is pumped steadily out of a flooded basement at 4.5 m/s through a hose of radius 1.0 cm, passing through a window 3.5 m above the waterline. What is the pump's power?

52. At a depth of 10.9 km, the Challenger Deep in the Marianas Trench of the Pacific Ocean is the deepest site in any ocean. Yet, in 1960, Donald Walsh and Jacques Piccard reached the Challenger Deep in the bathyscaph *Trieste*. Assuming that seawater has a uniform density of 1024 kg/m^3, approximate the hydrostatic pressure (in atmospheres) that the *Trieste* had to withstand. (Even a slight defect in the *Trieste* structure would have been disastrous.)

53. A water pipe having a 2.5 cm inside diameter carries water into the basement of a house at a speed of 0.90 m/s and a pressure of 190 kPa. If the pipe tapers to 1.2 cm and rises to the second floor 7.6 m above the input point, what are the (a) speed and (b) water pressure at the second floor?

54. A hollow sphere of inner radius 8.0 cm and outer radius 9.0 cm floats half-submerged in a liquid of density 820 kg/m^3. (a) What is the mass of the sphere? (b) Calculate the density of the material of which the sphere is made.

55. In Fig. 14-62, water flows through a horizontal pipe and then out into the atmosphere at a speed $v_1 = 23.0$ m/s. The diameters of the left and right sections of the pipe are 5.00 cm and 3.00 cm. (a) What volume of water flows into the atmosphere during a 20.0 min period? In the left section of the pipe, what are (b) the speed v_2 and (c) the gauge pressure?

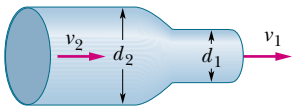

Figure 14-62 Problem 55.

56. Three liquids that will not mix are poured into a cylindrical container. The volumes and densities of the liquids are 1.50 L, 2.6 g/cm^3; 0.75 L, 1.0 g/cm^3; and 0.60 L, 0.80 g/cm^3. What is the force on the bottom of the container due to these liquids? One liter = 1 L = 1000 cm^3. (Ignore the contribution due to the atmosphere.)

57. Consider the venturi tube of Sample Problem 14.21 and Fig. 14-39 without the manometer. Let A equal 5a. Suppose the pressure p_1 at A is 3.0 atm. Compute the values of (a) the speed V at A and (b) the speed v at a that make the pressure p_2 at a equal to zero. (c) Compute the corresponding volume flow rate if the diameter at A is 5.0 cm. The phenomenon that occurs at a when p_2 falls to nearly zero is known as cavitation. The water vaporizes into small bubbles.

58. In 1654 Otto von Guericke, inventor of the air pump, gave a demonstration before the noble-men of the Holy Roman Empire in which two teams of eight horses could not pull apart two evacuated brass hemispheres. (a) Assuming the hemispheres have (strong) thin walls, so that R in Fig. 14-63 may be considered both the inside and outside radius, show that the force $\vec{F}$ required to pull apart the hemispheres has magnitude $F = \pi R^2 \Delta p$, where Δp is the difference between the pressures outside and inside the sphere.

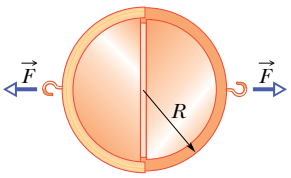

Figure 14-63 Problem 58.

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(b) Taking R as 40 cm, the inside pressure as 0.10 atm, and the outside pressure as 1.00 atm, find the force magnitude the teams of horses would have had to exert to pull apart the hemispheres. (c) Explain why one team of horses could have proved the point just as well if the hemispheres were attached to a sturdy wall.

59. A wood block (mass 3.67 kg, density 600 kg/m^3) is fitted with lead (density 1.14×10^4 kg/m^3) so that it floats in water with 0.700 of its volume submerged. Find the lead mass if the lead is fitted to the block's (a) top and (b) bottom.

60. A block of wood floats in fresh water with two-thirds of its volume V submerged and in oil with $0.92V$ submerged. Find the density of (a) the wood and (b) the oil.

61. Water is moving with a speed of 5.0 m/s through a pipe with a cross-sectional area of 4.0 cm^2. The water gradually descends 12 m as the pipe cross-sectional area increases to 8.0 cm^2. (a) What is the speed at the lower level? (b) If the pressure at the upper level is 1.5×10^5 Pa, what is the pressure at the lower level?

62. Crew members attempt to escape from a damaged submarine 100 m below the surface. What force must be applied to a pop-out hatch, which is 1.2 m by 0.60 m, to push it out at that depth? Assume that the density of the ocean water is 1024 kg/m^3 and the internal air pressure is at 1.00 atm.

63. A very simplified schematic of the rain drainage system for a home is shown in Fig. 14-64. Rain falling on the slanted roof runs off into gutters around the roof edge; it then drains through downspouts (only one is shown) into a main drainage pipe M below the basement, which carries the water to an even larger pipe below the street. In Fig. 14-58, a floor drain in the basement is also connected to drainage pipe M. Suppose the following apply:

(1) the downspouts have height $h_1 = 11$ m,
(2) the floor drain has height $h_2 = 1.2$ m,
(3) pipe M has radius 2.0 cm,
(4) the house has side width $w = 40$ m and front length $L = 70$ m,
(5) all the water striking the roof goes through pipe M,
(6) the initial speed of the water in a downspout is negligible, and
(7) the wind speed is negligible (the rain falls vertically).

At what rainfall rate, in centimeters per hour, will water from pipe M reach the height of the floor drain and threaten to flood the basement?

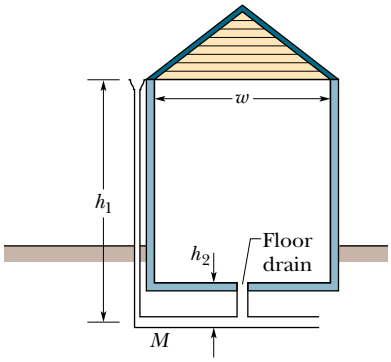

Figure 14-64 Problem 63.

64. An iron anchor of density 7870 kg/m^3 appears 200 N lighter in water than in air. (a) What is the volume of the anchor? (b) How much does it weigh in air?

65. An iron casting containing a number of cavities weighs 6000 N in air and 4200 N in water. What is the total cavity volume in the casting? The density of solid iron is 7.87 g/cm^3

66. Suppose that you release a small ball from rest at a depth of 0.400 m below the surface in a pool of water. If the density of the ball is 0.450 that of water and if the drag force on the ball from the water is negligible, how high above the water surface will the ball shoot as it emerges from the water? (Neglect any transfer of energy to the splashing and waves produced by the emerging ball.)

67. In Fig. 14-65, an open tube of length $L = 2.3$ m and cross-sectional area $A = 9.2$ cm^2 is fixed to the top of a cylindrical barrel of diameter $D = 1.2$ m and height $H = 2.3$ m. The barrel and tube are filled with water (to the top of the tube). Calculate the ratio of the hydrostatic force on the bottom of the barrel to the gravitational force on the water contained in the barrel. Why is that ratio not equal to 1.0? (You need not consider the atmospheric pressure.)

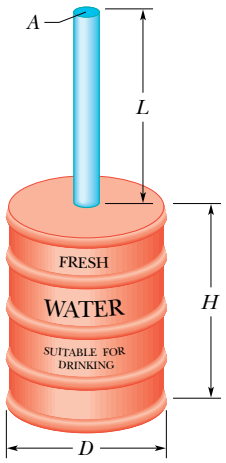

Figure 14-65 Problem 67.

68. In Fig. 14-66, a cube of edge length $L = 0.500$ m and mass 450 kg is suspended by a rope in an open tank of liquid of density 1030 kg/m^3. Find (a) the magnitude of the total downward force on the top of the cube from the liquid and the atmosphere, assuming atmospheric pressure is 1.00 atm, (b) the magnitude of the total upward force on the bottom of the cube, and (c) the tension in the rope. (d) Calculate the magnitude of the buoyant force on the cube using Archimedes' principle. What relation exists among all these quantities?

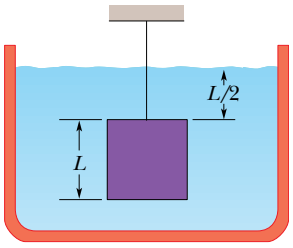

Figure 14-66 Problem 68.

69. *The bends during flight.* Anyone who scuba dives is advised not to fly within the next 24 h because the air mixture for diving can introduce nitrogen to the bloodstream. Without allowing the nitrogen to come out of solution slowly, any sudden air-pressure reduction (such as during airplane ascent) can result in the nitrogen forming bubbles in the blood, creating the *bends*, which can be painful and even fatal. Military special operation forces are especially at risk. What is the change in pressure on such a special-op soldier who must scuba dive at a depth of 25 m in seawater one day and parachute at an altitude of 8.1 km the next day? Assume that the average air density within the altitude range is 0.87 kg/m^3.

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70 In Fig. 14-67*a*, a rectangular block is gradually pushed facedown into a liquid. The block has height d; on the bottom and top the face area is $A = 8.00\ \text{cm}^2$. Figure 14-67*b* gives the apparent weight W_{app} of the block as a function of the depth h of its lower face. The scale on the vertical axis is set by $W_s = 0.20$ N. What is the density of the liquid?

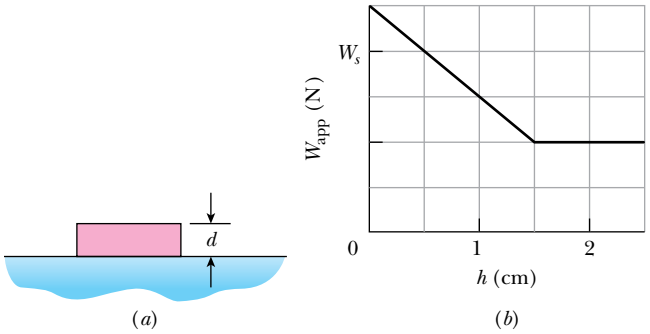

Figure 14-67 Problem 70.

71 The L-shaped fish tank shown in Fig. 14-68 is filled with water and is open at the top. Here, $d = 7.0$ m.

(a) What is the (total) force exerted by the water on face A?

(b) What is the (total) force exerted by the water on face B?

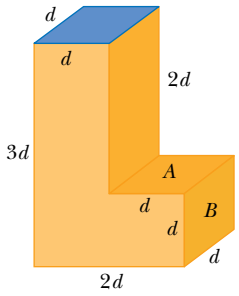

Figure 14-68 Problem 71.

PRACTICE QUESTIONS

Single Correct Choice Type

1. Which one of the following statements regarding a completely enclosed fluid is true?
- **(a)** Any change in the applied pressure of the fluid produces a change in pressure that depends on direction.
- **(b)** The pressure at all points within the fluid is independent of any pressure applied to it.
- **(c)** Any change in applied pressure produces an equal change in pressure at all points within the fluid.
- **(d)** An increase in pressure in one part of the fluid results in an equal decrease in pressure in another part.

2. Which one of the following statements regarding streamline flow is true?
- **(a)** At any given point in the fluid, the velocity is constant in time.
- **(b)** Streamline flow occurs when there are sharp obstacles in the path of a fast-moving fluid.
- **(c)** Streamline flow is described by Pascal's principle.
- **(d)** Streamline flow is described by Archimedes' principle.

3. Water enters through end A with a speed v_1 and leaves through end B with a speed v_2 of a cylindrical tube AB. The tube is always completely filled with water. In case I, the tube is horizontal, in case II, it is vertical with the end A upward, and in case III it is vertical with the end B upward. We have $v_1 = v_2$ for

- **(a)** Case I
- **(b)** Case II
- **(c)** Case III
- **(d)** Each case

4. Oil is flowing with a speed of 1.22 m/s through a pipeline with a radius of 0.305 m. How many gallons of oil (1 gal = $3.79 \times 10^{-3}\ \text{m}^3$) flow in 1 day?
- **(a)** 9.55×10^6 gal
- **(b)** 7.27×10^6 gal
- **(c)** 8.12×10^6 gal
- **(d)** 6.96×10^6 gal

5. The property utilized in the manufacture of lead shots is
- **(a)** specific weight of liquid lead.
- **(b)** specific gravity of liquid lead.
- **(c)** compressibility of liquid lead.
- **(d)** surface tension of liquid lead.

6. What mass of water (at 4.0°C) can be contained in a rectangular box whose dimensions are 10.0 cm by 5.00 cm by 1.00 cm? The density of water at 4.0°C is $1.000 \times 10^3\ \text{kg/m}^3$.
- **(a)** 5.0 g
- **(b)** 10.0 g
- **(c)** 25.0 g
- **(d)** 50.0 g

7. A film of water is formed between two straight parallel wires of length 10 cm each separated by 0.5 cm. If their separation is increased by 1 cm while still maintaining their parallelism, how much work will have to be done? (Surface tension of water = 7.2×10^{-2} N/m)
- **(a)** 7.22×10^{-6} J
- **(b)** 1.44×10^{-5} J
- **(c)** 2.88×10^{-5} J
- **(d)** 5.76×10^{-5} J

8. The amount of work done in blowing a soap bubble such that its diameter increases from d to D is (T is the surface tension of the solution)
- **(a)** $4\pi(D^2 - d^2)T$
- **(b)** $8\pi(D - d^2)T$
- **(c)** $\pi(D^2 - d^2)T$
- **(d)** $2\pi(D^2 - d^2)T$

9. A woman is enjoying a cold drink while lying on a beach. The acceleration due to gravity at her location is 9.85 m/s^2. What gauge pressure must exist in the woman's mouth if she is drinking through a straw extending 0.085 m above the surface of the drink? (Assume the drink has a density of 1015 kg/m^3.)
- **(a)** 850 Pa
- **(b)** 970 Pa
- **(c)** 1100 Pa
- **(d)** 2100 Pa

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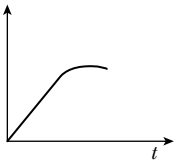
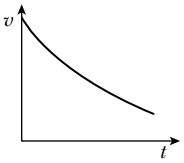
10. The density ρ of water of bulk modulus B at a depth y in the ocean is related to the density at surface ρ_0 by the relation

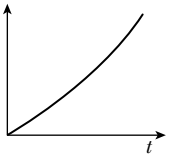
(a) $\rho = \rho_0\left(1 - \frac{\rho_0 gy}{B}\right)$ **(b)** $\rho = \rho_0\left(1 + \frac{\rho_0 gy}{B}\right)$

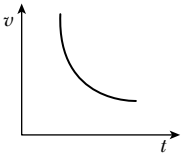
(c) $\rho = \rho_0\left(1 + \frac{B}{\rho_0 hgy}\right)$ **(d)** $\rho = \rho_0\left(1 - \frac{B}{\rho_0 gy}\right)$

11. Which of the following is the graph between the time and the velocity of a body moving in a viscous fluid?

(a) 

(b) 

(c) 

(d) 

12. The excess pressure inside a spherical drop of water is four times that of another drop. Then, their respective mass ratio is

(a) 1:16 **(b)** 8:1
(c) 1:4 **(d)** 1:64

13. The density of ice is 0.92 g/cm^3; and the density of seawater is 1.03 g/cm^3. A large iceberg floats in Arctic waters. What fraction of the volume of the iceberg is exposed?

(a) 0.080% **(b)** 11%
(c) 89% **(d)** 92%

14. A piece of wood floats in water kept in a beaker. If the beaker moves with a vertical acceleration a, the wood will

(a) sink deeper in the liquid if a is upward.
(b) sink deeper in the liquid if a is downward, with $a < g$.
(c) come out more than from the liquid if a is downward, with $a < g$.
(d) remain in the same position relative to the water.

15. Two soap bubbles coalesce to form a single bubble. If V is the subsequent change in volume of contained air and S the change in total surface area, T is the surface tension and P atmospheric pressure, which of the following relation is correct?

(a) $4PV + 3ST = 0$ **(b)** $3PV + 4ST = 0$
(c) $2PV + 3ST = 0$ **(d)** $3PV + 2ST = 0$

16. A full can of soda has a mass of 0.416 kg. It contains 3.54×10^{-4} m^3 of liquid. Assuming that the soda has the same density as water, find the volume of aluminum used to make the can.

(a) 6.2×10^{-5} m^3 **(b)** 4.2×10^{-5} m^3
(c) 2.3×10^{-5} m^3 **(d)** 1.8×10^{-5} m^3

17. Assume that a drop of liquid evaporates by decrease in its surface energy so that its temperature remains unchanged. What should be the minimum radius of the drop for this to be possible? The surface tension is T, density of liquid is ρ and L is its latent heat of vaporization.

(a) $\sqrt{\frac{T}{\rho L}}$ **(b)** $\frac{T}{\rho L}$

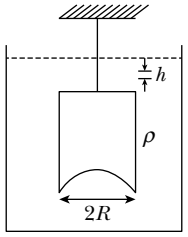
(c) $\frac{2T}{\rho L}$ **(d)** $\frac{\rho L}{T}$

18. Two capillary tubes of same radius r but of length l_1, l_2 are fitted in parallel to the bottom of a vessel. The pressure head is p. What should be the length of a single tube that can be replaced with the two tubes so that the rate of flow is same as before?

(a) $l_1 + l_2$ **(b)** $\frac{1}{l_1} + \frac{1}{l_2}$

(c) $\frac{l_1 l_2}{l_1 + l_2}$ **(d)** $\frac{1}{l_1 + l_2}$

19. A hemispherical portion of radius R is removed from the bottom of a cylinder of radius R. The volume of the remaining cylinder is V and mass M. It is suspended by a string in a liquid of density ρ where it stays vertical. The upper surface of the cylinder is at a depth h below the liquid surface. The force on the bottom of the cylinder by the liquid is

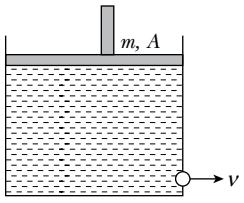


(a) Mg **(b)** $Mg - V\rho g$
(c) $Mg + \pi R^2 h\rho g$ **(d)** $\rho g(V + \pi R^2 h)$

20. A capillary tube of inner diameter 0.5 mm is dipped in a liquid of specific gravity 13.6 and having surface tension 545 dyn/cm (angle of contact 130°). Find the depression or elevation in the tube.

(a) depressed, 2.11 cm **(b)** elevated, 2.11 cm
(c) depressed, 3.71 cm **(d)** elevated, 3.71 cm

21. A cylindrical vessel contains a liquid of density ρ up to height h. The liquid is closed by a piston of mass m and of cross-section A. There is a small hole at the bottom of the vessel. The speed v with which the liquid comes out of the hole is



(a) $\sqrt{2gh}$ **(b)** $\sqrt{2\left(gh + \frac{mg}{\rho A}\right)}$

(c) $\sqrt{2\left(gh + \frac{mg}{A}\right)}$ **(d)** $\sqrt{2gh + \frac{mg}{A}}$

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22. In a car lift, compressed air with a gauge pressure of 4.0×10^5 Pa is used to raise a piston with a circular cross-sectional area. If the radius of the piston is 0.17 m, what is the maximum mass that can be raised using this piston?

(a) 530 kg **(b)** 3700 kg
(c) 9800 kg **(d)** 22 000 kg

23. A capillary tube of radius r is immersed in water and water rises in it to a height h. The mass of water in the capillary tube is 5 g. Another capillary tube of radius $2r$ is immersed in water. The mass of water that will rise in this tube is

(a) 2.5 g **(b)** 5.0 g
(c) 10 g **(d)** 20 g

24. The two dams are identical with the exception that the water reservoir behind dam A extends twice the horizontal distance behind it as that of dam B. Which one of the following statements regarding these two dams is correct?

(a) The force exerted by the water on dam A is greater than that on dam B.
(b) The force exerted by the water on dam B is greater than that on dam A.
(c) Dam A is more likely to collapse than dam B if the water level rises.
(d) The horizontal distance of the water behind the two dams does not determine the force on them.

25. A cylindrical air duct in an air conditioning system has a length of 5.5 m and a radius of 7.2×10^{-2} m. A fan forces air ($\eta = 1.8 \times 10^{-5}$ Pa·s) through the duct, such that the air in a room (volume = 280 m^3) is replenished every ten minutes. Determine the difference in pressure between the ends of the air duct.

(a) 14 Pa **(b)** 4.4 Pa
(c) 1.1 Pa **(d)** 0.47 Pa

26. A balloon inflated with helium gas (density = 0.2 kg/m^3) has a volume of 6×10^{-3} m^3. If the density of air is 1.3 kg/m^3, what is the buoyant force exerted on the balloon?

(a) 0.01 N **(b)** 0.8 N
(c) 0.08 N **(d)** 1.3 N

27. A glass bottle of soda is sealed with a screw cap. The absolute pressure of the carbon dioxide inside the bottle is 1.80×10^5 Pa. Assuming that the top and bottom surfaces of the cap each have an area of 4.10×10^{-4} m^2, obtain the magnitude of the force that the screw thread exerts on the cap in order to keep it on the bottle. The air pressure outside the bottle is one atmosphere.

(a) 18 N **(b)** 32 N
(c) 26 N **(d)** 58 N

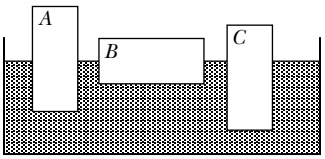
28. At a given instant, the blood pressure in the heart is 1.60×10^4 Pa. If an artery in the brain is 0.45 m above the heart, what is the pressure in the artery? Ignore any pressure changes due to blood flow?

(a) 1.6×10^4 Pa **(b)** 9.4×10^3 Pa
(c) 1.1×10^4 Pa **(d)** 2.1×10^4 Pa

29. Measured along the surface of the water, a rectangular swimming pool has a length of 15 m. Along this length, the flat bottom of the pool slopes downward at an angle of 11° below the horizontal, from one end to the other. By how much does the pressure at the bottom of the deep end exceed the pressure at the bottom of the shallow end?

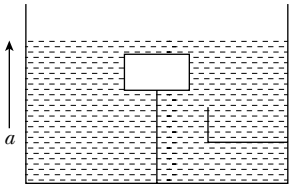
(a) 2.9×10^4 Pa **(b)** 6.1×10^4 Pa
(c) 4.3×10^4 Pa **(d)** 9.7×10^4 Pa

30. Three blocks, labeled A, B, and C, are floating in water as shown in the figure. Blocks A and B have the same mass and volume. Block C has the same volume, but is submerged to a greater depth than the other two blocks. Which one of the following statements concerning this situation is false?



(a) The density of block A is less than that of block C.
(b) The buoyant force acting on block A is equal to that acting on block B.
(c) The volume of water displaced by block C is greater than that displaced by block B.
(d) The volume of water displaced by block A is greater than that displaced by block B.

31. The tension in a string holding a solid block below the surface of a liquid (where $\rho_{\text{liquid}} > \rho_{\text{block}}$) as in the figure is T when the system is at rest. Then what will be the tension in the string if the system has upward acceleration a?



(a) $T\left(1-\frac{a}{g}\right)$ **(b)** $T\left(1+\frac{a}{g}\right)$
(c) $T\left(\frac{a}{g}-1\right)$ **(d)** $\frac{a}{g}T$

32. The main water line enters a house on the first floor. The line has a gauge pressure of 1.90×10^5 Pa. A faucet on the second floor, 6.50 m above the first floor, is turned off. What is the gauge pressure at this faucet?

(a) 1.45×10^5 Pa **(b)** 1.26×10^5 Pa
(c) 2.29×10^5 Pa **(d)** 2.55×10^5 Pa

33. The human lungs can function satisfactorily up to a limit where the pressure difference between the outside and inside of the lungs is one-twentieth of an atmosphere. If a diver uses a snorkel for breathing, how far below the water can she swim? Assume the diver is in salt water whose density is 1025 kg/m^3.

(a) 0.25 m **(b)** 0.35 m
(c) 0.50 m **(d)** 0.20 m

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34. The blood speed in a normal segment of a horizontal artery is 0.11 m/s. An abnormal segment of the artery is narrowed down by plaque to one-fourth the normal cross-sectional area. What is the difference in blood pressures between the normal and constricted segments of the artery? (Take density ρ of blood as 1060 kg/m^3)

(a) 96 Pa **(b)** 100 Pa
(c) 91 Pa **(d)** 87 Pa

35. Water flows through a pipe of diameter 8.0 cm with a speed of 10.0 m/s. It then enters a smaller pipe of diameter 3.0 cm. What is the speed of the water as it flows through the smaller pipe?

(a) 1.4 m/s **(b)** 71 m/s
(c) 2.8 m/s **(d)** 54 m/s

36. Water enters a pipe of diameter 3.0 cm with a velocity of 3.0 m/s. The water encounters a constriction where its velocity is 12 m/s. What is the diameter of the constricted portion of the pipe?

(a) 0.33 cm **(b)** 0.75 cm
(c) 1.0 cm **(d)** 1.5 cm

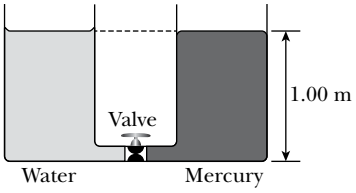
37. The vertical surface of a reservoir dam that is in contact with the water is 120 m wide and 12 m high. The air pressure is one atmosphere. Find the magnitude of the total force acting on this surface in a completely filled reservoir. (The pressure varies linearly with depth; so, you must use an average pressure.)

(a) 6.8×10^8 N **(b)** 3.1×10^8 N
(c) 4.6×10^8 N **(d)** 2.3×10^8 N

38. A pump and its horizontal intake pipe are located 12 m beneath the surface of a reservoir. The speed of the water in the intake pipe causes the pressure there to decrease, in accord with Bernoulli's principle. What is the maximum speed with which water can flow through the intake pipe (assuming non-viscous flow)?

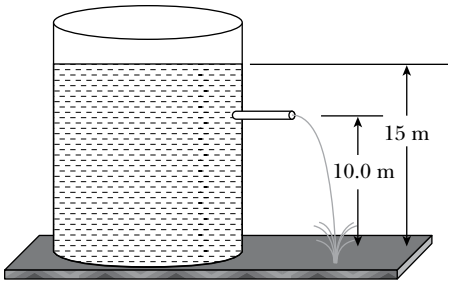
(a) 15 m/s **(b)** 18 m/s
(c) 21 m/s **(d)** 24 m/s

39. Two identical containers are open at the top and are connected at the bottom via a tube of negligible volume and a valve that is closed. Both containers are filled initially to the same height of 1.00 m, one with water, the other with mercury, as the figure indicates. The valve is then opened. Water and mercury are immiscible. Determine the fluid level in the left container when equilibrium is reestablished.



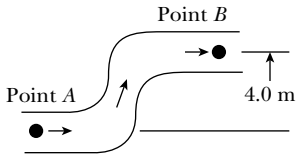
(a) 1.92 m **(b)** 1.22 m
(c) 1.46 m **(d)** 0.54 m

40. A large tank is filled with water to a depth of 15 m. A spout located 10.0 m above the bottom of the tank is then opened as shown in the figure. With what speed will water emerge from the spout?



(a) 3.1 m/s **(b)** 9.9 m/s
(c) 14 m/s **(d)** 17 m/s

41. Oil ($\rho = 925$ kg/m^3) is flowing through a pipeline at a constant speed when it encounters a vertical bend in the pipe raising it 4.0 m. The cross-sectional area of the pipe does not change. What is the difference in pressure ($p_B - p_A$) in the portions of the pipe before and after the rise?



(a) $+2.4 \times 10^4$ Pa **(b)** -7.2×10^5 Pa
(c) -3.6×10^4 Pa **(d)** -1.8×10^3 Pa

42. A 1.00 m tall container is filled to the brim, partway with mercury and the rest of the way with water. The container is open to the atmosphere. What must be the depth of the mercury so that the absolute pressure on the bottom of the container is twice the atmospheric pressure?

(a) 0.74 m **(b)** 0.88 m
(c) 0.43 m **(d)** 0.52 m

43. A dentist's chair with a patient in it weighs 2100 N. The output plunger of a hydraulic system begins to lift the chair when the dentist's foot applies a force of 55 N to the input piston. Neglect any height difference between the plunger and the piston. What is the ratio of the radius of the plunger to the radius of the piston?

(a) 9.3 **(b)** 2.6
(c) 6.2 **(d)** 0.64

44. A submarine is operating at 100.0 m below the surface of the ocean. If the air inside the submarine is maintained at a pressure of 1.0 atmosphere, what is the magnitude of the force that acts on the rectangular hatch 2.0 m × 1.0 m on the deck of the submarine?

(a) 980 N **(b)** 2.0×10^6 N
(c) 2.0×10^3 N **(d)** 9.8×10^5 N

45. A duck is floating on a lake with 25% of its volume beneath the water. What is the average density of the duck?

(a) 750 kg/m^3 **(b)** 130 kg/m^3
(c) 500 kg/m^3 **(d)** 250 kg/m^3

46. A large container is continually filled with a viscous liquid that flows from two equally long, cylindrical pipes, labeled A and B, onto a conveyer belt as shown in the figure. The diameter of pipe B is 1.75 times larger than that of pipe A.

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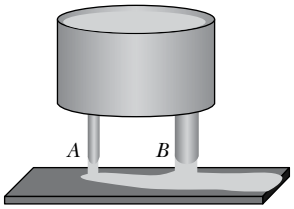
What is the ratio of the average volume flow rate of the liquid exiting pipe B to that of the liquid exiting pipe A?

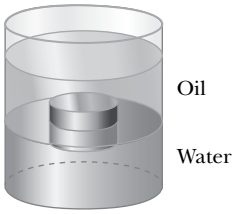


(a) 1.87 **(b)** 9.38
(c) 3.06 **(d)** 4.33

47. An object is solid throughout. When the object is completely submerged in ethyl alcohol, its apparent weight is 15.2 N. When completely submerged in water, its apparent weight is 13.7 N. What is the volume of the object?

(a) 1.6×10^{-3} m^3 **(b)** 7.9×10^{-4} m^3
(c) 1.0×10^{-3} m^3 **(d)** 8.6×10^{-4} m^3

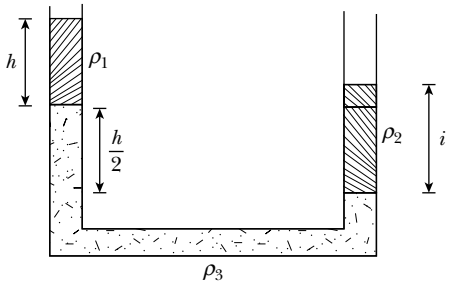
48. A solid cylinder (radius = 0.150 m, height = 0.120 m) has a mass of 7.00 kg. This cylinder is floating in water. Then oil ($\rho = 725$ kg/m^3) is poured on top of the water until the situation shown in the figure results. How much of the height of the cylinder is in the oil?



(a) 8.2×10^{-2} m **(b)** 7.6×10^{-2} m
(c) 9.5×10^{-3} m **(d)** 4.2×10^{-3} m

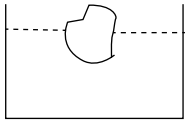
More than One Correct Choice Type

49. Three different liquids are filled in U-tube as shown in figure. Their densities are ρ_1, ρ_2 and ρ_3, respectively. From the figure we may conclude that:



(a) $\rho_2 > \rho_1$ **(b)** $\rho_1 > \rho_2$
(c) $\rho_3 = 2(\rho_2 - \rho_1)$ **(d)** $\rho_3 = \dfrac{\rho_2 + \rho_1}{2}$

50. Some pieces of impurity (density ρ) are embedded in ice. This ice is floating in water (density ρ_w). When ice melts, level of water will

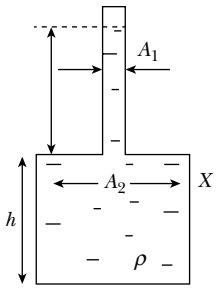


(a) fall if $\rho > \rho_w$.
(b) remain unchanged, if $\rho < \rho_w$.
(c) fall if $\rho < \rho_w$.
(d) rise if $\rho > \rho_w$.

51. An air bubble in a water tank rises from the bottom to the top. Which of the following statements are true?

(a) Bubble rises upward because pressure at the bottom is less than that at the top.
(b) Bubble rises upward because pressure at the bottom is greater than that at the top.
(c) As the bubble rises, its size increases.
(d) As the bubble rises, its size decreases.

52. The vessel shown in the figure has two sections of areas of cross-section A_1 and A_2. A liquid of density ρ fills both sections, up to a height h in each. Neglect atmospheric pressure.



(a) The pressure at the base of the vessel is $2h\rho g$.
(b) The force exerted by the liquid on the base of the vessel is $2h\rho gA_2$.
(c) The weight of the liquid is $< 2h\rho gA_2$.
(d) The walls of the vessel at the level X exert a downward force $h\rho g\,(A_1 - A_2)$ on the liquid.

53. A sample of metal weighs 210 g in air, 180 in water, and 120 in liquid. Then, relative density of

(a) metal is 3. **(b)** metal is 7.
(c) liquid is 3. **(d)** liquid is 1/3.

54. A vertical U-tube contains a liquid. The total length of the liquid column inside the tube is l. When the liquid is in equilibrium, the liquid surface in one of the arms of the U-tube is pushed down slightly and released. The entire liquid column will undergo a periodic motion.

(a) The motion is not simple harmonic motion.
(b) The motion is simple harmonic motion.
(c) If it undergoes simple harmonic motion, the time period will be $2\pi\sqrt{l/g}$.
(d) If it undergoes simple harmonic motion, the time period will be $2\pi\sqrt{l/2g}$.

55. A solid object floats in a liquid in a partially dipped position.

(a) The solid exerts a force equal to its weight on the liquid.
(b) The liquid exerts a force of buoyancy on the solid which is equal to the weight of the solid.
(c) The weight of the displaced liquid equals the weight of the solid.
(d) The weight of the dipped part of the solid is equal to the weight of the displaced liquid.

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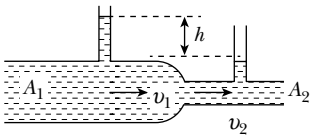
56. A liquid flows through a horizontal tube as shown in the figure. The velocities of the liquid in the two sections, which have areas of cross section A_1 and A_2, are v_1 and v_2, respectively. The difference in the levels of the liquid in the two vertical tubes is h. Therefore,



(a) the volume of the liquid flowing through the tube in unit time is A_1Y_1.

(b) $v_2 - v_1 = \sqrt{2gh}$.

(c) $v_2^2 - v_1^2 = 2gh$.

(d) the energy per unit mass of the liquid is the same in both sections of the tube.

57. The weight of an empty balloon on a spring balance is W_1, the weight becomes W_2 when the balloon is filling with air. Let the weight of the air itself be w. Neglect the thickness of the balloon when it is filled with air. Also neglect the difference in the densities of air inside and outside the balloon.

(a) $W_2 = W_1$ **(b)** $W_2 = W_1 + w$

(c) $W_2 < W_1 + w$ **(d)** $W_2 > W_1$

58. Water flows through two identical tubes A and B. A volume V_0 of water passes through the tube A and $2V_0$ through B in a given time. Which of the following may be correct?

(a) Flow in both the tubes is steady.

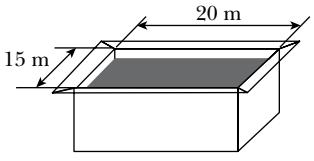
(b) Flow in both the tubes is turbulent.

(c) Flow is steady in A but turbulent in B.

(d) Flow is steady in B but turbulent in A.

Linked Comprehension

Paragraph for Questions 59 and 60: A swimming pool has the dimensions shown in the figure. It is filled with water to a uniform depth of 8.00 m. The density of water = 1.00×10^3 kg/m^3.



59. What is the total pressure exerted on the bottom of the swimming pool?

(a) 0.79×10^5 Pa **(b)** 1.80×10^5 Pa

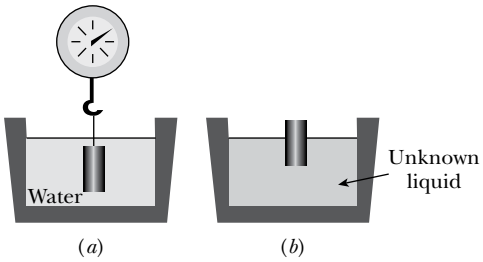
(c) 1.48×10^5 Pa **(d)** 1.97×10^5 Pa

60. What is the total force exerted on the bottom of the swimming pool?

(a) 2.40×10^7 N **(b)** 5.90×10^7 N

(c) 5.40×10^7 N **(d)** 7.50×10^7 N

Paragraph for Questions 61 and 62: When a block of volume 1.00×10^{-3} m^3 is hung from a spring scale as shown in Fig. (*a*), the scale reads 10.0 N. When the same block is then placed in an unknown liquid, it floats with two-third of its volume submerged as suggested in Fig. (*b*). The density of water is 1.00×10^{-3} kg/m^3.



61. Determine the mass of the block.

(a) 1.02 kg **(b)** 2.02 kg

(c) 3.02 kg **(d)** 4.04 kg

62. Determine the density of the unknown liquid.

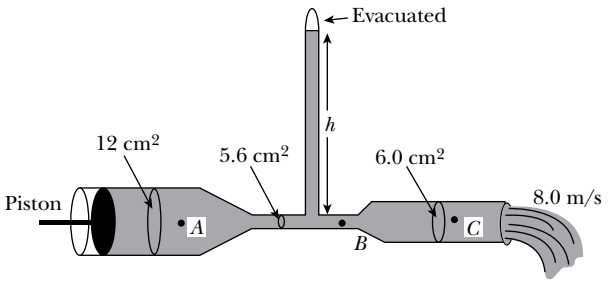
(a) 3.03×10^3 kg/m^3

(b) 4.62×10^3 kg/m^3

(c) 6.16×10^3 kg/m^3

(d) 8.01×10^3 kg/m^3

Paragraph for Questions 63–67: A glass tube has several different cross-sectional areas with the values indicated in the figure. A piston at the left end of the tube exerts pressure so that mercury within the tube flows from the right end with a speed of 8.0 m/s. Three points within the tube are labeled A, B, and C. (Atmospheric pressure is 1.01×10^5 N/m^2; Density of mercury is 13 600 kg/m^3.)



63. At what speed is mercury flowing past the point labeled A?

(a) 4.0 m/s **(b)** 2.0 m/s

(c) 8.0 m/s **(d)** 12 m/s

64. What is the total pressure at point A?

(a) 1.01×10^5 Pa **(b)** 4.27×10^5 Pa

(c) 2.02×10^5 Pa **(d)** 3.26×10^5 Pa

65. At what speed is mercury flowing past the point labeled B?

(a) 2.27 m/s **(b)** 7.47 m/s

(c) 4.27 m/s **(d)** 8.57 m/s

66. What is the total pressure at point B?

(a) 1.01×10^5 Pa **(b)** 4.27×10^5 Pa

(c) 3.64×10^4 Pa **(d)** 6.44×10^4 Pa

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67. What is the total pressure at point C?

(a) 1.01×10^5 Pa **(b)** 3.26×10^5 Pa
(c) 3.66×10^5 Pa **(d)** 6.44×10^5 Pa

Matrix-Match

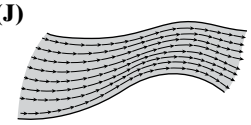
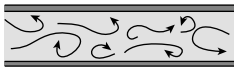
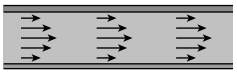
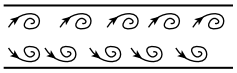
Directions for Question 68: Match the statements in Column I with the statements in Column II. One or more than one choice from Column II can match with a statement from Column I.

68.

Column I	Column II
(a) If the radius of a soap bubble A is 4 times that of another soap bubble B, then the ratio of excess pressure (P_B/P_A) is	**(p)** 16 : 9
(b) If two small drops of mercury, each of radius R, coalesce to form a single large drop, the ratio of the total surface energy before and after change is	**(q)** $2^{1/3}$: 1
(c) The energy required to blow a bubble of radius 4 cm and 3 cm in the same liquid is in the ratio of	**(r)** 4 : 1
(d) Two soap bubbles are blown. In the first bubble, excess pressure is four times that of the second soap bubble. The ratio of radii of first to second soap bubble is	**(s)** 1 : 4

Directions for Question 69 and 70: There is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

69. Types of flow of a liquid.

Column I	Column II	Column III
(I) Velocity of fluid particles that are crossing a particular point is the same at all the times.	**(i)** Flow revolves around an axis line.	**(J)** 
(II) Velocity exceeds beyond the critical velocity, the paths and velocities of liquid change continuously and haphazardly.	**(ii)** Motion of a particle varies rapidly in magnitude and direction.	**(K)** 
(III) Velocity at any point in the fluid remains constant with time.	**(iii)** Each particle takes the same path as taken by a previous particle through the point.	**(L)** 
(IV) Fluid flow velocity is greatest next to its axis.	**(iv)** All particles passing through a point in a steady flow follow the same path.	**(M)** 

(1) What are the characteristics of laminar flow?
(a) (IV) (iv) (J) **(b)** (III) (iv) (L)
(c) (II) (i) (M) **(d)** (II) (i) (K)

(2) What are the characteristics of turbulent flow?
(a) (I) (iii) (K) **(b)** (IV) (iii) (L)
(c) (III) (iii) (L) **(d)** (II) (ii) (K)

(3) What are the characteristics of steady flow?
(a) (III) (i) (K) **(b)** (I) (i) (L)
(c) (I) (iii) (J) **(d)** (I) (iv) (L)

70. Equation of continuity of a liquid that flows through a tube with four different specifications.

Column I	Column II	Column III
(I) $A_1 = 0.4\ m^2$	**(i)** $A_2 = 0.4\ m^2$	**(J)** $v_1 = 1$ m/s
(II) $A_1 = 2 \times 10^{-4}\ m^2$	**(ii)** $A_2 = 0.3\ m^2$	**(K)** $v_1 = 6$ m/s
(III) $A_1 = 0.3\ m^2$	**(iii)** $A_2 = 1.96 \times 10^{-5}\ m^2$	**(L)** $v_1 = 3$ m/s
(IV) $A_1 = 3 \times 10^{-4}\ m^2$	**(iv)** $A_2 = 1.5 \times 10^{-5}\ m^2$	**(M)** $v_1 = 0.4$ m/s

(1) Choose the correct choice of flow specifications for which $v_2 = 8$ m/s.
(a) (I) (iii) (L) **(b)** (IV) (i) (M)
(c) (II) (iv) (K) **(d)** (I) (ii) (K)

(2) Choose the correct choice of flow specifications for which $v_2 = 30.6$ m/s.
(a) (III) (ii) (J) **(b)** (II) (iii) (L)
(c) (II) (iii) (K) **(d)** (I) (i) (M)

(3) Choose the correct choice of flow specifications for which $v_2 = 20$ m/s.
(a) (IV) (iv) (J) **(b)** (I) (i) (K)
(c) (III) (iii) (L) **(d)** (I) (ii) (L)

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Integer Type

71. A soap bubble is blown to a diameter of 7 cm. If 36960 erg of work is done in blowing it further, find the new radius (in cm). The surface tension of soap solution is 40 dyn/cm.
72. Water rises in a capillary tube to a height 2 cm. In another capillary tube, whose radius is one-third of it, how much water will rise? If the first capillary tube is inclined at an angle of 60° with the vertical, then what will be the position of water in the tube?
73. The reading of pressure meter attached with a closed pipe is 3.5×10^4 N/m^2. On opening the valve of the pipe, the reading of the pressure meter is reduced to 3.0×10^5 N/m^2. Calculate the speed (in m/s) of the water flowing in the pipe.
74. Eight rain drops of radius 1 mm each falling downward with a terminal velocity of 5 cm/s coalesce to form a bigger drop. Find the terminal velocity of the bigger drop.
75. Find the depth at which an air bubble of radius 0.7 mm will remain in equilibrium in water. Given surface tension of water = 7.0×10^{-2} N/m.

ANSWER KEY

Checkpoints

1. All tie.
2. (a) All tie (the gravitation force on the penguin is the same); (b) $0.95r_0$, r_0, $1.1r_0$.
3. The level of water will not change because the quantity of water displaced will remain the same.
4. The system will not topple since according to Pascal's law the pressure on the bottom of the vessel will be same everywhere. Thus, the body will still remain in rotational equilibrium.
5. (a) The level of water in glass will drop; (b) the level of water will not change.
6. 1.5 m.
7. 13 cm^3/s outward.
8. (a) All tie; (b) 1 then 2 and 3 tie, 4 (wider means slower); (c) 4, 3, 2, 1 (wider and lower mean more pressure).
9. Air will leak into the hose and siphon stops.

Problems

1. (a) 109.8 torr ≈ 1.1×10^2 torr; (b) 406 torr ≈ 4.1×10^2 torr; (c) 296.2 torr ≈ 3.0×10^2 torr
2. (a) 74 N; (b) 1.5×10^2 m^3
3. (a) 1.10×10^9 N; (b) 1.10×10^{10} N·m; (c) 10.0 m.
4. (a) 1.80 m^3; (b) 4.71 m^3
5. 0.17 m
6. (a) 1.5 g/cm^3 (or 1500 kg/m^3); (b) 8.16×10^{-3} m^3 (or 8.16×10^{-3} m^3 in two significant digits)
7. (a) 0.25 m^2; (b) 8.7 m^3/s
8. –6.75 J
9. 1.2×10^6 Pa
10. (a) 0.0776 m^3/s; (b) 69.8 kg/s
11. 9.80 N
12. (a) 791.1 cm^3; (b) 6.329 m^3; (c) 6.329×10^3 kg
13. 44 km
14. 2.55 m/s ≈ 2.6 m/s
15. 4.0×10^3 Pa
16. 0.131 m; in upward direction.
17. (a) 8.6 10 torr; (b) 1482 torr ≈ 1.5×10^3 torr
18. (a) 50.1 kN; (b) 0.465 m^3
19. (a) 0.019 atm; (b) ≈0.34 atm
20. 4 logs
21. -3.5×10^4 Pa
22. 7.3 m/s
23. 15 torr
24. (a) $F = (A/a)f$; (b) 5.9×10^3 N
25. 2.0×10^5 J
26. 1.56×10^4 Pa
27. 1.1 cm
28. 7536 J ≈ 7.54 kJ
29. (a) 2; (b) 1/2; (c) 4.00 cm
30. 10.6 kg
31. 0.675 J
32. $r_{mid} = 4.35$ cm
33. 47.8 cm
34. 5.1×10^4 N
35. (a) 2.37 m/s ≈ 2.4 m/s; (b) 2.3 m/s
36. (a) 7.9 km; (b) 16 km
37. 310.7 m ≈ 311 m
38. (a) 1.5×10^{-3} m^3/s; (b) 0.90 m
39. (a) 2.2×10^2 kPa; (b) 9.3 kPa
40. 2.6 cm
41. 9.7 mm
42. (a) 63 L/min for the flow rate in the main (1.9 cm diameter) pipe; (b) ≈1.5
43. 0.167
44. (a) 37 cm; (b) 28 cm; (c) 20 cm
45. 5.28×10^5 N
46. (b) 55.5 m/s
47. 98 m/s
48. (a) 1.8848 m/s; (b) 223.76 Pa or about 224 Pa to three significant digits.
49. 739.21 torr
50. (a) Fraction of 10%; (b) Fraction of 8.3%
51. 63 W
52. ≈1.08×10^3 atm.
53. (a) 3.9 m/s; (b) 1.08×10^5 Pa
54. (a) 1.25 kg; (b) 1.4×10^3 kg/m^3
55. (a) 19.5 m^3; (b) 8.28 m/s; (c) 2.30×10^5 Pa
56. About 50 N
57. (a) 5.0 m/s; (b) 25 m/s; (c) 2.0×10^{-3} m^3/s.

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58. (b) 4.6×10^4 N; (c) One team of horses could be used if one half of the sphere is attached to a sturdy wall. The force of the wall on the sphere would balance the force of the horses.
59. (a) 0.612 kg; (b) 0.671 kg
60. (a) $\approx 6.7 \times 10^2$ kg/m^3; (b) 7.2×10^2 kg/m^3
61. (a) 2.5 m/s; (b) 2.8×10^5 Pa
62. 9.0×10^5 N
63. $\approx$2.2 cm/h
64. (a) 2.14×10^{-2} m^3; (b) 1.65×10^3 N
65. 7.2×10^5 N
66. 0.489 m
67. (a) 2.04×10^{-2} m^3 (b) 1.57×10^3 N
68. (a) 2.59×10^4 N; (b) 2.71×10^4 N; (c) 3.15×10^3 N; (d) 1.26×10^3 N, $T + F_b = mg$
69. $\approx 1.8 \times 10^5$ Pa
70. 1.3 g/cm^3
71. (a) 1.17×10^7 N; (b) 1.33×10^7 N

Practice Questions

Single Correct Choice Type

1. (c)
2. (a)
3. (d)
4. (c)
5. (d)
6. (d)
7. (b)
8. (d)
9. (a)
10. (b)
11. (a)
12. (c)
13. (b)
14. (d)
15. (b)
16. (c)
17. (c)
18. (c)
19. (d)
20. (a)
21. (b)
22. (b)
23. (c)
24. (d)
25. (b)
26. (c)
27. (b)
28. (c)
29. (a)
30. (d)
31. (b)
32. (b)
33. (c)
34. (a)
35. (b)
36. (d)
37. (d)
38. (c)
39. (c)
40. (b)
41. (c)
42. (a)
43. (c)
44. (b)
45. (d)
46. (b)
47. (b)
48. (b)

More than One Correct Choice Type

49. (a), (c)
50. (a), (b)
51. (b), (c)
52. (a), (b), (c), (d)
53. (b), (c)
54. (b), (d)
55. (a), (c),
56. (a), (c), (d)
57. (a), (c)
58. (a), (b), (c)

Linked Comprehension

59. (b)
60. (c)
61. (b)
62. (a)
63. (a)
64. (b)
65. (d)
66. (c)
67. (a)

Matrix-Match

68. (a) $\rightarrow$ (r); (b) $\rightarrow$ (q); (c) $\rightarrow$ (p); (d) $\rightarrow$ (s)
69. (1) $\rightarrow$ (b); (2) $\rightarrow$ (d); (3) $\rightarrow$ (c)
70. (1) $\rightarrow$ (d); (2) $\rightarrow$ (b); (3) $\rightarrow$ (a)

Integer Type

71. 7
72. 4
73. 10
74. 20
75. 4

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CHAPTER 15

Oscillations

15.1 | WHAT IS PHYSICS?

Our hearts beat, our lungs oscillate, we shiver when we are cold, we sometimes snore, we can hear and speak because our eardrums and larynges vibrate. We cannot even say the word "vibration" properly without the proper oscillation of our tip of the tongue.

Our world is filled with oscillations in which objects move back and forth repeatedly. Many oscillations are merely amusing or annoying, but many others are dangerous or financially important. Here are a few examples: When a bat hits a baseball, the bat may oscillate enough to sting the batter's hands or even to break apart. When wind blows past a power line, the line may oscillate ("gallop" in electrical engineering terms) so severely that it rips apart, shutting off the power supply to a community. When an airplane is in flight, the turbulence of the air flowing past the wings makes them oscillate, eventually leading to metal fatigue and even failure. When a train travels around a curve, its wheels oscillate horizontally ("hunt" in mechanical engineering terms) as they are forced to turn in new directions (you can hear the oscillations).

When an earthquake occurs near a city, buildings may be set oscillating so severely that they are shaken apart. When an arrow is shot from a bow, the feathers at the end of the arrow manage to snake around the bow staff without hitting it because the arrow oscillates. When a coin drops into a metal collection plate, the coin oscillates with such a familiar ring that the coin's denomination can be determined from the sound. When a rodeo cowboy rides a bull, the cowboy oscillates wildly as the bull jumps and turns (at least the cowboy hopes to be oscillating).

The study and control of oscillations are two of the primary goals of both physics and engineering. In this chapter we discuss a basic type of oscillation called *simple harmonic motion*.

Contents

15.1 What is Physics?
15.2 Simple Harmonic Motion
15.3 Force Law for Simple Harmonic Motion
15.4 Simple Harmonic Motion and Uniform Circular Motion
15.5 Energy in Simple Harmonic Motion
15.6 An Angular Simple Harmonic Oscillator
15.7 Pendulums
15.8 Damped Simple Harmonic Motion
15.9 Forced Oscillations and Resonance

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Periodic Motion

Any motion in which all the parameters of motion are repeated after a definite time interval is called periodic motion. The time which elapses between successive passages in the same direction through any point is termed the periodic time, or the period of the motion. The number of periods comprised in 1s is called the frequency; thus, if T is the period of motion and n is the frequency of a periodic motion, we have $n = 1/T$. The motion of the hands of a clock is periodic; the period of the motion of the minute hand being 1 h or 3600 s. The bob of a pendulum moves periodically, the period being equal to the time of one complete (to and fro) oscillation. Periodic motion can be along any path.

Oscillatory Motion

If in case of periodic motion, a particle moves "to and fro" on the same path, the motion is said to be oscillatory motion. A body that undergoes oscillatory motion always does so about a stable equilibrium position. When it is moved away from this position and released, it experiences a net force or torque to pull it back toward equilibrium position. However, by the time it gets there, the restoring force/torque would have done some positive work on it. Thus, it must have gained some kinetic energy, so it overshoots the equilibrium position. Now, it stops somewhere on the other side, and is again pulled back toward equilibrium. Imagine a ball rolling back and forth in a round bowl or a pendulum that swings back and forth past its lowest position.

CHECKPOINT 1

A particle hanging from a thread is launched horizontally such that it completes a circle. Is the motion periodic or oscillatory?

CHECKPOINT 2

A particle hanging from a thread is launched horizontally with small speed such that it does not even reach to the horizontal level. Is the motion periodic or oscillatory?

Important Characteristics of Oscillatory Motion

1. When a particle in stable equilibrium is disturbed, it has tendency to return to equilibrium position and this tendency is exhibited as oscillatory motion.
2. The force on the body acts toward the mean position, that is, force is always opposite to the displacement vector of the particle with respect to mean position. (This force is known as restoring force.)

$$\hat{F} = -\hat{r},\ \hat{\tau} = -\hat{\theta} \tag{15-1}$$

where $\hat{r}$ and $\hat{\theta}$ are displacement and angular displacement from mean position.
3. Energy is also conserved in the motion. If energy is not conserved, then the particle will not be able to repeat the parameters of the motion.

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15.2 | SIMPLE HARMONIC MOTION

Key Concepts

- The frequency f of periodic, or oscillatory, motion is the number of oscillations per second. In the SI system, it is measured in hertz: 1 Hz = 1/s.
- The period T is the time required for one complete oscillation, or cycle. It is related to the frequency by $T = 1/f$.
- In simple harmonic motion (SHM), the displacement $x(t)$ of a particle from its equilibrium position is described by the equation

$$x = x_m \cos(\omega t + \phi) \quad \text{(displacement)},$$

in which x_m is the amplitude of the displacement, $\omega t + \phi$ is the phase of the motion, and ϕ is the phase constant. The angular frequency ω is related to the period and frequency of the motion by $\omega = 2\pi/T = 2\pi f$.

- Differentiating $x(t)$ leads to equations for the particle's SHM velocity and acceleration as functions of time:

$$v = -\omega x_m \sin(\omega t + \phi) \quad \text{(velocity)}$$

and

$$a = -\omega^2 x_m \cos(\omega t + \phi) \quad \text{(acceleration)}.$$

In the velocity function, the positive quantity ωx_m is the velocity amplitude v_m. In the acceleration function, the positive quantity $\omega^2 x_m$ is the acceleration amplitude a_m.

Sinusoidal Vibrations

Our study will be mostly related to sinusoidal vibrations which we will show later, arise when net force experienced by an oscillating body has magnitude proportional to the distance from the mean position or torque is directly proportional to the angular displacement from mean position. There are two reasons for this—one physical, one mathematical, and both basic to the whole subject. The physical reason is that purely sinusoidal vibrations are common in many types of mechanical systems. Such motion is almost always possible if the displacements are small enough. If, for example, we have a body attached to a spring, the force exerted on it at a displacement x from equilibrium is actually

$$F(x) = -k_1 x + k_2 x^2 + k_3 x^3 + \cdots$$

where $k_1, k_2, k_3, \ldots$ are a set of constants, and we can always find a range of values of x within which the sum of the terms in x_2, x_3, etc., is negligible, compared to the term $-k_1 x$. If the body is of mass m and the mass of the spring is negligible, the equation of motion of the body then becomes

$$m\frac{d^2x}{dt^2} = -k_1 x \tag{15-2}$$

It is easy to verify that the above equation is satisfied by an equation of the form

$$x = x_m \sin(\omega t + \phi_0) \tag{15-3}$$

where $\omega = (k_1/m)^{1/2}$. Thus, sinusoidal vibration or simple harmonic motion (SHM) is possible in small vibrations. However, we should remember that in general it is only an approximation (although perhaps a very close one) to the true motion.

The second reason is the mathematical one. The actual importance of purely sinusoidal vibrations is proved in a famous theorem given by the French mathematician J. B. Fourier in 1807. According to Fourier's theorem, any periodic function with a period T can be considered as a sum of pure sinusoidal vibrations of periods T, $T/2$, $T/3$, etc., with appropriately chosen amplitudes. A thorough familiarity with sinusoidal vibrations will be a stepping stone for our understanding of every conceivable problem involving periodic phenomena.

Derivation of Equation of Simple Harmonic Motion

As we know, in SHM, magnitude of restoring force is directly proportional to distance from the mean position and is directed toward it. Considering origin at the mean position and displacing the particle along x axis, the net force on the particle F is given by

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$$F = -kx \quad \text{(where } k \text{ is a constant)}$$
$$ma = -kx$$
$$a = -\frac{k}{m}x = -\omega^2 x \quad \text{(considering } k/m = \omega^2\text{)}$$

This is a differential form of equation of SHM.

$$\frac{dv}{dt} = -\omega^2 x$$

Multiplying both sides by v, we get

$$v\frac{dv}{dt} = -\omega^2 xv$$
$$v\frac{dv}{dt} = -\omega^2 x\frac{dx}{dt}$$
$$vdv = -\omega^2 xdx$$

Integrating, we get

$$\frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + C$$

Here, C is constant of integration.

$$v^2 = C_1 - \omega^2 x^2 \quad \text{(where } C_1 = 2C\text{)}$$
$$v = \sqrt{C_1 - \omega^2 x^2}$$
$$\frac{dx}{dt} = \sqrt{C_1 - \omega^2 x^2}$$
$$\frac{dx}{\sqrt{C_1 - \omega^2 x^2}} = dt$$
$$\frac{dx}{\omega\sqrt{\left(\frac{C_1}{\omega}\right)^2 - x^2}} = dt$$

Integration of the above relation gives

$$\frac{1}{\omega}\sin^{-1}\left(\frac{x\omega}{\sqrt{C_1}}\right) = t + C_2$$

where C_2 is another constant of integration

$$\sin^{-1}\left(\frac{x\omega}{\sqrt{C_1}}\right) = \omega t + \omega C_2$$
$$x = \frac{\sqrt{C_1}}{\omega}\sin(\omega t + \omega C_2)$$
$$x = x_m \sin(\omega t + \phi_0) \quad \text{(where } x_m = \sqrt{C_1}/\omega \quad \text{and} \quad \phi_0 = \omega C_2\text{)}$$

or
$$x = x_m \cos(\omega t + \phi)$$

So, the sinusoidal function of the form of the above equations represent SHM.

Figure 15-1*a* shows a sequence of "snapshots" of a simple oscillating system, a particle moving repeatedly back and forth about the origin of an x axis. In this section we simply describe the motion. Later, we shall discuss how to attain such motion.

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One important property of oscillatory motion is its **frequency**, or number of oscillations that are completed each second. The symbol for frequency is f, and its SI unit is the **hertz** (abbreviated as Hz), where

$$1 \text{ hertz} = 1 \text{ Hz} = 1 \text{ oscillation per second} = 1/\text{s} \tag{15-4}$$

Related to the frequency is the **period** T of the motion, which is the time for one complete oscillation (or **cycle**); that is,

$$T = \frac{1}{f} \tag{15-5}$$

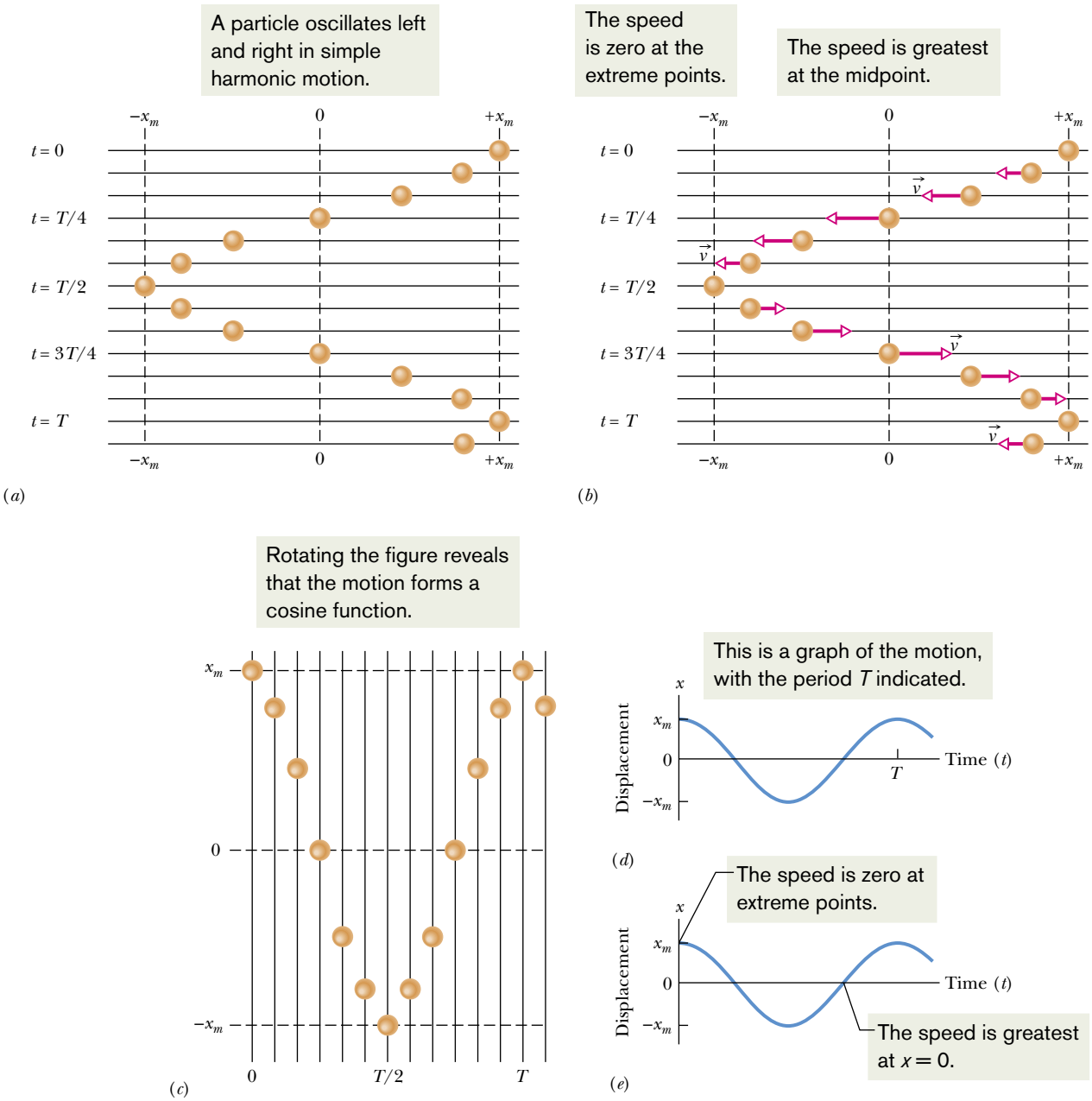

Figure 15-1 (*a*) A sequence of "snapshots" (taken at equal time intervals) showing the position of a particle as it oscillates back and forth about the origin of an x axis, between the limits $+x_m$ and $-x_m$. (*b*) The vector arrows are scaled to indicate the speed of the particle. The speed is maximum when the particle is at the origin and zero when it is at $\pm x_m$. If the time t is chosen to be zero when the particle is at $+x_m$, then the particle returns to $+x_m$ at $t = T$, where T is the period of the motion. The motion is then repeated. (*c*) Rotating the figure reveals the motion forms a cosine function of time, as shown in (*d*). (*e*) The speed (the slope) changes.

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Any motion that repeats itself at regular intervals is called **periodic motion** or **harmonic motion**. We are interested here in motion, that repeats itself in a particular way, namely, similar to one shown in Fig. 15-1*a*. For such motion, the displacement x of the particle from the origin is given as a function of time by

$$x(t) = x_m \cos(\omega t + \phi) \quad \text{(displacement)} \tag{15-6}$$

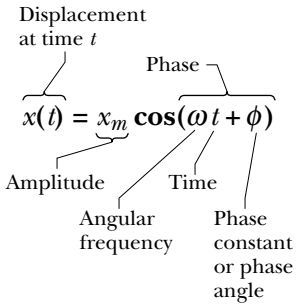

Figure 15-2 A handy guide to the quantities in Eq. 15-3 for simple harmonic motion.

in which x_m, ω, and ϕ are constants. This motion is called **simple harmonic motion** (SHM), a term that means the periodic motion is a sinusoidal function of time. Equation 15-3, in which the sinusoidal function is a cosine function, is graphed in Fig. 15-1*d*. (You can get that graph by rotating Fig. 15-1*a* counterclockwise by 90° and then connecting the successive locations of the particle with a curve.) The quantities that determine the shape of the graph are displayed in Fig. 15-2 with their names. We now shall define those quantities.

The quantity x_m, called the **amplitude** of the motion, is a positive constant whose value depends on how the motion was started. The subscript m stands for *maximum* because the amplitude is the magnitude of the maximum displacement of the particle in either direction. The cosine function in Eq. 15-3 varies between the limits ±1; so, the displacement $x(t)$ varies between the limits $\pm x_m$.

The time-varying quantity $\omega t + \phi$ in Eq. 15-3 is called the **phase** of the motion, and the constant ϕ is called the **phase constant** (or **phase angle**). The value of ϕ depends on the displacement and velocity of the particle at time $t = 0$. For the $x(t)$ plots of Fig. 15-3*a*, the phase constant ϕ is zero.

To interpret the constant ω, called the **angular frequency** of the motion, we first note that the displacement $x(t)$ must return to its initial value after one period T of the motion; that is, $x(t)$ must equal $x(t + T)$ for all t. To simplify this analysis, let us put $\phi = 0$ in Eq. 15-6. From that equation, we then can write as

$$x_m \cos \omega t = x_m \cos \omega(t + T). \tag{15-7}$$

The cosine function first repeats itself when its argument (the phase) has increased by 2π rad; so, Eq. 15-7 gives us

$$\omega(t + T) = \omega t + 2\pi$$

or

$$\omega T = 2\pi.$$

Thus, from Eq. 15-5, the angular frequency is

$$\omega = \frac{2\pi}{T} = 2\pi f \tag{15-8}$$

The SI unit of angular frequency is the radian per second. (To be consistent, then, f must be in radians.) Figure 15-3 compares $x(t)$ for two simple harmonic motions that differ either in amplitude, in period (and thus in frequency and angular frequency), or in phase constant.

CHECKPOINT 3

A particle undergoing simple harmonic oscillation of period T (like that in Fig. 15-1) is at $-x_m$ at time $t = 0$. Is it at $-x_m$, at $+x_m$, at 0, between $-x_m$ and 0, or between 0 and $+x_m$ when (a) $t = 2.00T$, (b) $t = 3.50T$, and (c) $t = 5.25T$?

Velocity of SHM

By differentiating Eq. 15-6, we can find an expression for the velocity of a particle moving with SHM, that is,

$$v(t) = \frac{dx(t)}{dt} = \frac{d}{dt}\left[x_m \cos(\omega t + \phi)\right]$$

or

$$v(t) = -\omega x_m \sin(\omega t + \phi) \quad \text{(velocity)} \tag{15-9}$$

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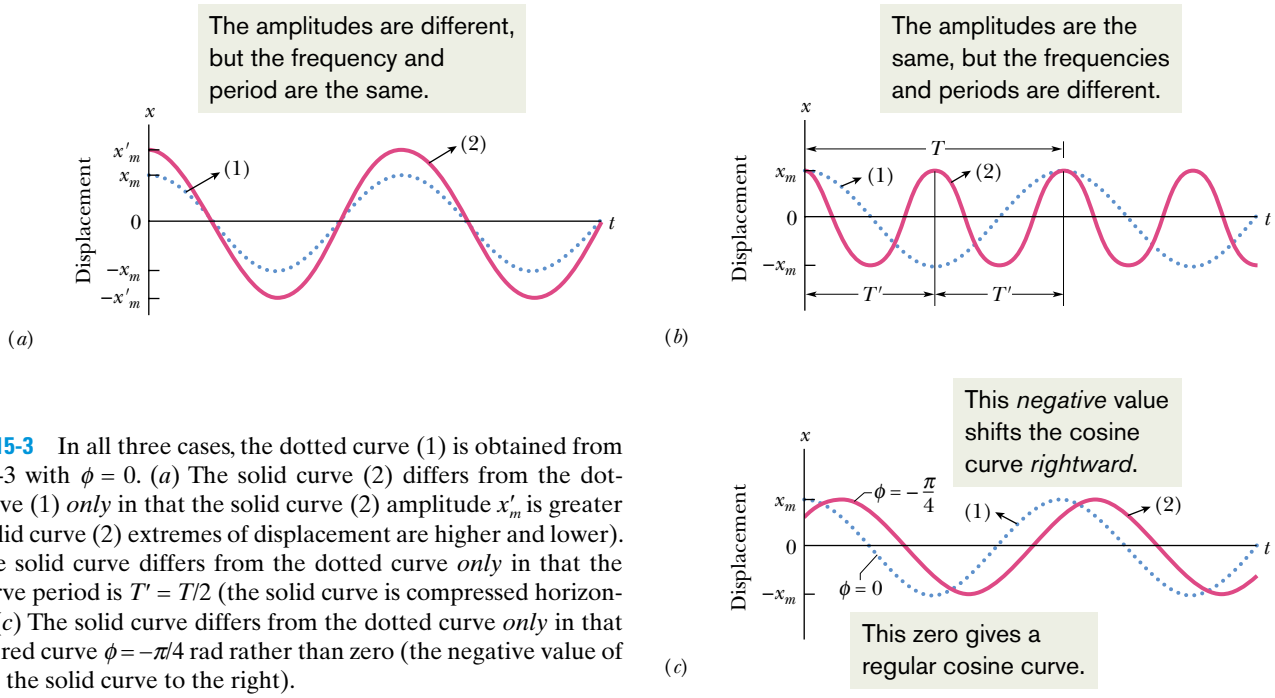

Figure 15-3 In all three cases, the dotted curve (1) is obtained from Eq. 15-3 with $\phi = 0$. (*a*) The solid curve (2) differs from the dotted curve (1) *only* in that the solid curve (2) amplitude x'_m is greater (the solid curve (2) extremes of displacement are higher and lower). (*b*) The solid curve differs from the dotted curve *only* in that the red-curve period is $T' = T/2$ (the solid curve is compressed horizontally). (*c*) The solid curve differs from the dotted curve *only* in that for the red curve $\phi = -\pi/4$ rad rather than zero (the negative value of ϕ shifts the solid curve to the right).

Figure 15-4*a* is a plot of Eq. 15-6 with $\phi = 0$. Figure 15-4*b* shows Eq. 15-9, also with $\phi = 0$. Analogous to the amplitude x_m in Eq. 15-3, the positive quantity ωx_m in Eq. 15-9 is called the **velocity amplitude** v_m. As you can see in Fig. 15-4*b*, the velocity of the oscillating particle varies between the limits $\pm v_m = \pm \omega x_m$. Note also in that figure that the curve of $v(t)$ is *shifted* (to the left) from the curve of $x(t)$ by one-quarter period; when the magnitude of the displacement is greatest [that is, $x(t) = x_m$], the magnitude of the velocity is least [that is, $v(t) = 0$]. When the magnitude of the displacement is least (that is, zero), the magnitude of the velocity is greatest (that is, $v_m = \omega x_m$).

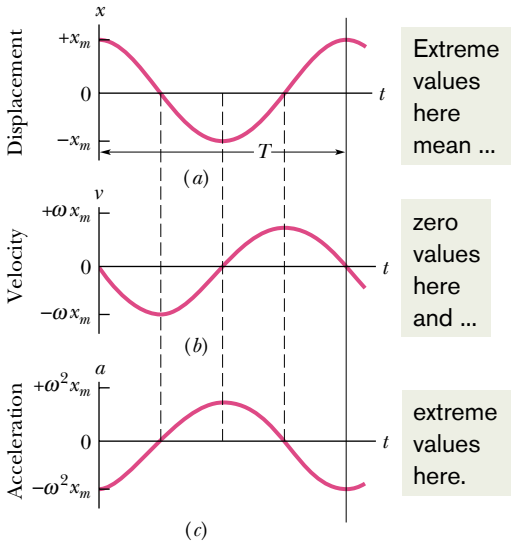

Figure 15-4 (*a*) The displacement $x(t)$ of a particle oscillating in SHM with phase angle ϕ equal to zero. The period T marks one complete oscillation. (*b*) The velocity $v(t)$ of the particle. (*c*) The acceleration $a(t)$ of the particle.

Acceleration of SHM

Knowing the velocity $v(t)$ for simple harmonic motion, we can find an expression for the acceleration of the oscillating particle by differentiating once more. Thus, from Eq. 15-9, we have

$$a(t) = \frac{dt(t)}{dt} = \frac{d}{dt}[-\omega x_m \sin(\omega t + \phi)]$$

or

$$a(t) = -\omega^2 x_m \cos(\omega t + \phi) \quad \text{(acceleration)} \qquad (15\text{-}10)$$

Figure 15-4*c* is a plot of Eq. 15-10 for the case $\phi = 0$. The positive quantity $\omega^2 x_m$ in Eq. 15-10 is called the **acceleration amplitude** a_m, that is, the acceleration of the particle varies between the limits $\pm a_m = \pm \omega^2 x_m$, as Fig. 15-4*c* shows. Note also that the acceleration curve $a(t)$ is shifted (to the left) by $\frac{1}{4}T$ relative to the velocity curve $v(t)$.

We can combine Eqs. 15-6 and 15-10 to yield

$$a(t) = -\omega^2 x(t) \qquad (15\text{-}11)$$

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which is the hallmark of SHM:

In SHM, the acceleration a is proportional to the displacement x but opposite in sign, and the two quantities are related by the square of the angular frequency ω.

Thus, as Fig. 15-4 shows, when the displacement has its greatest positive value, the acceleration has its greatest negative value, and conversely, when the displacement is zero, the acceleration is also zero.

CHECKPOINT 4

Which of the following relationships between a particle's acceleration a and its position x indicates simple harmonic oscillation: (a) $a = 3x^2$, (b) $a = 5x$, (c) $a = -4x$, (d) $a = -2/x$? For the SHM, what is the angular frequency (assume the unit of rad/s)?

PROBLEM-SOLVING TACTICS

Tactic 1: Phase Angles Note the effect of the phase angle ϕ on a plot of $x(t)$. When $\phi = 0$, $x(t)$ has a graph like that in Fig. 15-4*a*, a typical cosine curve. Increasing ϕ shifts the curve leftward along the t axis. (You might remember this with the symbol $\leftarrow\!\!\!\uparrow\phi$, where the up arrow indicates an increase in ϕ and the left arrow indicates the resulting shift in the curve.) Decreasing ϕ shifts the curve rightward, as shown in Fig. 15-3*c* for $\phi = -\pi/4$.

Two plots of SHM with different phase angles are said to have a *phase difference*; each is said to be *phase-shifted* from the other, or *out of phase* with the other. The curves in Fig. 15-3*c*, for example, have a phase difference of $\pi/4$ rad; that is, one curve is phase-shifted from the other by $\pi/4$ rad.

Because SHM repeats after each period T and the cosine function repeats after each 2π rad, one period T represents a phase difference of 2π rad. In Fig. 15-4, $x(t)$ is phase-shifted to the right from $v(t)$ by one-quarter period, or $-\pi/2$ rad; it is shifted to the right from $a(t)$ by one-half period, or $-\pi$ rad. A phase shift of 2π rad causes a curve of SHM to coincide with itself; that is, it looks unchanged.

15.3 | FORCE LAW FOR SIMPLE HARMONIC MOTION

Key Concept

- A particle with mass m that moves under the influence of a Hooke's law restoring force given by $F = -kx$ is a linear simple harmonic oscillator with

$$\omega = \sqrt{\frac{k}{m}} \quad \text{(angular frequency)}$$

and

$$T = 2\pi\sqrt{\frac{m}{k}} \quad \text{(period)}.$$

Once we know how the acceleration of a particle varies with time, we can use Newton's second law to learn what force must act on the particle to give it that acceleration. If we combine Newton's second law and Eq. 15-11, we find, for SHM,

$$F = ma = -(m\omega^2)x \tag{15-12}$$

This result – a restoring force that is proportional to the displacement but opposite in sign – is familiar. It is Hooke's law:

$$F = -kx \tag{15-13}$$

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for a spring, the spring constant here being

$$k = m\omega^2 \tag{15-14}$$

We can, in fact, take Eq. 15-13 as an alternative definition of SHM. It says the following:

Simple harmonic motion is the motion of a particle when the force acting on it is proportional to the particle's displacement but in the opposite direction.

The block–spring system of Fig. 15-5 forms a **linear simple harmonic oscillator** (linear oscillator, for short), where "linear" indicates that F is proportional to x rather than to some other power of x. The angular frequency ω of the SHM of the block is related to the spring constant k and the mass m of the block by Eq. 15-14, which yields

$$\omega = \sqrt{\frac{k}{m}} \quad \text{(angular frequency)} \tag{15-15}$$

By combining Eqs. 15-8 and 15-15, we can write, for the **period** of the linear oscillator of Fig. 15-5,

$$T = 2\pi\sqrt{\frac{m}{k}} \quad \text{(period)} \tag{15-16}$$

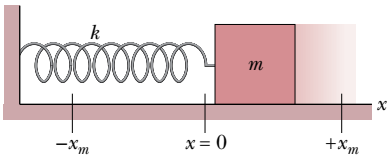

Figure 15-5 A linear simple harmonic oscillator. The surface is frictionless. Like the particle of Fig. 15-1, the block moves in simple harmonic motion once it has been either pulled or pushed away from the $x = 0$ position and released. Its displacement is then given by Eq. 15-3.

Equations 15-15 and 15-16 tell us that a large angular frequency (and thus a small period) goes with a stiff spring (large k) and a light block (small m). Every oscillating system, be it a diving board or a violin string, has some element of "springiness" and some element of "inertia" or mass, and thus resembles a linear oscillator. In the linear oscillator of Fig. 15-5, these elements are located in separate parts of the system: the springiness is entirely in the spring, which we assume to be massless, and the inertia is entirely in the block, which we assume to be rigid. In a violin string, however, the two elements are both within the string.

CHECKPOINT 5

Which of the following relationships between the force F on a particle and the particle's position x gives SHM: (a) $F = -5x$, (b) $F = -400x^2$, (c) $F = 10x$, (d) $F = 3x^2$?

SAMPLE PROBLEM 15.01

Block–spring SHM, amplitude, acceleration, phase constant

A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(a) What are the angular frequency, the frequency, and the period of the resulting motion?

KEY IDEA

The block–spring system forms a linear simple harmonic oscillator, with the block undergoing SHM.

Calculations: The angular frequency is given by Eq. 15-15:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{65\ \text{N/m}}{0.68\ \text{kg}}} = 9.78\ \text{rad/s}$$
$$\approx 9.8\ \text{rad/s}. \qquad \text{(Answer)}$$

The frequency follows from Eq. 15-8, which yields

$$f = \frac{\omega}{2\pi} = \frac{9.78\ \text{rad/s}}{2\pi\ \text{rad}} = 1.56\ \text{Hz} \approx 1.6\ \text{Hz}. \qquad \text{(Answer)}$$

The period follows from Eq. 15-5, which yields

$$T = \frac{1}{f} = \frac{1}{1.56\ \text{Hz}} = 0.64\ \text{s} = 640\ \text{ms}. \qquad \text{(Answer)}$$

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(b) What is the amplitude of the oscillation?

KEY IDEA

With no friction involved, the mechanical energy of the spring–block system is conserved.

Reasoning: The block is released from rest 11 cm from its equilibrium position, with zero kinetic energy and the elastic potential energy of the system at a maximum. Thus, the block will have zero kinetic energy whenever it is again 11 cm from its equilibrium position, which means it will never be farther than 11 cm from that position. Its maximum displacement is 11 cm:

$$x_m = 11 \text{ cm}. \qquad \text{(Answer)}$$

(c) What is the maximum speed v_m of the oscillating block, and where is the block when it has this speed?

KEY IDEA

The maximum speed v_m is the velocity amplitude ωx_m in Eq. 15-9.

Calculation: Thus, we have

$$v_m = \omega x_m = (9.78 \text{ rad/s})(0.11 \text{ m}) = 1.1 \text{ m/s}. \qquad \text{(Answer)}$$

This maximum speed occurs when the oscillating block is rushing through the origin; compare Figs. 15-4*a* and 15-4*b*, where you can see that the speed is a maximum whenever $x = 0$.

(d) What is the magnitude a_m of the maximum acceleration of the block?

KEY IDEA

The magnitude a_m of the maximum acceleration is the acceleration amplitude $\omega^2 x_m$ in Eq. 15-10.

Calculation: So, we have

$$a_m = \omega^2 x_m = (9.78 \text{ rad/s})^2 (0.11 \text{ m}) = 11 \text{ m/s}^2. \qquad \text{(Answer)}$$

This maximum acceleration occurs when the block is at the ends of its path, where the block has been slowed to a stop so that its motion can be reversed. At those extreme points, the force acting on the block has its maximum magnitude; compare Figs. 15-4*a* and 15-4*c*, where you can see that the magnitudes of the displacement and acceleration are maximum at the same times, when the speed is zero, as you can see in Fig. 15-4*b*.

(e) What is the phase constant ϕ for the motion?

Calculations: Equation 15-6 gives the displacement of the block as a function of time. We know that at time $t = 0$, the block is located at $x = x_m$. Substituting these *initial conditions*, as they are called, into Eq. 15-6 and canceling x_m give us

$$1 = \cos \phi.$$

Taking the inverse cosine then yields

$$\phi = 0 \text{ rad}. \qquad \text{(Answer)}$$

(Any angle that is an integer multiple of 2π rad also satisfies Eq. 15-14; we chose the smallest angle.)

(f) What is the displacement function $x(t)$ for the spring–block system?

Calculation: The function $x(t)$ is given in general form by Eq. 15-6. Substituting known quantities into that equation gives us

$$x(t) = x_m \cos(\omega t + \phi) = (0.11 \text{ m}) \cos[(9.8 \text{ rad/s})t + 0] = 0.11 \cos(9.8t), \qquad \text{(Answer)}$$

where x is in meters and t is in seconds.

SAMPLE PROBLEM 15.02

Finding SHM phase constant from displacement and velocity

At $t = 0$, the displacement $x(0)$ of the block in a linear oscillator like that of Fig. 15-5 is –8.50 cm. (Read $x(0)$ as "x at time zero.") The block's velocity $v(0)$ then is –0.920 m/s, and its acceleration $a(0)$ is +47.0 m/s^2.

(a) What is the angular frequency ω of this system?

KEY IDEA

With the block in SHM, Eqs. 15-6, 15-9, and 15-10 give its displacement, velocity, and acceleration, respectively, and each contains ω.

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Calculations: Let's substitute $t = 0$ into each to see whether we can solve any one of them for ω. We find

$$x(0) = x_m \cos\phi, \tag{15-17}$$

$$v(0) = -\omega x_m \sin\phi, \tag{15-18}$$

and

$$a(0) = -\omega^2 x_m \cos\phi. \tag{15-19}$$

In Eq. 15-17, ω has disappeared. In Eqs. 15-18 and 15-19, we know values for the left sides, but we do not know x_m and ϕ. However, if we divide Eq. 15-19 by Eq. 15-17, we neatly eliminate both x_m and ϕ and can then solve for ω as

$$\omega = \sqrt{-\frac{a(0)}{x(0)}} = \sqrt{-\frac{47.0\ \text{m/s}^2}{-0.0850\ \text{m}}}$$
$$= 23.5\ \text{rad/s}. \qquad \text{(Answer)}$$

(b) What are the phase constant ϕ and amplitude x_m?

Calculations: We know ω and want ϕ and x_m. If we divide Eq. 15-18 by Eq. 15-17, we eliminate one of those unknowns and reduce the other to a single trignometric function:

$$\frac{v(0)}{x(0)} = \frac{-\omega x_m \sin\phi}{x_m \cos\phi} = -\omega\tan\phi.$$

Solving for $\tan\phi$, we find

$$\tan\phi = -\frac{v(0)}{\omega x(0)} = -\frac{-0.920\ \text{m/s}}{(23.5\ \text{rad/s})(-0.0850\ \text{m})}$$
$$= -0.461.$$

This equation has two solutions:

$$\phi = -25° \quad \text{and} \quad \phi = -180° + (-25°) = 155°.$$

Normally only the first solution here is displayed by a calculator, but it may not be the physically possible solution. To choose the proper solution, we test them both by using them to compute values for the amplitude x_m. From Eq. 15-17, we find that if $\phi = -25°$, then

$$x_m = \frac{x(0)}{\cos\phi} = \frac{-0.0850\ \text{m}}{\cos(-25°)} = -0.094\ \text{m}.$$

We find similarly that if $\phi = -155°$, then $x_m = 0.094$ m. Because the amplitude of SHM must be a positive constant, the correct phase constant and amplitude here are

$$\phi = -155° \quad \text{and} \quad x_m = 0.094\ \text{m} = 9.4\ \text{cm}. \qquad \text{(Answer)}$$

15.4 | SIMPLE HARMONIC MOTION AND UNIFORM CIRCULAR MOTION

Key Concept

- Simple harmonic motion corresponds to the projection of uniform circular motion onto a diameter of the circle.

In 1610, Galileo, using his newly constructed telescope, discovered the four principal moons of Jupiter. Over weeks of observation, each moon seemed to him to be moving back and forth relative to the planet in what today we would call simple harmonic motion; the disk of the planet was the midpoint of the motion. The record of Galileo's observations, written in his own hand, is actually still available. A. P. French of MIT used Galileo's data to work out the position of the moon Callisto relative to Jupiter (actually, the angular distance from Jupiter as seen from Earth) and found that the data approximates the curve shown in Fig. 15-6. The curve strongly suggests Eq. 15-6, the displacement function for simple harmonic motion. A period of about 16.8 days can be measured from the plot, but it is a period of what exactly? After all, a moon cannot possibly be oscillating back and forth like a block on the end of a spring, and so why would Eq. 15-6 have anything to do with it?

Actually, Callisto moves with essentially constant speed in an essentially circular orbit around Jupiter. Its true motion—far from being simple harmonic—is uniform circular motion along that orbit. What Galileo saw—and what you can see with a good pair of binoculars and

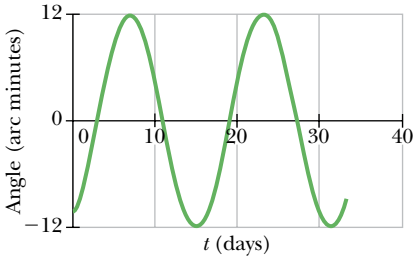

Figure 15-6 The angle between Jupiter and its moon Callisto as seen from Earth. Galileo's 1610 measurements approximate this curve, which suggests simple harmonic motion. At Jupiter's mean distance from Earth, 10 minutes of arc corresponds to about 2×10^6 km. (Based on A. P. French, *Newtonian Mechanics*, W.W. Norton & Company, New York, 1971, p. 288.)

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a little patience—is the projection of this uniform circular motion on a line in the plane of the motion. We are led by Galileo's remarkable observations to the conclusion that simple harmonic motion is uniform circular motion viewed edge-on. In more formal language:

Simple harmonic motion is the projection of uniform circular motion on a diameter of the circle in which the circular motion occurs.

Figure 15-7*a* gives an example. It shows a *reference particle* P' moving in uniform circular motion with (constant) angular speed ω in a *reference circle*. The radius x_m of the circle is the magnitude of the particle's position vector. At any time t, the angular position of the particle is $\omega t + \phi$, where ϕ is its angular position at $t = 0$.

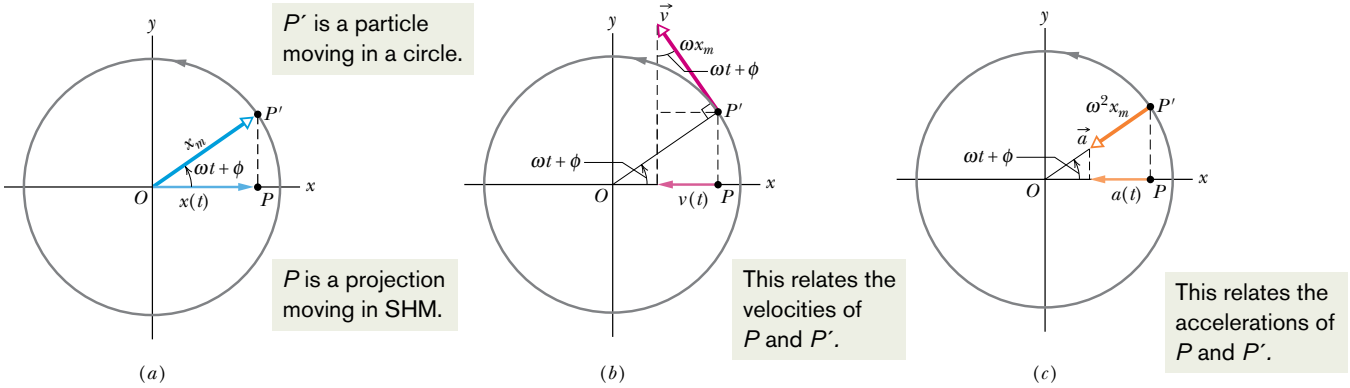
P′ is a particle moving in a circle.
P is a projection moving in SHM.
This relates the velocities of P and P′.
This relates the accelerations of P and P′.
(a)
(b)
(c)

Figure 15-7 (*a*) A reference particle P' moving with uniform circular motion in a reference circle of radius x_m. Its projection P on the x axis executes simple harmonic motion. (*b*) The projection of the velocity $\vec{v}$ of the reference particle is the velocity of SHM. (*c*) The projection of the radial acceleration $\vec{a}$ of the reference particle is the acceleration of SHM.

Position. The projection of particle P' onto the x axis is a point P, which we take to be a second particle. The projection of the position vector of particle P' onto the x axis gives the location $x(t)$ of P. (Can you see the x component in the triangle in Fig. 15-7*a*?) Thus, we find

$$x(t) = x_m \cos(\omega t + \phi), \tag{15-20}$$

which is precisely Eq. 15-6. Our conclusion is correct. If reference particle P' moves in uniform circular motion, its projection particle P moves in simple harmonic motion along a diameter of the circle.

Velocity. Figure 15-7*b* shows the velocity $\vec{v}$ of the reference particle. From Eq. 10-18 ($v = \omega r$), the magnitude of the velocity vector is ωx_m; its projection on the x axis is

$$v(t) = -\omega x_m \sin(\omega t + \phi), \tag{15-21}$$

which is exactly Eq. 15-9. The minus sign appears because the velocity component of P in Fig. 15-7*b* is directed to the left, in the negative direction of x. (The minus sign is consistent with the derivative of Eq. 15-20 with respect to time.)

Acceleration. Figure 15-7*c* shows the radial acceleration $\vec{a}$ of the reference particle. From Eq. 10-23 ($a_r = \omega^2 r$), the magnitude of the radial acceleration vector is $\omega^2 x_m$; its projection on the x axis is

$$a(t) = -\omega^2 x_m \cos(\omega t + \phi), \tag{15-22}$$

which is exactly Eq. 15-10. Thus, whether we look at the displacement, the velocity, or the acceleration, the projection of uniform circular motion is indeed simple harmonic motion.

One of the best ways of describing an SHM is to consider it as the projection of uniform circular motion. Imagine, for example, that a disk of radius A rotates about a horizontal axis at the rate of ω rad/s. Suppose that a peg P is

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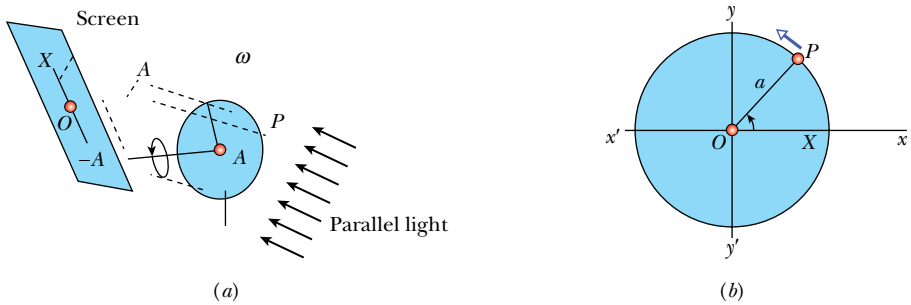

Figure 15-8 (*a*) Shadow of point *P* on a vertical wall undergoing SHM (*b*) Phasor diagram.

attached to the edge of the disk and that a horizontal beam of parallel light casts a shadow of the peg on a vertical screen, as shown in Fig. 15-8*a*. Then this shadow performs SHM with period $2\pi/\omega$ and amplitude x_m along a vertical line on the screen.

More abstractly, we can imagine SHM as being the geometrical projection of a uniform circular motion as shown in Fig. 15-8*b*. (By geometrical projection we mean simply the process of drawing a perpendicular to a given line from the instantaneous position of the point *P*.) Let the tracing point, *P*, revolve uniformly at a constant distance *OP* from *O*, the direction of motion being anticlockwise. Through *O* we draw the rectangular axes *X′OX* and *Y′OY*. From *P* draw *PQ* perpendicular to *OY* (Fig. 15-9). Thus, *OQ* is the component of *OP* resolved parallel to *OY*. As *P* revolves about *O*, the point *Q* moves up and down along *Y′OY*, its motion being of the simple harmonic type. When *P* passes across the axis *X′OX*, *Q* will pass through *O*. When *P* passes across the axis *Y′OY*, *Q* will be at its position of maximum displacement from its mean position, *O*.

The maximum displacement of *Q* will thus be equal to the radius *a* of the circular path of point *P*; it is termed the amplitude of the SHM.

The phase of the SHM at any instant is equal to the angle which has been swept out by the line *OP*. If we measure the phase from the particular position of *OP* when *Q* is moving in a certain direction (say upward) through *O*, the angle $XOP = \theta$ is equal to the phase of the SHM.

The displacements of *Q* for various values of the phase angle θ, are shown by the curve to the right of Fig. 15-9. The *x* coordinate is equal to the circular measure of the angle *XOP*, and the distance *OQ* is plotted along the *y* axis.

Let the amplitude $OP = a$, while $OQ = y$. Then $y/a = RP/OP = \sin\theta$. Therefore, $y = a \sin\theta$. The graph is plotted to the right of the figure as described by this equation.

Further, if *OP* completes a rotation (that is rotate through an angle 2π) in the period *T*, then we have $\theta = 2\pi t/T$, and therefore $y = a \sin(2\pi t/T)$.

When the tracing point *P* crosses the axis *OX*, it is moving, for that instant, parallel to the axis *OY*. Consequently, at this instant the point *Q* is moving along *OY* with a velocity equal to that of the tracing point *P* in its circular path. The velocity of *P* is equal $2\pi a/T$. Hence, the velocity of *Q* as it passes through its mean position *O* is equal to $2\pi a/T$.

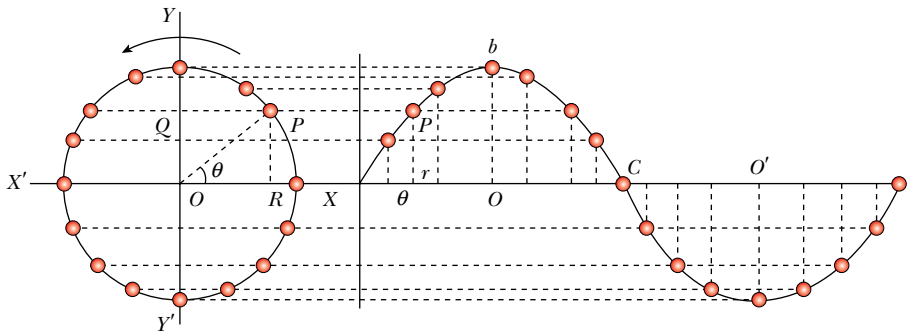

Figure 15-9 Location of *Q* plotted as function of time with corresponding positions of tracing point *P*.

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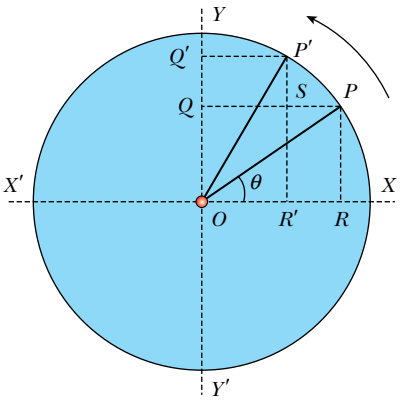

Figure 15-10 Taking shadow of tracing point P on X axis rather than Y axis.

When the tracing point P crosses the axis OY, it is moving, for that instant, in a direction perpendicular to OY. At this instant point Q will be stationary. Consequently, the point Q is stationary for an instant when point P is at the extreme, on either side of its mean position O.

The velocity corresponding to any phase of an SHM can be determined readily. Let the vector OP (Fig. 15-10), of length a, revolve uniformly about O in an anticlockwise direction, its period being equal to T; and let the projection Q of the point P on the axis YOY' represent simple harmonic motion. Thus, $OQ = y = a \sin(2\pi t/T)$. Differentiating it with respect to time we get

$$\frac{dy}{dt} = v\frac{2\pi a}{T}\cos\left(\frac{2\pi t}{T}\right)$$

This gives the instantaneous velocity of the simple harmonic motion at the time t. In terms of the phase angle θ, the velocity is equal to $(2\pi a/T)$ cos q. When $\theta = 0$, cos $\theta = 1$, and the velocity is equal to $2\pi a/T$, as already discussed; when $\theta = \pi/2$, cos $\theta = 0$, and the velocity is equal to zero and so on.

Thus, a point executing a SHM moves with a maximum velocity equal to $2\pi a/T$ on passing through its mean position. Its velocity diminishes as it moves away from its mean position, being equal to $(2\pi a/T)$ cos θ when the phase angle is θ and attaining the value zero at the extreme position. Subsequently, as the point returns toward its mean position, its velocity increases, and once more attains the value $2\pi a/T$ on moving through the mean position.

The use of a uniform circular motion as a purely geometrical basis for describing SHM embodies more than we have so far chosen to recognize. This circular motion, once we have set it up, defines SHM of amplitude x_m and angular frequency ω along any straight line in the plane of the circle.

PROBLEM-SOLVING TACTICS

Tactic 2: Identifying SHM In linear SHM, the acceleration a and displacement x of the system are related by an equation of the form

$$a = -(\text{a positive constant})x$$

which says that the acceleration is proportional to the displacement from the equilibrium position but is in the opposite direction. Once you find such an expression for an oscillating system, you can immediately compare it with Eq. 15-11, identify the positive constant as being equal to ω^2, and so quickly get an expression for the angular frequency of the motion. With Eq. 15-8 you then can find the period T and the frequency f.

In some problems you might derive an expression for the force F as a function of displacement x. If the motion is linear SHM, the force and displacement are related by

$$F = -(\text{a positive constant})x$$

which says that the force is proportional to the displacement but is in the opposite direction. Once you have found such an expression, you can immediately compare it with Eq. 15-13 and identify the positive constant as being k. If you know the mass that is involved, you can then use Eqs. 15-15, 15-16, and 15-8 to find the angular frequency ω, period T, and frequency f.

CHECKPOINT 6

Find the distance traveled by a particle executing SHM with time period T and amplitude x_m in time $T/12$ after starting from rest.

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SAMPLE PROBLEM 15.03

Finding position, velocity, and acceleration in SHM

If the position of the block in a block–spring oscillator is given by

$$x(t) = (3.00 \times 10^{-3}\ \text{m})\cos[(4.00\ \text{rad/s})t + 0.120\ \text{rad}], \tag{15-23}$$

what is the block's position, velocity, and acceleration at time $t = 2.30$ s?

KEY IDEAS

(1) The velocity $v(t)$ is the first time derivative of the position function $x(t)$.
(2) The acceleration $a(t)$ is the first time derivative of the velocity function $v(t)$.

Position: To find the position at the given time, we substitute that time into Eq. (15-23). Note that we use radians in the argument of the cosine function and thus your calculator should be in its radian mode.

$$\begin{aligned} x(t) &= (3.00 \times 10^{-3}\ \text{m})\cos[(4.00\ \text{rad/s})t + 0.120\ \text{rad}], \\ &= (3.00 \times 10^{-3}\ \text{m})(-0.99452) \\ &= -0.0029835\ \text{m} \approx -2.98\ \text{mm}. \end{aligned} \qquad \text{(Answer)}$$

Velocity: To find the velocity, we take the derivative of Eq. (15-23) with respect to time. Recall that we need to use the chain rule, in which we first take the derivative of the cosine function and then we take the derivative of the argument of the cosine function. Again, a calculator must be in its radian mode. We find

$$\begin{aligned} v(t) &= \frac{dx}{dt} = (3.00 \times 10^{-3}\ \text{m})\frac{d}{dt}\cos(4.00t + 0.120) \qquad (15\text{-}24) \\ &= (3.00 \times 10^{-3}\ \text{m})[-\sin(4.00t + 0.120)](4.00\ \text{rad/s}) \\ &= -(12.0 \times 10^{-3}\ \text{m/s})\sin[(4.00\ \text{rad/s})t + 0.120\ \text{rad}] \end{aligned}$$

Recall that the radian is a dimensionless unit that can be removed, as we do here to combine m and rad/s to get m/s. At $t = 2.3$ s,

$$\begin{aligned} v &= -(12.0 \times 10^{-3}\ \text{m/s})(0.1046) \\ &= -0.001255\ \text{m/s} \approx -1.26\ \text{mm/s}. \end{aligned}$$

Acceleration: To find the acceleration, we take the derivative of Eq. (15-24) with respect to time. Again, we must use the chain rule and radian mode. We obtain

$$\begin{aligned} a(t) &= \frac{dv}{dt} \\ &= -(12.0 \times 10^{-3}\ \text{m/s})[\cos(4.00t + 0.120)](4.00\ \text{rad/s}) \\ &= -(48.0 \times 10^{-3}\ \text{m/s})\cos(4.00t + 0.120). \end{aligned}$$

At $t = 2.3$ s,

$$\begin{aligned} a &= -(48.0 \times 10^{-3}\ \text{m})\cos[(4.00)(2.30) + 0.120], \\ &= -(48.0 \times 10^{-3}\ \text{m})(-0.99452) \\ &= 0.04774\ \text{m/s}^2 \approx 0.0477\ \text{m/s}^2. \end{aligned} \qquad \text{(Answer)}$$

We could have saved time by using the "hallmark" of simple harmonic motion: At any time and with the block in any position, the acceleration and position are related by

$$a(t) = -\omega^2 x(t).$$

The angular frequency ω is the quantity multiplying the symbol t in Eq. (15-23), namely, 4.0 rad/s. From above, the position at $t = 2.3$ s is –2.9835 mm. So,

$$\begin{aligned} a(t) &= -(4.0\ \text{rad/s})^2(-2.9835 \times 10^{-3}\ \text{m}) \\ &= 0.04774\ \text{m/s}^2 \end{aligned} \qquad \text{(Answer)}$$

SAMPLE PROBLEM 15.04

Angular frequency ω of a linear simple harmonic oscillator system

At $t = 0$, the displacement $x(0)$ of the block in a linear oscillator like that of Fig. 15-5 is –8.50 cm. (Read $x(0)$ as "x at time zero.") The block's velocity $v(0)$ then is –0.920 m/s, and its acceleration $a(0)$ is +47. 0 m/s^2.

(a) What is the angular frequency ω of this system?

KEY IDEAS

With the block in SHM, Eqs. 15-6, 15-9, and 15-10 give its displacement, velocity, and acceleration, respectively, and each contains v.

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Calculations: Let us substitute $t = 0$ into each to see whether we can solve any one of them for ω. We find

$$x(0) = x_m \cos \phi \qquad (15\text{-}25)$$

$$v(0) = -\omega x_m \sin \phi \qquad (15\text{-}26)$$

and

$$a(0) = -\omega^2 x_m \cos \phi \qquad (15\text{-}27)$$

In Eq. 15-25, ω has disappeared. In Eqs. 15-26 and 15-27, we know values for the left sides, but we do not know x_m and ϕ. However, if we divide Eq. 15-27 by Eq. 15-25, we neatly eliminate both x_m and ϕ and can then solve for ω as

$$\omega = \sqrt{\frac{a(0)}{x(0)}} = \sqrt{-\frac{47.0 \text{ m/s}^2}{-0.0850 \text{ m}}} \qquad \text{(Answer)}$$

(b) What is the phase constant ϕ and amplitude x_m?

Calculations: We know ω and want ϕ and x_m. If we divide Eq. 15-26 by Eq. 15-25, we find

$$\frac{v(0)}{x(0)} = \frac{-\omega x_m \sin \phi}{x_m \cos \phi}$$

Solving for $\tan \phi$, we find

$$\tan \phi = -\frac{v(0)}{\omega x(0)} = -\frac{-0.920 \text{ m/s}}{(23.5 \text{ rad/s})(-0.0850 \text{ m})}$$

This equation has two solutions:

$$\phi = -25° \quad \text{and} \quad \phi = 180° + (-25°) = 155°$$

(Normally, only the first solution here is displayed by a calculator.) To choose the proper solution, we test them both by using them to compute values for the amplitude x_m. From Eq. 15-25, we find that if $\phi = -25°$, then

$$x_m = \frac{x(0)}{\cos \phi} = \frac{-0.0850 \text{ m}}{\cos(-25°)} = -0.094 \text{ m}$$

We find similarly that if $\phi = 155°$, then $x_m = 0.094$ m. Because the amplitude of SHM must be a positive constant, the correct phase constant and amplitude here are

$$\phi = 155° \quad \text{and} \quad x_m = 0.094 \text{ m} = 9.4 \text{ cm} \qquad \text{(Answer)}$$

SAMPLE PROBLEM 15.05

Angular frequency and amplitude of a particle in SHM

Write equation of SHM of angular frequency ω and amplitude x_m if the particle is situated at $x_m/\sqrt{2}$ at $t = 0$ and is going toward mean position.

KEY IDEA

Since angular frequency and amplitude are given, we have two of the three quantities required for the equation of SHM. Thus, we need to find out the value of initial phase, ϕ from the given information about the initial conditions. We will use phasor diagram for this.

Calculation: At $t = 0$, particle was at $x_m/\sqrt{2}$ and was going toward mean position as shown in Fig. 15-11*a*.

The same situation is described by making reference point on a circle of radius x_m as shown in Fig. 15-11*b*.

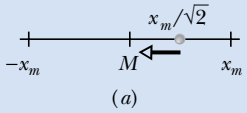

Figure 15-11 (*a*) Position and direction of particle at $t = 0$.

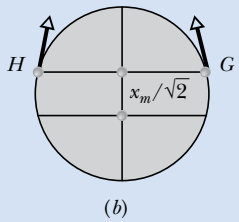

Figure 15-11 (*b*) Phasor diagram of particle at $t = 0$.

The reference point can be located at any of the two positions G and H as shown in Fig. 15-11*b* for having displacement $x_m/\sqrt{2}$, but to move toward mean position it should be H. Position H corresponds to angle $3\pi/4$.

Thus, $$\phi = \frac{3\pi}{4} \quad \text{and} \quad x = x_m \sin\left(\omega t + \frac{3\pi}{4}\right)$$

Alternative mathematical solution:
Putting $t = 0$ in the equation

$$x = x_m \sin(\omega t + \phi)$$

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We get

$$\frac{x_m}{\sqrt{2}} = x_m \sin\phi$$

Thus, $$\phi = \frac{\pi}{4}, \frac{3\pi}{4}$$

But at $t = 0$; $v < 0$ so $\pi/4$ is rejected, hence

$$x = x_m \sin\left(\omega t + \frac{3\pi}{4}\right)$$

Note: The ease of using phasor diagram is quite evident in this solution.

SAMPLE PROBLEM 15.06

Motion of particle moving under a variable force

A particle is moving on y axis under a variable force $F = -ky + C$.

(a) Can the motion of the particle be called SHM? Hence or otherwise write y position of particle as function of time.

KEY IDEA

The given force does not correspond to the standard equation of the SHM, that is, $F = -kx$. But still the relation between force and displacement is linear. The equation can be written in a standard format by just shifting the origin.

Calculation: We can write the given force as

$$F = -k\left(y - \frac{C}{k}\right)$$

Let us define

$$s = y - \frac{C}{k} \qquad (15\text{-}28)$$

where s is the displacement from mean position because at $y = C/k$, F is zero.

Thus, the mean position is at $y = C/k$.

Using Eq. 15-28, force can be written as

$$F = -ks \qquad (15\text{-}29)$$

which is the equation of SHM about $y = C/k$.

Differentiating Eq. 15-28 with respect to time twice, we get

$$\frac{d^2y}{dt^2} = \frac{d^2s}{dt^2}$$

Thus, force $F = m(d^2y/dt^2)$ can also be written as $F = m(d^2s/dt^2)$.

Substituting in Eq. 15-29

$$m\frac{d^2s}{dt^2} = -ks$$

Comparing with standard equation we know that the solution of the above SHM equation is

$$s = y_m \sin(\omega t + \phi)$$

where $$\omega = \sqrt{\frac{k}{m}}$$

Replacing with value of s from Eq. 15-28, we get

$$y = y_m \sin(\omega t + \phi) + \frac{C}{k}$$

Learn: This particle is undergoing SHM about $y = C/k$. Since no information is given about initial position and velocity of the particle, we have to keep initial phase ϕ f and amplitude y_m as unknown.

(b) If in part (a) $F = -10y + 20$, and mass m of the particle is 2.5 kg and it is released from rest from $y = +3$, at $t = 0$, write the equation of SHM.

KEY IDEA

Since information is given about initial position and velocity of the particle, we just need to find initial phase ϕ and amplitude.

Calculation: From previous discussion, we can write

$$y = y_m \sin(\omega t + \phi) + \frac{C}{k}$$

where $k = 10$, $m = 2.5$ kg, and $C = 20$.

Thus $\omega = \sqrt{\frac{k}{m}}$ gives $\omega = 2$ rad/s.

Mean position is at $y = 2$ m, as force at this point is zero.

$$F = -10 \times 2 + 20 = 0$$

$$y = y_m \sin(2t + \phi) + 2$$

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At $t = 0$, the particle is at $y = 3$ m and is at rest thus its extreme position is $y = 3$ m. The distance between the mean position and extreme position is 1 m. Thus, amplitude $y_m = 1$ m.

Thus, at $t = 0$

$$y = 1\sin\phi + 2 = 3 \text{ m}$$

$$\sin\phi = 1 \Rightarrow \phi = \frac{\pi}{2}$$

$$y = y_m \sin(2t + \phi) + 2 \qquad \left(\text{as } y_m = 1, \phi = \frac{\pi}{2}\right)$$

$$y = \cos 2t + 2 \qquad \text{(Answer)}$$

Learn: This particle is undergoing SHM about $y = 2$ m. Since information is given about initial position and velocity of the particle, we are able to find initial phase ϕ and amplitude y_m.

15.5 | ENERGY IN SIMPLE HARMONIC MOTION

Key Concept

- A particle in simple harmonic motion has, at any time, kinetic energy $K = 1/2mv^2$ and potential energy $U = 1/2kx^2$. If no friction is present, the mechanical energy $E = K + U$ remains constant even though K and U change.

We know that the energy in a linear oscillator transfers back and forth between kinetic energy and potential energy, while the sum of the two—the mechanical energy E of the oscillator—remains constant. The potential energy of a linear oscillator like that of Fig. 15-5 is associated entirely with the spring. Its value depends on how much the spring is stretched or compressed—that is, on $x(t)$. We can use Eqs. 8-52 and 15-6 to find

$$U(t) = \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2\cos^2(\omega t + \phi). \tag{15-30}$$

Caution: A function written in the form $\cos^2 A$ (as here) means $(\cos A)^2$ and is *not* the same as one written $\cos A^2$, which means $\cos(A^2)$.

The kinetic energy of the system of Fig. 15-5 is associated entirely with the block. Its value depends on how fast the block is moving—that is, on $v(t)$. We can use Eq. 15-9 to find

$$K(t) = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2x_m^2\sin^2(\omega t + \phi). \tag{15-31}$$

If we use Eq. 15-15 to substitute k/m for ω^2, we can write Eq. 15-31 as

$$K(t) = \frac{1}{2}mv^2 = \frac{1}{2}kx_m^2\sin^2(\omega t + \phi). \tag{15-32}$$

The mechanical energy follows from Eqs. 15-30 and 15-32 and is

$$\begin{aligned} E &= U + K \\ &= \frac{1}{2}kx_m^2\cos^2(\omega t + \phi) + \frac{1}{2}kx_m^2\sin^2(\omega t + \phi) \\ &= \frac{1}{2}kx_m^2[\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)]. \end{aligned}$$

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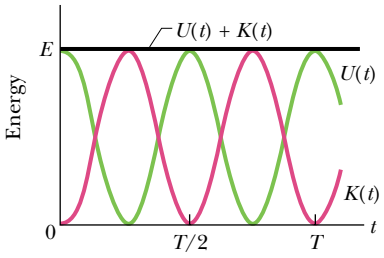
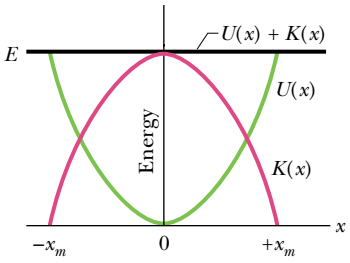


(a) As *time* changes, the energy shifts between the two types, but the total is constant.



(b) As *position* changes, the energy shifts between the two types, but the total is constant.

Figure 15-12 (*a*) Potential energy $U(t)$, kinetic energy $K(t)$, and mechanical energy E as functions of time t for a linear harmonic oscillator. Note that all energies are positive and that the potential energy and the kinetic energy peak twice during every period. (*b*) Potential energy $U(x)$, kinetic energy $K(x)$, and mechanical energy E as functions of position x for a linear harmonic oscillator with amplitude x_m. For $x = 0$ the energy is all kinetic, and for $x = \pm x_m$ it is all potential.

For any angle α,

$$\cos^2 \alpha + \sin^2 \alpha = 1.$$

Thus, the quantity in the square brackets above is unity and we have

$$E = U + K = \frac{1}{2} k x_m^2. \tag{15-33}$$

The mechanical energy of a linear oscillator is indeed constant and independent of time. The potential energy and kinetic energy of a linear oscillator are shown as functions of time t in Fig. 15-12*a* and as functions of displacement x in Fig. 15-12*b*. In any oscillating system, an element of springiness is needed to store the potential energy and an element of inertia is needed to store the kinetic energy.

CHECKPOINT 7

In Fig. 15-5, the block has a kinetic energy of 3 J and the spring has an elastic potential energy of 2 J when the block is at $x = +2.0$ cm. (a) What is the kinetic energy when the block is at $x = 0$? What is the elastic potential energy when the block is at (b) $x = -2.0$ cm and (c) $x = -x_m$?

SAMPLE PROBLEM 15.07

SHM potential energy, kinetic energy, mass dampers

Many tall buildings have *mass dampers*, which are anti-sway devices to prevent them from oscillating in a wind. The device might be a block oscillating at the end of a spring and on a lubricated track. If the building sways, say, eastward, the block also moves eastward but delayed enough so that when it finally moves, the building is then moving back westward. Thus, the motion of the oscillator is out of step with the motion of the building.

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Suppose the block has mass $m = 2.72 \times 10^5$ kg and is designed to oscillate at frequency $f = 10.0$ Hz and with amplitude $x_m = 20.0$ cm.

(a) What is the total mechanical energy E of the spring–block system?

KEY IDEA

The mechanical energy E (the sum of the kinetic energy $K = 1/2mv^2$ of the block and the potential energy $U = 1/2kx^2$ of the spring) is constant throughout the motion of the oscillator. Thus, we can evaluate E at any point during the motion.

Calculations: Because we are given amplitude x_m of the oscillations, let's evaluate E when the block is at position $x = x_m$, where it has velocity $v = 0$. However, to evaluate U at that point, we first need to find the spring constant k. From Eq. 15-15 ($\omega = \sqrt{k/m}$) and Eq. 15-5 ($\omega = 2\pi f$), we find

$$K = m\omega^2 = m(2\pi f)^2$$
$$= (2.72 \times 10^5 \text{ kg})(2\pi)^2 (10.0 \text{ Hz})^2$$
$$= 1.073 \times 10^9 \text{ N/m}.$$

We can now evaluate E as

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$
$$= 0 + \frac{1}{2}(1.073 \times 10^9 \text{ N/m})(0.20 \text{ m})^2$$
$$= 2.147 \times 10^7 \text{ J} \approx 2.1 \times 10^7 \text{ J}. \quad \text{(Answer)}$$

(b) What is the block's speed as it passes through the equilibrium point?

Calculations: We want the speed at $x = 0$, where the potential energy is $U = 1/2kx^2 = 0$ and the mechanical energy is entirely kinetic energy. So, we can write

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$
$$2.147 \times 10^7 \text{ J} = \frac{1}{2}(2.72 \times 10^5 \text{ kg})v^2 + 0,$$

or $$v = 12.6 \text{ m/s}. \quad \text{(Answer)}$$

Because E is entirely kinetic energy, this is the maximum speed v_m.

SAMPLE PROBLEM 15.08

Block attached to a spring undergoes SHM

A block of mass $4m$ is attached to a vertical spring of spring constant k. The block is made by gluing two blocks of mass $3m$ and m, respectively. Initially, the block is in equilibrium and at rest (Fig. 15-13*a*). At $t = 0$, the part of the block having mass $3m$ falls down. Considering hanging point on ceiling as $y = 0$ and downward direction as positive y, find y-coordinate of mass m as a function of time.

KEY IDEA

(1) As the original block of mass $4m$ is in equilibrium, the combined mass will be stretching the spring by $4\ mg/k$. Thus, the spring will be $4\ mg/k$ below the natural length. As the bigger block falls, the smaller block will experience a force larger than its weight in upward direction.

(2) The block remaining attached to the spring will be undergoing SHM. To find equation of SHM, we need to find angular frequency, amplitude, and initial phase

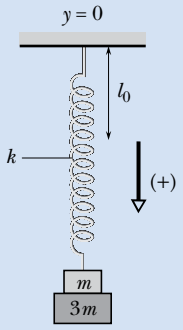

Figure15-13 (*a*) Two blocks, glued to each other hanging from the ceiling by a spring.

Calculations: When the extension in the spring is x, the net force on the block is

$$mg - kx = \frac{md^2x}{dt^2}$$

Again as $x = 0$ is not position of equilibrium, we have to shift origin to the position of equilibrium.

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Defining $x = (mg/k) + s$, as at $s = 0$, $x = mg/k$ is the position of equilibrium. Substituting for x in the equation we get

$$mg - k\left(\frac{mg}{k} + s\right) = \frac{md^2s}{dt^2}$$

On solving we get,

$$\frac{d^2s}{dt^2} = -\left(\frac{k}{m}\right)s$$

$$s = x_m \sin(\omega t + \phi) \qquad \left(\text{with } \omega = \sqrt{\frac{k}{m}}\right)$$

Now we have to find x_m. We have defined x_m as distance between $v = 0$ and $a = 0$. At $t = 0$, the block was at rest with extension of spring being 4 mg/k and at the new equilibrium position extension of spring is mg/k.

Thus, $$x_m = \frac{3mg}{k}$$

Now we have to find the initial phase. As the particle is at rest at $t = 0$ on positive side, the initial phase will be $\pi/2$.

$$\phi = \frac{\pi}{2}$$

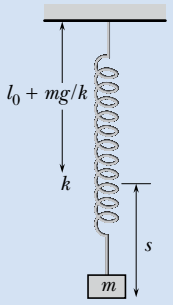

Figure 15-13 (b) At $t = 0$, one block falls down.

Thus, the y coordinate of the block

$$y = l_0 + \frac{mg}{k} + s$$

Substituting, all the values from above, we get

$$y = l_0 + \frac{mg}{k} + \frac{3mg}{k}\sin\left[\sqrt{\frac{k}{m}}t + \frac{\pi}{2}\right]$$

Note: At time $t = 0$, the distance of the block from the ceiling is $l_0 + 4\ mg/k$, which can be verified by putting $t = 0$ in the above equation.

SAMPLE PROBLEM 15.09

Motion of uniform bar, SHM, Angular frequency

In Fig. 15-14a, a uniform bar with mass m lies symmetrically across two rapidly rotating, fixed rollers, A and B, with distance $L = 2.0$ cm between the bar's center of mass and each roller. The rollers, whose directions of rotation are shown in the figure, slip against the bar with coefficient of kinetic friction $\mu_k = 0.40$. Suppose the bar is displaced horizontally by a distance x, as shown in Fig. 15-14b, and then released. What is the angular frequency ω of the resulting horizontal simple harmonic (back and forth) motion of the bar?

KEY IDEAS

(1) This situation seems very different from other SHM situations we have analyzed. However, the key idea here is that we can find ω if we find an expression for the horizontal acceleration a of the bar as a function of x and then compare it with $a = -\omega^2 x$.

(2) Since vertical forces (as shown in Fig. 15-14b), horizontal frictional forces, and torques all act on the bar, we shall apply Newton's second law both vertically and horizontally, and then in angular form about one of the contact points between bar and roller.

Calculations: The vertical forces acting on the bar are the gravitational force $\vec{F}_g$ (with magnitude mg) and supporting forces $\vec{F}_A$ due to roller A and $\vec{F}_B$ due to roller B. Since there is no vertical acceleration of the bar, Newton's second law for components along a vertical y axis ($F_{net,\,y} = ma_y$) gives us

$$F_A + F_B - mg = 0 \qquad (15\text{-}34)$$

The horizontal forces acting on the bar are the kinetic frictional forces $f_{kA} = \mu_k F_A$ (toward the right) due to roller A and $f_{kB} = \mu_k F_B$ (toward the left) due to roller B. Thus,

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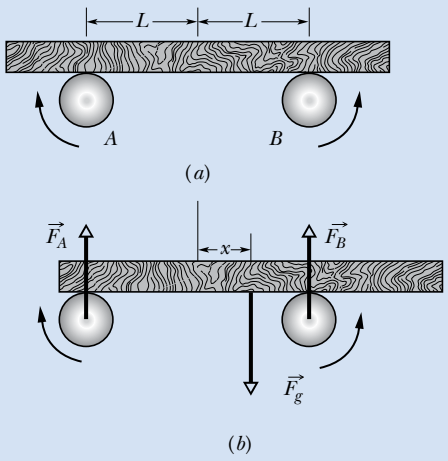

Figure 15-14 (*a*) A bar is in equilibrium on two rotating rollers, *A* and *B*, that slip beneath it. (*b*) The bar is displaced from equilibrium by a distance *x* and then released.

Newton's second law for components along a horizontal *x* axis ($F_{\text{net},x} = ma_x$) gives us

$$\mu_k F_A - \mu_k F_B = ma$$

$$a = \frac{\mu_k F_A - \mu_k F_B}{m} \qquad (15\text{-}35)$$

We now consider torques about an axis perpendicular to the plane of Fig. 15-14 and through the contact point between the bar and roller *A*. The bar experiences no angular acceleration about that axis, so Newton's second law for torque about such an axis ($\tau_{\text{net}} = I\alpha$) gives us

$$F_A(0) + F_B 2L - mg(L+x) + f_{kA}(0) + f_{kB}(0) = 0 \qquad (15\text{-}36)$$

where the forces $\vec{F}_A$, $\vec{F}_B$, $m\vec{g}$, $\vec{f}_{kA}$, and $\vec{f}_{kB}$ have moment arms about that axis of 0, $2L$, $L + x$, 0 and 0, respectively. Solving Eq. 15-36 for F_B and then solving Eq. 15-34 for F_A, we find

$$F_B = \frac{mg(L+x)}{2L} \quad \text{and} \quad F_A = \frac{mg(L-x)}{2L}$$

Substituting these results into Eq. 15-35 yields

$$a = \frac{\mu_k g}{L} x \qquad (15\text{-}37)$$

Comparison of Eq. 15-37 with $a = -\omega^2 x$ reveals that the bar must be undergoing SHM with angular frequency ω given by

$$\omega^2 = \frac{\mu_k g}{L}$$

Thus,

$$\omega = \sqrt{\frac{\mu_k g}{L}} = \sqrt{\frac{(0.40)(9.8\ \text{m/s}^2)}{0.020\ \text{m}}}$$

$$= 14\ \text{rad/s}$$

Learn: Another important point is that the rollers will be experiencing kinetic friction which can slow them down. There must be some external torque on the rollers to maintain the motion of the rollers.

15.6 | AN ANGULAR SIMPLE HARMONIC OSCILLATOR

Key Concept

- A torsion pendulum consists of an object suspended on a wire. When the wire is twisted and then released, the object oscillates in angular simple harmonic motion with a period given by

$$T = 2\pi\sqrt{\frac{I}{\kappa}},$$

where I is the rotational inertia of the object about the axis of rotation and κ is the torsion constant of the wire.

Figure 15-15 shows an angular version of a simple harmonic oscillator; the element of springiness or elasticity is associated with the twisting of a suspension wire rather than the extension and compression of a spring as we previously had. The device is called a **torsion pendulum**, with *torsion* referring to the twisting.

If we rotate the disk in Fig. 15-12 by some angular displacement θ from its rest position (where the reference line is at $\theta = 0$) and release it, it will oscillate about that position in **angular simple harmonic motion**. Rotating the disk through an angle θ in either direction introduces a restoring torque given by

$$\tau = -\kappa\theta. \qquad (15\text{-}38)$$

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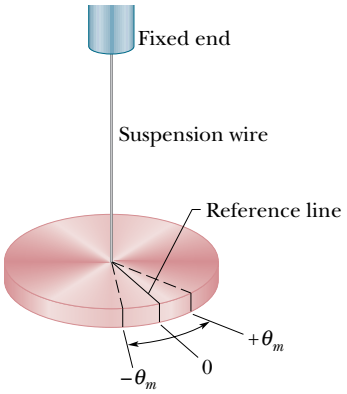

Figure 15-15 A torsion pendulum is an angular version of a linear simple harmonic oscillator. The disk oscillates in a horizontal plane; the reference line oscillates with angular amplitude θ_m. The twist in the suspension wire stores potential energy as a spring does and provides the restoring torque.

Here κ (Greek *kappa*) is a constant, called the **torsion constant**, that depends on the length, diameter, and material of the suspension wire.

Comparison of Eq. 15-38 with Eq. 15-13 leads us to suspect that Eq. 15-38 is the angular form of Hooke's law, and that we can transform Eq. 15-16, which gives the period of linear SHM, into an equation for the period of angular SHM: We replace the spring constant k in Eq. 15-16 with its equivalent, the constant κ of Eq. 15-38, and we replace the mass m in Eq. 15-38 with *its* equivalent, the rotational inertia I of the oscillating disk. These replacements lead to

$$T = 2\pi\sqrt{\frac{I}{\kappa}} \quad \text{(torsion pendulum)}. \tag{15-39}$$

SAMPLE PROBLEM 15.10

Angular simple harmonic oscillator, rotational inertia, period

Figure 15-16*a* shows a thin rod whose length L is 12.4 cm and whose mass m is 135 g, suspended at its midpoint from a long wire. Its period T_a of angular SHM is measured to be 2.53 s. An irregularly shaped object, which we call object X, is then hung from the same wire, as in Fig. 15-13*b*, and its period T_b is found to be 4.76 s. What is the rotational inertia of object X about its suspension axis?

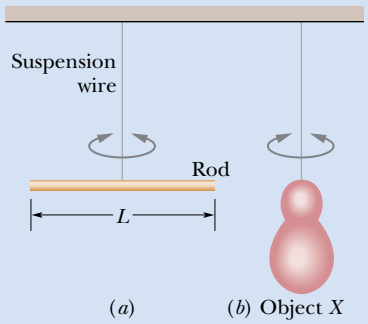

Figure 15-16 Two torsion pendulums, consisting of (*a*) a wire and a rod and (*b*) the same wire and an irregularly shaped object.

KEY IDEA

The rotational inertia of either the rod or object X is related to the measured period by Eq. 15-39.

Calculations: In Table 10-2*e*, the rotational inertia of a thin rod about a perpendicular axis through its midpoint is given as $1/12mL^2$. Thus, we have, for the rod in Fig. 15-16*a*,

$$I_a = \frac{1}{12}mL^2 = \left(\frac{1}{12}\right)(0.135\ \text{kg})(0.124\ \text{m})^2$$
$$= 1.73\times10^{-4}\ \text{kg}\cdot\text{m}^2.$$

Now let us write Eq. 15-39 twice, once for the rod and once for object X:

$$T_a = 2\pi\sqrt{\frac{I_a}{\kappa}} \quad \text{and} \quad T_b = 2\pi\sqrt{\frac{I_b}{\kappa}}.$$

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The constant κ, which is a property of the wire, is the same for both figures; only the periods and the rotational inertias differ.

Let us square each of these equations, divide the second by the first, and solve the resulting equation for I_b. The result is

$$I_b = I_a \frac{T_b^2}{T_a^2} = (1.73 \times 10^{-4}\ \text{kg} \cdot \text{m}^2)\frac{(4.76\ \text{s})^2}{(2.53\ \text{s})^2}$$
$$= 6.12 \times 10^{-4}\ \text{kg} \cdot \text{m}^2. \qquad \text{(Answer)}$$

15.7 | PENDULUMS

Key Concepts

- A simple pendulum consists of a rod of negligible mass that pivots about its upper end, with a particle (the bob) attached at its lower end. If the rod swings through only small angles, its motion is approximately simple harmonic motion with a period given by

$$T = 2\pi\sqrt{\frac{I}{mgL}} \quad \text{(simple pendulum)},$$

  where I is the particle's rotational inertia about the pivot, m is the particle's mass, and L is the rod's length.

- A physical pendulum has a more complicated distribution of mass. For small angles of swinging, its motion is simple harmonic motion with a period given by

$$T = 2\pi\sqrt{\frac{I}{mgh}} \quad \text{(physical pendulum)},$$

  where I is the pendulum's rotational inertia about the pivot, m is the pendulum's mass, and h is the distance between the pivot and the pendulum's center of mass.

We turn now to a class of simple harmonic oscillators in which the springiness is associated with the gravitational force rather than with the elastic properties of a twisted wire or a compressed or stretched spring.

The Simple Pendulum

If an apple swings on a long thread, does it have simple harmonic motion? If so, what is the period T? To answer, we consider a **simple pendulum**, which consists of a particle of mass m (called the *bob* of the pendulum) suspended from one end of an unstretchable, massless string of length L that is fixed at the other end, as in Fig. 15-17*a*. The bob is free to swing back and forth in the plane of the page, to the left and right of a vertical line through the pendulum's pivot point.

The Restoring Torque. The forces acting on the bob are the force $\vec{T}$ from the string and the gravitational force $\vec{F}_g$, as shown in Fig. 15-17*b*, where the string makes an angle θ with the vertical. We resolve $\vec{F}_g$ into a radial component $F_g \cos\theta$ and a component $F_g \sin\theta$ that is tangent to the path taken by the bob. This tangential component produces a restoring torque about the pendulum's pivot point because the component always acts opposite the displacement of the bob so as to bring the bob back toward its central location. That location is called the *equilibrium position* ($\theta = 0$) because the pendulum would be at rest there were it not swinging.

From Eq. 10-45, ($\tau = r_\perp F$), we can write this restoring torque as

$$\tau = -L\ (F_g \sin\theta), \qquad (15\text{-}40)$$

where the minus sign indicates that the torque acts to reduce θ and L is the moment arm of the force component $F_g \sin\theta$ about the pivot point. Substituting Eq. 15-40 into Eq. 10-64 ($\tau = I\alpha$) and then substituting mg as the magnitude of F_g, we obtain

$$-L\ (mg \sin\theta) = I\alpha, \qquad (15\text{-}41)$$

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where I is the pendulum's rotational inertia about the pivot point and α is its angular acceleration about that point.

We can simplify Eq. 15-41 if we assume the angle θ is small, for then we can approximate $\sin\theta$ with θ (expressed in radian measure). (As an example, if $\theta = 5.00° = 0.0873$ rad, then $\sin\theta = 0.0872$, a difference of only about 0.1%.) With that approximation and some rearranging, we then have

$$\alpha = -\frac{mgL}{I}\theta. \tag{15-42}$$

This equation is the angular equivalent of Eq. 15-11, the hallmark of SHM. It tells us that the angular acceleration α of the pendulum is proportional to the angular displacement θ but opposite in sign. Thus, as the pendulum bob moves to the right, as in Fig. 15-17*a*, its acceleration *to the left* increases until the bob stops and begins moving to the left. Then, when it is to the left of the equilibrium position, its acceleration to the right tends to return it to the right, and so on, as it swings back and forth in SHM. More precisely, the motion of a *simple pendulum swinging through only small angles* is approximately SHM. We can state this restriction to small angles another way: The **angular amplitude** θ_m of the motion (the maximum angle of swing) must be small.

Angular Frequency. Here is a neat trick. Because Eq. 15-42 has the same form as Eq. 15-11 for SHM, we can immediately identify the pendulum's angular frequency as being the square root of the constants in front of the displacement:

$$\omega = \sqrt{\frac{mgL}{I}}.$$

In the "problems" section, you might see oscillating systems that do not seem to resemble pendulums. However, if you can relate the acceleration (linear or angular) to the displacement (linear or angular), you can then immediately identify the angular frequency as we have just done here.

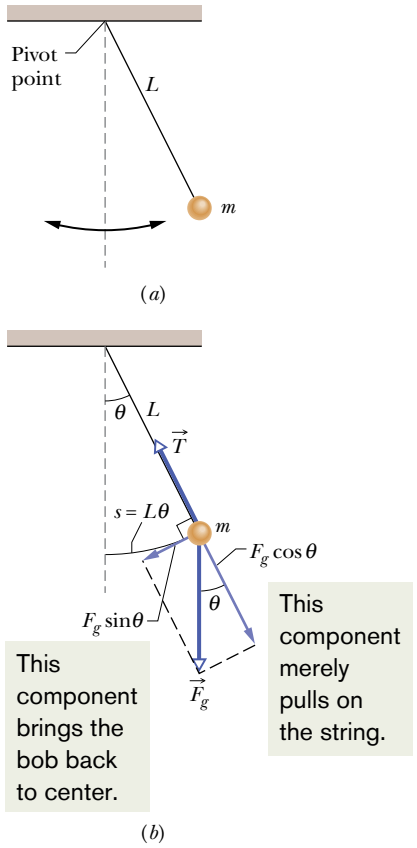

Figure 15-17 (*a*) A simple pendulum. (*b*) The forces acting on the bob are the gravitational force $\vec{F}_g$ and the force $\vec{T}$ from the string. The tangential component $F_g \sin\theta$ of the gravitational force is a restoring force that tends to bring the pendulum back to its central position.

Period. Next, if we substitute this expression for ω into Eq. 15-8 ($\omega = 2\pi/T$), we see that the period of the pendulum may be written as

$$T = 2\pi\sqrt{\frac{I}{mgL}}. \tag{15-43}$$

All the mass of a simple pendulum is concentrated in the mass m of the particle like bob, which is at radius L from the pivot point. Thus, we can use Eq. 10-33 ($I = mr^2$) to write $I = mL^2$ for the rotational inertia of the pendulum. Substituting this into Eq. 15-31 and simplifying then yield

$$T = 2\pi\sqrt{\frac{L}{g}} \quad \text{(simple pendulum, small amplitude).} \tag{15-44}$$

We assume small-angle swinging in this chapter.

The Physical Pendulum

A real pendulum, usually called a **physical pendulum**, can have a complicated distribution of mass. Does it also undergo SHM? If so, what is its period?

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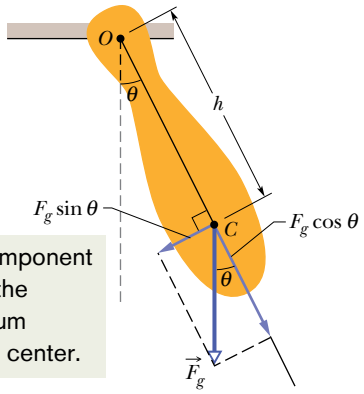

Figure 15-18 A physical pendulum. The restoring torque is $hF_g \sin\theta$. When $\theta = 0$, center of mass C hangs directly below pivot point O.

Figure 15-18 shows an arbitrary physical pendulum displaced to one side by angle θ. The gravitational force $\vec{F}_g$ acts at its center of mass C, at a distance h from the pivot point O. Comparison of Figs. 15-18 and 15-17*b* reveals only one important difference between an arbitrary physical pendulum and a simple pendulum. For a physical pendulum the restoring component $F_g \sin\theta$ of the gravitational force has a moment arm of distance h about the pivot point, rather than of string length L. In all other respects, an analysis of the physical pendulum would duplicate our analysis of the simple pendulum up through Eq. 15-43. Again (for small θ_m), we would find that the motion is approximately SHM.

If we replace L with h in Eq. 15-43, we can write the period as

$$T = 2\pi\sqrt{\frac{I}{mgh}} \quad \text{(physical pendulum, small amplitude).} \quad (15\text{-}45)$$

As with the simple pendulum, I is the rotational inertia of the pendulum about O. However, now I is not simply mL^2 (it depends on the shape of the physical pendulum), but it is still proportional to m.

A physical pendulum will not swing if it pivots at its center of mass. Formally, this corresponds to putting $h = 0$ in Eq. 15-45. That equation then predicts $T \to \infty$, which implies that such a pendulum will never complete one swing.

Corresponding to any physical pendulum that oscillates about a given pivot point O with period T is a simple pendulum of length L_0 with the same period T. We can find L_0 with Eq. 15-44. The point along the physical pendulum at distance L_0 from point O is called the *center of oscillation* of the physical pendulum for the given suspension point.

CHECKPOINT 8

Three physical pendulums, of masses m_0, $2m_0$, and $3m_0$, have the same shape and size and are suspended at the same point. Rank the masses according to the periods of the pendulums, greatest first.

SAMPLE PROBLEM 15.11

Physical pendulum, period and length

In Fig. 15-19*a*, a meter stick swings about a pivot point at one end, at distance h from the stick's center of mass.

(a) What is the period of oscillation T?

KEY IDEA

The stick is not a simple pendulum because its mass is not concentrated in a bob at the end opposite the pivot point—so the stick is a physical pendulum.

Calculations: The period for a physical pendulum is given by Eq. 15-45, for which we need the rotational inertia I of the stick about the pivot point. We can treat the stick as a uniform rod of length L and mass m. Then $I = 1/3mL^2$, and the distance h in Eq. 15-45 is $1/2L$. Substituting these quantities into Eq. 15-45, we find

$$T = 2\pi\sqrt{\frac{I}{mgh}} = 2\pi\sqrt{\frac{\frac{1}{3}mL^2}{mg(\frac{1}{2}L)}} \quad (15\text{-}46)$$

$$= 2\pi\sqrt{\frac{2L}{3g}} \quad (15\text{-}47)$$

$$= 2\pi\sqrt{\frac{(2)(1.00\text{ m})}{(3)(9.8\text{ m/s}^2)}} = 1.64\text{ s}. \quad \text{(Answer)}$$

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Note the result is independent of the pendulum's mass m.

(b) What is the distance L_0 between the pivot point O of the stick and the center of oscillation of the stick?

Calculations: We want the length L_0 of the simple pendulum (drawn in Fig. 15-19*b*) that has the same period as the physical pendulum (the stick) of Fig. 15-19*a*. Setting Eqs. 15-44 and 15-47 equal yields

$$T = 2\pi\sqrt{\frac{L_0}{g}} = 2\pi\sqrt{\frac{2L}{3g}}. \qquad (15\text{-}48)$$

You can see by inspection that

$$L_0 = \frac{2}{3}L \qquad (15\text{-}49)$$

$$= \left(\frac{2}{3}\right)(100\text{ cm}) = 66.7\text{ cm}. \qquad (\text{Answer})$$

In Fig. 15-19*a*, point P marks this distance from suspension point O. Thus, point P is the stick's center of oscillation for the given suspension point. Point P would be different for a different suspension choice.

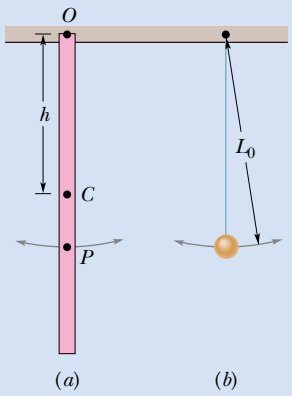

Figure 15-19 (*a*) A meter stick suspended from one end as a physical pendulum. (*b*) A simple pendulum whose length L_0 is chosen so that the periods of the two pendulums are equal. Point P on the pendulum of (*a*) marks the center of oscillation.

15.8 | DAMPED SIMPLE HARMONIC MOTION

Key Concepts

- The mechanical energy E in a real oscillating system decreases during the oscillations because external forces, such as a drag force, inhibit the oscillations and transfer mechanical energy to thermal energy. The real oscillator and its motion are then said to be damped.
- If the damping force is given by $\vec{F}_d = -d\vec{v}$, where $\vec{v}$ is the velocity of the oscillator and b is a damping constant, then the displacement of the oscillator is given by

$$x(t) = x_m e^{-bt/2m}\cos(\omega' t + \phi),$$

where ω', the angular frequency of the damped oscillator, is given by

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}.$$

- If the damping constant is small ($b \ll \sqrt{km}$), then $\omega' \approx \omega$, where ω is the angular frequency of the undamped oscillator. For small b, the mechanical energy E of the oscillator is given by

$$E(t) \approx \frac{1}{2}kx_m^2 e^{-bt/m}.$$

A pendulum will swing only briefly underwater, because the water exerts on the pendulum a drag force that quickly eliminates the motion. A pendulum swinging in air does better, but still the motion dies out eventually, because the air exerts a drag force on the pendulum (and friction acts at its support point), transferring energy from the pendulum's motion.

When the motion of an oscillator is reduced by an external force, the oscillator and its motion are said to be **damped**. An idealized example of a damped oscillator is shown in Fig. 15-20, where a block with mass m oscillates vertically on a spring with spring constant k. From the block, a rod extends to a vane (both assumed massless) that is submerged in a liquid. As the vane moves up and down, the liquid exerts an inhibiting drag force on it and thus on the entire oscillating system. With time, the mechanical energy of the block–spring system decreases, as energy is transferred to thermal energy of the liquid and vane.

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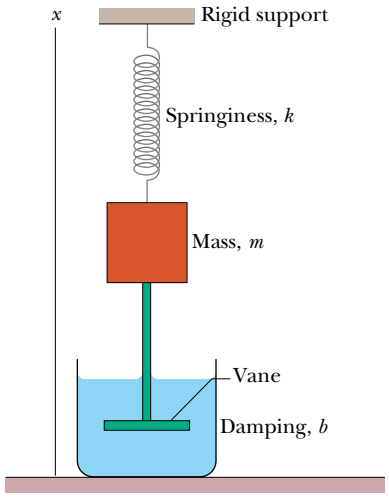

Figure 15-20 An idealized damped simple harmonic oscillator. A vane immersed in a liquid exerts a damping force on the block as the block oscillates parallel to the x axis.

Let us assume the liquid exerts a **damping force** $\vec{F}_d$ that is proportional to the Velocity $\vec{v}$ of the vane and block (an assumption that is accurate if the vane moves slowly). Then, for force and velocity components along the x axis in Fig. 15-20, we have

$$F_d = -bv, \tag{15-50}$$

where b is a **damping constant** that depends on the characteristics of both the vane and the liquid and has the SI unit of kilogram per second. The minus sign indicates that $\vec{F}_d$ opposes the motion.

Damped Oscillations. The force on the block from the spring is $F_s = -kx$. Let us assume that the gravitational force on the block is negligible relative to F_d and F_s. Then we can write Newton's second law for components along the x axis ($F_{\text{net},x} = ma_x$) as

$$-bv - kx = ma. \tag{15-51}$$

Substituting dx/dt for v and d^2x/dt^2 for a and rearranging give us the differential equation

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0. \tag{15-52}$$

The solution of this equation is

$$x(t) = x_m e^{-bt/2m}\cos(\omega' t + \phi), \tag{15-53}$$

where x_m is the amplitude and ω' is the angular frequency of the damped oscillator. This angular frequency is given by

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}. \tag{15-54}$$

If $b = 0$ (there is no damping), then Eq. 15-54 reduces to Eq. 15-15 ($\omega = \sqrt{k/m}$) for the angular frequency of an undamped oscillator, and Eq. 15-53 reduces to Eq. 15-6 for the displacement of an undamped oscillator. If the damping constant is small but not zero (so that $b \ll \sqrt{km}$), then $\omega' \approx \omega$.

Damped Energy. We can regard Eq. 15-53 as a cosine function whose amplitude, which is $x_m e^{-bt/2m}$, gradually decreases with time, as Fig. 15-21 suggests. For an undamped oscillator, the mechanical energy is constant and is given by Eq. 15-33 ($E = 1/2kx_m^2$). If the oscillator is damped, the mechanical energy is not constant but decreases

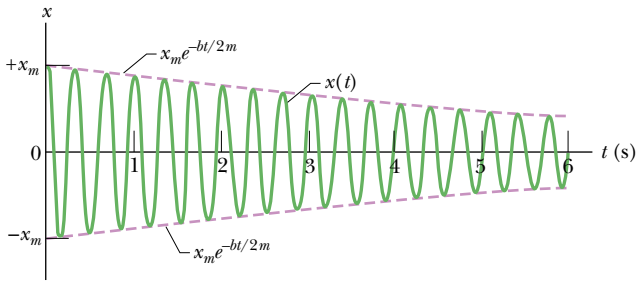

Figure 15-21 The displacement function $x(t)$ for the damped oscillator of Fig. 15-20. The amplitude, which is $x_m e^{-bt/2m}$, decreases exponentially with time.

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with time. If the damping is small, we can find $E(t)$ by replacing x_m in Eq. 15-33 with $x_m e^{-bt/2m}$, the amplitude of the damped oscillations. By doing so, we find that

$$E(t) \approx \frac{1}{2} k x_m^2 e^{-bt/m}, \tag{15-55}$$

which tells us that, like the amplitude, the mechanical energy decreases exponentially with time.

CHECKPOINT 9

Here are three sets of values for the spring constant, damping constant, and mass for the damped oscillator of Fig. 15-17. Rank the sets according to the time required for the mechanical energy to decrease to one-fourth of its initial value, greatest first.

Set 1	$2k_0$	b_0	m_0
Set 2	k_0	$6b_0$	$4m_0$
Set 3	$3k_0$	$3b_0$	m_0

SAMPLE PROBLEM 15.12

Damped harmonic oscillator, time to decay, energy

For the damped oscillator of Fig. 15-20, m = 250 g, k = 85 N/m, and b = 70 g/s.

(a) What is the period of the motion?

KEY IDEA

Because $b \ll \sqrt{km} = 4.6$ kg/s, the period is approximately that of the undamped oscillator.

Calculation: From Eq. 15-16, we then have

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.25\text{ kg}}{85\text{ N/m}}} = 0.34\text{ s}. \quad \text{(Answer)}$$

(b) How long does it take for the amplitude of the damped oscillations to drop to half its initial value?

KEY IDEA

The amplitude at time t is displayed in Eq. 15-53 as $x_m e^{-bt/2m}$.

Calculations: The amplitude has the value x_m at t = 0. Thus, we must find the value of t for which

$$x_m e^{-bt/2m} = \frac{1}{2} x_m.$$

Canceling x_m and taking the natural logarithm of the equation that remains, we have ln 1/2 on the right side and

$$\ln(e^{-bt/2m}) = -bt/2m$$

on the left side. Thus,

$$t = \frac{-2m \ln \frac{1}{2}}{b} = \frac{-(2)(0.25\text{ kg})(\ln \frac{1}{2})}{0.070\text{ kg/s}}$$
$$= 5.0\text{ s}. \quad \text{(Answer)}$$

Because T = 0.34 s, this is about 15 periods of oscillation.

(c) How long does it take for the mechanical energy to drop to one-half its initial value?

KEY IDEA

From Eq. 15-55, the mechanical energy at time t is $1/2kx_m^2 e^{-bt/m}$.

Calculations: The mechanical energy has the value $1/2kx_m^2$ at t = 0. Thus, we must find the value of t for which

$$\frac{1}{2} k x_m^2 e^{-bt/m} = \frac{1}{2}\left(\frac{1}{2} k x_m^2\right).$$

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If we divide both sides of this equation by $1/2kx_m^2$ and solve for t as we did above, we find

$$t = \frac{-m \ln \frac{1}{2}}{b} = \frac{-(0.25\ \text{kg})(\ln \frac{1}{2})}{0.070\ \text{kg/s}} = 2.5\ \text{s}. \quad \text{(Answer)}$$

This is exactly half the time we calculated in (b), or about 7.5 periods of oscillation. Figure 15-21 was drawn to illustrate this sample problem.

15.9 | FORCED OSCILLATIONS AND RESONANCE

Key Concepts

- If an external driving force with angular frequency ω_d acts on an oscillating system with natural angular frequency ω, the system oscillates with angular frequency ω_d.
- The velocity amplitude v_m of the system is greatest when

$$\omega_d = \omega,$$

a condition called resonance. The amplitude x_m of the system is (approximately) greatest under the same condition.

A person swinging in a swing without anyone pushing it is an example of *free oscillation*. However, if someone pushes the swing periodically, the swing has *forced,* or *driven, oscillations*. *Two* angular frequencies are associated with a system undergoing driven oscillations: (1) the *natural* angular frequency ω of the system, which is the angular frequency at which it would oscillate if it were suddenly disturbed and then left to oscillate freely, and (2) the angular frequency ω_d of the external driving force causing the driven oscillations.

We can use Fig. 15-20 to represent an idealized forced simple harmonic oscillator if we allow the structure marked "rigid support" to move up and down at a variable angular frequency ω_d. Such a forced oscillator oscillates at the angular frequency ω_d of the driving force, and its displacement $x(t)$ is given by

$$x(t) = x_m \cos(\omega_d t + \phi), \tag{15-56}$$

where x_m is the amplitude of the oscillations.

How large the displacement amplitude x_m is depends on a complicated function of ω_d and ω. The velocity amplitude v_m of the oscillations is easier to describe: it is greatest when

$$\omega_d = \omega \quad \text{(resonance)}, \tag{15-57}$$

a condition called **resonance**. Equation 15-57 is also *approximately* the condition at which the displacement amplitude x_m of the oscillations is greatest. Thus, if you push a swing at its natural angular frequency, the displacement and velocity amplitudes will increase to large values, a fact that children learn quickly by trial and error. If you push at other angular frequencies, either higher or lower, the displacement and velocity amplitudes will be smaller.

Figure 15-22 shows how the displacement amplitude of an oscillator depends on the angular frequency ω_d of the driving force, for three values of the damping coefficient b. Note that for all three the amplitude is approximately greatest when $\omega_d/\omega = 1$ (the resonance condition of Eq. 15-57). The curves of Fig. 15-22 show that less damping gives a taller and narrower *resonance peak*.

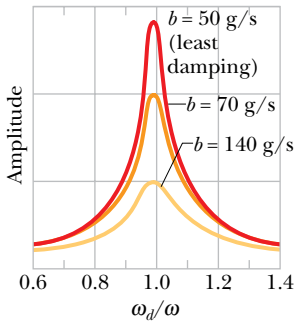

Figure 15-22 The displacement amplitude x_m of a forced oscillator varies as the angular frequency ω_d of the driving force is varied. The curves here correspond to three values of the damping constant b.

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REVIEW AND SUMMARY

Frequency The *frequency* f of periodic, or oscillatory, motion is the number of oscillations per second. In the SI system, it is measured in hertz:

$$1 \text{ hertz} = 1 \text{ Hz} = 1 \text{ oscillation per second} = 1/\text{s}. \quad (15\text{-}4)$$

Period The *period* T is the time required for one complete oscillation, or **cycle**. It is related to the frequency by

$$T = \frac{1}{f}. \quad (15\text{-}5)$$

Simple Harmonic Motion In *simple harmonic motion* (SHM), the displacement $x(t)$ of a particle from its equilibrium position is described by the equation

$$x = x_m \cos(\omega t + \phi) \quad \text{(displacement)}, \quad (15\text{-}6)$$

in which x_m is the **amplitude** of the displacement, $\omega t + \phi$ is the **phase** of the motion, and ϕ is the **phase constant**. The **angular frequency** ω is related to the period and frequency of the motion by

$$\omega = \frac{2\pi}{T} = 2\pi f \quad \text{(angular frequency)}. \quad (15\text{-}8)$$

Differentiating Eq. 15-6 leads to equations for the particle's SHM velocity and acceleration as functions of time:

$$v = -\omega x_m \sin(\omega t + \phi) \quad \text{(velocity)} \quad (15\text{-}9)$$

and

$$a = -\omega^2 x_m \cos(\omega t + \phi) \quad \text{(acceleration)}. \quad (15\text{-}10)$$

In Eq. 15-9, the positive quantity ωx_m is the **velocity amplitude** v_m of the motion. In Eq. 15-10, the positive quantity $\omega^2 x_m$ is the **acceleration amplitude** a_m of the motion.

The Linear Oscillator A particle with mass m that moves under the influence of a Hooke's law restoring force given by $F = -kx$ exhibits simple harmonic motion with

$$\omega = \sqrt{\frac{k}{m}} \quad \text{(angular frequency)} \quad (15\text{-}15)$$

and

$$T = 2\pi\sqrt{\frac{m}{k}} \quad \text{(period)}. \quad (15\text{-}16)$$

Such a system is called a **linear simple harmonic oscillator**.

Simple Harmonic Motion and Uniform Circular Motion Simple harmonic motion is the projection of uniform circular motion onto the diameter of the circle in which the circular motion occurs. Figure 15-17 shows that all parameters of circular motion (position, velocity, and acceleration) project to the corresponding values for simple harmonic motion.

Energy A particle in simple harmonic motion has, at any time, kinetic energy $K = 1/2mv^2$ and potential energy $U = 1/2kx^2$. If no friction is present, the mechanical energy $E = K + U$ remains constant even though K and U change.

Pendulums Examples of devices that undergo simple harmonic motion are the **torsion pendulum** of Fig. 15-15, the **simple pendulum** of Fig. 15-17, and the **physical pendulum** of Fig. 15-18. Their periods of oscillation for small oscillations are, respectively,

$$T = 2\pi\sqrt{I/\kappa} \quad \text{(torsion pendulum)}, \quad (15\text{-}39)$$

$$T = 2\pi\sqrt{L/g} \quad \text{(simple pendulum)}, \quad (15\text{-}44)$$

$$T = 2\pi\sqrt{I/mgh} \quad \text{(physical pendulum)}. \quad (15\text{-}45)$$

Damped Harmonic Motion The mechanical energy E in a real oscillating system decreases during the oscillations because external forces, such as a drag force, inhibit the oscillations and transfer mechanical energy to thermal energy. The real oscillator and its motion are then said to be **damped**. If the **damping force** is given by $\vec{F}_d = -b\vec{v}$, where $\vec{v}$ is the velocity of the oscillator and b is a **damping constant**, then the displacement of the oscillator is given by

$$x(t) = x_m e^{-bt/2m} \cos(\omega' t + \phi), \quad (15\text{-}53)$$

where ω', the angular frequency of the damped oscillator, is given by

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}. \quad (15\text{-}54)$$

If the damping constant is small ($b \ll \sqrt{km}$), then $\omega' \approx \omega$, where ω is the angular frequency of the undamped oscillator. For small b, the mechanical energy E of the oscillator is given by

$$E(t) \approx \frac{1}{2} k x_m^2 e^{-bt/m}. \quad (15\text{-}55)$$

Forced Oscillations and Resonance If an external driving force with angular frequency ω_d acts on an oscillating system with *natural* angular frequency ω, the system oscillates with angular frequency ω_d. The velocity amplitude v_m of the system is greatest when

$$\omega_d = \omega, \quad (15\text{-}57)$$

a condition called **resonance**. The amplitude x_m of the system is (approximately) greatest under the same condition.

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PROBLEMS

1. In Fig. 15-23, two identical springs of spring constant 7580 N/m are attached to a block of mass 0.270 kg. What is the frequency of oscillation on the frictionless floor?

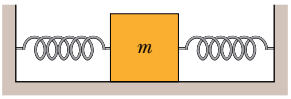

Figure 15-23 Problems 1 and 3.

2. Hanging from a horizontal beam are nine simple pendulums of the following lengths: (a) 0.10, (b) 0.30, (c) 0.70, (d) 0.80, (e) 1.2, (f) 2.6, (g) 3.5, (h) 5.0, and (i) 6.2 m. Suppose the beam undergoes horizontal oscillations with angular frequencies in the range from 2.00 rad/s to 4.00 rad/s. Which of the pendulums will be (strongly) set in motion?

3. In Fig. 15-23, two springs are attached to a block that can oscillate over a frictionless floor. If the left spring is removed, the block oscillates at a frequency of 30 Hz. If, instead, the spring on the right is removed, the block oscillates at a frequency of 50 Hz. At what frequency does the block oscillate with both springs attached?

4. Figure 15-24 gives the one-dimensional potential energy well for a 4.0 kg particle (the function $U(x)$ has the form bx^2 and the vertical axis scale is set by $U_s = 2.0$ J). (a) If the particle passes through the equilibrium position with a velocity of 85 cm/s, will it be turned back before it reaches $x = 15$ cm? (b) If yes, at what position, and if no, what is the speed of the particle at $x = 15$ cm?

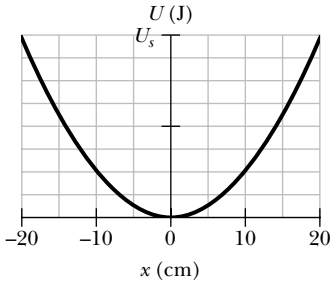

Figure 15-24 Problem 4.

5. An oscillator consists of a block of mass 0.500 kg connected to a spring. When set into oscillation with amplitude 35.0 cm, the oscillator repeats its motion every 0.350 s. Find the (a) period, (b) frequency, (c) angular frequency, (d) spring constant, (e) maximum speed, and (f) magnitude of the maximum force on the block from the spring.

6. In Fig. 15-25, a 5.00 kg disk of diameter $D = 42.0$ cm is supported by a rod of length $L = 76.0$ cm and negligible mass that is pivoted at its end. (a) With the massless torsion spring unconnected, what is the period of oscillation? (b) With the torsion spring connected, the rod is vertical at equilibrium. What is the torsion constant of the spring if the period of oscillation has been decreased by 0.500 s?

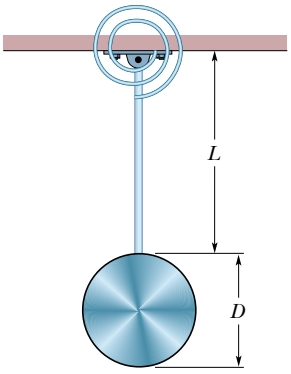

Figure 15-25 Problem 6.

7. In Fig. 15-26, the pendulum consists of a uniform disk with radius $r = 10.0$ cm and mass 500 g attached to a uniform rod with length $L = 500$ mm and mass 250 g. (a) Calculate the rotational inertia of the pendulum about the pivot point. (b) What is the distance between the pivot point and the center of mass of the pendulum? (c) Calculate the period of oscillation.

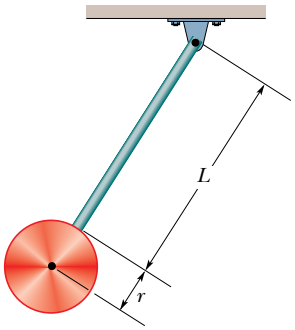

Figure 15-26 Problem 7.

8. A physical pendulum has a center of oscillation at distance $2L/3$ from its point of suspension. Show that the distance between the point of suspension and the center of oscillation for a physical pendulum of any form is I/mh, where I and h have the meanings in Eq. 15-43 and m is the mass of the pendulum.

9. In Fig. 15-27, a block weighing 14.0 N, which can slide without friction on an incline at angle $\theta = 40.0°$, is connected to the top of the incline by a massless spring of unstretched length 0.450 m and spring constant 135 N/m. (a) How far from the top of the incline is the block's equilibrium point? (b) If the block is pulled slightly down the incline and released, what is the period of the resulting oscillations?

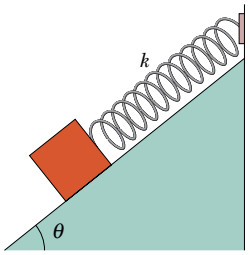

Figure 15-27 Problem 9.

10. A block of mass $M = 5.4$ kg, at rest on a horizontal frictionless table, is attached to a rigid support by a spring of constant $k = 6000$ N/m. A bullet of mass $m = 9.5$ g and velocity $\vec{v}$ of magnitude 680 m/s strikes and is embedded

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in the block (Fig. 15-28). Assuming the compression of the spring is negligible until the bullet is embedded, determine (a) the speed of the block immediately after the collision and (b) the amplitude of the resulting simple harmonic motion.

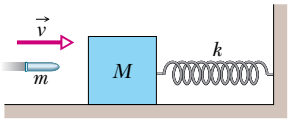

Figure 15-28 Problem 10.

11. In Fig. 15-29, two springs are joined and connected to a block of mass 0.490 kg that is set oscillating over a frictionless floor. The springs each have spring constant k = 5000 N/m. What is the frequency of the oscillations?

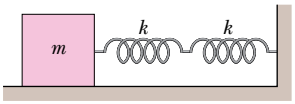

Figure 15-29 Problem 11.

12. In an electric shaver, the blade moves back and forth over a distance of 2.0 mm in simple harmonic motion, with frequency 100 Hz. Find (a) the amplitude, (b) the maximum blade speed, and (c) the magnitude of the maximum blade acceleration.

13. A rectangular block, with face lengths a = 35 cm and b = 45 cm, is to be suspended on a thin horizontal rod running through a narrow hole in the block. The block is then to be set swinging about the rod like a pendulum, through small angles so that it is in SHM. Figure 15-30 shows one possible position of the hole, at distance r from the block's center, along a line connecting the center with a corner. (a) Plot the period versus distance r along that line such that the minimum in the curve is apparent. (b) For what value of r does that minimum occur? There is a line of points around the block's center for which the period of swinging has the same minimum value. (c) What shape does that line make?

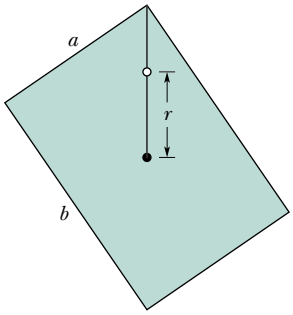

Figure 15-30 Problem 13.

14. A massless spring hangs from the ceiling with a small object attached to its lower end. The object is initially held at rest in a position y_i such that the spring is at its rest length. The object is then released from y_i and oscillates up and down, with its lowest position being 10 cm below y_i. (a) What is the frequency of the oscillation? (b) What is the speed of the object when it is 8.0 cm below the initial position? (c) An object of mass 600 g is attached to the first object, after which the system oscillates with half the original frequency. What is the mass of the first object? (d) How far below y_i is the new equilibrium (rest) position with both objects attached to the spring?

15. We arrange for two blocks to undergo simple harmonic motion along adjacent, parallel paths with amplitude A. What is their phase difference if they pass each other in opposite directions whenever their displacement is $A/2$?

16. Find the mechanical energy of a block–spring system with a spring constant of 1.8 N/cm and an amplitude of 2.4 cm.

17. An oscillating block–spring system has a mechanical energy of 2.00 J, an amplitude of 10.0 cm, and a maximum speed of 0.800 m/s. Find (a) the spring constant, (b) the mass of the block, and (c) the frequency of oscillation.

18. An automobile can be considered to be mounted on four identical springs as far as vertical oscillations are concerned. The springs of a certain car are adjusted so that the oscillations have a frequency of 3.00 Hz. (a) What is the spring constant of each spring if the mass of the car is 2110 kg and the mass is evenly distributed over the springs? (b) What will be the oscillation frequency if five passengers, averaging 85.0 kg each, ride in the car with an even distribution of mass?

19. The position function $x = (6.0\text{ m})\cos[(3\pi\text{ rad/s})\,t + \pi/3\text{ rad}]$ gives the simple harmonic motion of a body. At t = 2.1 s, what are the (a) displacement, (b) velocity, (c) acceleration, and (d) phase of the motion? Also, what are the (e) frequency and (f) period of the motion?

20. A 75 kg solid sphere with a 10 cm radius is suspended by a vertical wire. A torque of 0.35 N · m is required to rotate the sphere through an angle of 0.85 rad and then maintain that orientation. What is the period of the oscillations that result when the sphere is then released?

21. Two particles oscillate in simple harmonic motion along a common straight-line segment of length A. Each particle has a period of 1.5 s, but they differ in phase by $\pi/6$ rad. (a) How far apart are they (in terms of A) 0.60 s after the lagging particle leaves one end of the path? (b) Are they then moving in the same direction, toward each other, or away from each other?

22. Figure 15-31 shows block 1 of mass 0.200 kg sliding to the right over a frictionless elevated surface at a speed of 10.0 m/s. The block undergoes an elastic collision with stationary block 2, which is attached to a spring of spring constant 1208.5 N/m. (Assume that the spring does not affect the collision.) After the collision, block 2 oscillates in SHM with a period of 0.140 s, and block 1 slides off the opposite end of the elevated surface, landing a distance d from the base of that surface after falling height h = 6.20 m. What is the value of d?

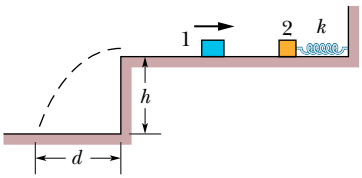

Figure 15-31 Problem 22.

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23. A 10 g particle undergoes SHM with an amplitude of 2.0 mm, a maximum acceleration of magnitude 6.5×10^3 m/s^2, and an unknown phase constant ϕ. What are (a) the period of the motion, (b) the maximum speed of the particle, and (c) the total mechanical energy of the oscillator? What is the magnitude of the force on the particle when the particle is at (d) its maximum displacement and (e) half its maximum displacement?

24. A physical pendulum consists of a meter stick that is pivoted at a small hole drilled through the stick a distance d from the 50 cm mark. The period of oscillation is 5.0 s. Find d.

25. Figure 15-32*a* is a partial graph of the position function $x(t)$ for a simple harmonic oscillator with an angular frequency of 1.00 rad/s; Fig. 15-32*b* is a partial graph of the corresponding velocity function $v(t)$. The vertical axis scales are set by $x_s = 5.00$ cm and $v_s = 10.0$ cm/s. What is the phase constant of the SHM if the position function $x(t)$ is in the general form $x = x_m \cos(\omega t + \phi)$?

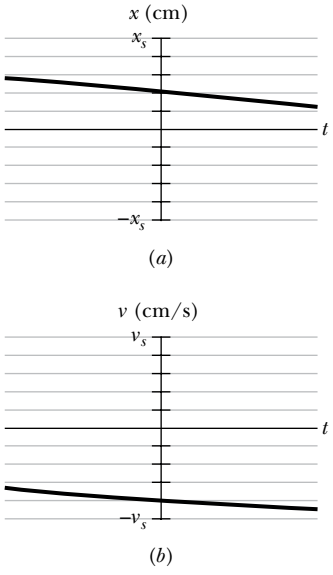

Figure 15-32 Problem 25.

26. In Fig. 15-33, a stick of length $L = 1.65$ m oscillates as a physical pendulum. (a) What value of distance x between the stick's center of mass and its pivot point O gives the least period? (b) What is that least period?

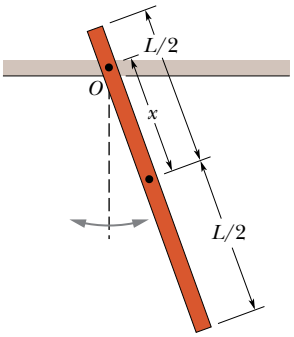

Figure 15-33 Problem 26.

27. What is the phase constant for the harmonic oscillator with the velocity function $v(t)$ given in Fig. 15-34 if the position function $x(t)$ has the form $x = x_m \cos(\omega t + \phi)$? The vertical axis scale is set by $v_s = 8.0$ cm/s.

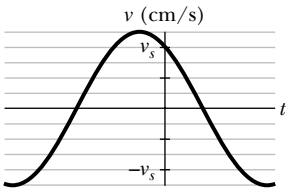

Figure 15-34 Problem 27.

28. (a) What is the oscillation amplitude of a 4.00 kg box oscillating on a spring with spring constant 100 N/m if at time $t = 1.00$ s the position is $x = 0.129$ m and the velocity is $v = 5.00$ m/s? At $t = 0$, what are (a) the position and (b) the velocity?

29. A 5.00 kg object on a horizontal frictionless surface is attached to a spring with $k = 1000$ N/m. The object is displaced from equilibrium 40.0 cm horizontally and given an initial velocity of 10.0 m/s back toward the equilibrium position. What are (a) the motion's frequency, (b) the initial potential energy of the block–spring system, (c) the initial kinetic energy, and (d) the motion's amplitude?

30. In Fig. 15-35, two blocks ($m = 1.8$ kg and $M = 5.0$ kg) and a spring ($k = 200$ N/m) are arranged on a horizontal, frictionless surface. The coefficient of static friction between the two blocks is 0.40. What amplitude of simple harmonic motion of the spring–blocks system puts the smaller block on the verge of slipping over the larger block?

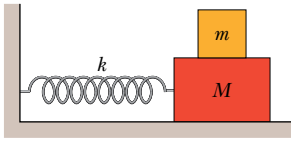

Figure 15-35 Problem 30.

31. What is the maximum acceleration of a platform that oscillates at amplitude 2.50 cm and frequency 6.60 Hz?

32. What is the phase constant for the harmonic oscillator with the position function $x(t)$ given in Fig. 15-36 if the position function has the form $x = x_m \cos(\omega t + \phi)$? The vertical axis scale is set by $x_s = 9.0$ cm.

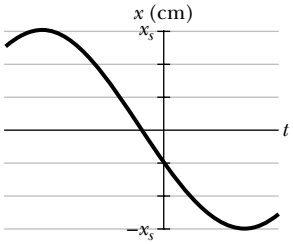

Figure 15-36 Problem 32.

33. At a certain harbor, the tides cause the ocean surface to rise and fall a distance d (from highest level to lowest level) in simple harmonic motion, with a period of 12.5 h. How long does it take for the water to fall a distance $0.15d$ from its highest level?

34. A body undergoes simple harmonic motion of amplitude 4.25 cm and period 0.200 s. The magnitude of the maximum force acting on it is 10.0 N. (a) What is the mass? (b) If the oscillations are produced by a spring, what is the spring constant?

35. A thin uniform rod (mass = 0.90 kg) swings about an axis that passes through one end of the rod and is perpendicular to the plane of the swing. The rod swings with a period of 1.5 s and an angular amplitude of 10°. (a) What is the length of the rod? (b) What is the maximum kinetic energy of the rod as it swings?

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36. In the overhead view of Fig. 15-37, a long uniform rod of mass 0.600 kg is free to rotate in a horizontal plane about a vertical axis through its center. A spring with force constant $k = 1530$ N/m is connected horizontally between one end of the rod and a fixed wall. When the rod is in equilibrium, it is parallel to the wall. What is the period of the small oscillations that result when the rod is rotated slightly and released?

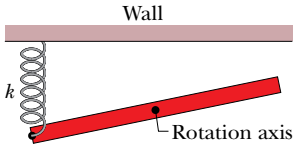

Figure 15-37 Problem 36.

37. For the damped oscillator system shown in Fig. 15-20, with $m = 250$ g, $k = 85$ N/m, and $b = 70$ g/s, what is the ratio of the oscillation amplitude at the end of 10 cycles to the initial oscillation amplitude?

38. A pendulum is formed by pivoting a long thin rod about a point on the rod. In a series of experiments, the period is measured as a function of the distance x between the pivot point and the rod's center. (a) If the rod's length is $L = 2.20$ m and its mass is $m = 22.1$ g, what is the minimum period? (b) If x is chosen to minimize the period and then L is increased, does the period increase, decrease, or remain the same? (c) If, instead, m is increased without L increasing, does the period increase, decrease, or remain the same?

39. The 0.800 kg cube in Fig. 15-38 has edge lengths $d = 6.00$ cm and is mounted on an axle through its center. A spring ($k = 3600$ N/m) connects the cube's upper corner to a rigid wall. Initially the spring is at its rest length. If the cube is rotated 3° and released, what is the period of the resulting SHM?

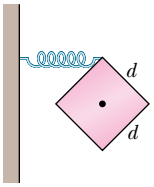

Figure 15-38 Problem 39.

40. Figure 15-39 shows the kinetic energy K of a simple harmonic oscillator versus its position x. The vertical axis scale is set by $K_s = 8.0$ J. What is the spring constant?

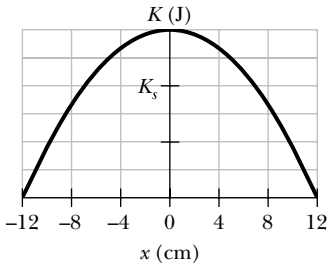

Figure 15-39 Problem 40.

41. An object undergoing simple harmonic motion takes 0.25 s to travel from one point of zero velocity to the next such point. The distance between those points is 32 cm. Calculate the (a) period, (b) frequency, and (c) amplitude of the motion.

42. If the phase angle for a block–spring system in SHM is $\pi/8$ rad and the block's position is given by $x = x_m \cos(\omega t + \phi)$, what is the ratio of the kinetic energy to the potential energy at time $t = 0$?

43. An oscillator consists of a block attached to a spring ($k = 425$ N/m). At some time t, the position (measured from the system's equilibrium location), velocity, and acceleration of the block are $x = 0.100$ m, $v = -13.6$ m/s, and $a = -123$ m/s^2. Calculate (a) the frequency of oscillation, (b) the mass of the block, and (c) the amplitude of the motion.

44. An oscillating block–spring system takes 0.25 s to begin repeating its motion. Find (a) the period, (b) the frequency in hertz, and (c) the angular frequency in radians per second.

45. A particle with a mass of 3.00×10^{-20} kg is oscillating with simple harmonic motion with a period of 2.00×10^{-5} s and a maximum speed of 1.00×10^{3} m/s. Calculate (a) the angular frequency and (b) the maximum displacement of the particle.

46. When the displacement in SHM is 0.40 times the amplitude x_m, what fraction of the total energy is (a) kinetic energy and (b) potential energy? (c) At what displacement, in terms of the amplitude, is the energy of the system half kinetic energy and half potential energy?

47. In Fig. 15-40, block 2 of mass 2.0 kg oscillates on the end of a spring in SHM with a period of 14.1 ms. The block's position is given by $x = (1.0 \text{ cm}) \cos(\omega t + \pi/2)$. Block 1 of mass 4.0 kg slides toward block 2 with a velocity of magnitude 6.0 m/s, directed along the spring's length. The two blocks undergo a completely inelastic collision at time $t = 5.0$ ms. (The duration of the collision is much less than the period of motion.) What is the amplitude of the SHM after the collision?

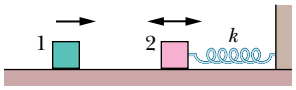

Figure 15-40 Problem 47.

48. A physical pendulum consists of two meter-long sticks joined together as shown in Fig. 15-41. What is the pendulum's period of oscillation about a pin inserted through point A at the center of the horizontal stick?

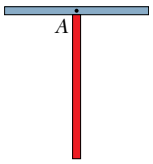

Figure 15-41 Problem 48.

49. A block rides on a piston (a squat cylindrical piece) that is moving vertically with simple harmonic motion. (a) If the SHM has period 1.0 s, at what amplitude of motion will the block and piston separate? (b) If the piston has an amplitude of 3.0 cm, what is the maximum frequency for which the block and piston will be in contact continuously?

50. Suppose that a simple pendulum consists of a small 60.0 g bob at the end of a cord of negligible mass. If the angle θ between the cord and the vertical is given by

$$\theta = (0.0800 \text{ rad}) \cos[(6.80 \text{ rad/s})t + \phi],$$

what are (a) the pendulum's length and (b) its maximum kinetic energy?

51. A block is on a horizontal surface (a shake table) that is moving back and forth horizontally with simple harmonic motion of frequency 2.0 Hz. The coefficient of static friction between block and surface is 0.45. How great can the amplitude of the SHM be if the block is not to slip along the surface?

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PRACTICE QUESTIONS

Single Correct Choice Type

1. Which one of the following statements is true concerning an object executing simple harmonic motion?
- **(a)** The object's velocity is never zero.
- **(b)** The object's acceleration is never zero.
- **(c)** The object's velocity and acceleration are simultaneously zero.
- **(d)** The object's velocity is zero when its acceleration is a maximum.

2. Resonance occurs in harmonic motion when
- **(a)** the system is overdamped.
- **(b)** the system is critically damped.
- **(c)** the energy in the system is a minimum.
- **(d)** the driving frequency is the same as the natural frequency of the system.

3. A simple pendulum has length L and period T. As it passes through its equilibrium position, the string is suddenly clamped at its midpoint. The period then becomes
- **(a)** $2T$
- **(b)** T
- **(c)** $\frac{T}{2}$
- **(d)** none of these

4. An object of mass m, oscillating on the end of a spring with spring constant k has amplitude A. Its maximum speed is
- **(a)** $A\sqrt{k/m}$
- **(b)** A^2k/m
- **(c)** $A\sqrt{m/k}$
- **(d)** Am/k

5. The velocity of a certain simple harmonic oscillator is given by $v = -(12\text{ m/s})\sin[(6.0\text{ rad/s})t]$. What is the amplitude of the simple harmonic motion?
- **(a)** 2.0 m
- **(b)** 4.0 m
- **(c)** 6.0 m
- **(d)** 8.0 m

6. A 0.20 kg object mass attached to a spring whose spring constant is 500 N/m executes simple harmonic motion. If its maximum speed is 5.0 m/s, the amplitude of its oscillation is
- **(a)** 0.0020 m
- **(b)** 0.10 m
- **(c)** 0.20 m
- **(d)** 25 m

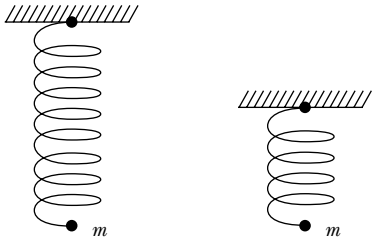
7. A ball hung from a vertical spring oscillates in simple harmonic motion with an angular frequency of 2.6 rad/s and an amplitude of 0.075 m. What is the maximum acceleration of the ball?
- **(a)** 0.13 m/s^2
- **(b)** 0.20 m/s^2
- **(c)** 0.51 m/s^2
- **(d)** 2.6 m/s^2

8. A 1.2 kg mass is oscillating without friction on a spring whose spring constant is 3400 N/m. When the mass's displacement is 7.2 cm, what is its acceleration?
- **(a)** –3.8 m/s^2
- **(b)** –200 m/s^2
- **(c)** –240 m/s^2
- **(d)** -2.0×10^4 m/s^2

9. The displacement of an object oscillating on a spring is given by $x(t) = x_m\cos(\omega t + \phi)$. If the initial displacement is zero and the initial velocity is in the negative x direction, then the phase constant ϕ is
- **(a)** 0 rad
- **(b)** $\frac{\pi}{2}$ rad
- **(c)** π rad
- **(d)** $\frac{3\pi}{2}$ rad

10. The displacement of an object oscillating on a spring is given by $x(t) = x_m\cos(\omega t + \phi)$. If the object is initially displaced in the negative x direction and given a negative initial velocity, then the phase constant ϕ is between
- **(a)** 0 and $\frac{\pi}{2}$ rad
- **(b)** $\frac{\pi}{2}$ and π rad
- **(c)** π and $\frac{3\pi}{2}$ rad
- **(d)** $\frac{3\pi}{2}$ and 2π rad

11. A simple harmonic oscillator consists of a mass m and an ideal spring with spring constant k. The particle oscillates as shown in (i) with period T. If the spring is cut into half and used with the same particle, as shown in (ii), the period will be



(i) (ii)

- **(a)** $2T$
- **(b)** $\sqrt{2}T$
- **(c)** $T/\sqrt{2}$
- **(d)** T

12. A 0.25-kg block oscillates on the end of the spring with a spring constant of 200 N/m. If the oscillation is started by elongating the spring 0.15 m and giving the block a speed of 3.0 m/s, then the maximum speed of the block is
- **(a)** 0.13 m/s
- **(b)** 0.18 m/s
- **(c)** 3.7 m/s
- **(d)** 5.2 m/s

13. The amplitude of oscillation of a simple pendulum is increased from 1° to 4°. Its maximum acceleration changes by a factor of
- **(a)** $\frac{1}{4}$
- **(b)** $\frac{1}{2}$
- **(c)** 2
- **(d)** 4

14. A particle undergoes damped harmonic motion. The spring constant is 100 N/m; the damping constant is 8.0×10^{-3} kg m/s, and the mass is 0.050 kg. If the particle starts at its maximum displacement, $x = 1.5$ m, at time $t = 0$, what is the particle's position at $t = 5.0$ s?
- **(a)** –1.5 m
- **(b)** –0.73 m
- **(c)** 0 m
- **(d)** 0.73 m

15. Five particles undergo damped harmonic motion. Values for the spring constant k, the damping constant b, and the mass m are given below. Which leads to the smallest rate of loss of mechanical energy?
- **(a)** $k = 100$ N/m, $m = 50$ g, $b = 8$ g m/s
- **(b)** $k = 150$ N/m, $m = 50$ g, $b = 5$ g m/s
- **(c)** $k = 150$ N/m, $m = 10$ g, $b = 8$ g m/s
- **(d)** $k = 200$ N/m, $m = 8$ g, $b = 6$ g m/s

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16. A particle is oscillating according to the equation $X = 7\cos(0.5\pi t)$, where t is in second. The point moves from the position of equilibrium to maximum displacement in time
(a) 4.0 s **(b)** 2.0 s
(c) 1.0 s **(d)** 0.5 s

17. An archer pulls the bowstring back for a distance of 0.470 m before releasing the arrow. The bow and string act like a spring whose spring constant is 425 N/m. (a) What is the elastic potential energy of the drawn bow?
(a) 93.8 J **(b)** 49.9 J
(c) 46.9 J **(d)** 61.8 J

18. A simple pendulum is made from a 0.65-m-long string and a small ball attached to its free end. The ball is pulled to one side through a small angle and then released from rest. After the ball is released, how much time elapses before it attains its greatest speed?
(a) 0.40 s **(b)** 0.10 s
(c) 0.80 s **(d)** 0.20 s

19. The length of a simple pendulum is 0.79 m and the mass of the particle (the "bob") at the end of the cable is 0.24 kg. The pendulum is pulled away from its equilibrium position by an angle of 8.50° and released from rest. Assume that friction can be neglected and that the resulting oscillatory motion is simple harmonic motion. What is the angular frequency of the motion?
(a) 9.1 rad/s
(b) 12 rad/s
(c) 8.2 rad/s
(d) 3.5 rad/s

20. A block is attached to a horizontal spring and oscillates back and forth on a frictionless horizontal surface at a frequency of 3.00 Hz. The amplitude of the motion is 5.08×10^{-2} m. At the point where the block has its maximum speed, it suddenly splits into two identical parts, only one part remaining attached to the spring. What is the amplitude and the frequency of the simple harmonic motion that exists after the block splits?
(a) 5.08×10^{-2} m; 4.24 Hz
(b) 3.59×10^{-2} m; 4.24 Hz
(c) 2.54×10^{-2} m; 3.00 Hz
(d) 3.59×10^{-2} m; 6.00 Hz

21. A copper rod (length = 2.0 m, radius = 3.0×10^{-3} m) hangs down from the ceiling. A 9.0 kg object is attached to the lower end of the rod. The rod acts as a "spring" and the object oscillates vertically with a small amplitude. Ignoring the rod's mass, find the frequency f of the simple harmonic motion.
(a) 66 Hz **(b)** 13 Hz
(c) 93 Hz **(d)** 37 Hz

22. Two physical pendulums (not simple pendulums) are made from meter sticks that are suspended from the ceiling at one end. They are identical, except that one is made of wood and the other of metal. They are set into oscillation and execute simple harmonic motion. The wood and metal pendulums have masses of 0.17 and 0.85 kg, respectively. Determine the period of the wood pendulum.
(a) 1.64 s **(b)** 1.98 s
(c) 1.02 s **(d)** 1.35 s

23. The position of a simple harmonic oscillator is given by $x(t) = (0.50\text{ m})\cos\left(\frac{\pi}{3}t\right)$, where t is measured in seconds. What is the maximum velocity of this oscillator?
(a) 0.17 m/s **(b)** 0.52 m/s
(c) 0.67 m/s **(d)** 1.0 m/s

24. The acceleration of a certain simple harmonic oscillator is given by $a = -(15.8\text{ m/s}^2)\cos(2.51t)$. What is the amplitude of the simple harmonic motion?
(a) 2.51 m **(b)** 4.41 m
(c) 6.30 m **(d)** 11.1 m

25. A 1.0-kg block oscillates with a frequency of 10 Hz at the end of a certain spring. The spring is then cut into two halves. The 1.0-kg block is then made to oscillate at the end of one of the halves. What is the frequency of oscillation of the block?
(a) 5 Hz **(b)** 10 Hz
(c) 14 Hz **(d)** 20 Hz

26. A simple harmonic oscillator with a period of 2.0 s is subject to damping so that it loses one percent of its amplitude per cycle. About how much energy does this oscillator lose per cycle?
(a) 0.5% **(b)** 1.0%
(c) 2.0% **(d)** 3.0%

27. A pendulum is transported from sea-level, where the acceleration due to gravity g is 9.80 m/s^2, to the bottom of Death Valley. At this location, the period of the pendulum is decreased by 3.00%. What is the value of g in Death Valley?
(a) 9.22 m/s^2 **(b)** 9.51 m/s^2
(c) 9.80 m/s^2 **(d)** 10.4 m/s^2

28. In a certain clock, a pendulum of length L_1 has a period $T_1 = 0.95$ s. The length of the pendulum is adjusted to a new value L_2 such that $T_2 = 1.0$ s. What is the ratio L_2/L_1?
(a) 0.90 **(b)** 0.95
(c) 1.0 **(d)** 1.1

29. A simple pendulum consists of a ball of mass m suspended from the ceiling using a string of length L. The ball is displaced from its equilibrium position by a small angle θ. What is the magnitude of the restoring force that moves the ball toward its equilibrium position and produces simple harmonic motion?
(a) kx **(b)** mg
(c) $mg(\cos\theta)$ **(d)** $mg(\sin\theta)$

30. An iron ball hangs from a 21.5-m steel cable and is used in the demolition of a building at a location where the acceleration due to gravity is 9.78 m/s^2. The ball is swung outward from its equilibrium position for a distance of 4.20 m. Assuming the system behaves as a simple pendulum, find the maximum speed of the ball during its swing.
(a) 4.45 m/s **(b)** 2.83 m/s
(c) 17.8 m/s **(d)** 1.71 m/s

31. For a particle undergoing SHM, the displacement x is related to time t as $x = x_m \cos\omega$. In the graphs shown in the figure, its potential energy (P.E.) is plotted against t (1st graph) and x (second graph). Two options are shown in each graph, marked as 1 and 2 in the first graph and 3 and 4 in the second graph. Which of these are correct?

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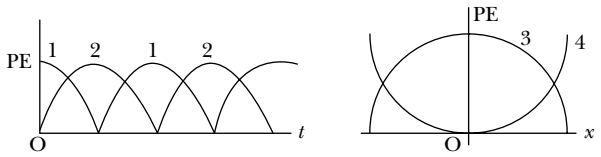


(a) 1 and 3 **(b)** 2 and 4
(c) 2 and 3 **(d)** 1 and 4

32. The motion of a particle is given by $x = A\sin\omega t + B\cos\omega t$. The motion of the particle is
(a) not simple harmonic.
(b) simple harmonic with amplitude $A + B$.
(c) simple harmonic with amplitude $(A + B)/2$.
(d) simple harmonic with amplitude $\sqrt{A^2 + B^2}$.

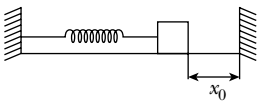
33. A particle executes simple harmonic motion with a frequency f. The frequency with which the kinetic energy oscillates is
(a) $f/2$ **(b)** f
(c) $2f$ **(d)** Zero

34. A thin uniform rod of length l is pivoted at its upper end. It is free to swing in a vertical plane. Its time period for oscillations of small amplitude is
(a) $2\pi\sqrt{l/g}$ **(b)** $2\pi\sqrt{2l/3g}$
(c) $2\pi\sqrt{3l/2g}$ **(d)** $2\pi\sqrt{l/2g}$

35. A particle moves on the x-axis according to the equation $x = x_0\sin^2\omega t$. The motion is simple harmonic
(a) with amplitude x_0. **(b)** with amplitude $2x_0$.
(c) with time period $2\pi/\omega$. **(d)** with time period π/ω.

36. This question consists of Statements 1 and 2. Of the four choices given, choose the one that best describes the two statements.

A block is attached with a spring and placed on a smooth horizontal surface. Spring is not stretched initially. Block is displaced in left direction by Δx and released from rest. Assume elastic collisions with the wall.



Statement 1: Time period of oscillation of block depends upon initial displacement (Δx) of block along left direction, for all values of Δx.

Statement 2: If initial displacement (Δx) along left direction increases, $(\Delta x > x_0)$, time period of oscillation decreases.
(a) Statement 1 is true, Statement 2 is true; Statement 2 is the correct explanation of Statement 1.
(b) Statement 1 is true, Statement 2 is true; Statement 2 is not the correct explanation of Statement 1.
(c) Statement 1 is true, Statement 2 is false.
(d) Statement 1 is false, Statement 2 is true.

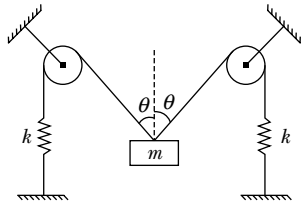
37. This question consists of Statements 1 and 2. Of the four choices given, choose the one that best describes the two statements.

Statement 1: While crossing a bridge, soldiers are asked to break steps.

Statement 2: When natural frequency of an oscillating system equals frequency of external impulse, its amplitude of oscillation may become very high.
(a) Statement 1 is true, Statement 2 is true; Statement 2 is the correct explanation of Statement 1.
(b) Statement 1 is true, Statement 2 is true; Statement 2 is not the correct explanation of Statement 1.
(c) Statement 1 is true, Statement 2 is false.
(d) Statement 1 is false, Statement 2 is true.

38. Two springs of spring constant 6 N/m and 4 N/m are attached to a block of mass 1 kg and to fixed support. Then time period of the given oscillation is
(Consider $\sqrt{10} = \pi$.)
(a) 1 s **(b)** 2 s
(c) 0.5 s **(d)** 4 s

39. Time period of vertical oscillation of the block in the system shown in the figure is



(a) $2\pi\sec\theta\sqrt{\frac{m}{2k}}$ **(b)** $2\pi\cos\theta\sqrt{\frac{m}{2k}}$
(c) $2\pi\sin\theta\sqrt{\frac{m}{2k}}$ **(d)** $2\pi\,\text{cosec}\,\theta\sqrt{\frac{m}{2k}}$

More than One Correct Choice Type

40. A particle moves in the xy plane according to the equation

$$\vec{r} = (\hat{i} + 2\hat{j})A\cos\omega t$$

The motion of the particle is
(a) on a straight line. **(b)** on an ellipse.
(c) periodic. **(d)** simple harmonic.

41. An object is released from rest. The time it takes to fall through a distance h and the speed of the object as it falls through this distance are measured with a pendulum clock. The entire apparatus is taken on the Moon and the experiment is repeated. Then,
(a) the measured times are same.
(b) the measured speeds are same.
(c) the actual times in the fall are equal.
(d) the actual speeds are equal.

42. Which of the following quantities are always negative in a simple harmonic motion?
(a) $\vec{F}\cdot\vec{a}$ **(b)** $\vec{v}\cdot\vec{r}$
(c) $\vec{a}\cdot\vec{r}$ **(d)** $\vec{F}\cdot\vec{r}$

43. The amplitude of a particle executing SHM about point O is 10 cm. Then,
(a) when the kinetic energy is 0.64 of its maximum kinetic energy, its displacement is 6 cm from point O.

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(b) when the displacement is 5 cm from point O, its kinetic energy is 0.75 of its maximum potential energy.
(c) its total energy at any point is equal to its maximum kinetic energy.
(d) its velocity is half the maximum velocity when its displacement is half the maximum displacement.

Linked Comprehension

Paragraph for Questions 44 and 45: A (hypothetical) large slingshot is stretched 2.30 m to launch a 170 g projectile with speed sufficient to escape from Earth (11.2 km/s). Assume the elastic bands of the slingshot obey Hooke's law.

44. What is the spring constant of the device if all the elastic potential energy is converted to kinetic energy?
(a) 4.03×10^6 N/m **(b)** 2.02×10^6 N/m
(c) 8.06×10^6 N/m **(d)** 6.09×10^6 N/m

45. Assume that an average person can exert a force of 490 N. How many people are required to stretch the elastic bands?
(a) 9.5×10^3 **(b)** 1.89×10^4
(c) 3.78×10^4 **(d)** 2.85×10^4

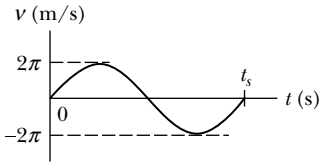
Paragraph for Questions 46—48: A 2.0-kg block executes SHM while attached to a horizontal spring of spring constant 200 N/m. The maximum speed of the block as it slides on a horizontal frictionless surface is 3.0 m/s.

46. What is the amplitude of the block's motion?
(a) 0.15 m **(b)** 0.45 m
(c) 0.30 m **(d)** 0.60 m

47. What is the magnitude of its maximum and minimum accelerations?
(a) 30 m/s^2; 15 m/s^2 **(b)** 30 m/s^2; 5 m/s^2
(c) 15 m/s^2; 0 m/s^2 **(d)** 30 m/s^2; 0 m/s^2

48. How long does the block take to complete 7.0 cycles of its motion?
(a) 4.0 s **(b)** 4.4 s
(c) 3.8 s **(d)** 3.5 s

Paragraph for Questions 49–51: A simple harmonic oscillator consists of a block attached to a spring with k = 200 N/m. The block slides on a frictionless surface, with equilibrium point $x = 0$ and amplitude 0.20 m. A graph of the block's velocity v as a function of time t is shown in the figure. The horizontal scale is set by $t_s = 0.20$ s.



49. What is the period of the SHM and mass of the block?
(a) 0.20 s and 0.20 kg **(b)** 0.40 s and 0.20 kg
(c) 0.20 s and 0.40 kg **(d)** 0.40 s and 0.40 kg

50. What is its displacement and acceleration at $t = 0$?
(a) 0.20 m and 1.0×10^2 m/s^2
(b) 0.15 m and -1.0×10^2 m/s^2
(c) -0.20 m and -2.0×10^2 m/s^2
(d) -0.15 m and 2.0×10^2 m/s^2

51. What is its maximum kinetic energy?
(a) 3.0 J **(b)** 4.0 J
(c) 5.0 J **(d)** 6.0 J

Paragraph for Questions 52—54: A block weighing 10.0 N is attached to the lower end of a vertical spring (k = 200.0 N/m), the other end of which is attached to a ceiling. The block oscillates vertically and has a kinetic energy of 2.00 J as it passes through the point at which the spring is unstretched.

52. What is the period of the oscillation?
(a) 0.45 s **(b)** 0.90 s
(c) 0.15 s **(d)** 0.303 s

53. Use the law of conservation of energy to determine the maximum distance the block moves both above and below the point at which the spring is unstretched (these are not necessarily the same)
(a) 0.05 m and 0.25 m **(b)** 0.10 m and 0.20 m
(c) 0.15 m and 0.15 m **(d)** 0.13 m and 0.17 m

54. What is the amplitude of the oscillation and the maximum kinetic energy of the block as it oscillates?
(a) 0.20 m and 4.50 J **(b)** 0.30 m and 3.35 J
(c) 0.25 m and 3.25 J **(d)** 0.15 m and 2.25 J

Matrix-Match

Direction for Question 55: Match the statements in Column I with the statements in Column II. One or more than one choice from Column II can match with a statement from Column I.

55.

Column I	Column II
(a) Velocity–displacement graph	**(p)** Straight line
(b) Acceleration–velocity graph	**(q)** Circle
(c) Acceleration–displacement	**(r)** Ellipse
(d) Acceleration–time graph	**(s)** Sine curve
	(t) Cosine curve

Directions for Question 56 and 57: There is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

56. In the given table, Column I gives type of motion, Column II gives the path traced and Column III gives other characteristic of the motion.

Column I	Column II	Column III
(I) Back and forth type of motion	**(i)** along a circular path	**(J)** after regular intervals of time
(II) A body or a moving particle that changes its motion	**(ii)** restoring force is proportional to displacement from the mean position	**(K)** does not oppose the increase in motion

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Column I	Column II	Column III
(III) A body or a moving particle that repeats its motion	**(iii)** need not be periodic	**(L)** need not have fixed extreme positions
(IV) Motion in which	**(iv)** along a definite path	**(M)** opposes the increase in motion

(1) Choose the combination that defines the characteristics of periodic motion.
(a) (III) (ii) (L) **(b)** (III) (iv) (J)
(c) (II) (i) (M) **(d)** (II) (i) (K)

(2) Choose the combination that defines the characteristics of oscillatory motion.
(a) (I) (iii) (K) **(b)** (IV) (iii) (J)
(c) (II) (iii) (M) **(d)** (I) (iii) (L)

(3) Choose the combination that defines the characteristics of simple harmonic motion.
(a) (III) (i) (K) **(b)** (I) (i) (L)
(c) (IV) (ii) (M) **(d)** (I) (iv) (L)

57. In the given table, Column I gives type of oscillation, Column II gives change in amplitude of oscillation and Column III gives examples of these types of oscillation.

Column I	Column II	Column III
(I) Oscillation of a particle with fundamental frequency under the influence of restoring force	**(i)** Amplitude of oscillation decreases due to damping forces but on account of the energy gained from the external source it remains constant.	**(J)** Balancing wheel of watch.
(II) Oscillation in which a body oscillates under the influence of an external periodic force.	**(ii)** Amplitude of oscillation remains constant.	**(K)** Shock absorbers in automobiles.
(III) Oscillation of a body whose amplitude goes on decreasing with time.	**(iii)** Amplitude of oscillation keeps increasing.	**(L)** Flicking a wine glass.
(IV) The oscillation in which the loss of oscillator is compensated by the supplying energy from an external source.	**(iv)** Amplitude of oscillation decreases exponentially due to damping forces like frictional force, viscous force, and hysteresis.	**(M)** Striking a tuning fork.

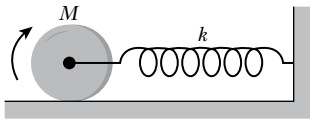
(1) The combination that defines the features of damped oscillation is
(a) (I) (iii) (L) **(b)** (IV) (i) (M)
(c) (II) (iv) (K) **(d)** (III) (iv) (K)

(2) The combination that defines the features of forced oscillation is
(a) (III) (ii) (L) **(b)** (II) (i) (L)
(c) (II) (iii) (K) **(d)** (I) (i) (M)

(3) The combination that defines the features of free oscillation is
(a) (I) (ii) (M)
(b) (I) (i) (K)
(c) (III) (iii) (L)
(d) (I) (ii) (L)

Integer Type

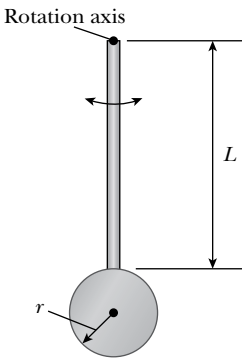
58. In the given figure, a solid cylinder attached to a horizontal spring $k = 3.00$ N/m rolls without slipping along a horizontal surface. If the system is released from rest when the spring is stretched by 0.250 m, find the translational kinetic energy (in $\times 10^{-2}$ J).



59. When a 20 N can is hung from the bottom of a vertical spring, it causes the spring to stretch 20 cm. This spring is now placed horizontally on a frictionless table. One end of it is held fixed, and the other end is attached to a 5.0 N can. The can is then moved (stretching the spring) and released from rest. What is the period (in $\times 10^{-2}$ s) of the resulting oscillation?

60. An engineer has an odd-shaped 10 kg object and needs to find its rotational inertia about an axis through its center of mass. The object is supported on a wire stretched along the desired axis. The wire has a torsion constant $k = 0.50$ N·m. If this torsion pendulum oscillates through 20 cycles in 50 s, what is the rotational inertia (in $\times 10^{-2}$ kg m^2) of the object?

61. A grandfather clock has a pendulum that consists of a thin brass disk of radius $r = 15.00$ cm and mass 1.000 kg that is attached to a long thin rod of negligible mass. The pendulum swings freely about an axis perpendicular to the rod and through the end of the rod opposite the disk, as shown in the figure. If the pendulum is to have a period of 2.000 s for small oscillations at a place where g = 9.800 m/s^2, what must be the rod length L (in $\times 10^{-1}$ m) to the nearest tenth of a millimeter?



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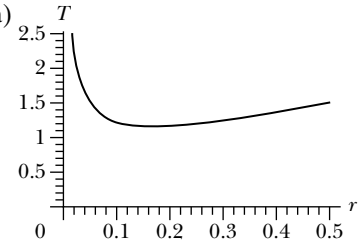
ANSWER KEY

Checkpoints

1. Periodic
2. Oscillatory
3. Draw x versus t; (a) $-x_m$; (b) $+x_m$; (c) 0
4. (c) (a must have the form of Eq. 15-11)
5. (a) (F must have the form of Eq. 15-13.)
6. $(\sqrt{3}/2)x_m$
7. (a) 5 J; (b) 2 J; (c) 5 J
8. All tie (in Eq. 15-37, m is included in I)
9. Set 1, Set 2, Set 3 (the ratio m/b matters; k does not)

Problems

1. 37.7 Hz
2. Pendulum discussed in (c), (d), and (e)
3. 58 Hz
4. (a) 15 cm; (b) 0.40 m/s
5. (a) 0.350 s; (b) 2.86 Hz; (c) 18.0 rad/s; (d) 161 N/m; (e) 6.28 m/s; (f) 56.4 N
6. (a) 2.00 s; (b) 37.0 N m/rad
7. (a) 0.203 kg·m^2; (b) 0.483 m; (c) 1.50 s
9. (a) 0.517 m; (b) 0.646 s
10. (a) 1.2 m/s; (b) 3.6×10^{-2} m
11. 11.4 Hz
12. (a) 1.0 mm; (b) 0.63 m/s; (c) 3.9×10^2 m/s^2
13. (a)



(b) 0.16 m; (c) the locus of points is a circle around the center.

14. (a) 2.2 Hz; (b) 0.56 m/s; (c) 0.200 kg; (d) 0.200 m
15. $2\pi/3$
16. 5.2×10^{-2} J.
17. (a) 400 N/m; (b) 6.25 kg; (c) 1.27 Hz
18. (a) 1.87×10^5 N/m; (b) 2.74 Hz
19. (a) –2.4 m; (b) –52 m/s; (c) -2.2×10^2 m/s^2; (d) ≈21 rad; (e) 1.5 Hz; (f) ≈0.67 s
20. 5.4 s
21. (a) ≈0.093A; (b) Moving in the same direction.
22. 5.62 m
23. (a) 3.5×10^{-3} s; (b) 3.6 m/s; (c) 0.065 J; (d) 65 N; (e) 33 N
24. 1.3 cm
25. 1.33 rad (or 4.96 rad)
26. $x = 0.48$ m; (b) $T = 2.0$ s
27. –0.927 rad or +5.36 rad
28. (a) 1.01 m; (b) 0.996 m; (c) 0.800 m/s
29. (a) 2.25 Hz; (b) 80.0 J; (c) 250 J; (d) 0.812 m
30. 13 cm
31. 43.0 m/s^2
32. +1.91 rad or –4.37 rad [The other "root" (+4.37 rad) can be rejected on the grounds that it would lead to a positive slope at $t = 0$.]
33. 1.58 h
34. (a) 0.238 kg; (b) 235 N/m
35. (a) 0.84 m; (b) 0.056 J
36. 0.0718 s.
37. 0.62
38. (a) 2.26 s; (b) increases; (c) same
39. 5.4×10^{-2} s
40. 1.7 10 N/m
41. (a) 0.50 s; (b) 2.0 Hz; (c) 16 cm
42. 0.172
43. (a) 5.58 Hz; (b) 0.346 kg; (c) 0.401 m
44. (a) 0.25 s; (b) 4.0 Hz, where the SI unit abbreviation Hz stands for Hertz, which means a cycle-per-second; (c) 8π rad/s ≈ 25 rad/s.
45. (a) 3.14×10^5 rad/s; (b) 3.18×10^{-3} m
46. (a) 0.84; (b) 0.16; (c) $x = x_m/\sqrt{2}$
47. 0.021 m
48. 1.83 s
49. (a) 0.25; (b) 2.9 Hz
50. (a) $L = 0.212$ m; (b) 3.99×10^{-4} J
51. 0.028 m; A larger amplitude requires a larger force at the end points of the motion. The surface cannot supply the larger force and the block slips.

Practice Questions

Single Correct Choice Type

1. (d)
2. (d)
3. (d)
4. (a)
5. (a)
6. (b)
7. (c)
8. (b)
9. (b)
10. (b)

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11. (c) 12. (d) 13. (d) 14. (b) 15. (b)
16. (c) 17. (c) 18. (a) 19. (d) 20. (b)
21. (a) 22. (a) 23. (b) 24. (a) 25. (c)
26. (c) 27. (d) 28. (d) 29. (d) 30. (b)
31. (d) 32. (d) 33. (c) 34. (b) 35. (d)
36. (d) 37. (a) 38. (b) 39. (a)

More than One Correct Choice Type

40. (a), (c), (d) 41. (a), (b) 42. (c), (d) 43. (a), (b), (c)

Linked Comprehension

44. (a) 45. (b) 46. (c) 47. (d) 48. (b)
49. (a) 50. (c) 51. (b) 52. (a) 53. (b)
54. (d)

Matrix-Match

55. (a) → (q), (r); (b) → (q), (r); (c) → (p); (d) → (s), (t) 56. (1) → (b); (2) → (d); (3) → (c)
57. (1) → (d); (2) → (b); (3) → (a)

Integer Type

58. 6 59. 4 60. 8 61. 8

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CHAPTER 16

Waves – I

16.1 | WHAT IS PHYSICS?

One of the primary subjects of physics is waves. To see how important waves are in the modern world, just consider the music industry. Every piece of music you hear, from some retro-punk band playing in a campus dive to the most eloquent concerto playing on the web, depends on performers producing waves and your detecting those waves. In between production and detection, the information carried by the waves might need to be transmitted (as in a live performance on the web) or recorded and then reproduced (as with CDs, DVDs, or the other devices currently being developed in engineering labs worldwide). The financial importance of controlling music waves is staggering, and the rewards to engineers who develop new control techniques can be rich.

This chapter focuses on waves traveling along a stretched string, such as on a guitar. The next chapter focuses on sound waves, such as those produced by a guitar string being played. Before we do all this, though, our first job is to classify the countless waves of the everyday world into basic types.

16.2 | TYPES OF WAVES

Key Concept

- Mechanical waves can exist only in material media and are governed by Newton's laws. Transverse mechanical waves, like those on a stretched string, are waves in which the particles of the medium oscillate perpendicular to the wave's direction of travel. Waves in which the particles of the medium oscillate parallel to the wave's direction of travel are longitudinal waves.

Waves are of three main types:

1. ***Mechanical waves.*** These waves are most familiar because we encounter them almost constantly; common examples include water waves, sound waves, and seismic waves. All these waves have two central features: They are governed by Newton's laws, and they can exist only within a material medium, such as water, air, and rock.

Contents

16.1 What is Physics?
16.2 Types of Waves
16.3 Wave Characteristics
16.4 Wave Speed on a Stretched String
16.5 Energy and Power of a Wave Traveling along a String
16.6 General Differential Equation of a Wave
16.7 Principle of Superposition for Waves
16.8 Interference of Waves
16.9 Phasors
16.10 Standing Waves
16.11 Standing Waves and Resonance
16.12 Propagation of Wave in Two Connected Strings

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2. ***Electromagnetic waves.*** These waves are less familiar, but you use them constantly; common examples include visible and ultraviolet light, radio and television waves, microwaves, X rays, and radar waves. These waves require no material medium to exist. Light waves from stars, for example, travel through the vacuum of space to reach us. All electromagnetic waves travel through a vacuum at the same speed $c = 299\,792\,458$ m/s.
3. ***Matter waves.*** Although these waves are commonly used in modern technology, they are probably very unfamiliar to you. These waves are associated with electrons, protons, and other fundamental particles, and even atoms and molecules. Because we commonly think of these particles as constituting matter, such waves are called matter waves.

Much of what we discuss in this chapter applies to waves of all kinds. However, for specific examples we shall refer to mechanical waves.

Transverse and Longitudinal Waves

A wave sent along a stretched, taut string is the simplest mechanical wave. If you give one end of a stretched string a single up-and-down jerk, a wave in the form of a single *pulse* travels along the string. This pulse and its motion can occur because the string is under tension. When you pull your end of the string upward, it begins to pull upward on the adjacent section of the string via tension between the two sections. As the adjacent section moves upward, it begins to pull the next section upward, and so on. Meanwhile, you have pulled down on your end of the string. As each section moves upward in turn, it begins to be pulled back downward by neighboring sections that are already on the way down. The net result is that a distortion in the string's shape (a pulse, as in Fig. 16-1*a*) moves along the string at some velocity $\vec{v}$.

If you move your hand up and down in continuous simple harmonic motion, a continuous wave travels along the string at velocity $\vec{v}$. Because the motion of your hand is a sinusoidal function of time, the wave has a sinusoidal shape at any given instant, as in Fig. 16-1*b*; that is, the wave has the shape of a sine curve or a cosine curve.

We consider here only an "ideal" string, in which no friction-like forces within the string cause the wave to die out as it travels along the string. In addition, we assume that the string is so long that we need not consider a wave rebounding from the far end.

One way to study the waves of Fig. 16-1 is to monitor the **wave forms** (shapes of the waves) as they move to the right. Alternatively, we could monitor the motion of an element of the string as the element oscillates up and down while a wave passes through it. We would find that the displacement of every such oscillating string element is *perpendicular* to the direction of travel of the wave, as indicated in Fig. 16-1*b*. This motion is said to be **transverse**, and the wave is said to be a **transverse wave**.

Longitudinal Waves. Figure 16-2 shows how a sound wave can be produced by a piston in a long, air-filled pipe. If you suddenly move the piston rightward and then leftward, you can send a pulse of sound along the pipe. The rightward motion of the piston moves the elements of air next to it rightward, changing the air pressure there. The increased air pressure then pushes rightward on the elements of air somewhat farther along the pipe. Moving the piston leftward reduces the air pressure next to it. As a result, first the elements nearest the piston and then farther elements move leftward. Thus, the motion of the air and the change in air pressure travel rightward along the pipe as a pulse.

If you push and pull on the piston in simple harmonic motion, as is being done in Fig. 16-2, a sinusoidal wave travels along the pipe. Because the motion of the elements of air is parallel to the direction of the wave's travel, the

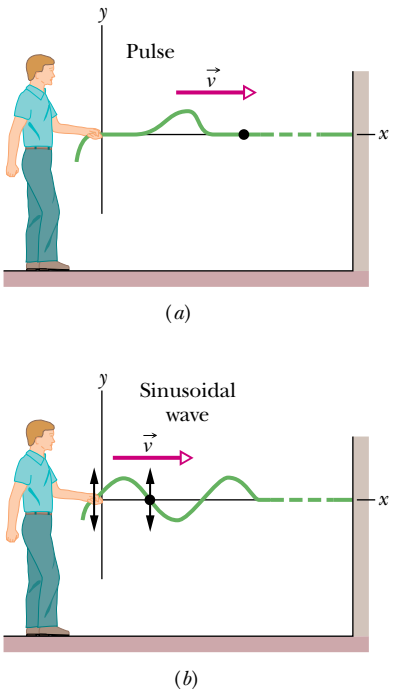

Figure 16-1 (*a*) A single pulse is sent along a stretched string. A typical string element (marked with a dot) moves up once and then down as the pulse passes. The element's motion is perpendicular to the wave's direction of travel, so the pulse is a *transverse wave*. (*b*) A sinusoidal wave is sent along the string. A typical string element moves up and down continuously as the wave passes. This too is a transverse wave.

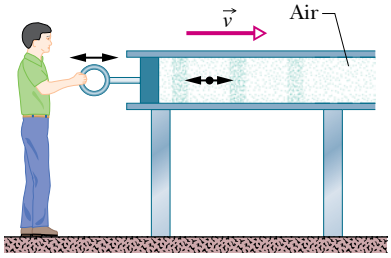

Figure 16-2 A sound wave is set up in an air-filled pipe by moving a piston back and forth. Because the oscillations of an element of the air (represented by the dot) are parallel to the direction in which the wave travels, the wave is a *longitudinal wave*.

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motion is said to be **longitudinal**, and the wave is said to be a **longitudinal wave**. In this chapter we focus on transverse waves, and string waves in particular; in Chapter 17 we focus on longitudinal waves, and sound waves in particular.

Both a transverse wave and a longitudinal wave are said to be **traveling waves** because they both travel from one point to another, as from one end of the string to the other end in Fig. 16-1 and from one end of the pipe to the other end in Fig. 16-2. Note that it is the wave that moves from end to end, not the material (string or air) through which the wave moves.

16.3 | WAVE CHARACTERISTICS

Key Concepts

- A sinusoidal wave moving in the positive direction of an x axis has the mathematical form

$$y(x, t) = y_m \sin(kx - \omega t),$$

where y_m is the amplitude (magnitude of the maximum displacement) of the wave, k is the angular wave number, ω is the angular frequency, and $kx - \omega t$ is the phase. The wavelength λ is related to k by

$$k = \frac{2\pi}{\lambda}.$$

- The period T and frequency f of the wave are related to ω by

$$\frac{\omega}{2\pi} = f = \frac{1}{T}.$$

- The wave speed v (the speed of the wave along the string) is related to these other parameters by

$$v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f.$$

- Any function of the form

$$y(x, t) = h(kx \pm \omega t),$$

can represent a traveling wave with a wave speed as given above and a wave shape given by the mathematical form of h. The plus sign denotes a wave traveling in the negative direction of the x axis, and the minus sign a wave traveling in the positive direction.

Wave Equation

Suppose we have a continuous, homogeneous string stretching from $x = 0$ to infinity. The string is attached at $x = 0$ to an oscillator, vibrating at constant frequency, that is shaking the string and thus producing a traveling wave along the string. Suppose that the displacement $y(t)$ of the output terminal is given by the harmonic oscillation

$$y(t) = y_m \sin \omega t \tag{16-1}$$

We wish to find the displacement $\psi(x, t)$ of a moving part located at position x, where x is anywhere between $x = 0$ and infinity. We can easily find $\psi(x, t)$ at $x = 0$. Since the string is tied directly to the oscillator, the displacement of the string at $x = 0$ is equal to $y(t)$

$$\psi(0, t) = y(t) = y_m \sin \omega t \tag{16-2}$$

We assume that the wave travels with constant velocity as long as the properties of the medium remain constant. We also understand that the motion of a part of string at position x, at time t is the same as that of the string at $x = 0$ at some earlier time t'. Here t' is earlier than t by the time that the wave takes to travel the distance x at speed of wave, v

$$t' = t - \frac{x}{v}$$

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Thus, we have the form of a travelling sinusoidal wave,

$$\begin{aligned}\psi(x,t) &= \psi(0,t') \\ &= y_m \sin \omega t' \\ &= y_m \sin \omega\left(t - \frac{x}{v}\right) \\ &= y_m \sin\left(\omega t - \frac{\omega x}{v}\right)\end{aligned}$$

Note that at fixed x, $\psi(x,t)$ is a harmonic oscillation in time.

Wave Parameters

Wavelength and Frequency

To completely describe a wave on a string (and the motion of any element along its length), we need a function that gives the shape of the wave. This means that we need a relation in the form

$$y = h(x,t), \tag{16-3}$$

in which y is the transverse displacement of any string element as a function h of the time t and the position x of the element along the string. In general, a sinusoidal shape like the wave in Fig. 16-1*b* can be described with h being either a sine or cosine function; both give the same general shape for the wave. In this chapter we use the sine function.

Sinusoidal Function. Imagine a sinusoidal wave like that of Fig. 16-1*b* traveling in the positive direction of an x axis. As the wave sweeps through succeeding elements (that is, very short sections) of the string, the elements oscillate parallel to the y axis. At time t, the displacement y of the element located at position x is given by

$$y(x,t) = y_m \sin(kx - \omega t). \tag{16-4}$$

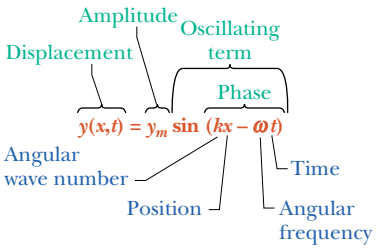

Figure 16-3 The names of the quantities in Eq. 16-4, for a transverse sinusoidal wave.

Because this equation is written in terms of position x, it can be used to find the displacements of all the elements of the string as a function of time. Thus, it can tell us the shape of the wave at any given time.

The names of the quantities in Eq. 16-4 are displayed in Fig. 16-3 and defined next. Before we discuss them, however, let us examine Fig. 16-4, which shows five "snapshots" of a sinusoidal wave traveling in the positive direction of an x axis. The movement of the wave is indicated by the rightward progress of the short arrow pointing to a high point of the wave. From snapshot to snapshot, the short arrow moves to the right with the wave shape, but the string moves *only* parallel to the y axis. To see that, let us follow the motion of the red-dyed string element at $x = 0$. In the first snapshot (Fig. 16-4*a*), this element is at displacement $y = 0$. In the next snapshot, it is at its extreme downward displacement because a *valley* (or extreme low point) of the wave is passing through it. It then moves back up through $y = 0$. In the fourth snapshot, it is at its extreme upward displacement because a *peak* (or extreme high point) of the wave is passing through it. In the fifth snapshot, it is again at $y = 0$, having completed one full oscillation.

Amplitude and Phase

The **amplitude** y_m of a wave, such as that in Fig. 16-4, is the magnitude of the maximum displacement of the elements from their equilibrium positions as the wave passes through them. (The subscript m stands for maximum.) Because y_m is a magnitude, it is always a positive quantity, even if it is measured downward instead of upward as drawn in Fig. 16-4*a*.

The **phase** of the wave is the *argument* $kx - \omega t$ of the sine in Eq. 16-4. As the wave sweeps through a string element at a particular position x, the phase changes linearly with time t. This means that the sine also changes, oscillating

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between +1 and –1. Its extreme positive value (+1) corresponds to a peak of the wave moving through the element; at that instant the value of y at position x is y_m. Its extreme negative value (–1) corresponds to a valley of the wave moving through the element; at that instant the value of y at position x is $-y_m$. Thus, the sine function and the time-dependent phase of a wave correspond to the oscillation of a string element, and the amplitude of the wave determines the extremes of the element's displacement.

Caution: When evaluating the phase, rounding off the numbers before you evaluate the sine function can throw of the calculation considerably.

Wavelength and Angular Wave Number

The **wavelength** λ of a wave is the distance (parallel to the direction of the wave's travel) between repetitions of the shape of the wave (or *wave shape*). A typical wavelength is marked in Fig. 16-4*a*, which is a snapshot of the wave at time $t = 0$. At that time, Eq. 16-4 gives, for the description of the wave shape,

$$y(x, 0) = y_m \sin kx. \tag{16-5}$$

By definition, the displacement y is the same at both ends of this wavelength—that is, at $x = x_1$ and $x = x_1 + \lambda$. Thus, by Eq. 16-5,

$$\begin{aligned} y_m \sin kx_1 &= y_m \sin k(x_1 + \lambda) \\ &= y_m \sin(kx_1 + k\lambda). \end{aligned} \tag{16-6}$$

A sine function begins to repeat itself when its angle (or argument) is increased by 2π rad, so in Eq. 16-6 we must have $k\lambda = 2\pi$, or

$$k = \frac{2\pi}{\lambda} \quad \text{(angular wave number)}. \tag{16-7}$$

We call k the **angular wave number** of the wave; its SI unit is the radian per meter, or the inverse meter. (Note that the symbol k here does *not* represent a spring constant as previously.)

Notice that the wave in Fig. 16-4 moves to the right by $1/4\lambda$ from one snapshot to the next. Thus, by the fifth snapshot, it has moved to the right by 1λ.

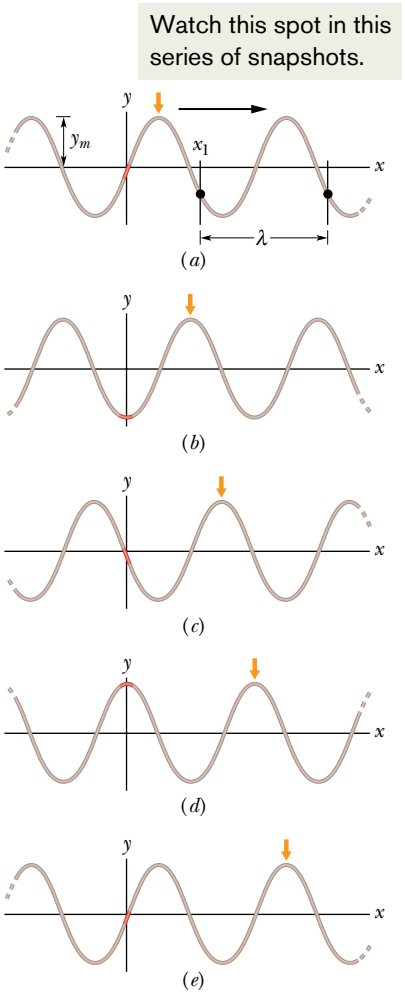

Figure 16-4 Five "snapshots" of a string wave traveling in the positive direction of an x axis. The amplitude y_m is indicated. A typical wavelength λ, measured from an arbitrary position x_1, is also indicated.

Period, Angular Frequency, and Frequency

Figure 16-5 shows a graph of the displacement y of Eq. 16-4 versus time t at a certain position along the string, taken to be $x = 0$. If you were to monitor the string, you would see that the single element of the string at that position moves up and down in simple harmonic motion given by Eq. 16-4 with $x = 0$:

$$\begin{aligned} y(0, t) &= y_m \sin(-\omega t) \\ &= -y_m \sin \omega t \quad (x = 0). \end{aligned} \tag{16-8}$$

Here we have made use of the fact that $\sin(-\alpha) = -\sin \alpha$, where α is any angle. Figure 16-5 is a graph of this equation, with displacement plotted versus time; it *does not* show the shape of the wave. (Figure 16-4 shows the shape and is a picture of reality; Fig. 16-5 is a graph and thus an abstraction.)

We define the **period** of oscillation T of a wave to be the time any string element takes to move through one full oscillation. A typical period is

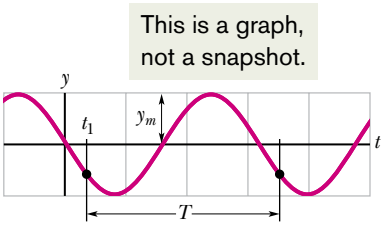

Figure 16-5 A graph of the displacement of the string element at $x = 0$ as a function of time, as the sinusoidal wave of Fig. 16-4 passes through the element. The amplitude y_m is indicated. A typical period T, measured from an arbitrary time t_1, is also indicated.

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marked on the graph of Fig. 16-5. Applying Eq. 16-8 to both ends of this time interval and equating the results yield

$$-y_m \sin \omega t_1 = -y_m \sin \omega(t_1 + T)$$
$$= -y_m \sin(\omega t_1 + \omega T). \tag{16-9}$$

This can be true only if $\omega T = 2\pi$, or if

$$\omega = \frac{2\pi}{T} \quad \text{(angular frequency)}. \tag{16-10}$$

We call ω the **angular frequency** of the wave; its SI unit is the radian per second.

Look back at the five snapshots of a traveling wave in Fig. 16-4. The time between snapshots is $1/4T$. Thus, by the fifth snapshot, every string element has made one full oscillation.

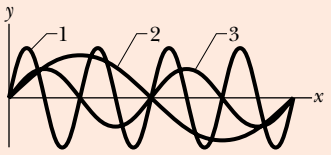
The **frequency** f of a wave is defined as $1/T$ and is related to the angular frequency ω by

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \quad \text{(frequency)}. \tag{16-11}$$

Like the frequency of simple harmonic motion in Chapter 15, this frequency f is a number of oscillations per unit time—here, the number made by a string element as the wave moves through it. As in Chapter 15, f is usually measured in hertz or its multiples, such as kilohertz.

CHECKPOINT 1

The figure is a composite of three snapshots, each of a wave traveling along a particular string. The phases for the waves are given by (a) $2x - 4t$, (b) $4x - 8t$, and (c) $8x - 16t$. Which phase corresponds to which wave in the figure?



Phase Constant

When a sinusoidal traveling wave is given by the wave function of Eq. 16-4, the wave near $x = 0$ looks like Fig. 16-6*a* when $t = 0$. Note that at $x = 0$, the displacement is $y = 0$ and the slope is at its maximum positive value. We can generalize Eq. 16-4 by inserting a **phase constant** ϕ in the wave function:

$$y = y_m \sin(kx - \omega t + \phi). \tag{16-12}$$

The value of ϕ can be chosen so that the function gives some other displacement and slope at $x = 0$ when $t = 0$. For example, a choice of $\phi = +\pi/5$ rad gives the displacement and slope shown in Fig. 16-6*b* when $t = 0$. The wave is still sinusoidal with the same values of y_m, k, and ω, but it is now shifted from what you see in Fig. 16-6*a* (where $\phi = 0$). Note also the direction of the shift. A positive value of ϕ shifts the curve in the negative direction of the x axis; a negative value shifts the curve in the positive direction.

The effect of the phase constant ϕ is to shift the wave.

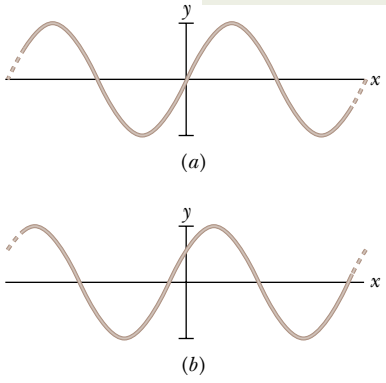


Figure 16-6 A sinusoidal traveling wave at $t = 0$ with a phase constant ϕ of (*a*) 0 and (*b*) $\pi/5$ rad.

Phase Difference

Suppose the oscillation of the particle (Fig. 16-7) at $x = 0$ is simple harmonic motion of angular frequency ω. Its displacement with time t will be given by $y = y_m \sin\omega t$, where y_m is the amplitude of the oscillation. If the wave generated travels from left to right, a particle at P at a distance x from O, will lag in phase by some phase angle β.

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All the disturbances reaching the point P take time $t_1 = x/v$, after passing through O. For the displacement y of the particle at P, we can write

$$y = y_m \sin(\omega t - \beta)$$

thus $$\beta = \omega t_1 = \omega x/v.$$

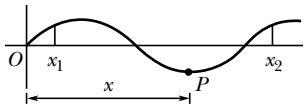

Figure 16-7 Particles at x_1 and at x_2 are in phase.

Substituting for β in the above equation, we get

$$y = y_m \sin(\omega t - \beta x/v)$$
$$= y_m \sin(\omega t - kx)$$

where $k = \omega/v$. This is the equation for a wave traveling from left to right. If the wave travelled in the opposite direction, the vibration at P would lead on the vibration at O by β and the displacement y at P would be written as

$$y = y_m \sin(\omega t + kx)$$

This is the equation for a wave traveling from right to left.

Phase difference between particles at position x_2 and x_1 is $\Delta\phi = \omega/v(x_2 - x_1)$.

Note that for wave traveling in positive direction x_2 lags behind x_1 by this amount and vice versa, that is, for wave traveling in negative direction x_1 lags behind x_2 by this amount.

The Speed of Wave on a String

Figure 16-8 shows two snapshots of the wave of Eq. 16-4, taken a small time interval Δt apart. The wave is traveling in the positive direction of x (to the right in Fig. 16-8), the entire wave pattern moving a distance Δx in that direction during the interval Δt. The ratio $\Delta x/\Delta t$ (or, in the differential limit, dx/dt) is the **wave speed** v. How can we find its value?

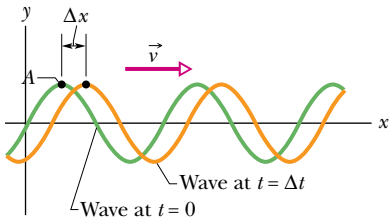

Figure 16-8 Two snapshots of the wave of Fig. 16-4, at time $t = 0$ and then at time $T = \Delta t$. As the wave moves to the right at velocity $\vec{v}$, the entire curve shifts a distance Δx during Δt. Point A "rides" with the wave form, but the string elements move only up and down.

As the wave in Fig. 16-8 moves, each point of the moving wave form, such as point A marked on a peak, retains its displacement y. (Points on the string do not retain their displacement, but points on the wave *form* do.) If point A retains its displacement as it moves, the phase in Eq. 16-4 giving it that displacement must remain a constant:

$$kx - \omega t = \text{a constant}. \tag{16-13}$$

Note that although this argument is constant, both x and t are changing. In fact, as t increases, x must also, to keep the argument constant. This confirms that the wave pattern is moving in the positive direction of x.

To find the wave speed v, we take the derivative of Eq. 16-13, getting

$$k\frac{dx}{dt} - \omega = 0$$

or $$\frac{dx}{dt} = v = \frac{\omega}{k}. \tag{16-14}$$

Using Eq. 16-7 ($k = 2\pi/\lambda$) and Eq. 16-10 ($\omega = 2\pi/T$), we can rewrite the wave speed as

$$v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f \quad \text{(wave speed)}. \tag{16-15}$$

The equation $v = \lambda/T$ tells us that the wave speed is one wavelength per period; the wave moves a distance of one wavelength in one period of oscillation.

Equation 16-4 describes a wave moving in the positive direction of x. We can find the equation of a wave traveling in the opposite direction by replacing t in Eq. 16-4 with $-t$. This corresponds to the condition

$$kx + \omega t = \text{a constant}, \tag{16-16}$$

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which (compare Eq. 16-13) requires that x *decrease* with time. Thus, a wave traveling in the negative direction of x is described by the equation

$$y(x, t) = y_m \sin(kx + \omega t). \tag{16-17}$$

If you analyze the wave of Eq. 16-17 as we have just done for the wave of Eq. 16-4, you will find for its velocity

$$\frac{dx}{dt} = -\frac{\omega}{k}. \tag{16-18}$$

The minus sign (compare Eq. 16-14) verifies that the wave is indeed moving in the negative direction of x and justifies our switching the sign of the time variable.

Consider now a wave of arbitrary shape, given by

$$y(x, t) = h(kx \pm \omega t), \tag{16-19}$$

where h represents *any* function, the sine function being one possibility. Our previous analysis shows that all waves in which the variables x and t enter into the combination $kx \pm \omega t$ are traveling waves. Furthermore, all traveling waves *must* be of the form of Eq. 16-19. Thus, $y(x, t) = \sqrt{ax + bt}$ represents a possible (though perhaps physically a little bizarre) traveling wave. The function $y(x, t) = \sin(ax^2 - bt)$, on the other hand, does *not* represent a traveling wave.

CHECKPOINT 2

Here are the equations of three waves:
(1) $y(x, t) = 2 \sin(4x - 2t)$, (2) $y(x, t) = \sin(3x - 4t)$, (3) $y(x, t) = 2 \sin(3x - 3t)$. Rank the waves according to their (a) wave speed and (b) maximum speed perpendicular to the wave's direction of travel (the transverse speed), greatest first.

Velocity of Particle and Slope

Let us study Fig. 16-9 which shows a moving particle. In time Δt particle at P will move $y_1 - y_2$ downward or in other words wave would have traveled Δx in the same time.

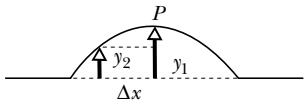

Figure 16-9 Disturbance reaches P traveling distance Δx and taking it from y_1 to y_2 height.

$$\Delta t = \frac{y_1 - y_2}{v_p} = -\frac{\Delta y}{v_p} \quad \text{and} \quad \Delta t = \frac{\Delta x}{v}$$

Equating the time durations and applying limit Δt approach zero, we get

$$v_p = -\left[\frac{\partial y}{\partial x}\right] v$$

Partial derivative is being used as y depends on both x and t.

SAMPLE PROBLEM 16.01

Determining the quantities in an equation for a transverse wave

A transverse wave traveling along an x axis has the form given by

$$y = y_m \sin(kx \pm \omega t + \phi). \tag{16-20}$$

Figure 16-10*a* gives the displacements of string elements as a function of x, all at time $t = 0$. Figure 16-10*b* gives the displacements of the element at $x = 0$ as a function of t. Find the values of the quantities shown in Eq. 16-20, including the correct choice of sign.

KEY IDEAS

(1) Figure 16-10*a* is effectively a snapshot of reality (something that we can see), showing us motion spread

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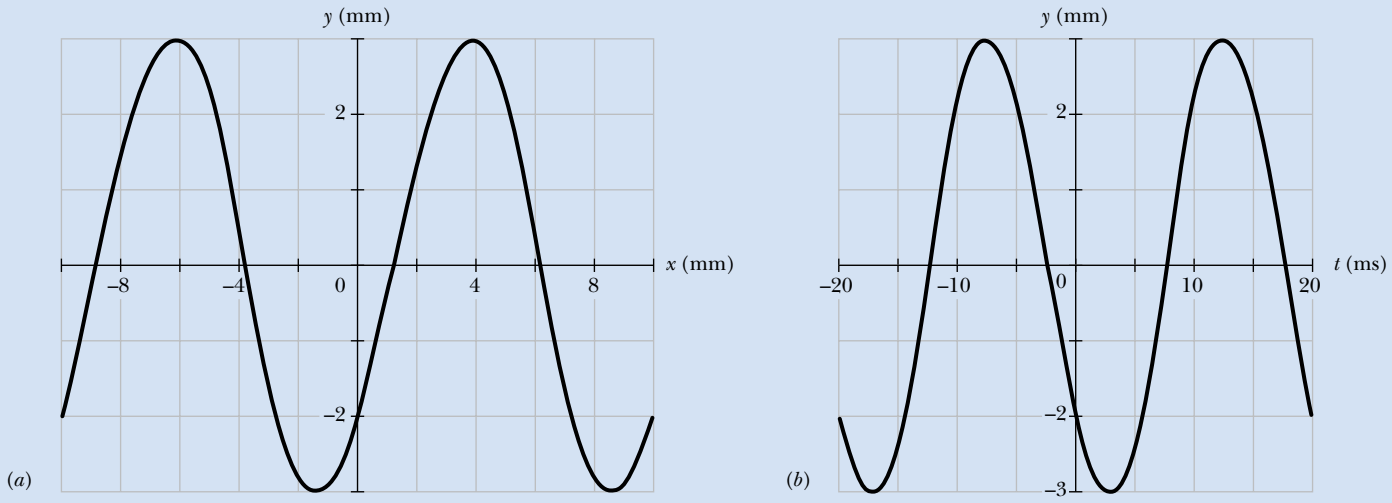

Figure 16-10 (*a*) A snapshot of the displacement y versus position x along a string, at time $t = 0$. (*b*) A graph of displacement y versus time t for the string element at $x = 0$.

out over the x axis. From it we can determine the wavelength λ of the wave along that axis, and then we can find the angular wave number k ($= 2\pi/\lambda$) in Eq. 16-20. (2) Figure 16-10*b* is an abstraction, showing us motion spread out over time. From it we can determine the period T of the string element in its SHM and thus also of the wave itself. From T we can then find angular frequency ω ($= 2\pi/T$) in Eq. 16-20. (3) The phase constant ϕ is set by the displacement of the string at $x = 0$ and $t = 0$.

Amplitude: From either Fig. 16-10*a* or 16-10*b* we see that the maximum displacement is 3.0 mm. Thus, the wave's amplitude $x_m = 3.0$ mm.

Wavelength: In Fig. 16-10*a*, the wavelength λ is the distance along the x axis between repetitions in the pattern. The easiest way to measure λ is to find the distance from one crossing point to the next crossing point where the string has the same slope. Visually we can roughly measure that distance with the scale on the axis. Instead, we can lay the edge of a paper sheet on the graph, mark those crossing points, slide the sheet to align the left-hand mark with the origin, and then read off the location of the right-hand mark. Either way we find $\lambda = 10$ mm. From Eq. 16-7, we then have

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.010\text{ m}} = 200\pi\text{ rad/m}.$$

Period: The period T is the time interval that a string element's SHM takes to begin repeating itself. In Fig. 16-10*b*, T is the distance along the t axis from one crossing point to the next crossing point where the plot has the same slope. Measuring the distance visually or with the aid of a sheet of paper, we find $T = 20$ ms. From Eq. 16-10, we then have

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.020\text{ s}} = 100\pi\text{ rad/s}.$$

Direction of travel: To find the direction, we apply a bit of reasoning to the figures. In the snapshot at $t = 0$ given in Fig. 16-10*a*, note that if the wave is moving rightward, then just after the snapshot, the depth of the wave at $x = 0$ should increase (mentally slide the curve slightly rightward). If, instead, the wave is moving leftward, then just after the snapshot, the depth at $x = 0$ should decrease. Now let's check the graph in Fig. 16-10*b*. It tells us that just after $t = 0$, the depth increases. Thus, the wave is moving rightward, in the positive direction of x, and we choose the minus sign in Eq. 16-20.

Phase constant: The value of ϕ is set by the conditions at $x = 0$ at the instant $t = 0$. From either figure we see that at that location and time, $y = -2.0$ mm. Substituting these three values and also $y_m = 3.0$ mm into Eq. 16-20 gives us

$$-2.0\text{ mm} = (3.0\text{ mm})\sin(0 + 0 + \phi)$$

or

$$\phi = \sin^{-1}\left(-\frac{2}{3}\right) = -0.73\text{ rad}.$$

Note that this is consistent with the rule that on a plot of y versus x, a negative phase constant shifts the normal sine function rightward, which is what we see in Fig. 16-10*a*.

Equation: Now we can fill out Eq. 16-20:

$$y = (3.0\text{ mm})\sin(200\pi x - 100\pi t - 0.73\text{ rad}), \quad \text{(Answer)}$$

with x in meters and t in seconds.

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SAMPLE PROBLEM 16.02

Transverse velocity and transverse acceleration of a string element

A wave traveling along a string is described by

$$y(x,t) = (0.00327\text{ m})\sin(72.1x - 2.72t), \qquad (16\text{-}21)$$

in which the numerical constants are in SI units (72.1 rad/m and 2.72 rad/s).

(a) What is the velocity of this wave?

KEY IDEA

The speed of the wave is given by Eq. 16-14:

$$\frac{dx}{dt} = v = \frac{\omega}{k}$$

Calculation: The speed of the wave is

$$v = \frac{\omega}{k} = \frac{2.72\text{ rad/s}}{72.1\text{ rad/m}} = 0.0377\text{ m/s}$$
$$= 3.77\text{ cm/s} \qquad \text{(Answer)}$$

Because the phase in Eq. 16-21 contains the position variable x, the wave is moving along the x axis. Also, because the wave equation is written in the form of Eq. 16-4, the *minus* sign in front of the ωt term indicates that the wave is moving in the *positive* direction of the x axis.

(b) What is the displacement y at x = 22.5 cm and t = 18.9 s?

Calculations: Equation 16-21 gives the displacement as a function of position x and time t. Substituting the given values into the equation yields

$$y = 0.00327\sin(72.1\times 0.225 - 2.72\times 18.9)$$
$$= (0.00327\text{ m})\sin(-35.1855\text{ rad})$$
$$= (0.00327\text{ m})(0.588)$$
$$= 0.00192\text{ m} = 1.92\text{ mm} \qquad \text{(Answer)}$$

Thus, the displacement is positive.

(c) What is the transverse velocity u of the string element at x = 22.5 cm at time t = 18.9 s? (This velocity, which is associated with the transverse oscillation of a string element, is parallel to the y axis. Do not confuse it with v, the constant velocity at which the wave form moves along the x axis.)

KEY IDEA

The transverse velocity u is the rate at which the displacement y of the element is changing. In general, that displacement is given by

$$y(x,t) = y_m\sin(kx - \omega t). \qquad (16\text{-}22)$$

For an element at a certain location x, we find the rate of change of y by taking the derivative of Eq. 16-21 with respect to t while treating x as a constant. A derivative taken while one (or more) of the variables is treated as a constant is called a partial derivative and is represented by a symbol such as $\partial/\partial t$ rather than d/dt.

Calculations: Here we have

$$u = \frac{\partial y}{\partial t} = -\omega y_m\cos(kx - \omega t). \qquad (16\text{-}23)$$

Next, substituting numerical values but suppressing the units, which are SI, we write

$$u = (-2.72)(0.00327)\cos[(72.1)(0.225) - (2.72)(18.9)]$$
$$= 0.00720\text{ m/s} = 7.20\text{ mm/s}. \qquad \text{(Answer)}$$

Thus, at t = 18.9 s our string element is moving in the positive direction of y with a speed of 7.20 mm/s. (*Caution:* In evaluating the cosine function, we keep all the significant figures in the argument or the calculation can be off considerably. For example, round off the numbers to two significant figures and then see what you get for u.)

(d) What is the transverse acceleration a_y of our string element at t = 18.9 s?

KEY IDEA

The transverse acceleration a_y is the rate at which the element's transverse velocity is changing.

Calculations: From Eq. 16-23, again treating x as a constant but allowing t to vary, we find

$$a_y = \frac{\partial u}{\partial t} = -\omega^2 y_m\sin(kx - \omega t). \qquad (16\text{-}24)$$

Substituting numerical values but suppressing the units, which are SI, we have

$$a_y = -(2.72)^2(0.00327)\sin[(72.1)(0.225) - (2.72)(18.9)]$$
$$= -0.0142\text{ m/s}^2 = -14.2\text{ mm/s}^2. \qquad \text{(Answer)}$$

From part (a) we learn that at t = 18.9 s our string element is moving in the positive direction of y, and here we learn that it is slowing because its acceleration is in the opposite direction of u.

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PROBLEM-SOLVING TACTIC

Tactic 1: Evaluating Large Phases Sometimes, as in Sample Problem 16.02 *b*, *c* and *d*, an angle much greater than 2π rad (or 360°) crops up and you are asked to find its sine or cosine. Adding or subtracting an integral multiple of 2π rad to such an angle does not change the value of any of its trigonometric functions. In Sample Problem 16.02*b*, for example, we used the angle –35.1855 rad in a sine function. Adding (6)(2π rad) to this angle yields

$$-35.1855 \text{ rad} + (6)(2\pi \text{ rad}) = 2.51361 \text{ rad}$$

an angle of less than 2π rad that has the same trigonometric functions as –35.1855 rad (Fig. 16-11). As an example, the sine of both 2.51361 rad and –35.1855 rad is 0.588.

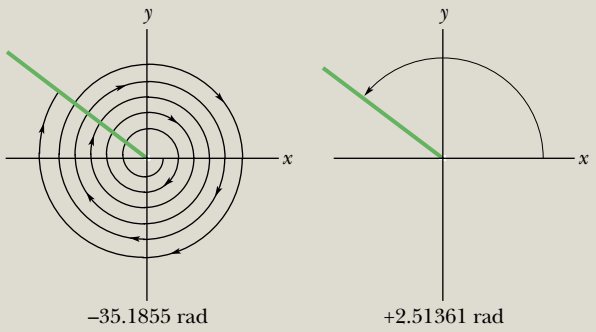

Figure 16-11 These two angles are different, but all their trigonometric functions are identical.

Caution: Do not round off large angles if you intend to take their sines or cosines. In taking the sine of a very large angle, you are throwing away most of the angle and taking the sine of what is left over. If, for example, you were to round –35.1855 rad to –35 rad (a change of 0.5% and normally a reasonable step), you would be changing the sine of the angle by 27%. Also, if you change a large angle from degrees to radians, be sure to use an exact conversion factor (such as 180° = π rad) rather than an approximate one (such as 57.3° ≈ 1 rad).

16.4 | WAVE SPEED ON A STRETCHED STRING

Key Concepts

- The speed of a wave on a stretched string is set by properties of the string, not properties of the wave such as frequency or amplitude.
- The speed of a wave on a string with tension τ and linear density μ is $v = \sqrt{\frac{\tau}{\mu}}$.

The speed of a wave is related to the wave's wavelength and frequency by Eq. 16-15, but *it is set by the properties of the medium*. If a wave is to travel through a medium such as water, air, steel, or a stretched string, it must cause the particles of that medium to oscillate as it passes, which requires both mass (for kinetic energy) and elasticity (for potential energy). Thus, the mass and elasticity determine how fast the wave can travel. Here, we find that dependency in two ways.

Dimensional Analysis

In dimensional analysis we carefully examine the dimensions of all the physical quantities that enter into a given situation to determine the quantities they produce. In this case, we examine mass and elasticity to find a speed v, which has the dimension of length divided by time, or LT^{-1}.

For the mass, we use the mass of a string element, which is the mass m of the string divided by the length l of the string. We call this ratio the *linear density* μ of the string. Thus, $\mu = m/l$, its dimension being mass divided by length, ML^{-1}.

You cannot send a wave along a string unless the string is under tension, which means that it has been stretched and pulled taut by forces at its two ends. The tension τ in the string is equal to the common magnitude of those two forces. As a wave travels along the string, it displaces elements of the string by causing additional stretching, with adjacent sections of string pulling on each other because of the tension. Thus, we can associate the tension in the string with the stretching (elasticity) of the string. The tension and the stretching forces it produces have the dimension of a force—namely, MLT^{-2} (from $F = ma$).

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We need to combine μ (dimension ML^{-1}) and τ (dimension MLT^{-2}) to get v (dimension LT^{-1}). A little juggling of various combinations suggests

$$v = C\sqrt{\frac{\tau}{\mu}}, \tag{16-25}$$

in which C is a dimensionless constant that cannot be determined with dimensional analysis. In our second approach to determining wave speed, you will see that Eq. 16-25 is indeed correct and that $C = 1$.

Derivation from Newton's Second Law

Instead of the sinusoidal wave of Fig. 16-1*b*, let us consider a single symmetrical pulse such as that of Fig. 16-12, moving from left to right along a string with speed v. For convenience, we choose a reference frame in which the pulse remains stationary; that is, we run along with the pulse, keeping it constantly in view. In this frame, the string appears to move past us, from right to left in Fig. 16-12, with speed v.

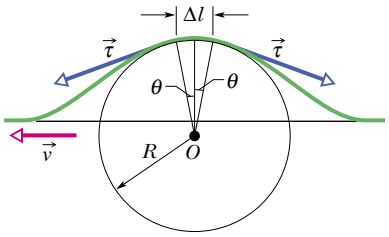

Figure 16-12 A symmetrical pulse, viewed from a reference frame in which the pulse is stationary and the string appears to move right to left with speed v. We find speed v by applying Newton's second law to a string element of length Δl, located at the top of the pulse.

Consider a small string element of length Δl within the pulse, an element that forms an arc of a circle of radius R and subtending an angle 2θ at the center of that circle. A force $\vec{\tau}$ with a magnitude equal to the tension in the string pulls tangentially on this element at each end. The horizontal components of these forces cancel, but the vertical components add to form a radial restoring force $\vec{F}$. In magnitude,

$$F = 2(\tau \sin\theta) \approx \tau(2\theta) = \tau\frac{\Delta l}{R} \quad \text{(force)}, \tag{16-26}$$

where we have approximated $\sin\theta$ as θ for the small angles θ in Fig. 16-12. From that figure, we have also used $2\theta = \Delta l/R$. The mass of the element is given by

$$\Delta m = \mu \Delta l \quad \text{(mass)}, \tag{16-27}$$

where μ is the string's linear density.

At the moment shown in Fig. 16-12, the string element Δl is moving in an arc of a circle. Thus, it has a centripetal acceleration toward the center of that circle, given by

$$a = \frac{v^2}{R} \quad \text{(acceleration)}. \tag{16-28}$$

Equations 16-26, 16-27, and 16-28 contain the elements of Newton's second law. Combining them in the form

$$\text{force} = \text{mass} \times \text{acceleration}$$

gives

$$\frac{\tau \Delta l}{R} = (\mu \Delta l)\frac{v^2}{R}.$$

Solving this equation for the speed v yields

$$v = \sqrt{\frac{\tau}{\mu}} \quad \text{(speed)}, \tag{16-29}$$

in exact agreement with Eq. 16-25 if the constant C in that equation is given the value unity. Equation 16-29 gives the speed of the pulse in Fig. 16-12 and the speed of *any* other wave on the same string under the same tension.

Equation 16-29 tells us:

The speed of a wave along a stretched ideal string depends only on the tension and linear density of the string and not on the frequency of the wave.

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The *frequency* of the wave is fixed entirely by whatever generates the wave (for example, the person in Fig. 16-1*b*). The *wavelength* of the wave is then fixed by Eq. 16-15 in the form $\lambda = v/f$.

CHECKPOINT 3

You send a traveling wave along a particular string by oscillating one end. If you increase the frequency of the oscillations, do (a) the speed of the wave and (b) the wavelength of the wave increase, decrease, or remain the same? If, instead, you increase the tension in the string, do (c) the speed of the wave and (d) the wavelength of the wave increase, decrease, or remain the same?

SAMPLE PROBLEM 16.03

Speed of pulse in a stretched string

In Fig. 16-13, two strings have been tied together with a knot and then stretched between two rigid supports. The strings have linear densities $\mu_1 = 1.4 \times 10^{-4}$ kg/m and $\mu_2 = 2.8 \times 10^{-4}$ kg/m. Their lengths are $L_1 = 3.0$ m and $L_2 = 2.0$ m, and string 1 is under a tension of 400 N. Simultaneously, on each string a pulse is sent from the rigid support end, toward the knot. Which pulse reaches the knot first?

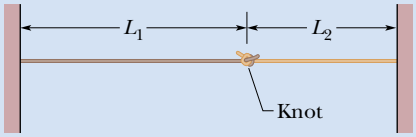

Figure 16-13 Two strings, of lengths L_1 and L_2, tied together with a knot and stretched between two rigid supports.

KEY IDEA

(1) The time t taken by a pulse to travel a length L is $t = L/v$, where v is the constant speed of the pulse.

(2) The speed of a pulse on a stretched string depends on the string's tension τ and linear density μ, and is given by equation $v = \sqrt{\tau/\mu}$.

(3) Because the two strings are stretched together, they must both be under the same tension τ (= 400 N).

Calculations: Putting these three ideas together gives us, as the time for the pulse on string 1 to reach the knot,

$$t_1 = \frac{L_1}{v_1} = L_1\sqrt{\frac{\mu_1}{\tau}} = (3.0\,\text{m})\sqrt{\frac{1.4\times10^{-4}\,\text{kg/m}}{400\,\text{N}}}$$
$$= 1.77\times10^{-3}\,\text{s}.$$

Similarly, the data for the pulse on string 2 give us

$$t_2 = L_2\sqrt{\frac{\mu_2}{\tau}} = 1.67\times10^{-3}\,\text{s}$$

Thus, the pulse on string 2 reaches the knot first.

Now look back at the second point. The linear density of string 2 is greater than that of string 1, so the pulse on string 2 must be slower than that on string 1. Could we have guessed the answer from that fact alone? No, because from the first point we see that the distance traveled by a pulse also matters.

16.5 | ENERGY AND POWER OF A WAVE TRAVELING ALONG A STRING

Key Concept

- The average power of, or average rate at which energy is transmitted by, a sinusoidal wave on a stretched string is given by

$$P_{\text{avg}} = \frac{1}{2}\mu v\omega^2 y_m^2.$$

When we set up a wave on a stretched string, we provide energy for the motion of the string. As the wave moves away from us, it transports that energy as both kinetic energy and elastic potential energy. Let us consider each form in turn.

Kinetic Energy

A string element of mass dm, oscillating transversely in simple harmonic motion as the wave passes through it, has kinetic energy associated with its transverse velocity $\vec{u}$. When the element is rushing through its $y = 0$ position

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(element *b* in Fig. 16-14), its transverse velocity—and thus its kinetic energy—is a maximum. When the element is at its extreme position $y = y_m$ (as is element *a*), its transverse velocity—and thus its kinetic energy—is zero.

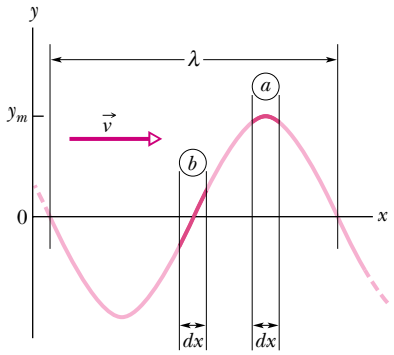

Figure 16-14 A snapshot of a traveling wave on a string at time $t = 0$. String element *a* is at displacement $y = y_m$, and string element *b* is at displacement $y = 0$. The kinetic energy of the string element at each position depends on the transverse velocity of the element. The potential energy depends on the amount by which the string element is stretched as the wave passes through it.

Elastic Potential Energy

To send a sinusoidal wave along a previously straight string, the wave must necessarily stretch the string. As a string element of length *dx* oscillates transversely, its length must increase and decrease in a periodic way if the string element is to fit the sinusoidal wave form. Elastic potential energy is associated with these length changes, just as for a spring.

When the string element is at its $y = y_m$ position (element *a* in Fig. 16-14), its length has its normal undisturbed value *dx*, so its elastic potential energy is zero. However, when the element is rushing through its $y = 0$ position, it has maximum stretch and thus maximum elastic potential energy.

Energy Transport

The oscillating string element thus has both its maximum kinetic energy and its maximum elastic potential energy at $y = 0$. In the snapshot of Fig. 16-14, the regions of the string at maximum displacement have no energy, and the regions at zero displacement have maximum energy. As the wave travels along the string, forces due to the tension in the string continuously do work to transfer energy from regions with energy to regions with no energy.

As in Fig. 16-1*b*, let's set up a wave on a string stretched along a horizontal *x* axis such that Eq. 16-4 applies. As we oscillate one end of the string, we continuously provide energy for the motion and stretching of the string—as the string sections oscillate perpendicularly to the *x* axis, they have kinetic energy and elastic potential energy. As the wave moves into sections that were previously at rest, energy is transferred into those new sections. Thus, we say that the wave *transports* the energy along the string.

The Rate of Energy Transmission

The kinetic energy *dK* associated with a string element of mass *dm* is given by

$$dK = \frac{1}{2} dm u^2, \tag{16-30}$$

where *u* is the transverse speed of the oscillating string element. To find *u*, we differentiate Eq. 16-4 with respect to time while holding *x* constant:

$$u = \frac{\partial y}{\partial t} = -\omega y_m \cos(kx - \omega t). \tag{16-31}$$

Using this relation and putting $dm = \mu\, dx$, we rewrite Eq. 16-30 as

$$dK = \frac{1}{2}(\mu\, dx)(-\omega y_m)^2 \cos^2(kx - \omega t). \tag{16-32}$$

Dividing Eq. 16-32 by *dt* gives the rate at which kinetic energy passes through a string element, and thus the rate at which kinetic energy is carried along by the wave. The *dx*/*dt* that then appears on the right of Eq. 16-32 is the wave speed *v*, so

$$\frac{dK}{dt} = \frac{1}{2}\mu v \omega^2 y_m^2 \cos^2(kx - \omega t). \tag{16-33}$$

The *average* rate at which kinetic energy is transported is

$$\left(\frac{dK}{dt}\right)_{\text{avg}} = \frac{1}{2}\mu v \omega^2 y_m^2 [\cos^2(kx - \omega t)]_{\text{avg}}$$

$$= \frac{1}{4}\mu v \omega^2 y_m^2. \tag{16-34}$$

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Here we have taken the average over an integer number of wavelengths and have used the fact that the average value of the square of a cosine function over an integer number of periods is 1/2.

Elastic potential energy is also carried along with the wave, and at the same average rate given by Eq. 16-34. The proof for this is available in Box 1 but, you should recall that, in an oscillating system such as a pendulum or a spring–block system, the average kinetic energy and the average potential energy are equal.

Imagine that one end of a very long string is being oscillated transversely to generate a sinusoidal wave traveling out along the string. The process must involve a continuing input of energy and the work equivalent to this energy must therefore be supplied at the end of the string. This can be verified by the expression derived for instantaneous power in Box 2 which is used to find average power over a time period.

BOX 1

Derivation of Expression for Potential Energy of a Wave

The **potential energy** can be calculated by finding the amount by which the deformed string is longer than when it is straight. This extension, multiplied by tension T, is the work done in the deformation (Fig. B1-1). Thus, for the segment, we have potential energy = $T\,(ds - dx)$

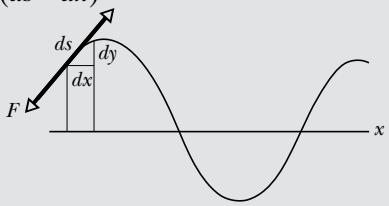

Figure B1-1 A small element in a string.

where
$$ds = (dx^2 + dy^2)^{1/2}$$
$$= dx\left[1 + \left(\frac{\partial y}{\partial x}\right)^2\right]^{1/2}$$

If we assume that the transverse displacements are small, so that $\partial y/\partial x \ll 1$, we can approximate the above expression using the binomial expansion, thus getting

$$ds - dx \approx \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx$$

Therefore,

$$\text{potential energy} \approx \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 dx$$

Hence, we have

$$\text{potential-energy density} \equiv \frac{dU}{dx} \equiv \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 \qquad \text{(B1-1)}$$

It is very interesting to note that both kinetic energy and potential energy are varying similarly, that is, when kinetic energy becomes zero potential energy also becomes zero and when kinetic energy becomes maximum potential energy also becomes maximum. We can see in Fig. B1-2 that part (a) has maximum kinetic energy as it is passing through mean position and it also has maximum potential energy as the stretch is maximum.

Looking at part (b) we can see it has minimum kinetic energy as it is at extreme position and it also has minimum potential energy as the stretch at (b) is minimum.

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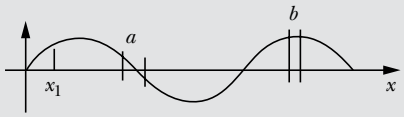

Figure B1-2 A small element in a string is having maximum kinetic and potential energy at mean position and minimum kinetic and potential energy at extreme position.

The total kinetic energy in the segment of string between $x = 0$ and $x = \lambda$ is thus

$$K = \frac{1}{2}\mu u_0^2 \int_0^{\lambda} \cos^2\left(\frac{2\pi x}{\lambda}\right) dx$$

where u_0 represents maximum particle velocity, $u_0 = wy_m$

that is,
$$K = \frac{1}{4}(\lambda\mu)u_0^2$$

This, then, is the kinetic energy associated with one complete wavelength of the disturbance. (You can easily verify that the same answer is obtained by integrating the kinetic-energy density between any two values of x separated by λ at a given instant.)

The potential energy over the same portion of the string must, as we have already seen, be equal to the kinetic energy.

$$U = \pi^2 A^2 \mu v^2 \lambda$$

which is equal in magnitude to K, as given by Eq. 16-32.

The total energy per wavelength, E, can be written as

$$E = \frac{1}{2}(\lambda\mu)u_0^2 \qquad \text{(B1-2)}$$

and is thus equal to the kinetic energy that a piece of the string of length λ would have if all of it were moving with the maximum transverse velocity u_0 associated with the wave.

Although we have chosen to make the calculation for a sinusoidal wave, equivalent results can be calculated for other kinds of waveform.

The **average power**, which is the average rate at which energy of both kinds is transmitted by the wave, is then

$$P_{\text{avg}} = 2\left(\frac{dK}{dt}\right)_{\text{avg}} \qquad \text{(16-35)}$$

or, from Eq. 16-34,

$$P_{\text{avg}} = \frac{1}{2}\mu v\omega^2 y_m^2 \quad \text{(average power).} \qquad \text{(16-36)}$$

The factors μ and v in this equation depend on the material and tension of the string. The factors ω and y_m depend on the process that generates the wave. The dependence of the average power of a wave on the square of its amplitude and also on the square of its angular frequency is a general result, true for waves of all types.

BOX 2

Derivation of Expression for Power of a Wave

Power flowing along x direction will be given by product of y-component of tension and velocity of particle at x (Fig. B2-1). Thus, instantaneous power transferred from left part to the right part is P_i.

$$P_i = (T\sin\theta)\left(\frac{\partial y}{\partial t}\right)$$

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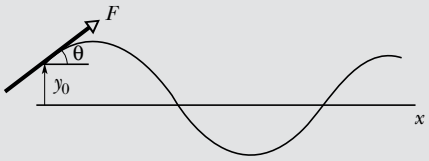

Figure B2-1 Force doing work.

$$P_i = -T\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right) \tag{B2-1}$$

The minus sign shows that the tension is acting downward.

Suppose now that we have, in particular, a sinusoidal wave described by equation

$$y(x,t) = y_m \sin 2\pi v\left(t - \frac{x}{v}\right)$$

Then at any given value of x we have on differentiating with respect to x

$$\frac{\partial y}{\partial x} = -\frac{2\pi v\, y_m}{v}\cos 2\pi v\left(t - \frac{x}{v}\right)$$

Therefore,

$$F_y = -\frac{2\pi v\, y_m T}{v}\cos 2\pi v\left(t - \frac{x}{v}\right) = -\frac{1}{m}Tk\cos(\omega t - kx)$$

We can find instantaneous power for a sine wave $y = y_m \sin(\omega t - kx)$ by using Eq. B2-1,

$$P_i = -T\,(y_m k\cos(\omega t - kx))\,[y_m \omega\cos(\omega t - kx)]$$

$$P_{\text{inst.}} = T y_m^2\, k\omega\cos^2(\omega t - kx)$$

To find average power over a time period, we use $P_{\text{avg}} = \dfrac{\int_0^i P dt}{T_0}$

$$P_{\text{avg}} = \frac{y_m^2\,\omega^2 v}{2}$$

This is the average energy flowing through any point in one time period.

SAMPLE PROBLEM 16.04

Average power of a transverse wave

A string has linear density $\mu = 525$ g/m and is under tension $\tau = 45$ N. We send a sinusoidal wave with frequency $f = 120$ Hz and amplitude $y_m = 8.5$ mm along the string. At what average rate does the wave transport energy?

KEY IDEA

The average rate of energy transport is the average power P_{avg} as given by Eq. 16-36.

Calculations: To use Eq. 16-36, we first must calculate angular frequency ω and wave speed v. From Eq. 16-11,

$$\omega = 2\pi f = (2\pi)(120\text{ Hz}) = 754\text{ rad/s}.$$

From Eq. 16-29, we have

$$v = \sqrt{\frac{\tau}{\mu}} = \sqrt{\frac{45\text{N}}{0.525\text{ kg/m}}} = 9.26\text{ m/s}.$$

Equation 16-36 then yields

$$P_{\text{avg}} = \frac{1}{2}\mu v\omega^2 y_m^2$$

$$= \left(\frac{1}{2}\right)(0.525\text{ kg/m})(9.26\text{ kg/s})(754\text{ red/s})^2\,(0.0085\text{ m})^2$$

$$\approx 100\text{ W}. \qquad \text{(Answer)}$$

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16.6 | GENERAL DIFFERENTIAL EQUATION OF A WAVE

Key Concept

- The general differential equation that governs the travel of waves of all types is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}.$$

Here the waves travel along an x axis and oscillate parallel to the y axis, and they move with speed v, in either the positive x direction or the negative x direction.

As a wave passes through any element on a stretched string, the element moves perpendicularly to the wave's direction of travel (we are dealing with a transverse wave). By applying Newton's second law to the element's motion, we can derive a general differential equation, called the *wave equation*, that governs the travel of waves of any type.

Figure 16-15*a* shows a snapshot of a string element of mass dm and length ℓ as a wave travels along a string of linear density μ that is stretched along a horizontal x axis. Let us assume that the wave amplitude is small so that the element can be tilted only slightly from the x axis as the wave passes. The force $\vec{F}_2$ on the right end of the element has a magnitude equal to tension τ in the string and is directed slightly upward. The force $\vec{F}_1$ on the left end of the element also has a magnitude equal to the tension τ but is directed slightly downward. Because of the slight curvature of the element, these two forces are not simply in opposite direction so that they cancel. Instead, they combine to produce a net force that causes the element to have an upward acceleration a_y. Newton's second law written for y components ($F_{\text{net},y} = ma_y$) gives us

$$F_{2y} - F_{1y} = dm\, a_y. \tag{16-37}$$

Let's analyze this equation in parts, first the mass dm, then the acceleration component a_y, then the individual force components F_{2y} and F_{1y}, and then finally the net force that is on the left side of Eq. 16-37.

Mass. The element's mass dm can be written in terms of the string's linear density μ and the element's length ℓ as $dm = \mu\ell$. Because the element can have only a slight tilt, $\ell \approx dx$ (Fig. 16-15*a*) and we have the approximation

$$dm = \mu\, dx. \tag{16-38}$$

Acceleration. The acceleration a_y in Eq. 16-37 is the second derivative of the displacement y with respect to time:

$$a_y = \frac{d^2 y}{dt_2}. \tag{16-39}$$

Forces. Figure 16-15*b* shows that $\vec{F}_2$ is tangent to the string at the right end of the string element. Thus, we can relate the components of the force to the string slope S_2 at the right end as

$$\frac{F_{2y}}{F_{2x}} = S_2. \tag{16-40}$$

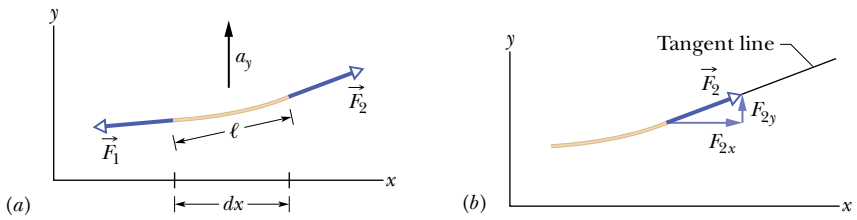

Figure 16-15 (*a*) A string element as a sinusoidal transverse wave travels on a stretched string. Forces $\vec{F}_1$ and $\vec{F}_2$ act at the left and right ends, producing acceleration $\vec{a}$ having a vertical component a_y. (*b*) The force at the element's right end is directed along a tangent to the element's right side.

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We can also relate the components to the magnitude F_2 ($= \tau$) with

$$F_2 = \sqrt{F_{2x}^2 + F_{2y}^2}$$

or

$$\tau = \sqrt{F_{2x}^2 + F_{2y}^2}. \tag{16-41}$$

However, because we assume that the element is only slightly tilted, $F_{2y} \ll F_{2x}$ and therefore we can rewrite Eq. 16-41 as

$$\tau = F_{2x}. \tag{16-42}$$

Substituting this into Eq. 16-40 and solving for F_{2y} yield

$$F_{2y} = \tau S_2. \tag{16-43}$$

Similar analysis at the left end of the string element gives us

$$F_{1y} = \tau S_1. \tag{16-44}$$

Net Force. We can now substitute Eqs. 16-38, 16-39, 16-43, and 16-44 into Eq. 16-37 to write

$$\tau S_2 - \tau S_1 = (\mu\, dx)\frac{d^2y}{dt^2},$$

or

$$\frac{S_2 - S_1}{dx} = \frac{\mu}{\tau}\frac{d^2y}{dt^2}. \tag{16-45}$$

Because the string element is short, slopes S_2 and S_1 differ by only a differential amount dS, where S is the slope at any point:

$$S = \frac{dy}{dx}. \tag{16-46}$$

First replacing $S_2 - S_1$ in Eq. 16-45 with dS and then using Eq. 16-46 to substitute dy/dx for S, we find

$$\frac{dS}{dx} = \frac{\mu}{\tau}\frac{d^2y}{dt^2},$$

$$\frac{d(dy/dx)}{dx} = \frac{\mu}{\tau}\frac{d^2y}{dt^2},$$

and

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{\tau}\frac{\partial^2 y}{\partial t^2}. \tag{16-47}$$

In the last step, we switched to the notation of partial derivatives because on the left we differentiate only with respect to x and on the right we differentiate only with respect to t. Finally, substituting from Eq. 16-29 ($v = \sqrt{\tau/\mu}$), we find

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \quad \text{(wave equation)}. \tag{16-48}$$

This is the general differential equation that governs the travel of waves of all types.

16.7 | PRINCIPLE OF SUPERPOSITION FOR WAVES

Key Concept

- When two or more waves traverse the same medium, the displacement of any particle of the medium is the sum of the displacements that the individual waves would give it, an effect known as the principle of superposition for waves.

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It often happens that two or more waves pass simultaneously through the same region. When we listen to a concert, for example, sound waves from many instruments fall simultaneously on our eardrums. The electrons in the antennas of our radio and television receivers are set in motion by the net effect of many electromagnetic waves from many different broadcasting centers. The water of a lake or harbor may be churned up by waves in the wakes of many boats.

Suppose that two waves travel simultaneously along the same stretched string. Let $y_1(x, t)$ and $y_2(x, t)$ be the displacements that the string would experience if each wave traveled alone. The displacement of the string when the waves overlap is then the algebraic sum

$$y'(x, t) = y_1(x, t) + y_2(x, t). \tag{16-49}$$

This summation of displacements along the string means that

Overlapping waves algebraically add to produce a **resultant wave** (or **net wave**).

This is another example of the **principle of superposition**, which says that when several effects occur simultaneously, their net effect is the sum of the individual effects. (We should be thankful that only a simple sum is needed. If two effects somehow amplified each other, the resulting nonlinear world would be very difficult to manage and understand.)

Figure 16-16 shows a sequence of snapshots of two pulses traveling in opposite directions on the same stretched string. When the pulses overlap, the resultant pulse is their sum. Moreover,

Overlapping waves do not in any way alter the travel of each other.

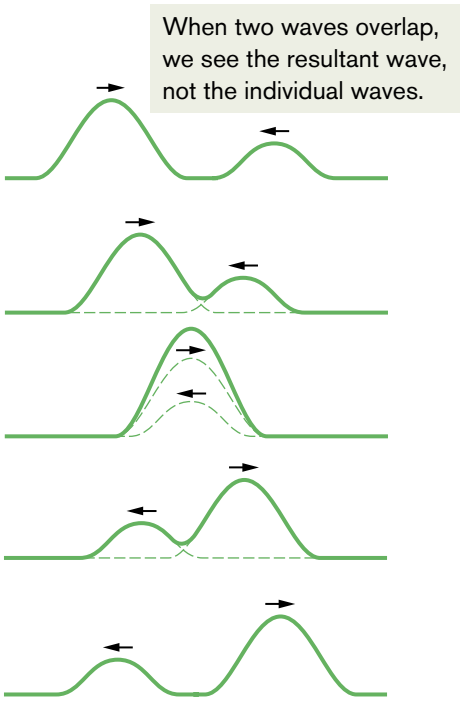

Figure 16-16 A series of snapshots that show two pulses traveling in opposite directions along a stretched string. The superposition principle applies as the pulses move through each other.

The most important and amazing features of the wave is that the two pulses traveling in opposite directions, can pass right through each other and emerge from the encounter with their separate original identities. Figure 16-17 shows such superpositions. There are two symmetrical pulses traveling in opposite directions. These pulses have exact same shape, except that one is in positive y direction and the other is in negative y direction. As they pass through each other, there is an instant at which the whole string is straight; it appears as if the pulses have destroyed each other. But we know that each pulse is carrying some energy, which does not disappear unless some dissipative force is present. According to principle of superposition, the pulses do reappear with exact same shape as before. The question arises that how can the pulses remember the original shape after the string became straight, (and apparently, they were dead) in a manner that they recover intact in their original form? The answer lies in the different velocities of the different parts of the string. The velocities also add as the displacements add. The velocity distribution of a symmetrical positive pulse traveling to the right is exactly the same as that of a similar negative pulse traveling to the left. This can also be understood from the fact that the signs of both $\partial y/\partial x$ and $\partial x/\partial t$ will be reversed hence, v_y will not change. This can also be seen by making sketches of the two pulses as they appear at two successive instants as shown in Fig. 16-17*b*. Thus, the transverse displacements cancel, but the transverse velocities add, and for this one instant, the whole energy of the system resides in the kinetic energy associated with these velocities.

It is convenient to assume simple geometric shapes such as the rectangle, triangle, and trapezoid etc. as shown in Fig. 16-17*c* for pulses. With a triangular pulse, for example, the transverse velocity is the same for all points along

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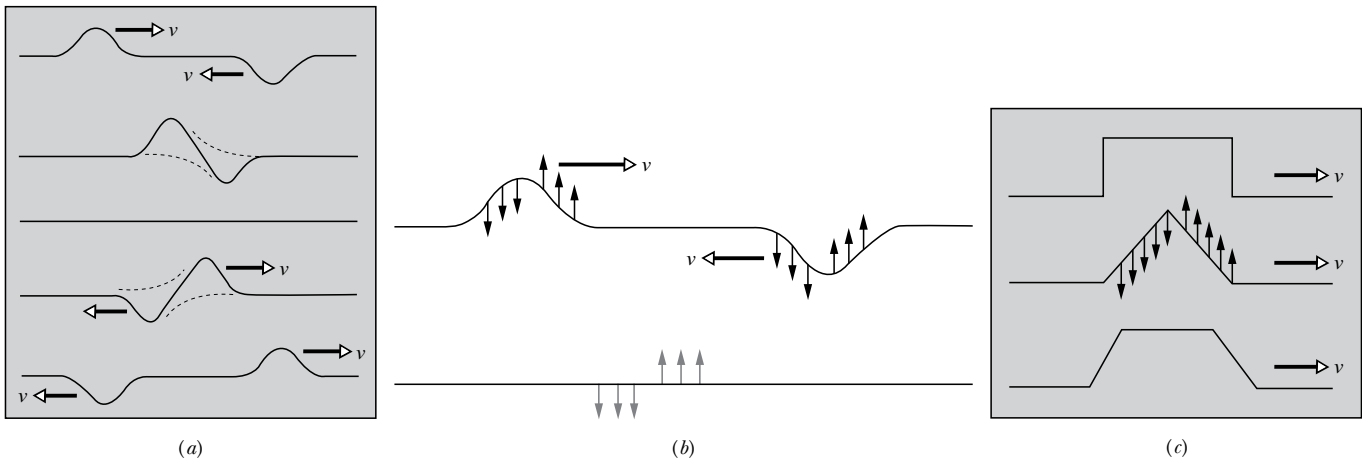

Figure 16-17 (*a*) Two waves traveling in opposite directions. (*b*) Superposition of velocities. (*c*) Superposition of velocities and shape of pulse.

each side of the pulse, and the effects of superposing such pulses are easily analyzed. It should be realized, however, that such shapes and corners (such as the trapezoid) mean discontinuous changes in transverse velocity, which in turn means infinite accelerations which is not possible.

Any real pulse, therefore, has rounded corners and sloping sides, whatever shape it may have otherwise.

16.8 | INTERFERENCE OF WAVES

Key Concept

- Two sinusoidal waves on the same string exhibit interference, adding or canceling according to the principle of superposition. If the two are traveling in the same direction and have the same amplitude y_m and frequency (hence the same wavelength) but differ in phase by a phase constant ϕ, the result is a single wave with this same frequency:

$$y'(x,y) = \left[2y_m \cos\frac{1}{2}\phi\right] \sin\left(kx - \omega t + \frac{1}{2}\phi\right).$$

If $\phi = 0$, the waves are exactly in phase and their interference is fully constructive; if $\phi = \pi$ rad, they are exactly out of phase and their interference is fully destructive.

Suppose we send two sinusoidal waves of the same wavelength and amplitude in the same direction along a stretched string. The superposition principle applies. What resultant wave does it predict for the string?

The resultant wave depends on the extent to which the waves are *in phase* (in step) with respect to each other—that is, how much one wave form is shifted from the other wave form. If the waves are exactly in phase (so that the peaks and valleys of one are exactly aligned with those of the other), they combine to double the displacement of either wave acting alone. If they are exactly out of phase (the peaks of one are exactly aligned with the valleys of the other), they combine to cancel everywhere, and the string remains straight. We call this phenomenon of combining waves **interference**, and the waves are said to **interfere**. (These terms refer only to the wave displacements; the travel of the waves is unaffected.)

Let one wave traveling along a stretched string be given by

$$y_1(x,t) = y_m \sin(kx - \omega t) \tag{16-50}$$

and another, shifted from the first, by

$$y_2(x,t) = y_m \sin(kx - \omega t + \phi). \tag{16-51}$$

These waves have the same angular frequency ω (and thus the same frequency f), the same angular wave number k (and thus the same wavelength λ), and the same amplitude y_m. They both travel in the positive direction of the x axis, with the same speed, given by Eq. 16-29. They differ only by a constant angle ϕ, the phase constant. These waves

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are said to be *out of phase* by ϕ or to have a *phase difference* of ϕ, or one wave is said to be *phase-shifted* from the other by ϕ.

From the principle of superposition (Eq. 16-49), the resultant wave is the algebraic sum of the two interfering waves and has displacement

$$\begin{aligned} y'(x,t) &= y_1(x,t) + y_2(x,t) \\ &= y_m \sin(kx - \omega t) + y_m \sin(kx - \omega t + \phi). \end{aligned} \tag{16-52}$$

In Appendix E we see that we can write the sum of the sines of two angles α and β as

$$\sin\alpha + \sin\beta = 2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta). \tag{16-53}$$

Applying this relation to Eq. 16-52 leads to

$$y'(x,t) = \left[2y_m \cos\frac{1}{2}\phi\right]\sin\left(kx - \omega t + \frac{1}{2}\phi\right). \tag{16-54}$$

As Fig. 16-18 shows, the resultant wave is also a sinusoidal wave traveling in the direction of increasing x. It is the only wave you would actually see on the string (you would *not* see the two interfering waves of Eqs. 16-50 and 16-51).

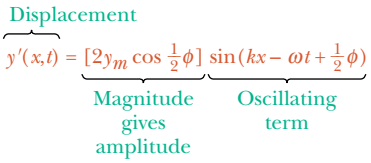

Figure 16-18 The resultant wave of Eq. 16-54, due to the interference of two sinusoidal transverse waves, is also a sinusoidal transverse wave, with an amplitude and an oscillating term.

If two sinusoidal waves of the same amplitude and wavelength travel in the *same* direction along a stretched string, they interfere to produce a resultant sinusoidal wave traveling in that direction.

The resultant wave differs from the interfering waves in two respects: (1) its phase constant is $(1/2)\phi$, and (2) its amplitude y'_m is the magnitude of the quantity in the brackets in Eq. 16-54:

$$y'_m = \left|2y_m \cos\frac{1}{2}\phi\right| \quad \text{(amplitude)}. \tag{16-55}$$

If $\phi = 0$ rad (or 0°), the two interfering waves are exactly in phase and Eq. 16-54 reduces to

$$y'(x,t) = 2y_m \sin(kx - \omega t) \quad (\phi = 0). \tag{16-56}$$

The two waves are shown in Fig. 16-19*a*, and the resultant wave is plotted in Fig. 16-19*d*. Note from both that plot and Eq. 16-56 that the amplitude of the resultant wave is twice the amplitude of either interfering wave. That is the greatest amplitude the resultant wave can have, because the cosine term in Eqs. 16-54 and 16-55 has its greatest value (unity) when $\phi = 0$. Interference that produces the greatest possible amplitude is called *fully constructive interference*.

If $\phi = \pi$ rad (or 180°), the interfering waves are exactly out of phase as in Fig. 16-19*b*. Then cos $(1/2)\phi$ becomes cos $\pi/2 = 0$, and the amplitude of the resultant wave as given by Eq. 16-55 is zero. We then have, for all values of x and t,

$$y'(x,t) = 0 \quad (\phi = \pi \text{ rad}). \tag{16-57}$$

The resultant wave is plotted in Fig. 16-19*e*. Although we sent two waves along the string, we see no motion of the string. This type of interference is called *fully destructive interference*.

Because a sinusoidal wave repeats its shape every 2π rad, a phase difference of $\phi = 2\pi$ rad (or 360°) corresponds to a shift of one wave relative to the other wave by a distance equivalent to one wavelength. Thus, phase differences can be described in terms of wavelengths as well as angles. For example, in Fig. 16-19*b* the waves may be said to be 0.50 wavelength out of phase. Table 16-1 shows some other examples of phase differences and the interference they produce. Note that when interference is neither fully constructive nor fully destructive, it is called *intermediate interference*. The amplitude of the resultant wave is then intermediate between 0 and $2y_m$. For example, from Table 16-1, if the interfering waves have a phase difference of 120° ($\phi = 2\pi/3$ rad = 0.33 wavelength), then the resultant wave has an amplitude of y_m, the same as that of the interfering waves (see Figs. 16-19*c* and *f*).

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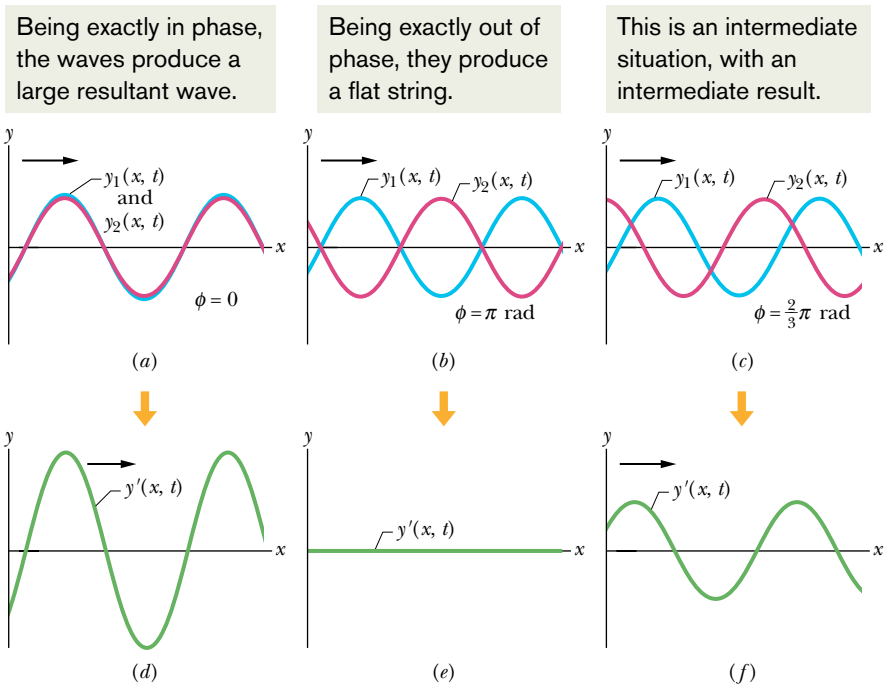

Figure 16-19 Two identical sinusoidal waves, $y_1(x, t)$ and $y_2(x, t)$, travel along a string in the positive direction of an x axis. They interfere to give a resultant wave $y'(x, t)$. The resultant wave is what is actually seen on the string. The phase difference ϕ between the two interfering waves is (*a*) 0 rad or 0°, (*b*) π rad or 180°, and (*c*) $2/3\pi$ rad or 120°. The corresponding resultant waves are shown in (*d*), (*e*), and (*f*).

Table 16-1 Phase Difference and Resulting Interference Types[a]

Phase Difference, in				
Degrees	**Radians**	**Wavelengths**	**Amplitude of Resultant Wave**	**Type of Interference**
0	0	0	$2y_m$	Fully constructive
120	$2/3\pi$	0.33	y_m	Intermediate
180	π	0.50	0	Fully destructive
240	$4/3\pi$	0.67	y_m	Intermediate
360	2π	1.00	$2y_m$	Fully constructive
865	15.1	2.40	$0.60y_m$	Intermediate

[a]The phase difference is between two otherwise identical waves, with amplitude y_m, moving in the same direction.

Two waves with the same wavelength are in phase if their phase difference is zero or any integer number of wavelengths. Thus, the integer part of any phase difference *expressed in wavelengths* may be discarded. For example, a phase difference of 0.40 wavelength (an intermediate interference, close to fully destructive interference) is equivalent in every way to one of 2.40 wavelengths, and so the simpler of the two numbers can be used in computations. Thus, by looking at only the decimal number and comparing it to 0, 0.5, or 1.0 wavelength, you can quickly tell what type of interference two waves have.

CHECKPOINT 4

Here are four possible phase differences between two identical waves, expressed in wavelengths: 0.20, 0.45, 0.60, and 0.80. Rank them according to the amplitude of the resultant wave, greatest first.

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SAMPLE PROBLEM 16.05

Interference of two waves, same direction, same amplitude

Two identical sinusoidal waves, moving in the same direction along a stretched string, interfere with each other. The amplitude y_m of each wave is 9.8 mm, and the phase difference ϕ between them is 100°.

(a) What is the amplitude y'_m of the resultant wave due to the interference, and what is the type of this interference?

KEY IDEA

These are identical sinusoidal waves traveling in the *same direction* along a string, so they interfere to produce a sinusoidal traveling wave.

Calculations: Because they are identical, the waves have the *same amplitude*. Thus, the amplitude y'_m of the resultant wave is given by Eq. 16-55:

$$y'_m = \left|2y_m \cos\frac{1}{2}\phi\right| = |(2)(9.8\text{ mm})\cos(100°/2)|$$

$$= 13\text{ mm}. \qquad \text{(Answer)}$$

We can tell that the interference is *intermediate* in two ways. The phase difference is between 0 and 180°, and, correspondingly, the amplitude y'_m is between 0 and $2y_m$ (= 19.6 mm).

(b) What phase difference, in radians and wavelengths, will give the resultant wave an amplitude of 4.9 mm?

Calculations: Now we are given y'_m and seek ϕ. From Eq. 16-55,

$$y'_m = \left|2y_m \cos\frac{1}{2}\phi\right|,$$

we now have

$$4.9\text{ mm} = (2)(9.8\text{ mm})\cos\frac{1}{2}\phi,$$

which gives us (with a calculator in the radian mode)

$$\phi = 2\cos^{-1}\frac{4.9\text{ mm}}{(2)(9.8\text{ mm})}$$

$$= \pm 2.636\text{ rad} \approx \pm 2.6\text{ rad}. \qquad \text{(Answer)}$$

There are two solutions because we can obtain the same resultant wave by letting the first wave *lead* (travel ahead of) or *lag* (travel behind) the second wave by 2.6 rad. In wavelengths, the phase difference is

$$\frac{\phi}{2\pi\text{ rad/wavelength}} = \frac{\pm 2.636\text{ rad}}{2\pi\text{ rad/wavelength}}$$

$$= \pm 0.42\text{ wavelength}. \qquad \text{(Answer)}$$

SAMPLE PROBLEM 16.06

Equations for two interfering sinusoidal waves

Two sinusoidal waves with the same amplitude of 4.0 mm and the same wavelength traveled together along a string that is stretched along an x axis; the *resultant wave* due to their interference was recorded on video tape. The curves in Fig. 16-20 represent the *resultant* wave in two freeze frames, first as the solid curve and then, 1.0 ms later, as the dotted curve. The grid lines along the x axis are 1.0 cm apart, and the string elements oscillated vertically (perpendicular to the x axis) by 6.0 mm as the resultant wave passed through them. That wave moved a distance d = 4.20 cm to the right in the 1.0 ms time interval. Write equations for the two interfering waves and for their resultant wave.

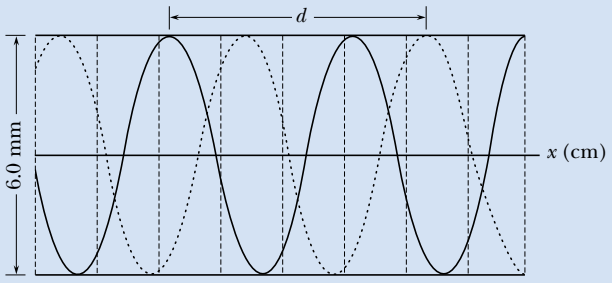

Figure 16-20 The resultant wave of two sinusoidal string waves traveling along an x axis is shown at two instants, between which the resultant wave travels distance d.

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KEY IDEAS AND CALCULATIONS

From Fig. 16-20, we see that the resultant wave is a sinusoidal wave that moves in the positive direction of the x axis. The interfering waves must both move in that direction and we can write them as

$$y_1(x,t) = y_m \sin(kx - \omega t) \tag{16-58}$$

and

$$y_2(x,t) = y_m \sin(kx - \omega t + \phi) \tag{16-59}$$

where y_m, = 4.0 mm. We can now write the resultant wave as

$$y'(x,t) = \left[2y'_m, \cos\frac{1}{2}\phi\right]\sin\left(kx - \omega t + \frac{1}{2}\phi\right) \tag{16-60}$$

The amplitude $2y_m \cos 1/2\phi$ of the resultant wave is half the total oscillation distance of 6.0 mm, so we have

$$2y_m \cos\frac{1}{2}\phi = 3.0 \text{ mm}$$

Substituting $y_m = 4.0$ mm and solving for ϕ, we get

$$\phi = 2\cos^{-1}\frac{3.0 \text{ mm}}{2(4.0 \text{ mm})} = 136° \approx 2.4 \text{ rad}$$

Thus, the phase constant in Eq. 16-59 is $\phi = 2.4$ rad and the phase constant in Eq. 16-60 is $1/2\phi = 1/2(2.4 \text{ rad}) \approx 1.2$ rad.

To find the angular wave number k, we use two key ideas. First, the common value of k in Eqs. 16-58, 16-59, and 16-60 is related to the common wavelength λ by equation $k = 2\pi/\lambda$. Second, the wavelength λ can be measured in Fig. 16-20 as the distance (parallel to the x axis) between repetitions of the wave shape. Let's use the solid curve and pick any point at which it crosses the x axis. That curve makes an identical crossing 3.0 cm to the right (or left) from the first point. Thus, $\lambda = 3.0$ cm, and

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.030 \text{ m}} = 209 \text{ m}^{-1}$$

To find the common angular frequency ω, we use two more key ideas: The common value of ω in Eqs. 16-58, 16-59, and 16-60 is related to k and the common wave speed v by equation $v = \omega/k$. That wave speed v is the ratio of the distance d traveled by the resultant wave to the time interval Δt required for that travel. Thus, we have

$$\omega = kv = k\frac{d}{\Delta t} = (209 \text{ m}^{-1})\frac{0.0420 \text{ m}}{0.0010 \text{ s}} = 8778 \text{ s}^{-1} \approx 8800 \text{ s}^{-1}$$

We can now write Eqs. 16-58 and 16-59 for the interfering waves as

$$y_1(x,t) = (4.0 \text{ mm}) \sin(209x - 88007)$$

and

$$y_2(x,t) = (4.0 \text{ mm}) \sin(209x - 8800t + 2.4 \text{ rad}) \quad \text{(Answer)}$$

with x in meters and t in seconds. We can also write Eq. 16-60 for the resultant wave as

$$y'(x,t) = (3.0 \text{ mm}) \sin(209x - 8800t + 1.2 \text{ rad}) \quad \text{(Answer)}$$

again with x in meters and t in seconds.

The phase difference between two waves must be 2.4 rad, although the two wave equations can have any value of initial phase.

16.9 | PHASORS

Key Concept

- A wave $y(x,t)$ can be represented with a phasor. This is a vector that has a magnitude equal to the amplitude y_m of the wave and that rotates about an origin with an angular speed equal to the angular frequency ω of the wave. The projection of the rotating phasor on a vertical axis gives the displacement y of a point along the wave's travel.

Adding two waves as discussed in the preceding section is strictly limited to waves with *identical* amplitudes. If we have such waves, that technique is easy enough to use, but we need a more general technique that can be applied to any waves, whether or not they have the same amplitudes. One neat way is to use phasors to represent the waves. Although this may seem bizarre at first, it is essentially a graphical technique that uses the vector addition rules of Chapter 3 instead of messy trig additions.

A **phasor** is a vector that rotates around its tail, which is pivoted at the origin of a coordinate system. The magnitude of the vector is equal to the amplitude y_m of the wave that it represents. The angular speed of the rotation is equal to the angular frequency ω of the wave. For example, the wave

$$y_1(x,t) = y_{m1} \sin(kx - \omega t) \tag{16-61}$$

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is represented by the phasor shown in Figs. 16-21*a* to *d*. The magnitude of the phasor is the amplitude y_{m1} of the wave. As the phasor rotates around the origin at angular speed ω, its projection y_1 on the vertical axis varies sinusoidally, from a maximum of y_{m1} through zero to a minimum of $-y_{m1}$ and then back to y_{m1}. This variation

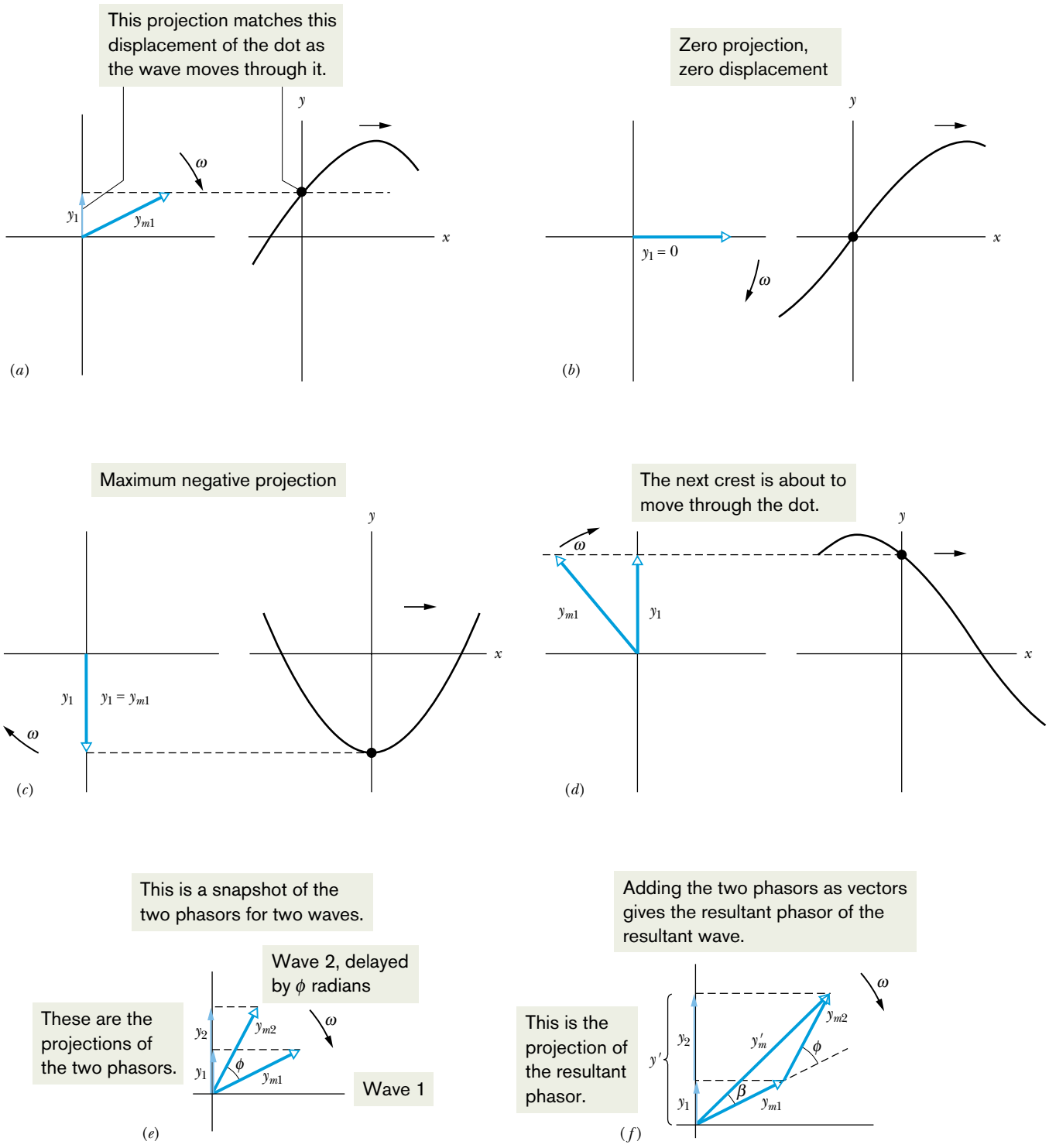
This projection matches this displacement of the dot as the wave moves through it.
Zero projection, zero displacement
Maximum negative projection
The next crest is about to move through the dot.
This is a snapshot of the two phasors for two waves.
Wave 2, delayed by ϕ radians
These are the projections of the two phasors.
Wave 1
Adding the two phasors as vectors gives the resultant phasor of the resultant wave.
This is the projection of the resultant phasor.
(a)
(b)
(c)
(d)
(e)
(f)

Figure 16-21 (*a*)–(*d*) A phasor of magnitude y_{m1} rotating about an origin at angular speed ω represents a sinusoidal wave. The phasor's projection y_1 on the vertical axis represents the displacement of a point through which the wave passes. (*e*) A second phasor, also of angular speed ω but of magnitude y_{m2} and rotating at a constant angle ϕ from the first phasor, represents a second wave, with a phase constant ϕ. (*f*) The resultant wave is represented by the vector sum y'_m of the two phasors.

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corresponds to the sinusoidal variation in the displacement y_1 of any point along the string as the wave passes through that point.

When two waves travel along the same string in the same direction, we can represent them and their resultant wave in a *phasor diagram*. The phasors in Fig. 16-21*e* represent the wave of Eq. 16-61 and a second wave given by

$$y_2(x, t) = y_{m2} \sin(kx - \omega t + \phi). \tag{16-62}$$

This second wave is phase-shifted from the first wave by phase constant ϕ. Because the phasors rotate at the same angular speed ω, the angle between the two phasors is always ϕ. If ϕ is a *positive* quantity, then the phasor for wave 2 *lags* the phasor for wave 1 as they rotate, as drawn in Fig. 16-21*e*. If ϕ is a negative quantity, then the phasor for wave 2 *leads* the phasor for wave 1.

Because waves y_1 and y_2 have the same angular wave number k and angular frequency ω, we know from Eqs. 16-54 and 16-55 that their resultant is of the form

$$y'(x, t) = y'_m \sin(kx - \omega t + \beta), \tag{16-63}$$

where y'_m is the amplitude of the resultant wave and β is its phase constant. To find the values of y'_m and β, we would have to sum the two combining waves, as we did to obtain Eq. 16-54. To do this on a phasor diagram, we vectorially add the two phasors at any instant during their rotation, as in Fig. 16-21*f* where phasor y_{m2} has been shifted to the head of phasor y_{m1}. The magnitude of the vector sum equals the amplitude y'_m in Eq. 16-63. The angle between the vector sum and the phasor for y_1 equals the phase constant β in Eq. 16-63.

Note that, in contrast to the method of Section 16.7:

We can use phasors to combine waves *even if their amplitudes are different*.

CHECKPOINT 5

Find two particles closest to $x = 0.1$ having same velocity as particle at $x = 0.1$ m at $t = 0$.

CHECKPOINT 6

At what time after $t = 0$, $x = 0.2$ m will be at $y = 0.25$ mm and going up for first time.

SAMPLE PROBLEM 16.07

Wave equation and plot using phasors

A sinusoidal wave with amplitude $y_m = 0.5$ mm and wavelength $\lambda = 0.4$ m, is traveling on a string with speed $v = 12$ m/s. At $t = 0$ particle at $x = 0.1$ m is located at $y = 0.25$ mm and going down.

(a) Find equation of wave.

KEY IDEA

This problem can be solved using phasors when the velocity of the particle at $x = 0.1$ is given (Fig. 16-22). We have to find the initial phase of this particle as in simple harmonic motion because the particle will be undergoing simple harmonic motion. The amplitude and frequency of its oscillation will be same as that of the wave.

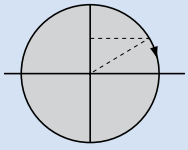

Figure 16-22 Phasor of particle is coming to be in first quadrant.

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Calculations: Angular frequency

$$\omega = 2\pi\frac{v}{\lambda} = 2\pi\frac{12}{0.4}\text{ rad/s} = 60\pi\text{ rad/s}$$

Wave number

$$k = 2\pi/\lambda = 5\pi\text{ rad/s}$$

Amplitude

$$y_m = 0.5\text{ mm}$$

Thus, equation of a wave traveling in positive x direction with above parameters is $y = 0.5(\text{mm})\sin(60\pi t - 5\pi x + \phi)$, here ϕ is unknown initial phase.

Now, putting $x = 0.1$, $t = 0$, and $y = 0.25$ mm in above equation we get $\sin(\phi - \pi/2) = 0.5$.

We can have two values $\pi/6$ and $5\pi/6$ satisfying argument of sine function. By making phasor we realize that for particle to have negative velocity $5\pi/6$ is the right choice. Therefore,

$$\phi - \frac{\pi}{2} = \frac{5\pi}{6}$$

$$\phi = \frac{5\pi}{6} + \frac{\pi}{2} = \frac{5\pi + 3\pi}{6} = \frac{4\pi}{3}$$

Hence equation of wave is

$$y = y_m \sin\left(60\pi t - 5\pi x + \frac{4\pi}{3}\right)$$

(b) Plot the sinusoidal wave at $t = 0$.

Calculations: To plot the wave function at $t = 0$, we need to locate the point $x = 0$. Thus,

$$y = y_m \sin\frac{4\pi}{3} \quad \text{or} \quad y = -\frac{\sqrt{3}y_m}{2}$$

If we proceed with this information we have two possible shapes. To decide which of them is the actual possible shape we should see the motion of the particle at $x = 0$.

Therefore,

$$\frac{dy}{dt} = y_m \cos\left(\frac{4\pi}{3} - 5\pi x\right)$$

at $x = 0$; velocity is downward; hence Fig. 16-23 is the correct graph.

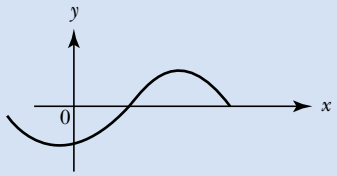

Figure 16-23 Shape of string at $t = 0$.

SAMPLE PROBLEM 16.08

Interference of two waves, same direction, phasors, any amplitudes

Two sinusoidal waves $y_1(x, t)$ and $y_2(x, t)$ have the same wavelength and travel together in the same direction along a string. Their amplitudes are $y_{m1} = 4.0$ mm and $y_{m2} = 3.0$ mm, and their phase constants are 0 and $\pi/3$ rad, respectively. What are the amplitude y'_m and phase constant β of the resultant wave? Write the resultant wave in the form of Eq. 16-63.

KEY IDEAS

(1) The two waves have a number of properties in common: Because they travel along the same string, they must have the same speed v, as set by the tension and linear density of the string according to Eq. 16-29. With the same wavelength λ, they have the same angular wave number k $(= 2\pi/\lambda)$. Also, because they have the same wave number k and speed v, they must have the same angular frequency ω $(= kv)$.

(2) The waves (call them waves 1 and 2) can be represented by phasors rotating at the same angular speed ω about an origin. Because the phase constant for wave 2 is *greater* than that for wave 1 by $\pi/3$, phasor 2 must *lag* phasor 1 by $\pi/3$ rad in their clockwise rotation, as shown in Fig. 16-24*a*. The resultant wave due to the interference of waves 1 and 2 can then be represented by a phasor that is the vector sum of phasors 1 and 2.

Calculations: To simplify the vector summation, we drew phasors 1 and 2 in Fig. 16-24*a* at the instant when phasor 1 lies along the horizontal axis. We then drew lagging phasor 2 at positive angle $\pi/3$ rad. In Fig. 16-24*b* we shifted phasor 2 so its tail is at the head of phasor 1. Then

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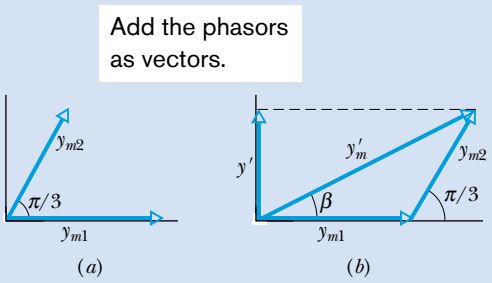

Figure 16-24 (*a*) Two phasors of magnitudes y_{m1} and y_{m2} and with phase difference $\pi/3$. (*b*)Vector addition of these phasors at any instant during their rotation gives the magnitude y'_m of the phasor for the resultant wave.

we can draw the phasor y'_m of the resultant wave from the tail of phasor 1 to the head of phasor 2. The phase constant β is the angle phasor y'_m makes with phasor 1.

To find values for y'_m and β, we can sum phasors 1 and 2 as vectors on a vector-capable calculator. However, here we shall sum them by components. (They are called horizontal and vertical components, because the symbols x and y are already used for the waves themselves.) For the horizontal components we have

$$y'_{mh} = y_{m1}\cos 0 + y_{m2}\cos \pi/3$$
$$= 4.0\text{ mm} + (3.0\text{ mm})\cos \pi/3 = 5.50\text{ mm}.$$

For the vertical components we have

$$y'_{mv} = y_{m1}\cos 0 + y_{m2}\sin \pi/3$$
$$= 0 + (3.0\text{ mm})\sin \pi/3 = 2.60\text{ mm}.$$

Thus, the resultant wave has an amplitude of

$$y'_m = \sqrt{(5.50\text{ mm})^2 + (2.60\text{ mm})^2}$$
$$= 6.1\text{ mm} \qquad \text{(Answer)}$$

and a phase constant of

$$\beta = \tan^{-1}\frac{2.60\text{ mm}}{5.50\text{ mm}} = 0.44\text{ rad}. \qquad \text{(Answer)}$$

From Fig. 16-24*b*, phase constant β is a *positive* angle relative to phasor 1. Thus, the resultant wave *lags* wave 1 in their travel by phase constant $\beta = +0.44$ rad. From Eq. 16-63, we can write the resultant wave as

$$y'(x,t) = (6.1\text{ mm})\sin(kx - \omega t + 0.44\text{ rad}). \qquad \text{(Answer)}$$

Learn: Use of phasor has made the solution very simple as compared to direct trigonometric equation.

16.10 | STANDING WAVES

Key Concept

- The interference of two identical sinusoidal waves moving in opposite directions produces standing waves. For a string with fixed ends, the standing wave is given by

$$y'(x,t) = [2y_m \sin kx]\cos \omega t.$$

Standing waves are characterized by fixed locations of zero displacement called nodes and fixed locations of maximum displacement called antinodes.

In Section 16.7, we discussed two sinusoidal waves of the same wavelength and amplitude traveling *in the same direction* along a stretched string. What if they travel in opposite directions? We can again find the resultant wave by applying the superposition principle.

Figure 16-25 suggests the situation graphically. It shows the two combining waves, one traveling to the left in Fig. 16-25*a*, the other to the right in Fig. 16-25*b*. Figure 16-25*c* shows their sum, obtained by applying the superposition principle graphically. The outstanding feature of the resultant wave is that there are places along the string, called **nodes**, where the string never moves. Four such nodes are marked by dots in Fig. 16-25*c*. Halfway between adjacent nodes are **anti-nodes**, where the amplitude of the resultant wave is a maximum. Wave patterns such as that of Fig. 16-25*c* are called **standing waves** because the wave patterns do not move left or right; the locations of the maxima and minima do not change.

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As the waves move through each other, some points never move and some move the most.

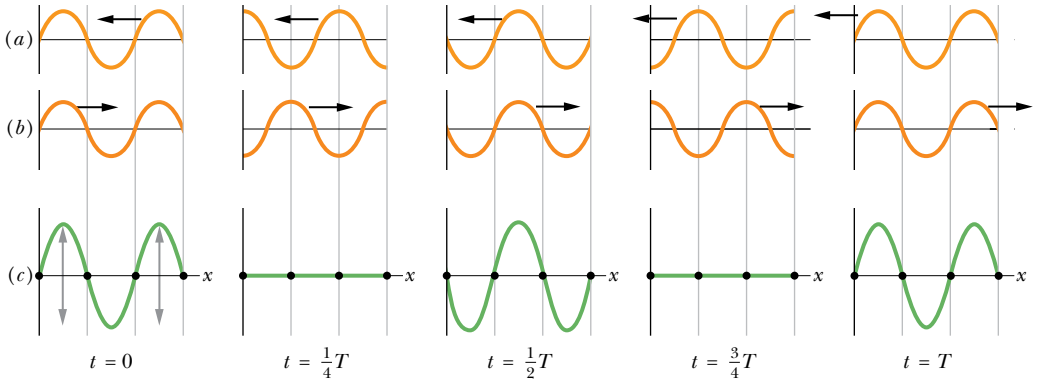

Figure 16-25 (*a*) Five snapshots of a wave traveling to the left, at the times *t* indicated below part (*c*) (*T* is the period of oscillation). (*b*) Five snapshots of a wave identical to that in (*a*) but traveling to the right, at the same times *t*. (*c*) Corresponding snapshots for the superposition of the two waves on the same string. At $t = 0, 1/2T$, and T, fully constructive interference occurs because of the alignment of peaks with peaks and valleys with valleys. At $t = 1/4T$ and $3/4T$, fully destructive interference occurs because of the alignment of peaks with valleys. Some points (the nodes, marked with dots) never oscillate; some points (the antinodes) oscillate the most.

If two sinusoidal waves of the same amplitude and wavelength travel in *opposite* directions along a stretched string, their interference with each other produces a standing wave.

To analyze a standing wave, we represent the two waves with the equations

$$y_1(x,t) = y_m \sin(kx - \omega t) \tag{16-64}$$

and

$$y_2(x,t) = y_m \sin(kx + \omega t). \tag{16-65}$$

The principle of superposition gives, for the combined wave,

$$y'(x,t) = y_1(x,t) + y_2(x,t) = y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t).$$

Applying the trigonometric relation of Eq. 16-53 leads to Fig. 16-26 and

$$y'(x,t) = [2y_m \sin kx] \cos \omega t. \tag{16-66}$$

Displacement

$y'(x,t) = [2y_m \sin kx] \cos \omega t$

Magnitude gives amplitude at position x

Oscillating term

Figure 16-26 The resultant wave of Eq. 16-66 is a standing wave and is due to the interference of two sinusoidal waves of the same amplitude and wavelength that travel in opposite directions.

This equation does not describe a traveling wave because it is not of the form of Eq. 16-19. Instead, it describes a standing wave.

The quantity $2y_m \sin kx$ in the brackets of Eq. 16-66 can be viewed as the amplitude of oscillation of the string element that is located at position x. However, since an amplitude is always positive and $\sin kx$ can be negative, we take the absolute value of the quantity $2y_m \sin kx$ to be the amplitude at x.

In a traveling sinusoidal wave, the amplitude of the wave is the same for all string elements. That is not true for a standing wave, in which the amplitude *varies with position*. In the standing wave of Eq. 16-66, for example, the amplitude is zero for values of kx that give $\sin kx = 0$. Those values are

$$kx = n\pi, \quad \text{for } n = 0, 1, 2, \ldots. \tag{16-67}$$

Substituting $k = 2\pi/\lambda$ in this equation and rearranging, we get

$$x = n\frac{\lambda}{2}, \quad \text{for } n = 0, 1, 2, \ldots \quad \text{(nodes)}, \tag{16-68}$$

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as the positions of zero amplitude—the nodes—for the standing wave of Eq. 16-66. Note that adjacent nodes are separated by $\lambda/2$, half a wavelength.

The amplitude of the standing wave of Eq. 16-66 has a maximum value of $2y_m$, which occurs for values of kx that give $|\sin kx| = 1$. Those values are

$$kx = \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \ldots$$

$$= \left(n + \frac{1}{2}\right)\pi, \quad \text{for } n = 0, 1, 2, \ldots. \tag{16-69}$$

Substituting $k = 2\pi/\lambda$ in Eq. 16-69 and rearranging, we get

$$x = \left(n + \frac{1}{2}\right)\frac{\lambda}{2}, \quad \text{for } n = 0, 1, 2, \ldots. \quad \text{(antinodes)}, \tag{16-70}$$

as the positions of maximum amplitude—the antinodes—of the standing wave of Eq. 16-66. Antinodes are separated by $\lambda/2$ and are halfway between nodes.

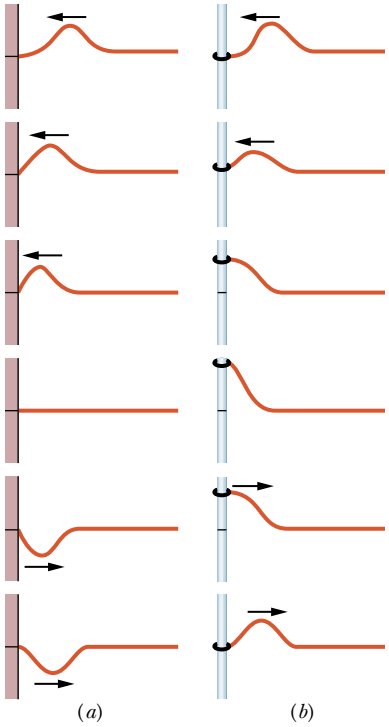
Reflections at a Boundary

We can set up a standing wave in a stretched string by allowing a traveling wave to be reflected from the far end of the string so that the wave travels back through itself. The incident (original) wave and the reflected wave can then be described by Eqs. 16-64 and 16-65, respectively, and they can combine to form a pattern of standing waves.

In Fig. 16-27, we use a single pulse to show how such reflections take place. In Fig. 16-27*a*, the string is fixed at its left end. When the pulse arrives at that end, it exerts an upward force on the support (the wall). By Newton's third law, the support exerts an opposite force of equal magnitude on the string. This second force generates a pulse at the support, which travels back along the string in the direction opposite that of the incident pulse. In a "hard" reflection of this kind, there must be a node at the support because the string is fixed there. The reflected and incident pulses must have opposite signs, so as to cancel each other at that point.

In Fig. 16-27*b*, the left end of the string is fastened to a light ring that is free to slide without friction along a rod. When the incident pulse arrives, the ring moves up the rod. As the ring moves, it pulls on the string, stretching the string and producing a reflected pulse with the same sign and amplitude as the incident pulse. Thus, in such a "soft" reflection, the incident and reflected pulses reinforce each other, creating an antinode at the end of the string; the maximum displacement of the ring is twice the amplitude of either of these two pulses.

There are two ways a pulse can reflect from the end of a string.



Figuer 16-27 (*a*) A pulse incident from the right is reflected at the left end of the string, which is tied to a wall. Note that the reflected pulse is inverted from the incident pulse. (*b*) Here the left end of the string is tied to a ring that can slide without friction up and down the rod. Now the pulse is not inverted by the reflection.

CHECKPOINT 7

Two waves with the same amplitude and wavelength interfere in three different situations to produce resultant waves with the following equations:

(1) $y'(x, t) = 4 \sin(5x - 4t)$

(2) $y'(x, t) = 4 \sin(5x) \cos(4t)$

(3) $y'(x, t) = 4 \sin(5x + 4t)$

In which situation are the two combining waves traveling (a) toward positive x, (b) toward negative x, and (c) in opposite directions?

Generation of Standing Wave Due to Reflection of a Traveling Wave

Rigid Wall: Not Capable of Transferring Energy

Suppose we have a continuous, homogeneous string stretching from $x = 0$ to $x = l$ as shown in Fig. 16-28. The string is attached at $x = 0$ to an oscillator, vibrating at constant frequency, that is shaking the string and thus producing the traveling wave along the string.

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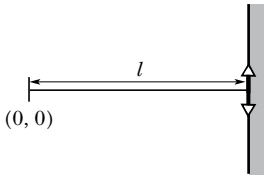

Figure 16-28 A string attached to a rigid wall.

Suppose that the displacement $y(t)$ of the output terminal is given by the harmonic oscillation

$$y(t) = y_m \omega \sin t$$

Since the string is tied directly to the oscillator, the displacement of the string at $x = 0$ is equal to $y(t)$:

$$y_i(0, t) = y(t) = y_m \sin(\omega t + \phi)$$

The displacement $y_i(x, t)$ of a moving part located at position x, where x is anywhere between $x = 0$ and l.

$$y_i(x, t) = y_m \sin\left(\omega t - \frac{\omega x}{v} + \phi\right)$$

$$y_i(l, t) = y_m \sin\left(\omega t - \frac{\omega l}{v} + \phi\right)$$

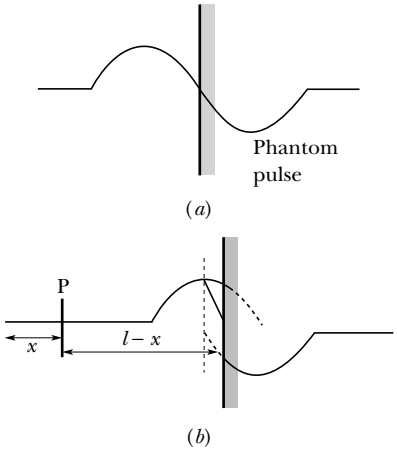

Figure 16-29 Pulse on the right-hand side of the wall represents phantom pulse which later on superimposes with the original pulse.

When this disturbance reaches $x = l$, it will get inverted upon reflection as shown in Fig. 16-29. This can also be understood from the concept of phantom pulse.

$$y_r(l, t) = -y_i(l, t)$$

and the point on the wall will become source of a new wave.

Now we have to find the equation of the reflected wave, $y_r(x, t)$.

$$y_r(l, t) = -y_m \sin\left(\omega t - \frac{\omega l}{v} + \phi\right)$$

$$= y_m \sin\left(\omega t - \frac{\omega l}{v} + \phi + \pi\right)$$

Particle at x is at distance of $(l - x)$ from the wall which is the source of inverted wave. Thus, it is $\omega[(l - x)/v]$ behind in phase with the source. Therefore,

$$y_r(x, t) = y_m \sin\left[\left\{\omega t - \left(\frac{\omega l}{v}\right) + \phi + \pi\right\} - \omega\frac{(l - x)}{v}\right]$$

After some time both waves will exist in the medium and we will have superposition of the two waves.

$$y = y_i + y_r$$

$$= 2y_m \cos\left(kx - \frac{\pi}{2} - kl\right)\sin\left(\omega t + \frac{\pi}{2} - kl\right)$$

This is an equation of a standing wave. We can verify that at $x = l$, that is at the wall net displacement y is zero at all times by putting $x = l$ in this equation.

Free End: Not Capable of Transferring Energy

If wall is replaced by free end with frictionless rod and massless ring as shown in Fig. 16-30. then $y_r(l, t) = y_i(l, t)$

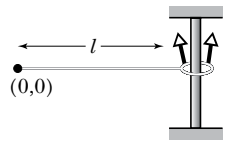

Figure 16-30 A string tied to a free end.

$$y_i(l, t) = y_m \sin\left(\omega t - \frac{\omega l}{v} + \phi\right)$$

Thus,

$$y_r(l, t) = y_m \sin\left(\omega t - \frac{\omega l}{v} + \phi\right)$$

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Particle at x is at distance of $(l-x)$ from the wall which is the source of inverted wave. Thus, it is $\omega[(l-x)/v]$ behind in phase with the source.

$$y_i(x,t) = y_m \sin(\omega t - kl + \phi - k(l-x))$$
$$y_r(x,t) = y_m \sin(\omega t - kx + \phi - 2kl)$$

After some time both waves will exist in the medium and we will have superposition of the two waves.

$$y = y_i + y_r$$
$$= 2y_m \cos(kx - kl)\sin(\omega t + \phi - kl)$$

This is an equation of a standing wave. We can verify that at $x = l$, that is, at the free end net displacement y is maximum (it is antinode) by putting $x = l$ in this equation.

SAMPLE PROBLEM 16.09

Expressions for the two superposition waves and for their resultant wave

Two sinusoidal waves with the same amplitude and wavelength travel along a string that is stretched along an x axis; the *resultant wave* due to their superposition was recorded on video tape. The curves in Fig. 16-31 represent the resultant wave in four freeze frames in the sequence of a, b, c, and d, with 1.0 ms elapsing between curves a and d. The grid lines along the x axis are 1.0 cm apart, and the string elements oscillated perpendicular to the x axis by 16.0 mm (between the extreme displacements shown by curves a and d) as the resultant wave passed through them. Write equations for the two superposition waves and for their resultant wave.

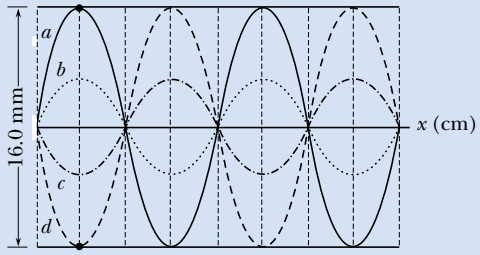

Figure 16-31 The resultant wave of two sinusoidal string waves traveling along an x axis is shown at four instants (a, b, c, and d).

KEY IDEA

Figure 16-31, shows a standing wave and so the two waves must be traveling in opposite directions.

The two waves have the forms given by

$$y_1(x,t) = y_m \sin(kx - \omega t) \quad (16\text{-}71)$$

and

$$y_2(x,t) = y_m \sin(kx + \omega t) \quad (16\text{-}72)$$

Hence, we can write the resultant wave as

$$y'(x,t) = (2y_m, \sin kx)\cos\omega t \quad (16\text{-}73)$$

We need to find the common values of y_m, k, and ω in the three equations.

Calculations: To find y_m, we use the key idea that the amplitude of a standing wave is a maximum at an antinode, and there the amplitude is $2y_m$. Here the maximum amplitude of the standing wave is 8.0 mm, so

$$y_m = 4.0 \text{ mm}$$

To find the angular wave number k, we use two more key ideas. First, the common angular wave number k in Eqs. 16-71, 16-72 and 16-73 is related to the common wavelength λ by equation $k = 2\pi/\lambda$. Second, the nodes in a standing wave are separated by 0.50λ. Because the antinodes in Fig. 16-31 are separated by 2.0 cm, we have $\lambda = 2 \times 2.0 \text{ cm} = 4.0 \text{ cm}$. Then

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.040 \text{ m}} = 157 \text{ m}^{-1} \approx 160 \text{ m}^{-1}$$

To find the common angular frequency ω we again use two more key ideas: We can relate the common value of ω in Eqs. 16-71, 16-72, and 16-73 to the period T of the standing wave with equation $\omega = 2\pi/T$. A string element at an antinode takes time T to move through a full oscillation from one extreme and back to the first extreme. According to the given data, the string element at the dot on curve a in Fig. 16-31 takes 1.0 ms to move to the position of the dot on curve d, which is one-half a full oscillation. Thus, $T = 2.0$ ms, and

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.0020 \text{ s}} = 3142 \text{ s}^{-1} \approx 3100 \text{ s}^{-1}$$

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Now we can write Eqs. 16-71 and 16-72 for the interfering waves as

$$y_1(x,t) = (4.0\text{ mm})\sin(160x - 3100t)$$

and $$y_2(x,t) = (4.0\text{ mm})\sin(160x - 3100t) \quad \text{(Answer)}$$

with x in meters and t in seconds. We can also write the standing wave of Eq. 16-73 as

$$y'(x,t) = (8.0\text{ mm})(\sin 160x)(\cos 3100t) \quad \text{(Answer)}$$

again with x in meters and t in seconds.

16.11 | STANDING WAVES AND RESONANCE

Key Concept

- Standing waves on a string can be set up by reflection of traveling waves from the ends of the string. If an end is fixed, it must be the position of a node. This limits the frequencies at which standing waves will occur on a given string. Each possible frequency is a resonant frequency, and the corresponding standing wave pattern is an oscillation mode. For a stretched string of length L with fixed ends, the resonant frequencies are

$$f = \frac{v}{\lambda} = n\frac{v}{2L}, \quad \text{for } n = 1, 2, 3, \ldots.$$

The oscillation mode corresponding to $n = 1$ is called the *fundamental mode* or the *first harmonic*; the mode corresponding to $n = 2$ is the *second harmonic*; and so on.

Normal Modes of Vibrations

Consider a string, such as a guitar string, that is stretched between two clamps. Suppose we send a continuous sinusoidal wave of a certain frequency along the string, say, toward the right. When the wave reaches the right end, it reflects and begins to travel back to the left. That left-going wave then overlaps the wave that is still traveling to the right. When the left-going wave reaches the left end, it reflects again and the newly reflected wave begins to travel to the right, overlapping the left-going and right-going waves. In short, we very soon have many overlapping traveling waves, which interfere with one another.

For certain frequencies, the interference produces a standing wave pattern (or **oscillation mode**) with nodes and large antinodes like those in Fig. 16-32. Such a standing wave is said to be produced at **resonance**, and the string is said to *resonate* at these certain frequencies, called **resonant frequencies**. If the string is oscillated at some frequency other than a resonant frequency, a standing wave is not set up. Then the interference of the right-going and left-going traveling waves results in only small, temporary (perhaps even imperceptible) oscillations of the string.

Let a string be stretched between two clamps separated by a fixed distance L. To find expressions for the resonant frequencies of the string, we note that a node must exist at each of its ends, because each end is fixed and cannot oscillate. The simplest pattern that meets this key requirement is that in Fig. 16-33*a*, which shows the string at both its extreme displacements (one solid and one dashed, together forming a single "loop"). There is only one antinode, which is at the center of the string. Note that half a wavelength spans the length L, which we take to be the string's length. Thus, for this pattern, $\lambda/2 = L$. This condition tells us that if the left-going and right-going traveling waves are to set up this pattern by their interference, they must have the wavelength $\lambda = 2L$.

A second simple pattern meeting the requirement of nodes at the fixed ends is shown in Fig. 16-33*b*. This pattern has three nodes and two antinodes and is said to be a two-loop pattern. For the left-going and right-going waves to

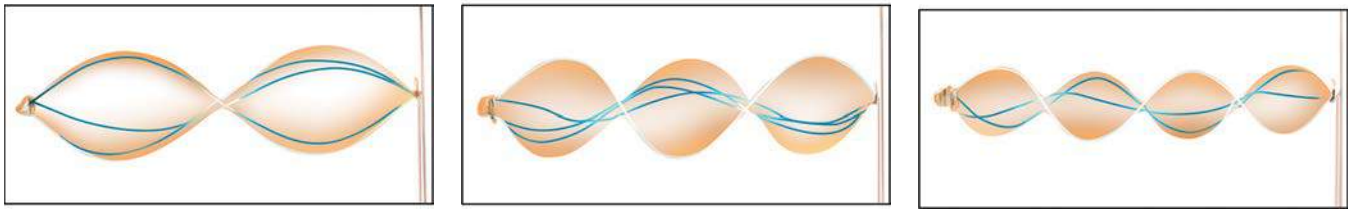

Figure 16-32 Stroboscopic photographs reveal (imperfect) standing wave patterns on a string being made to oscillate by an oscillator at the left end. The patterns occur at certain frequencies of oscillation.

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set it up, they must have a wavelength $\lambda = L$. A third pattern is shown in Fig. 16-33*c*. It has four nodes, three antinodes, and three loops, and the wavelength is $\lambda = 2/3L$. We could continue this progression by drawing increasingly more complicated patterns. In each step of the progression, the pattern would have one more node and one more antinode than the preceding step, and an additional $\lambda/2$ would be fitted into the distance L.

Thus, a standing wave can be set up on a string of length L by a wave with a wavelength equal to one of the values

$$\lambda = \frac{2L}{n}, \quad \text{for } n = 1, 2, 3, \ldots. \tag{16-74}$$

The resonant frequencies that correspond to these wavelengths follow from Eq. 16-15:

$$f = \frac{v}{\lambda} = n\frac{v}{2L}, \quad \text{for } n = 1, 2, 3, \ldots. \tag{16-75}$$

Here v is the speed of traveling waves on the string.

Equation 16-75 tells us that the resonant frequencies are integer multiples of the lowest resonant frequency, $f = v/2L$, which corresponds to $n = 1$. The oscillation mode with that lowest frequency is called the *fundamental mode* or the *first harmonic*. The *second harmonic* is the oscillation mode with $n = 2$, the *third harmonic* is that with $n = 3$, and so on. The frequencies associated with these modes are often labeled f_1, f_2, f_3, and so on. The collection of all possible oscillation modes is called the **harmonic series**, and n is called the **harmonic number** of the nth harmonic.

For a given string under a given tension, each resonant frequency corresponds to a particular oscillation pattern. Thus, if the frequency is in the audible range, you can hear the shape of the string. Resonance can also occur in two dimensions (such as on the surface of the kettledrum in Fig. 16-34) and in three dimensions (such as in the wind-induced swaying and twisting of a tall building).

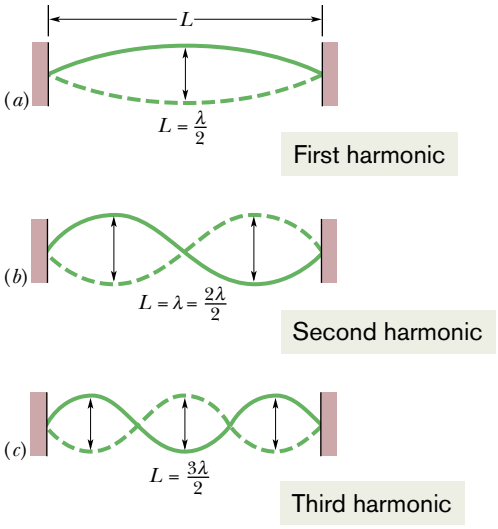
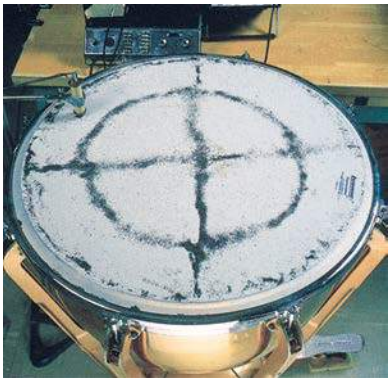


Figure 16-33 A string, stretched between two clamps, is made to oscillate in standing wave patterns. (*a*) The simplest possible pattern consists of one *loop*, which refers to the composite shape formed by the string in its extreme displacements (the solid and dashed lines). (*b*) The next simplest pattern has two loops. (*c*) The next has three loops.



Courtesy Thomas D. Rossing, Northern Illinois University

Figure 16-34 One of many possible standing wave patterns for a kettledrum head, made visible by dark powder sprinkled on the drumhead. As the head is set into oscillation at a single frequency by a mechanical oscillator at the upper left of the photograph, the powder collects at the nodes, which are circles and straight lines in this two-dimensional example.

Footbridges

When the Millennium Bridge over the Thames River was first opened to pedestrians, the bridge did not initially exhibit oscillations. The footsteps of the pedestrians produced vertical and horizontal forces on the bridge that tended to set up the second harmonic on the bridge (which is much like the second harmonic on a string), but the pedestrians were few and their walking was uncoordinated. However, once their number exceeded a critical value, the second harmonic suddenly became noticeable and walking became difficult. To keep their balance, many pedestrians began to time their steps to the bridge's swaying, which then became even worse and led to the bridge being closed until damping devices could be added.

Similar structural oscillations occur when an audience on a football stadium deck or in a concert hall begins to sway or stomp in a coordinated fashion.

Sonometer

Equation 16-75 may be verified experimentally using a sonometer. Sonometer consists of a wire under tension, arranged on a hollow wooden box as in Fig. 16-35. The vibrations of the wire are passed by the movable bridges to the box and then to the air inside it. To verify the relationship between f and l, the bridges are moved so that different lengths between them excite at their different fundamental frequencies when plucked at the center.

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The frequencies may be found by measuring length of wire which vibrates in resonance with tuning forks of known frequencies. The product $f \times l$ should be constant within the limits of experimental error.

To check that $f \propto \sqrt{T}$, the same length of wire is subjected to different tensions by changing the hanging mass.

The relation between f and $l/\sqrt{\mu}$ requires the use of wires of different diameters and materials but of the same vibrating length and under the same tension.

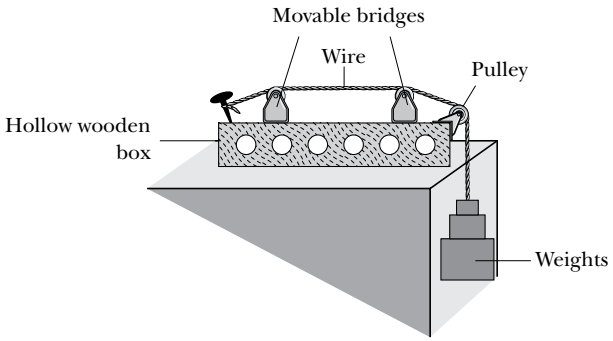

Figure 16-35 Sonometer.

Energy and Power in a Standing Wave

For a standing wave, we can still use equations for kinetic and potential energy and power derived in Section 16.5 because during the derivation it was never assumed that the wave is progressive. Only assumption was that string remains elastic.

Let's consider a standing wave $y = y_m \cos kx \sin \omega t$. Thus, we get kinetic energy of small element of length dx as

$$dK = \frac{1}{2}(\mu dx) y_m^2\, \omega^2 \cos^2 kx \cos^2 \omega t$$

Similarly, potential energy

$$dU = \frac{1}{2} T dx y_m^2 k^2 \sin^2 kx \sin^2 \omega t$$

Total energy of a small element of length dx is

$$dE = dK + dU$$

$$dE = \frac{\mu y_m^2\, \omega^2}{2}(\cos^2 kx \cos^2 \omega t + \sin^2 kx \sin^2 \omega t)\, dx \qquad (16\text{-}76)$$

We can see that energy of a small element is dependent on time, that is, it is varying. We may have expected that energy of an element is constant, but it is not, so we will show in next part that power is transmitted in stationary wave.

$$P_i = -T\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right)$$

$$P_i = -T\,[-y_m K \sin \omega t \sin kx]\,(y_m \omega \cos kx \cos \omega t)$$

$$P_i = T y_m^2\, \omega k \sin \omega t \cos \omega t \sin kx \cos kx$$

Again, you can see that energy is flowing through all parts except nodes and anti-nodes, as at nodes $\cos kx = 0$ and at antinodes $\sin kx = 0$.

Only at nodes and antinodes power transmitted is zero. This can be explained simply by observing that node is always at rest, so no energy can be transferred across it. At the antinode slope is zero so the force of tension is always along x axis and hence perpendicular to the velocity of particle which is always along y axis.

Thus, it is very important to understand that energy is flowing in stationary wave except at nodes and antinodes. Total energy from one node to the adjacent node remains constant and we can find energy of half loop from a antinode to node by integrating Eq. (16-76) from 0 to $\lambda/4$

$$\int dE = E = \frac{\mu y_m^2\, \omega^2}{2}\left[\cos^2 \omega t \int_0^{\lambda/4} \cos^2 kx\, dx + \sin^2 \omega t \int_0^{\lambda/4} \sin^2 kx\, dx\right]$$

$$= \frac{\mu y_m^2\, \omega^2}{2}\left[\cos^2 \omega t\left(\frac{1}{2}\right)\left(\frac{\lambda}{4}\right) + \sin^2 \omega t\left(\frac{1}{2}\right)\left(\frac{\lambda}{4}\right)\right]$$

$$= \frac{\mu^2 y_m^2\, \omega^2 \lambda}{16}$$

We can see that it is constant for half loop measured from a node to antinode or from antinode to a node.

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CHECKPOINT 8

In the following series of resonant frequencies, one frequency (lower than 400 Hz) is missing: 150, 225, 300, 375 Hz. (a) What is the missing frequency? (b) What is the frequency of the seventh harmonic?

SAMPLE PROBLEM 16.10

Finding wavelength at resonance

Consider a string connected at both ends with rigid wall (Fig. 16-36). Find the second longest resonance wavelength for which $3l/8$ is an antinode.

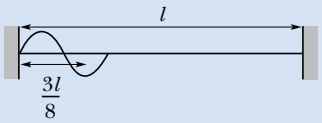

Figure 16-36 A string connected at both ends with rigid wall.

KEY IDEA

If we use normal frequencies for string with both rigid ends we can solve this problem simply by trial and error. We can consider rope of length $3l/8$ with one end rigid and one end free. Also, we can consider a rope of length l with both ends rigid. The frequencies which satisfy both equations simultaneously are the required frequencies.

Calculations: All possible frequencies which will satisfy condition of both ends rigid:

$$f_1 = \frac{m}{2f}\sqrt{\frac{T}{\mu}}$$

All possible frequencies which will satisfy condition of rope of length $3l/8$ with one end rigid and one end free:

$$f_2 = \frac{2n+1}{4(3l/8)}\sqrt{\frac{T}{\mu}}$$

Only those vibrations will have resonance which satisfy $f_1 = f_2$, so

$$\frac{2n+1}{4(3l/8)}\sqrt{\frac{T}{\mu}} = f = \frac{m}{2f}\sqrt{\frac{T}{\mu}}$$

Therefore

$$8n + 4 = 3m$$
$$3m - 8n = 4$$

m	n
4	1
12	4
20	7

The second row has the second lowest frequency, hence second largest wavelength.

Therefore, on substituting value, $\lambda = 6l$. We can verify the answer by checking for the above value that results, in satisfying both the conditions.

SAMPLE PROBLEM 16.11

Resonance of transverse waves, standing waves, harmonics

Figure 16-37 shows resonant oscillation of a string of mass $m = 2.500$ g and length $L = 0.800$ m and that is under tension $\tau = 325.0$ N. What is the wavelength λ of the transverse waves producing the standing wave pattern, and what is the harmonic number n? What is the frequency f of the transverse waves and of the oscillations of the moving string elements? What is the maximum magnitude of the transverse velocity u_m of the element oscillating at coordinate $x = 0.180$ m? At what point during the element's oscillation is the transverse velocity maximum?

KEY IDEAS

(1) The traverse waves that produce a standing wave pattern must have a wavelength such that an integer number n of half-wavelengths fit into the length L of the string.

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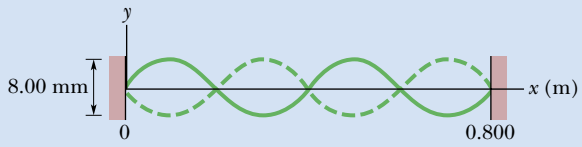

Figure 16-37 Resonant oscillation of a string under tension.

(2) The frequency of those waves and of the oscillations of the string elements is given by Eq. 16-75 ($f = nv/2L$).

(3) The displacement of a string element as a function of position x and time t is given by Eq. 16-66:

$$y'(x, t) = [2y_m \sin kx] \cos \omega t. \qquad (16\text{-}77)$$

Wavelength and harmonic number: In Fig. 16-37, the solid line, which is effectively a snapshot (or freeze-frame) of the oscillations, reveals that 2 full wavelengths fit into the length $L = 0.800$ m of the string. Thus, we have

$$2\lambda = L,$$

or

$$\lambda = \frac{L}{2}. \qquad (16\text{-}78)$$

$$= \frac{0.800 \text{ m}}{2} = 0.400 \text{ m}. \qquad \text{(Answer)}$$

By counting the number of loops (or half-wavelengths) in Fig. 16-37, we see that the harmonic number is

$$n = 4. \qquad \text{(Answer)}$$

We also find $n = 4$ by comparing Eqs. 16-78 and 16-74 ($\lambda = 2L/n$). Thus, the string is oscillating in its fourth harmonic.

Frequency: We can get the frequency f of the transverse waves from Eq. 16-15 ($v = \lambda f$) if we first find the speed v of the waves. That speed is given by Eq. 16-29, but we must substitute m/L for the unknown linear density μ. We obtain

$$v = \sqrt{\frac{\tau}{\mu}} = \sqrt{\frac{\tau}{m/L}} = \sqrt{\frac{\tau L}{m}}$$

$$= \sqrt{\frac{(325 \text{ N})(0.800 \text{ m})}{2.50 \times 10^{-3} \text{ kg}}} = 322.49 \text{ m/s}.$$

After rearranging Eq. 16-15, we write

$$f = \frac{v}{\lambda} = \frac{322.49 \text{ m/s}}{0.400 \text{ m}}$$

$$= 806.2 \text{ Hz} \approx 806 \text{ Hz}. \qquad \text{(Answer)}$$

Note that we get the same answer by substituting into Eq. 16-75:

$$f = n\frac{v}{2L} = 4\frac{322.49 \text{ m/s}}{2(0.800 \text{ m})}$$

$$= 806 \text{ Hz}. \qquad \text{(Answer)}$$

Now note that this 806 Hz is not only the frequency of the waves producing the fourth harmonic but also it is said to *be* the fourth harmonic, as in the statement, "The fourth harmonic of this oscillating string is 806 Hz." It is also the frequency of the string elements as they oscillate vertically in the figure in simple harmonic motion, just as a block on a vertical spring would oscillate in simple harmonic motion. Finally, it is also the frequency of the sound you would hear as the oscillating string periodically pushes against the air.

Transverse velocity: The displacement y' of the string element located at coordinate x is given by Eq. 16-77 as a function of time t. The term $\cos \omega t$ contains the dependence on time and thus provides the "motion" of the standing wave. The term $2y_m \sin kx$ sets the extent of the motion—that is, the amplitude. The greatest amplitude occurs at an antinode, where $\sin kx$ is +1 or −1 and thus the greatest amplitude is $2y_m$. From Fig. 16-37, we see that $2y_m = 4.00$ mm, which tells us that $y_m = 2.00$ mm.

We want the transverse velocity—the velocity of a string element parallel to the y axis. To find it, we take the time derivative of Eq. 16-77:

$$u(x, t) = \frac{\partial y'}{\partial t} = \frac{\partial}{\partial t}[(2y_m \sin kx)\cos \omega t]$$

$$= [-2y_m \omega \sin kx] \sin \omega t. \qquad (16\text{-}79)$$

Here the term $\sin \omega t$ provides the variation with time and the term $-2y_m \omega \sin kx$ provides the extent of that variation. We want the absolute magnitude of that extent:

$$u_m = |-2y_m \omega \sin kx|.$$

To evaluate this for the element at $x = 0.180$ m, we first note that $y_m = 2.00$ mm, $k = 2\pi/\lambda = 2\pi/(0.400 \text{ m})$, and $\omega = 2\pi f = 2\pi(806.2 \text{ Hz})$. Then the maximum speed of the element at $x = 0.180$ m is

$$u_m = \left| -2(2.00 \times 10^{-3} \text{ m})(2\pi)(806.2 \text{ Hz}) \times \sin\left(\frac{2\pi}{0.400 \text{ m}}(0.180 \text{ m})\right)\right|$$

$$= 6.26 \text{ m/s}. \qquad \text{(Answer)}$$

To determine when the string element has this maximum speed, we could investigate Eq. 16-79. However, a little thought can save a lot of work. The element is undergoing SHM and must come to a momentary stop at its extreme upward position and extreme downward position. It has the greatest speed as it zips through the midpoint of its oscillation, just as a block does in a block–spring oscillator.

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16.12 | PROPAGATION OF WAVE IN TWO CONNECTED STRINGS

Key Concept

- For a transverse wave propagating with different speeds on a string which is further connected to another string, the amplitudes of the transmitted and reflected waves in terms of the amplitude of the incident wave is given by

$$\frac{y_{mr}}{y_{mi}} = \frac{k_1 - k_2}{k_1 + k_2} \quad \text{and} \quad \frac{y_{mt}}{y_{mi}} = \frac{2k_1}{k_1 + k_2}$$

Consider a transverse wave propagating on a string which is further connected to another string, on which the wave has different speeds of propagation. Let the strings, before displacement, lie along the x direction and the point where the strings are joined be at position x equal to zero. The string on the left-hand side has a wave of angular frequency ω propagating at speed v_1 having wave number k_1, and the string on the right-hand side of the joint propagates waves at a speed v_2 corresponding to wave number k_2. Some important points are as follows:

1. The frequency of the waves on the two separate parts of the string must be the same; otherwise the joint would not stay intact. At $x = 0$ the strings must go up and down together.
2. The tension in the strings must also be the same everywhere; otherwise the joint would move in the x direction.
3. The total disturbance on the left-hand string is the sum of the incident wave $y_{mi}\cos(\omega t - k_1 x)$ and the reflected wave $y_{mr}\cos(\omega t + k_1 x)$, where y_{mi} and y_{mr} are the respective amplitudes. The total wave on the left-hand string is thus given by

$$y_L = y_{mi}\cos(\omega t - k_1 x) + y_{mr}\cos(\omega t + k_1 x)$$

4. The disturbance on the right-hand string is a progressive wave given by

$$y_R = y_{mt}\cos(\omega t - k_2 x) \tag{16-80}$$

where y_{mt} is the amplitude of the transmitted wave.

We assume that there are no phase differences between the incident, reflected, and transmitted waves at $x = 0$. At $x = 0$, where the strings join, the string is continuous so at all times the displacement on the left at $x = 0$ must be equal to the displacement on the right at $x = 0$. Also, in order to make the forces on a very small section at the joint equal and opposite, the slopes of the string on either side of the joint must be the same, that is $(\partial y/\partial x)_L = (\partial y/\partial x)_R$ at $x = 0$. The conditions that the displacement and its derivative be continuous at the joint are the boundary conditions that must be satisfied when the propagating wave meets the discontinuity.

The boundary condition on string continuity applied at time $t = 0$ gives

$$y_{mi} + y_{mr} = y_{mt}$$

and the equal-slope condition applied at time $t = \pi/2\omega$ gives

$$k_1 y_{mi} - k_1 y_{mr} = k_2 y_{mt}$$

Manipulation of these two equations gives

$$\frac{y_{mr}}{y_{mi}} = \frac{k_1 - k_2}{k_1 + k_2} \tag{16-81}$$

and

$$\frac{y_{mt}}{y_{mi}} = \frac{2k_1}{k_1 + k_2}$$

We have now determined the amplitudes of the transmitted and reflected waves in terms of the amplitude of the incident wave. This was done by assuming that the phases of all three waves were equal at $x = 0$. Here y_{mt} has the same sign as y_{mi} since their ratio is always positive. However, y_{mr} may have a different sign then y_{mi} depending on the values of k_1 and k_2. If $k_2 > k_1$, there is a phase difference of π between incident and reflected waves; adding π to the phase of the reflected wave in Eq. 16-80 is equivalent to changing the sign of y_{mr}.

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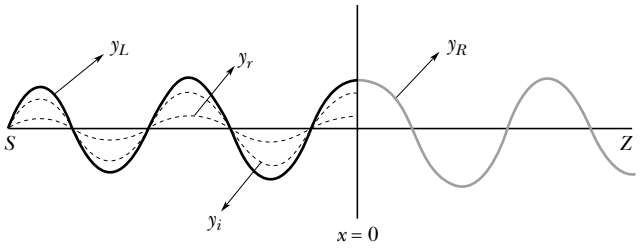

Figure 16-38 Two different mass density strings connected.

Whether or not there is a phase change of π on reflection is determined by the sign of $(k_1 - k_2)$. The wave number on the right is greater than that on the left if the speed of the waves on the right is less than that on the left. Thus, from Eq. 16-81, the phase change is zero if the string on the right is lighter than on the left, or π if it is heavier.

Figure 16-38 shows at time $t = 0$, a wave of amplitude y_{mi} incident on the join at $x = 0$ between two strings. The string on the right has a smaller value of mass per unit length μ than the string on the left.

REVIEW AND SUMMARY

Transverse and Longitudinal Waves Mechanical waves can exist only in material media and are governed by Newton's laws. **Transverse** mechanical waves, like those on a stretched string, are waves in which the particles of the medium oscillate perpendicular to the wave's direction of travel. Waves in which the particles of the medium oscillate parallel to the wave's direction of travel are **longitudinal** waves.

Sinusoidal Waves A sinusoidal wave moving in the positive direction of an x axis has the mathematical form

$$y(x, t) = y_m \sin(kx - \omega t), \tag{16-4}$$

where y_m is the **amplitude** of the wave, k is the **angular wave number**, ω is the **angular frequency**, and $kx - \omega t$ is the **phase**. The **wavelength** λ is related to k by

$$k = \frac{2\pi}{\lambda}. \tag{16-7}$$

The **period** T and **frequency** f of the wave are related to ω by

$$\frac{\omega}{2\pi} = f = \frac{1}{T}. \tag{16-11}$$

Finally, the **wave speed** v is related to these other parameters by

$$v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f. \tag{16-15}$$

Equation of a Traveling Wave Any function of the form

$$y(x, t) = h(kx \pm \omega t) \tag{16-19}$$

can represent a **traveling wave** with a wave speed given by Eq. 16-15 and a wave shape given by the mathematical form of h. The plus sign denotes a wave traveling in the negative direction of the x axis, and the minus sign a wave traveling in the positive direction.

Wave Speed on Stretched String The speed of a wave on a stretched string is set by properties of the string. The speed on a string with tension τ and linear density μ is

$$v = \sqrt{\frac{\tau}{\mu}}. \tag{16-29}$$

Power The **average power** of, or average rate at which energy is transmitted by, a sinusoidal wave on a stretched string is given by

$$P_{\text{avg}} = \frac{1}{2}\mu v \omega^2 y_m^2. \tag{16-36}$$

Superposition of Waves When two or more waves traverse the same medium, the displacement of any particle of the medium is the sum of the displacements that the individual waves would give it.

Interference of Waves Two sinusoidal waves on the same string exhibit **interference**, adding or canceling according to the principle of superposition. If the two are traveling in the same direction and have the same amplitude y_m and frequency (hence the same wavelength) but differ in phase by a **phase constant** ϕ, the result is a single wave with this same frequency:

$$y'(x, t) = \left[2y_m \cos\frac{1}{2}\phi\right] \sin\left(kx - \omega t + \frac{1}{2}\phi\right). \tag{16-54}$$

If $\phi = 0$, the waves are exactly in phase and their interference is fully constructive; if $\phi = \pi$ rad, they are exactly out of phase and their interference is fully destructive.

Phasors A wave $y(x, t)$ can be represented with a **phasor**. This is a vector that has a magnitude equal to the amplitude y_m of the wave and that rotates about an origin with an angular speed equal to the angular frequency ω of the wave. The projection of the rotating phasor on a vertical axis gives the displacement y of a point along the wave's travel.

Standing Waves The interference of two identical sinusoidal waves moving in opposite directions produces **standing waves**. For a string with fixed ends, the standing wave is given by

$$y'(x, t) = [2y_m \sin kx] \cos \omega t. \tag{16-66}$$

Standing waves are characterized by fixed locations of zero displacement called **nodes** and fixed locations of maximum displacement called **antinodes**.

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Resonance Standing waves on a string can be set up by reflection of traveling waves from the ends of the string. If an end is fixed, it must be the position of a node. This limits the frequencies at which standing waves will occur on a given string. Each possible frequency is a **resonant frequency**, and the corresponding standing wave pattern is an **oscillation mode**. For a stretched string of length L with fixed ends, the resonant frequencies are

$$f = \frac{v}{\lambda} = n\frac{v}{2L}, \quad \text{for } n = 1, 2, 3, \ldots. \tag{16-75}$$

The oscillation mode corresponding to $n = 1$ is called the *fundamental mode* or the *first harmonic*; the mode corresponding to $n = 2$ is the *second harmonic*; and so on.

Propagation of Wave in Two Connected Strings For a transverse wave propagating with different speeds on a string which is further connected to another string, the amplitudes of the transmitted and reflected waves in terms of the amplitude of the incident wave is given by

$$\frac{y_{mr}}{y_{mi}} = \frac{k_1 - k_2}{k_1 + k_2} \quad \text{and} \quad \frac{y_{mt}}{y_{mi}} = \frac{2k_1}{k_1 + k_2} \tag{16-81}$$

PROBLEMS

1. A stretched string has a mass per unit length of 5.00 g/cm and a tension of 10.0 N. A sinusoidal wave on this string has an amplitude of 0.16 mm and a frequency of 100 Hz and is traveling in the negative direction of an x axis. If the wave equation is of the form $y(x, t) = y_m \sin(kx \pm \omega t)$, what are (a) y_m, (b) k, (c) ω, and (d) the correct choice of sign in front of ω?
2. The heaviest and lightest strings on a certain violin have linear densities of 3.2 and 0.26 g/m. What is the ratio of the diameter of the heaviest string to that of the lightest string, assuming that the strings are of the same material?
3. A string fixed at both ends is 7.50 m long and has a mass of 0.120 kg. It is subjected to a tension of 96.0 N and set oscillating. (a) What is the speed of the waves on the string? (b) What is the longest possible wavelength for a standing wave? (c) Give the frequency of that wave.
4. The equation of a transverse wave on a string is
$$y = (2.0 \text{ mm}) \sin[(15 \text{ m}^{-1})x - (900 \text{ s}^{-1})t].$$
The linear density is 4.17 g/m. (a) What is the wave speed? (b) What is the tension in the string?
5. Two waves are generated on a string of length 4.0 m to produce a three-loop standing wave with an amplitude of 1.0 cm. The wave speed is 100 m/s. Let the equation for one of the waves be of the form $y(x, t) = y_m \sin(kx + \omega t)$. In the equation for the other wave, what are (a) y_m, (b) k, (c) ω, and (d) the sign in front of ω?
6. What phase difference between two identical traveling waves, moving in the same direction along a stretched string, results in the combined wave having an amplitude 0.852 times that of the common amplitude of the two combining waves? Express your answer in (a) degrees, (b) radians, and (c) wavelengths.
7. A 100 g wire is held under a tension of 220 N with one end at $x = 0$ and the other at $x = 10.0$ m. At time $t = 0$, pulse 1 is sent along the wire from the end at $x = 10.0$ m. At time $t = 30.0$ ms, pulse 2 is sent along the wire from the end at $x = 0$. At what position x do the pulses begin to meet?
8. String A is stretched between two clamps separated by distance L. String B, with the same linear density and under the same tension as string A, is stretched between two clamps separated by distance $3L$. Consider the first eight harmonics of string B. For which of these eight harmonics of B (if any) does the frequency match the frequency of (a) A's first harmonic, (b) A's second harmonic, and (c) A's third harmonic?
9. Two sinusoidal waves with the same amplitude of 6.00 mm and the same wavelength travel together along a string that is stretched along an x axis. Their resultant wave is shown twice in Fig. 16-39, as valley A travels in the negative direction of the x axis by distance $d = 56.0$ cm in 8.0 ms. The tick marks along the axis are separated by 10 cm, and height H is 8.0 mm. Let the equation for one wave be of the form $y(x, t) = y_m \sin(kx \pm \omega t + \phi_1)$, where $\phi_1 = 0$ and you must choose the correct sign in front of ω. For the equation for the other wave, what are (a) y_m, (b) k, (c) ω, (d) ϕ_2, and (e) the sign in front of ω?

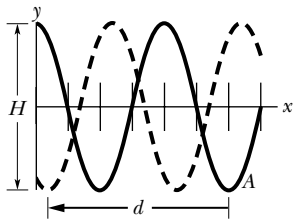

Figure 16-39 Problem 9.

10. The tension in a wire clamped at both ends is halved without appreciably changing the wire's length between the clamps. What is the ratio of the new to the old wave speed for transverse waves traveling along this wire?
11. Two identical traveling waves, moving in the same direction, are out of phase by 0.70π rad. What is the amplitude of the resultant wave in terms of the common amplitude y_m of the two combining waves?
12. A rope, with mass 1.39 kg and fixed at both ends, oscillates in a second-harmonic standing wave pattern. The displacement of the rope is given by
$$y = (0.10 \text{ m})(\sin \pi x/2) \sin 12\pi t,$$
where $x = 0$ at one end of the rope, x is in meters, and t is in seconds. What are (a) the length of the rope, (b) the speed of the waves on the rope, and (c) the tension of the rope? (d) If the rope oscillates in a third-harmonic standing wave pattern, what will be the period of oscillation?
13. A sinusoidal wave travels along a string. The time for a particular point to move from maximum displacement to zero is 0.135 s. What are the (a) period and (b) frequency? (c) The wavelength is 1.40 m; what is the wave speed?

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14. For a particular transverse standing wave on a long string, one of the antinodes is at $x = 0$ and an adjacent node is at $x = 0.10$ m. The displacement $y(t)$ of the string particle at $x = 0$ is shown in Fig. 16-40, where the scale of the y axis is set by $y_s = 4.0$ cm. When $t = 0.50$ s, what is the displacement of the string particle at (a) $x = 0.20$ m and (b) $x = 0.30$ m? What is the transverse velocity of the string particle at $x = 0.20$ m at (c) $t = 0.50$ s and (d) $t = 1.0$ s? (e) Sketch the standing wave at $t = 0.50$ s for the range $x = 0$ to $x = 0.40$ m.

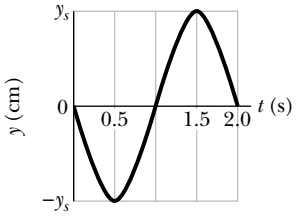

Figure 16-40 Problem 14.

15. A sinusoidal transverse wave of wavelength 18 cm travels along a string in the positive direction of an x axis. The displacement y of the string particle at $x = 0$ is given in Fig. 16-41 as a function of time t. The scale of the vertical axis is set by $y_s = 4.0$ cm. The wave equation is to be in the form $y(x, t) = y_m \sin(kx \pm \omega t + \phi)$. (a) At $t = 0$, is a plot of y versus x in the shape of a positive sine function or a negative sine function? What are (b) y_m, (c) k, (d) ω, (e) ϕ, (f) the sign in front of ω, and (g) the speed of the wave? (h) What is the transverse velocity of the particle at $x = 0$ when $t = 5.0$ s?

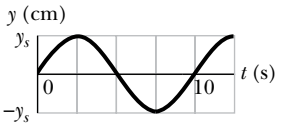

Figure 16-41 Problem 15.

16. Two sinusoidal waves of the same frequency are to be sent in the same direction along a taut string. One wave has an amplitude of 5.50 mm, the other 12.0 mm. (a) What phase difference ϕ_1 between the two waves results in the smallest amplitude of the resultant wave? (b) What is that smallest amplitude? (c) What phase difference ϕ_2 results in the largest amplitude of the resultant wave? (d) What is that largest amplitude? (e) What is the resultant amplitude if the phase angle is $(\phi_1 - \phi_2)/2$?

17. A nylon guitar string has a linear density of 7.20 g/m and is under a tension of 180 N. The fixed supports are distance $D = 90.0$ cm apart. The string is oscillating in the standing wave pattern shown in Fig. 16-42. Calculate the (a) speed, (b) wavelength, and (c) frequency of the traveling waves whose superposition gives this standing wave.

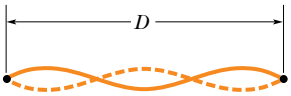

Figure 16-42 Problem 17.

18. A sinusoidal wave of angular frequency 1200 rad/s and amplitude 3.00 mm is sent along a cord with linear density 4.00 g/m and tension 1200 N. (a) What is the average rate at which energy is transported by the wave to the opposite end of the cord? (b) If, simultaneously, an identical wave travels along an adjacent, identical cord, what is the total average rate at which energy is transported to the opposite ends of the two cords by the waves? If, instead, those two waves are sent along the *same* cord simultaneously, what is the total average rate at which they transport energy when their phase difference is (c) 0, (d) 0.4π rad, and (e) π rad?

19. A generator at one end of a very long string creates a wave given by

$$y = (6.0 \text{ cm}) \cos \frac{\pi}{2} [(2.00 \text{ m}^{-1})x + (6.00 \text{ s}^{-1})t],$$

and a generator at the other end creates the wave

$$y = (6.0 \text{ cm}) \cos \frac{\pi}{2} [(2.00 \text{ m}^{-1})x - (6.00 \text{ s}^{-1})t].$$

Calculate the (a) frequency, (b) wavelength, and (c) speed of each wave. For $x \geq 0$, what is the location of the node having the (d) smallest, (e) second smallest, and (f) third smallest value of x? For $x \geq 0$, what is the location of the antinode having the (g) smallest, (h) second smallest, and (i) third smallest value of x?

20. A string under tension τ_i oscillates in the third harmonic at frequency f_3, and the waves on the string have wavelength λ_3. If the tension is increased to $\tau_f = 8\,\tau_i$ and the string is again made to oscillate in the third harmonic, what then are (a) the frequency of oscillation in terms of f_3 and (b) the wavelength of the waves in terms of λ_3?

21. In Fig. 16-43, an aluminum wire, of length $L_1 = 60.0$ cm, cross-sectional area 1.25×10^{-2} cm^2, and density 2.60 g/cm^3, is joined to a steel wire, of density 7.80 g/cm^3 and the same cross-sectional area. The compound wire, loaded with a block of mass $m = 10.0$ kg, is arranged so that the distance L_2 from the joint to the supporting pulley is 86.6 cm. Transverse waves are set up on the wire by an external source of variable frequency; a node is located at the pulley. (a) Find the lowest frequency that generates a standing wave having the joint as one of the nodes. (b) How many nodes are observed at this frequency?

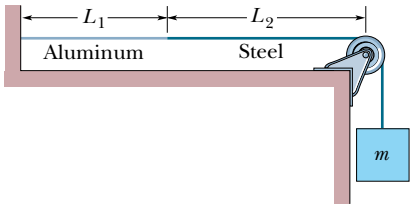

Figure 16-43 Problem 21.

22. *A human wave.* During sporting events within large, densely packed stadiums, spectators will send a wave (or pulse) around the stadium (Fig. 16-44). As the wave reaches a group of spectators, they stand with a cheer and then sit. At any instant, the width w of the wave is the distance from the leading edge (people are just about to stand) to the trailing edge (people have just sat down). Suppose a human wave travels a distance of 853 seats around a stadium in 51 s, with spectators requiring about 1.8 s to respond to the wave's passage by standing and then sitting. What are (a) the wave speed v (in seats per second) and (b) width w (in number of seats)?

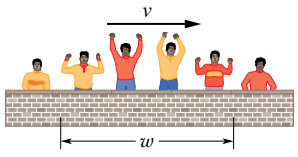

Figure 16-44 Problem 22.

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23. The linear density of a string is 1.9×10^{-4} kg/m. A transverse wave on the string is described by the equation

$$y = (0.021 \text{ m}) \sin[(2.0 \text{ m}^{-1})x + (30 \text{ s}^{-1})t].$$

What are (a) the wave speed and (b) the tension in the string?

24. Two sinusoidal waves with identical wavelengths and amplitudes travel in opposite directions along a string with a speed of 15 cm/s. If the time interval between instants when the string is flat is 0.20 s, what is the wavelength of the waves?

25. A string that is stretched between fixed supports separated by 75.0 cm has resonant frequencies of 450 and 308 Hz, with no intermediate resonant frequencies. What are (a) the lowest resonant frequency and (b) the wave speed?

26. If a transmission line in a cold climate collects ice, the increased diameter tends to cause vortex formation in a passing wind. The air pressure variations in the vortexes tend to cause the line to oscillate (*gallop*), especially if the frequency of the variations matches a resonant frequency of the line. In long lines, the resonant frequencies are so close that almost any wind speed can set up a resonant mode vigorous enough to pull down support towers or cause the line to *short out* with an adjacent line. If a transmission line has a length of 310 m, a linear density of 3.35 kg/m, and a tension of 90.1 MN, what are (a) the frequency of the fundamental mode and (b) the frequency difference between successive modes?

27. In Fig. 16-45, a string, tied to a sinusoidal oscillator at P and running over a support at Q, is stretched by a block of mass m. Separation $L = 1.20$ m, linear density $\mu = 1.20$ g/m, and the oscillator frequency $f = 120$ Hz. The amplitude of the motion at P is small enough for that point to be considered a node. A node also exists at Q. (a) What mass m allows the oscillator to set up the fourth harmonic on the string? (b) What standing wave mode, if any, can be set up if $m = 1.00$ kg?

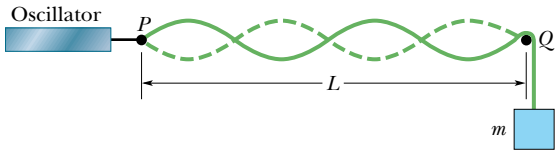

Figure 16-45 Problems 27 and 29.

28. In Fig. 16-46, a sinusoidal wave moving along a string is shown twice as crest A travels in the positive direction of an x axis by distance $d = 6.0$ cm in 3.0 ms. The tick marks along the axis are separated by 10 cm; height $H = 6.00$ mm. The equation for the wave is in the form $y(x, t) = y_m \sin(kx \pm \omega t)$, so what are (a) y_m, (b) k, (c) ω, and (d) the correct choice of sign in front of ω?

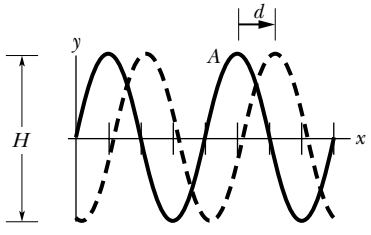

Figure 16-46 Problem 28.

29. In Fig. 16-45, a string, tied to a sinusoidal oscillator at P and running over a support at Q, is stretched by a block of mass m. The separation L between P and Q is 1.20 m, and the frequency f of the oscillator is fixed at 120 Hz. The amplitude of the motion at P is small enough for that point to be considered a node. A node also exists at Q. A standing wave appears when the mass of the hanging block is 286.1 g or 447.0 g, but not for any intermediate mass. What is the linear density of the string?

30. A sinusoidal wave of frequency 500 Hz has a speed of 320 m/s. (a) How far apart are two points that differ in phase by $\pi/3$ rad? (b) What is the phase difference between two displacements at a certain point at times 1.00 ms apart?

31. Use the wave equation to find the speed of a wave given by

$$y(x, t) = (3.00 \text{ mm}) \sin[(3.00 \text{ m}^{-1})x - (8.00 \text{ s}^{-1})t].$$

32. A string has mass 2.00 g, wave speed 120 m/s, and tension 7.00 N. (a)What is its length? (b) What is the lowest resonant frequency of this string?

33. One of the harmonic frequencies for a particular string under tension is 310 Hz. The next higher harmonic frequency is 400 Hz. What harmonic frequency is next higher after the harmonic frequency 850 Hz?

34. In Fig. 16-47*a*, string 1 has a linear density of 3.00 g/m, and string 2 has a linear density of 5.00 g/m. They are under tension due to the hanging block of mass $M = 800$ g. Calculate the wave speed on (a) string 1 and (b) string 2. (*Hint:* When a string loops halfway around a pulley, it pulls on the pulley with a net force that is twice the tension in the string.) Next the block is divided into two blocks (with $M_1 + M_2 = M$) and the apparatus is rearranged as shown in Fig. 16-47*b*. Find (c) M_1 and (d) M_2 such that the wave speeds in the two strings are equal.

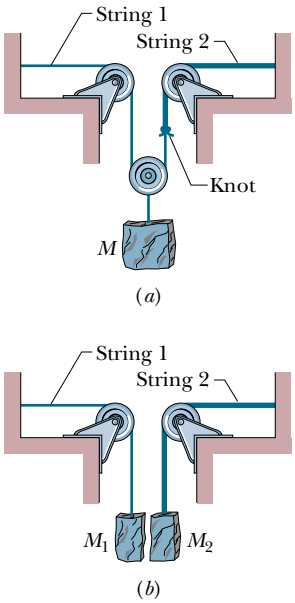

Figure 16-47 Problem 34.

35. Two sinusoidal waves of the same frequency travel in the same direction along a string. If $y_{m1} = 2.0$ cm, $y_{m2} = 4.0$ cm,

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$\phi_1 = 0$, and $\phi_2 = \pi/2$ rad, what is the amplitude of the resultant wave?

36. Figure 16-48 shows the transverse velocity u versus time t of the point on a string at $x = 0$, as a wave passes through it. The scale on the vertical axis is set by $u_s = 12$ m/s. The wave has the generic form $y(x, t) = y_m \sin(kx - \omega t + \phi)$. What then is ϕ?

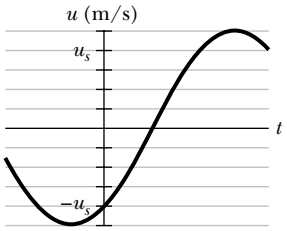

Figure 16-48 Problem 36.

37. These two waves travel along the same string:

$$y_1(x, t) = (4.00 \text{ mm}) \sin(2\pi x - 650\pi t)$$
$$y_2(x, t) = (6.20 \text{ mm}) \sin(2\pi x - 650\pi t + 0.60\pi \text{ rad}).$$

What are (a) the amplitude and (b) the phase angle (relative to wave 1) of the resultant wave? (c) If a third wave of amplitude 5.00 mm is also to be sent along the string in the same direction as the first two waves, what should be its phase angle in order to maximize the amplitude of the new resultant wave?

38. A standing wave pattern on a string is described by

$$y(x, t) = 0.040 \text{ (sin } 4\pi x)(\cos 40\pi t),$$

where x and y are in meters and t is in seconds. For $x \geq 0$, what is the location of the node with the (a) smallest, (b) second smallest, and (c) third smallest value of x? (d) What is the period of the oscillatory motion of any (nonnode) point? What are the (e) speed and (f) amplitude of the two traveling waves that interfere to produce this wave? For $t \geq 0$, what are the (g) first, (h) second, and (i) third time that all points on the string have zero transverse velocity?

39. A sinusoidal wave is sent along a string with a linear density of 5.0 g/m. As it travels, the kinetic energies of the mass elements along the string vary. Figure 16-49*a* gives the rate dK/dt at which kinetic energy passes through the string elements at a particular instant, plotted as a function of distance x along the string. Figure 16-49*b* is similar except that it gives the rate at which kinetic energy passes through a particular mass element (at a particular location), plotted as a function of time t. For both figures, the scale on the vertical (rate) axis is set by $R_s = 10$ W. What is the amplitude of the wave?

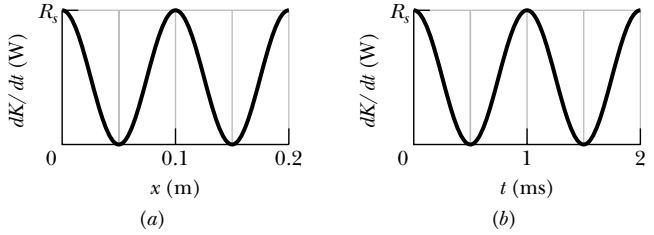

Figure 16-49 Problem 39.

40. The equation of a transverse wave traveling along a very long string is $y = 3.0 \sin(0.020\pi x - 4.0\pi t)$, where x and y are expressed in centimeters and t is in seconds. Determine (a) the amplitude, (b) the wavelength, (c) the frequency, (d) the speed, (e) the direction of propagation of the wave, and (f) the maximum transverse speed of a particle in the string. (g) What is the transverse displacement at $x = 3.5$ cm when $t = 0.26$ s?

41. What is the speed of a transverse wave in a rope of length 1.75 m and mass 60.0 g under a tension of 500 N?

42. The function $y(x, t) = (15.0 \text{ cm}) \cos(\pi x - 15\pi t)$, with x in meters and t in seconds, describes a wave on a taut string. What is the transverse speed for a point on the string at an instant when that point has the displacement $y = +6.00$ cm?

43. What are (a) the lowest frequency, (b) the second lowest frequency, and (c) the third lowest frequency for standing waves on a wire that is 10.0 m long, has a mass of 100 g, and is stretched under a tension of 275 N?

44. A sand scorpion can detect the motion of a nearby beetle (its prey) by the waves the motion sends along the sand surface (Fig. 16-50). The waves are of two types: transverse waves traveling at $v_t = 50$ m/s and longitudinal waves traveling at $v_l = 150$ m/s. If a sudden motion sends out such waves, a scorpion can tell the distance of the beetle from the difference Δt in the arrival times of the waves at its leg nearest the beetle. What is that time difference if the distance to the beetle is 37.5 cm?

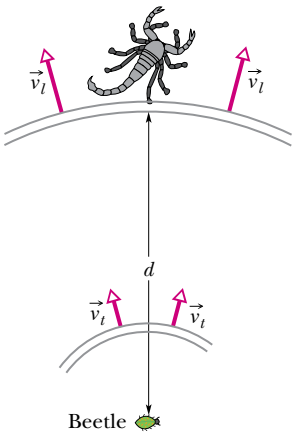

Figure 16-50 Problem 44.

45. Two sinusoidal waves of the same period, with amplitudes of 5.0 and 7.0 mm, travel in the same direction along a stretched string; they produce a resultant wave with an amplitude of 10.0 mm. The phase constant of the 5.0 mm wave is 0. What is the phase constant of the 7.0 mm wave?

46. Four waves are to be sent along the same string, in the same direction:

$$y_1(x, t) = (5.00 \text{ mm}) \sin(4\pi x - 400\pi t)$$
$$y_2(x, t) = (5.00 \text{ mm}) \sin(4\pi x - 400\pi t + 0.8\pi)$$

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$$y_3(x, t) = (5.00\text{ mm})\sin(4\pi x - 400\pi t + \pi)$$
$$y_4(x, t) = (5.00\text{ mm})\sin(4\pi x - 400\pi t + 1.8\pi).$$

What is the amplitude of the resultant wave?

47. If a wave $y(x,t) = (5.0\text{ mm})\sin(kx + (600\text{ rad/s})t + \phi)$ travels along a string, how much time does any given point on the string take to move between displacements $y = +2.0$ mm and $y = -2.0$ mm?

48. A string along which waves can travel is 2.70 m long and has a mass of 130 g. The tension in the string is 36.0 N. What must be the frequency of traveling waves of amplitude 7.70 mm for the average power to be 170 W?

49. A uniform rope of mass m and length L hangs from a ceiling. (a) Show that the speed of a transverse wave on the rope is a function of y, the distance from the lower end, and is given by $v = \sqrt{gy}$. (b) Show that the time a transverse wave takes to travel the length of the rope is given by $t = 2\sqrt{L/g}$.

50. The speed of a transverse wave on a string is 115 m/s when the string tension is 200 N. To what value must the tension be changed to raise the wave speed to 223 m/s?

51. A sinusoidal transverse wave is traveling along a string in the negative direction of an x axis. Figure 16-51 shows a plot of the displacement as a function of position at time $t = 0$; the scale of the y axis is set by $y_s = 4.0$ cm. The string tension is 3.6 N, and its linear density is 28 g/m. Find the (a) amplitude, (b) wavelength, (c) wave speed, and (d) period of the wave. (e) Find the maximum transverse speed of a particle in the string. If the wave is of the form $y(x,t) = y_m \sin(kx \pm \omega t + \phi)$, what are (f) k, (g) ω, (h) ϕ, and (i) the correct choice of sign in front of ω?

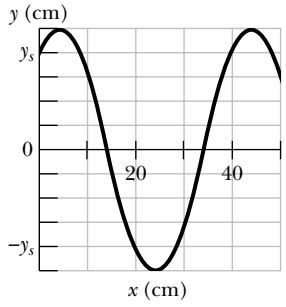

Figure 16-51 Problem 51.

52. Use the wave equation to find the speed of a wave given in terms of the general function $h(x,t)$:

$$y(x, t) = (4.00\text{ mm})\, h[(22.0\text{ m}^{-1})x + (8.00\text{ s}^{-1})t].$$

53. A transverse sinusoidal wave is moving along a string in the positive direction of an x axis with a speed of 70 m/s. At $t = 0$, the string particle at $x = 0$ has a transverse displacement of 4.0 cm and is not moving. The maximum transverse speed of the string particle at $x = 0$ is 16 m/s. (a) What is the frequency of the wave? (b) What is the wavelength of the wave? If $y(x, t) = y_m \sin(kx \pm \omega t + \phi)$ is the form of the wave equation, what are (c) y_m, (d) k, (e) ω, (f) ϕ, and (g) the correct choice of sign in front of ω?

54. A sinusoidal wave travels along a string under tension. Figure 16-52 gives the slopes along the string at time $t = 0$. The scale of the x axis is set by $x_s = 0.40$ m. What is the amplitude of the wave?

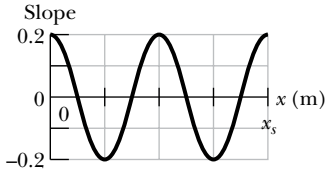

Figure 16-52 Problem 54.

55. A string oscillates according to the equation

$$y' = (0.80\text{ cm})\sin\left[\left(\frac{\pi}{3}\text{ cm}^{-1}\right)x\right]\cos[(40\pi\text{ s}^{-1})t].$$

What are the (a) amplitude and (b) speed of the two waves (identical except for direction of travel) whose superposition gives this oscillation? (c) What is the distance between nodes? (d) What is the transverse speed of a particle of the string at the position $x = 2.1$ cm when $t = 0.50$ s?

56. Two sinusoidal waves with the same amplitude and wavelength travel through each other along a string that is stretched along an x axis. Their resultant wave is shown twice in Fig. 16-53, as the antinode A travels from an extreme upward displacement to an extreme downward displacement in 6.0 ms. The tick marks along the axis are separated by 15 cm; height H is 1.20 cm. Let the equation for one of the two waves be of the form $y(x, t) = y_m \sin(kx + \omega t)$. In the equation for the other wave, what are (a) y_m, (b) k, (c) ω, and (d) the sign in front of ω?

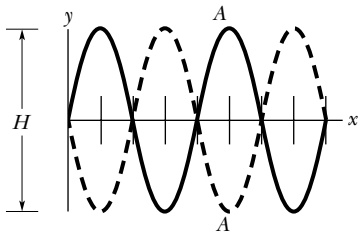

Figure 16-53 Problem 56.

57. A 40 cm wire having a mass of 3.2 g is stretched between two fixed supports 40.05 cm apart. In its fundamental mode, the wire vibrates at 220 Hz. If the area of cross section of the wire is 1.0 mm^2, find its Young modulus?

58. A string, fixed at both ends, vibrates in a resonant mode with a separation of 2.0 cm between the consecutive nodes. For the next higher resonant frequency, this separation is reduced to 1.6 cm. Find the length of the string.

59. Figure 16-54 shows the displacement y versus time t of the point on a string at $x = 0$, as a wave passes through that

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point. The scale of the y axis is set by $y_s = 6.0$ mm. The wave has form $y(x, t) = y_m \sin(kx - \omega t + \phi)$. What is ϕ?

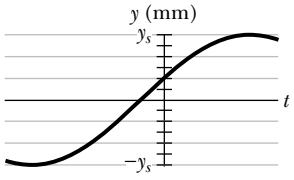


Figure 16-54 Problem 59.

60. Figure 16-55 shows an aluminum wire of length 60 cm joined to a steel wire of length 80 cm and stretched between two fixed supports. The tension produced is 40 N. The cross-sectional area of the steel wire is 1.0 mm^2 and that of the aluminum wire is 3.0 mm^2. What could be the minimum frequency of a tuning fork which can produce standing waves in the system with the joint as a node? The density of aluminum is 2.6 g/cm^3 and that of steel is 7.8 g/cm^3.

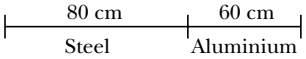
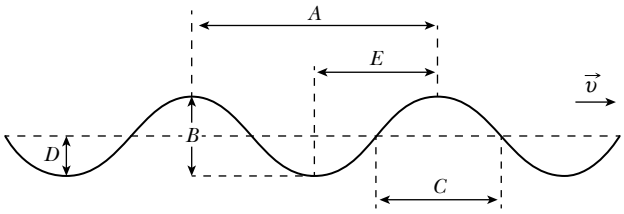


Figure 16-55 Problem 60.

PRACTICE QUESTIONS

Single Correct Choice Type

1. A sinusoidal wave is traveling toward the right as shown in the figure. Which letter correctly labels the wavelength of the wave?



(a) A **(b)** B
(c) C **(d)** D

2. Which one of the following waves is purely longitudinal?
(a) Sound waves in air.
(b) Radio waves traveling through air.
(c) Light waves traveling through vacuum.
(d) Waves on a plucked guitar string.

3. A periodic wave is produced on a stretched string. Which one of the following properties is *not* related to the speed of the wave?
(a) Frequency
(b) Amplitude
(c) Period
(d) Wavelength

4. A sinusoidal transverse wave is traveling on a string. Any point on the string
(a) moves in the same direction as the wave.
(b) moves in simple harmonic motion with a different frequency than that of the wave.
(c) moves in simple harmonic motion with the same angular frequency as the wave.
(d) moves in uniform circular motion with a different angular speed than the wave.

5. A transverse periodic wave described by the expression

$$y = \sin\left[2\pi\left(\frac{x}{2} + \frac{t}{10}\right)\right]$$

(where y and x are in meters and t is in seconds) is established on a string. Which one of the following statements concerning this wave is false?
(a) The wave is traveling in the negative x direction.
(b) The amplitude is 1.0 m.
(c) The frequency of the wave is 0.10 Hz.
(d) The wave travels with speed 5.0 m/s.

6. When a certain string is clamped at both ends, the lowest four resonant frequencies are measured to be 100, 150, 200, and 250 Hz. One of the resonant frequencies (below 200 Hz) is missing. What is it?
(a) 25 Hz **(b)** 50 Hz
(c) 75 Hz **(d)** 125 Hz

7. A string of length 0.4 m and mass 10^{-2} kg is clamped at its ends. The tension in the string is 1.6 N. When a pulse travels along the string, the shape of the string is found to be the same at times t and $t + \Delta t$. The value of Δt is
(a) 0.05 s **(b)** 0.1 s
(c) 0.2 s **(d)** 0.4 s

8. A woman is standing in the ocean, and she notices that after a wave crest passes, five more crests pass in a time of 50.0 s. The distance between two successive crests is 32 m. Determine, if possible, the wave's period and wavelength.
(a) 0.0200 s, 64 m
(b) 0.100 s, 16 m
(c) 10.0 s, 32 m
(d) 50.0 s, 8.0 m

9. What is the wavelength of a wave with a speed of 12 m/s and a period of 0.25 s?
(a) 0.25 m **(b)** 1.5 m
(c) 3.0 m **(d)** 24 m

10. The mass of a string is 5.0×10^{-3} kg, and it is stretched so that the tension in it is 180 N. A transverse wave traveling on this string has a frequency of 260 Hz and a wavelength of 0.60 m. What is the length of the string?
(a) 1.1 m **(b)** 0.68 m
(c) 0.47 m **(d)** 0.22 m

11. A steel cable of cross-sectional area 2.83×10^{-3} m^2 is kept under a tension of 1.00×10^4 N. The density of steel is

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7860 kg/m^3 (this is *not* the linear density). At what speed does a transverse wave move along the cable?

(a) 22.7 m/s **(b)** 21.2 m/s
(c) 19.9 m/s **(d)** 17.6 m/s

12. A copper wire, whose cross-sectional area is 1.1×10^{-6} m^2, has a linear density of 7.0×10^{-3} kg/m and is strung between two walls. At the ambient temperature, a transverse wave travels with a speed of 46 m/s on this wire. The coefficient of linear expansion for copper is 17×10^{-6}/(°C), and Young's modulus for copper is 1.1×10^{11} N/m^2. What will be the speed of the wave when the temperature is lowered by 14°C? Ignore any change in the linear density caused by the change in temperature.

(a) 84 m/s **(b)** 79 m/s
(c) 61 m/s **(d)** 55 m/s

13. A transverse wave on a string has an amplitude of 0.20 m and a frequency of 175 Hz. Consider the particle of the string at $x = 0$ m. It begins with a displacement of $y = 0$ m when $t = 0$ s. How much time passes between the first two instants when this particle has a displacement of $y = 0.10$ m?

(a) 1.9×10^{-3} s **(b)** 2.9×10^{-3} s
(c) 3.8×10^{-3} s **(d)** 5.7×10^{-3} s

14. The displacement (in meters) of a wave is given according to $y = 0.26 \sin(\pi t - 3.7\pi x)$, where t is in seconds and x is in meters. What is the displacement y when $t = 38$ s and $x = 13$ m?

(a) −0.14 m **(b)** −0.080 m
(c) −0.041 m **(d)** −0.019 m

15. A wire is stretched between two posts. Another wire is stretched between two posts that are twice as far apart. The tension in the wires is the same, and they have the same mass. A transverse wave travels on the shorter wire with a speed of 240 m/s. What would be the speed of the wave on the longer wire?

(a) 170 m/s **(b)** 240 m/s
(c) 290 m/s **(d)** 340 m/s

16. A person lying on an air mattress in the ocean rises and falls through one complete cycle every five seconds. The crests of the wave causing the motion are 20.0 m apart. Determine the frequency and the speed of the wave.

(a) 0.150 Hz, 15.0 m/s
(b) 0.100 Hz, 8.00 m/s
(c) 0.350 Hz, 2.00 m/s
(d) 0.200 Hz, 4.00 m/s

17. The amplitude of a transverse wave on a string is 4.5 cm. The ratio of the maximum particle speed to the speed of the wave is 3.1. What is the wavelength (in cm) of the wave?

(a) 4.5 cm **(b)** 7.0 cm
(c) 9.1 cm **(d)** 14 cm

18. A jet skier is moving at 8.4 m/s in the direction in which the waves on a lake are moving. Each time he passes over a crest, he feels a bump. The bumping frequency is 1.2 Hz, and the crests are separated by 5.8 m. What is the wave speed?

(a) 1.4 m/s **(b)** 2.1 m/s
(c) 7.0 m/s **(d)** 8.4 m/s

19. A certain string on a piano is tuned to produce musical note of frequency $f = 261.63$ Hz by carefully adjusting the tension in the string. For a fixed wavelength, what is the frequency when this tension is doubled?

(a) 130.08 Hz
(b) 185.00 Hz
(c) 370.00 Hz
(d) 446.63 Hz

20. Of the three traveling waves listed below, which one(s) is(are) traveling in the + x direction?

(1) $y = +3.2 \sin[4.1t + 2.3x]$
(2) $y = -6.8 \sin[-3.0t + 1.5x]$
(3) $y = +4.9 \sin[12.0t + 18x]$

(a) 1 only **(b)** 2 only
(c) 3 only **(d)** 1, 2, and 3

21. A transverse periodic wave is established on a string. The wave is described by the expression

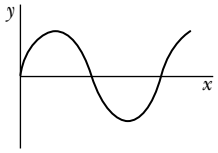
$$y = 0.005 \sin(20.0x - 2\pi ft)$$

where y is in meters when x and t are in meters and seconds, respectively. If the wave travels with a speed of 20.0 m/s, what is its frequency, f?

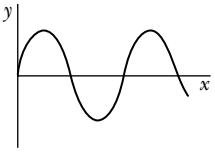
(a) 0.16 Hz **(b)** 0.64 Hz
(c) 31.9 Hz **(d)** 63.7 Hz

22. Three traveling sinusoidal waves are on identical strings, with the same tension. The mathematical forms of the waves are $y_1(x, t) = y_m \sin(3x - 6t)$, $y_2(x, t) = y_m \sin(4x - 8t)$, and $y_3(x, t) = y_m \sin(6x - 12t)$, where x is in meters and t is in seconds. Match each mathematical form to the appropriate graph below.

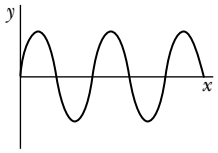
(i)



(ii)



(iii)



(a) $y_1 \rightarrow$ (i); $y_2 \rightarrow$ (ii); $y_3 \rightarrow$ (iii)
(b) $y_1 \rightarrow$ (iii); $y_2 \rightarrow$ (ii); $y_3 \rightarrow$ (i)
(c) $y_1 \rightarrow$ (i); $y_2 \rightarrow$ (iii); $y_3 \rightarrow$ (ii)
(d) $y_1 \rightarrow$ (ii); $y_2 \rightarrow$ (i); $y_3 \rightarrow$ (iii)

23. The displacement of a string carrying a traveling sinusoidal wave is given by

$$y(x, t) = y_m \sin(kx - \omega t - \phi)$$

At time $t = 0$ the point at $x = 0$ has a velocity of 0 and a positive displacement. The phase constant ϕ is

(a) 90° **(b)** 135°
(c) 180° **(d)** 270°

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24. The tension in a string with a linear density of 0.0010 kg/m is 0.40 N. A 100 Hz sinusoidal wave on this string has a wavelength of
(a) 0.05 cm **(b)** 2.0 cm
(c) 5.0 cm **(d)** 20 cm

25. A long string is constructed by joining the ends of two shorter strings. The tension in the strings is the same but string I has 4 times the linear mass density of string II. When a sinusoidal wave passes from string I to string II:
(a) the frequency decreases by a factor of 4.
(b) the frequency decreases by a factor of 2.
(c) the wavelength decreases by a factor of 4.
(d) the wavelength decreases by a factor of 2.

26. Two identical but separate strings, with the same tension, carry sinusoidal waves with the same frequency. Wave A has an amplitude that is twice that of wave B and transmits energy at a rate that is _____ that of wave B.
(a) half
(b) twice
(c) one-fourth
(d) four times

27. Fully constructive interference between two sinusoidal waves of the same frequency occurs only if they
(a) travel in opposite directions and are in phase.
(b) travel in opposite directions and are 180° out of phase.
(c) travel in the same direction and are in phase.
(d) travel in the same direction and are 180° out of phase.

28. Two sinusoidal waves have the same angular frequency, the same amplitude y_m, and travel in the same direction in the same medium. If they differ in phase by 50°, the amplitude of the resultant wave is given by
(a) $0.64y_m$ **(b)** $1.3y_m$
(c) $0.91y_m$ **(d)** $1.8y_m$

29. A wave on a stretched string is reflected from a fixed end P of the string. The phase difference, at P, between the incident and reflected waves is
(a) zero.
(b) π rad.
(c) $\pi/2$ rad.
(d) depends on the velocity of the wave.

30. Which of the following represents a standing wave?
(a) $y = (6.0\text{ mm})\sin[(3.0\text{ m}^{-1})x + (2.0\text{ s}^{-1})t] - (6.0\text{ mm})\cos[(3.0\text{ m}^{-1})x + 2.0]$
(b) $y = (6.0\text{ mm})\cos[(3.0\text{ m}^{-1})x - (2.0\text{ s}^{-1})t] + (6.0\text{ mm})\cos[(2.0\text{ s}^{-1})t + 3.0\text{ m}^{-1})x]$
(c) $y = (6.0\text{ mm})\cos[(3.0\text{ m}^{-1})x - (2.0\text{ s}^{-1})t] - (6.0\text{ mm})\sin[(2.0\text{ s}^{-1})t - 3.0]$
(d) $y = (6.0\text{ mm})\sin[(3.0\text{ m}^{-1})x - (2.0\text{ s}^{-1})t] - (6.0\text{ mm})\cos[(2.0\text{ s}^{-1})t + 3.0\text{ m}^{-1})x]$

31. Standing waves are produced by the interference of two traveling sinusoidal waves, each of frequency 100 Hz. The distance from the 2nd node to the 5th node is 60 cm. The wavelength of each of the two original waves is
(a) 50 cm. **(b)** 40 cm.
(c) 30 cm. **(d)** 20 cm.

32. When a string is vibrating in a standing wave pattern the power transmitted across an antinode, compared to the power transmitted across a node, is
(a) more. **(b)** less.
(c) the same (zero). **(d)** the same (non-zero).

33. Two strings A and B, made of same material, are stretched by same tension. The radius of string A is double of the radius of B. A transverse wave travels on A with speed v_A and on B with speed v_B. The ratio v_A/v_B is
(a) 1/2 **(b)** 2
(c) 1/4 **(d)** 4

More than One Correct Choice Type

34. At the nodes in stationary waves, the
(a) change in pressure and density are maximum.
(b) change in pressure and density are minimum.
(c) strain is zero.
(d) energy is maximum.

35. Two particles A and B have a phase difference of π when a sine wave passes through the region
(a) A oscillates at half the frequency of B.
(b) A and B move in opposite direction.
(c) A and B must be separated by half of the wavelength.
(d) The displacement at A and B have equal magnitudes.

36. A plane progressive wave of frequency 25 Hz, amplitude 2.5×10^{-5} m and initial phase is zero moves along the negative x direction with a velocity of 300 m/s. A and B are two points 6 m apart on the line of propagation of the wave. At any instant the phase difference between A and B is ϕ. The maximum difference in the displacement at A and B is Δ. Then,
(a) $\phi = \pi$ **(b)** $\phi = 0$
(c) $\Delta = 0$ **(d)** $\Delta = 5 \times 10^{-5}$ m

37. In a stationary wave system, all the particles of the medium
(a) have zero displacement simultaneously at some instant.
(b) have maximum displacement simultaneously at some instant.
(c) are at rest simultaneously at some instant.
(d) reach maximum velocity simultaneously at some instant.

38. A string of length L is stretched along x-axis is rigidly clamped as it ends, it undergoes transverse vibration. If n is an integer, which of the following relations may represent the shape of the string at any time t?
(a) $y = y_m \sin\left(\frac{n\pi x}{L}\right)\cos\omega t$
(b) $y = y_m \sin\left(\frac{n\pi x}{L}\right)\sin\omega t$
(c) $y = y_m \cos\left(\frac{n\pi x}{L}\right)\cos\omega t$
(d) $y = y_m \cos\left(\frac{n\pi x}{L}\right)\sin\omega t$

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39. It is desired to increase the fundamental resonance frequency in a tube that is closed at one end. This can be achieved by
- **(a)** replacing the air in the tube by hydrogen gas.
- **(b)** increasing the length of the tube.
- **(c)** decreasing the length of the tube.
- **(d)** opening the closed end of the tube.

40. A wave disturbance in a medium is described by $y(x, t) = 0.02\cos 50\pi t + (\pi/2)\cos(10\pi x)$, where x and y are in meters and t in seconds.
- **(a)** A displacement node occurs at $x = 0.15$ m.
- **(b)** An antinode occurs at $x = 0.3$ m.
- **(c)** The wavelength of the wave is 0.2 m.
- **(d)** The speed of the wave is 5.0 m/s.

41. As a wave propagates,
- **(a)** the wave intensity remains constant for a plane wave.
- **(b)** the wave intensity decreases as the inverse of the distance from the source for a spherical wave.
- **(c)** the wave intensity decreases as the inverse square of the distance from the source for a spherical wave.
- **(d)** total intensity of the spherical wave over the spherical surface centered at the source remains constant at all times.

42. A transverse sinusoidal wave of amplitude a, wavelength λ, and frequency n is traveling on a stretched string. The maximum speed of any point on the string is $v/10$, where v is the speed of propagation of the wave. If $a = 10^{-3}$ m and $v = 10$ m/s, then λ and n are given by

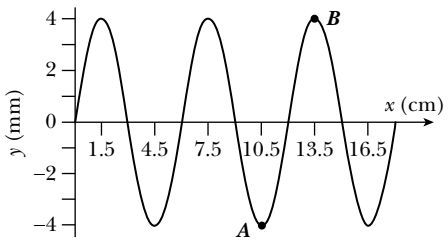
(a) $\lambda = 2\pi \times 10^{-2}$ m **(b)** $\lambda = 10^{-3}$ m

(c) $n = \dfrac{10^3}{2\pi}$ Hz **(d)** $n = 10^4$ Hz

43. A wave is represented by the given equation $y = A\sin[10\pi x + 15\pi x + (\pi/3)]$ where x is in meter and t is in second. The expression represents
- **(a)** a wave traveling in the positive x direction with a velocity of 1.5 m/s.
- **(b)** a wave traveling in the negative x direction with a velocity of 1.5 m/s.
- **(c)** a wave traveling in the negative x direction with a wavelength of 0.2 m.
- **(d)** a wave traveling in the positive x direction with a wavelength of 0.2 m.

Linked Comprehension

Paragraph for Questions 44–47: The displacement of a vibrating string versus position along the string is shown in the figure. The periodic waves have a speed of 10.0 cm/s. A and B are two points on the string.



44. What is the amplitude of the wave?
- **(a)** 2 mm **(b)** 4 mm
- **(c)** 12 mm **(d)** 8 mm

45. What is the wavelength of the wave?
- **(a)** 3.0 cm **(b)** 6.0 cm
- **(c)** 9.0 cm **(d)** 12 cm

46. What is the frequency of the wave?
- **(a)** 0.60 Hz **(b)** 0.90 Hz
- **(c)** 1.7 Hz **(d)** 1.3 Hz

47. What is the difference in phase between the points A and B?
- **(a)** $(\pi/4)$ radians **(b)** $(\pi/2)$ radians
- **(c)** π radians **(d)** $(3\pi/4)$ radians

Paragraph for Questions 48–51: A periodic transverse wave is established on a string such that there are exactly two cycles traveling along a 3.0-m section of the string. The crests move at 20.0 m/s.

48. What is the frequency of the wave?
- **(a)** 0.67 Hz **(b)** 1.33 Hz
- **(c)** 13 Hz **(d)** 30 Hz

49. What is the shortest horizontal distance from a crest to a point of zero acceleration?
- **(a)** 0.38 m **(b)** 0.75 m
- **(c)** 1.5 m **(d)** 3.0 m

50. How long does it take a particle at the top of a crest to reach the bottom of an adjacent trough?
- **(a)** 0.018 s **(b)** 0.038 s
- **(c)** 0.075 s **(d)** 0.150 s

51. How could the speed of the wave be increased?
- **(a)** By increasing the period.
- **(b)** By increasing the tension in the string.
- **(c)** By decreasing the amplitude.
- **(d)** By decreasing the frequency.

Matrix-Match

52. One end of a steel wire is attached to a fixed support and the other end is attached to a string, which is going over a pulley and is connected to a block, which is hanging vertically. It is observed that the wire vibrates in its fundamental mode with frequency of 200 Hz. When the block is submerged in water, the same wire vibrates in its 1 harmonic with frequency of 100 Hz. Density of steel wire is 8000 kg/m^3 and length of wire is 1 m.

Column I	Column II
(a) Longitude stress in the wire (in 10^7 N/m^2)	**(p)** 400
(b) Velocity of transverse waves along this wire (in m/s)	**(q)** 128
(c) Longitudinal stress in the wire in submerged condition in (10^7 N/m^2)	**(r)** 0.002
(d) If maximum acceleration of the wire is 800 m/s^2, amplitude of vibration at the midpoint (in m)	**(s)** 32

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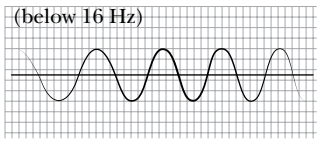
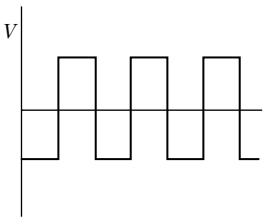
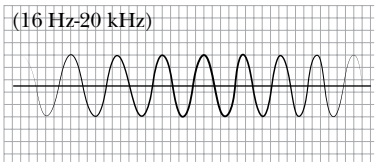
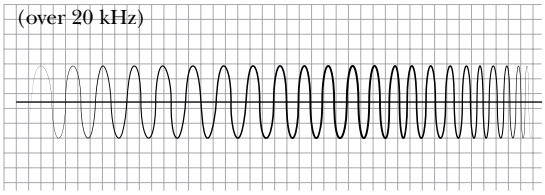
Directions for Questions 53–54: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

1. In the given table, Column I gives frequencies of different waves, Column II gives their examples and Column III gives their graphical representation.

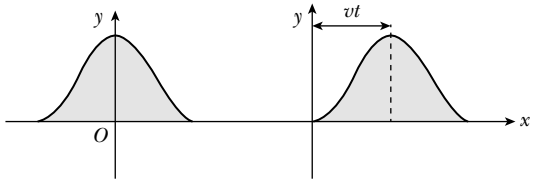
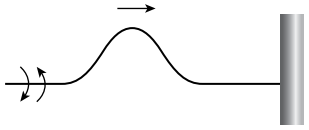
Column I	Column II	Column III
(I) Frequency range = 20 Hz to 20 kHz	**(i)** waves produced by ocean waves	**(J)** **Infrasoud** 
(II) Frequency range = 20 Hz	**(ii)** dog whistle that cannot be heard by humans	**(K)** 
(III) Frequency range < 20 Hz	**(iii)** waves produced during earth quake	**(L)** **Audible frequencies** 
(IV) Frequency range >20 KHz	**(iv)** waves produced by vibrating sitar	**(M)** **Ultrasound** 

(1) What are the characteristics of audible waves?
- **(a)** (I) (iv) (L)
- **(b)** (II) (iv) (J)
- **(c)** (II) (i) (M)
- **(d)** (II) (i) (K)

(2) What are the characteristics of ultrasonic waves?
- **(a)** (IV) (ii) (M)
- **(b)** (I) (iii) (L)
- **(c)** (II) (iv) (L)
- **(d)** (III) (ii) (M)

(3) What are the characteristics of infrasonic waves?
- **(a)** (II) (i) (K)
- **(b)** (I) (i) (L)
- **(c)** (III) (iii) (J)
- **(d)** (I) (iv) (L)

54. In the given table, Column I gives description of waves characteristic, Column II gives their characteristic and Column III shows their graphical representation.

Column I	Column II	Column III
(I) Short wave produced in a medium	**(i)** displacement for sound waves	**(J)** 
(II) Mathematical description of the disturbance created by a wave	**(ii)** the disturbance has the same phase at all points.	**(K)** 

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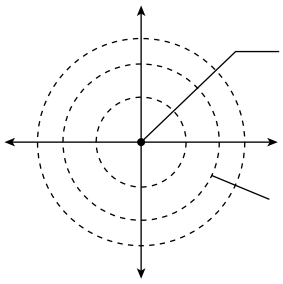
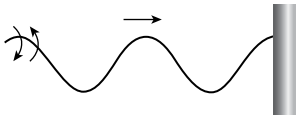
Column I	Column II	Column III
(III) A group of waves of equal or similar wavelengths	**(iii)** traveling in the same direction.	**(L)** 
(IV) A line or surface on which	**(iv)** when the disturbance is created for a short time	**(M)** 

(1) What are the characteristics of a wave pulse?
(a) (I) (iii) (L) **(b)** (IV) (ii) (M)
(c) (II) (iv) (J) **(d)** (I) (iv) (K)

(2) What are the characteristics of a wave front?
(a) (III) (ii) (J) **(b)** (IV) (ii) (L)
(c) (II) (iii) (K) **(d)** (I) (i) (M)

(3) What are the characteristics of a wave function?
(a) (II) (i) (J) **(b)** (I) (i) (K)
(c) (III) (iii) (L) **(d)** (I) (ii) (L)

Integer Type

55. A progressive and a stationary simple harmonic wave each has the same frequency 250 Hz and the same velocity of 30 m/s. Calculate the distance between consecutive nodes (in cm) in the stationary wave.

56. When a force F_1 acts on a particle, frequency is 6 Hz and when a force F_2 acts, frequency is 8 Hz. What is the frequency when both the forces act simultaneously in the same direction?

57. The speed of a transverse wave on a string is 450 m/s, and the wavelength is 0.18 m. The amplitude of the wave is 2.0 mm. How much time is required for a particle of the string to move through a total distance of 1.0 km?

58. A sinusoidal wave of angular frequency 1200 rad/s and amplitude 3.00 mm is sent along a cord with linear density 2.00 g/m and tension 1200 N. What is the average rate at which energy is transported by the wave to the opposite end of the cord?

ANSWER KEY

Checkpoints

1. (a) 2; (b) 3; (c) 1 (compare with the phase in Eq. 16-4, then see Eq. 16-7)
2. (a) 2, 3, 1; (b) 3 then 2 and 1 tie (find amplitude of dy/dt)
3. (a) same (independent of f); (b) decreases ($\lambda = v/f$); (c) increase; (d) increase
4. 0.20 and 0.80 tie and then 0.60, 0.45 **5.** 0.133 **6.** 11/369 s **7.** (a) 1; (b) 3; (c) 2
8. (a) 75 Hz; (b) 525 Hz

Problems

1. (a) 0.160 mm; (b) 141 m^{-1}; (c) 628 rad/s; (d) plus sign is used since the wave is traveling in the negative x direction
2. 3.5 **3.** (a) 77.5 m/s; (b) 15.0 m; (c) 5.17 Hz **4.** (a) 60 m/s; (b) 15 N
5. (a) 5.0×10^{-3} m.; (b) 2.4 m^{-1}; (c) 38 Hz; (d) minus **6.** (a) 130°; (b) 2.26 rad; (c) 0.360 wavelength
7. 7.22 m (distance from right end of wire) or 2.78 (distance from left end of wire)
8. (a) third harmonic of B; (b) sixth harmonic of B; (c) None of the first eight harmonics of B would match A's third harmonic
9. (a) 6.00 mm; (b) 16 rad/m; (c) 1.1×10^3 rad/s; (d) 2.5 rad; (e) plus sign
10. $1/\sqrt{2}$ **11.** 0.91 y_m
12. (a) 4.0 m; (b) 24 m/s; (c) 200 N; (d) 0.11 s **13.** (a) 0.540 s; (b) 1.85 Hz; (c) 2.59 m/s

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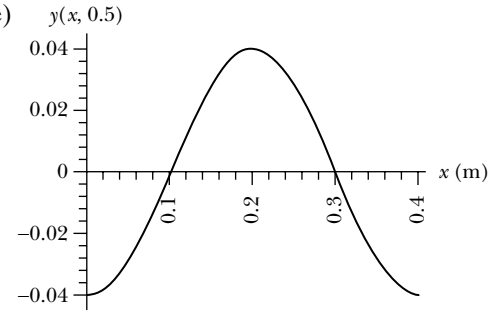
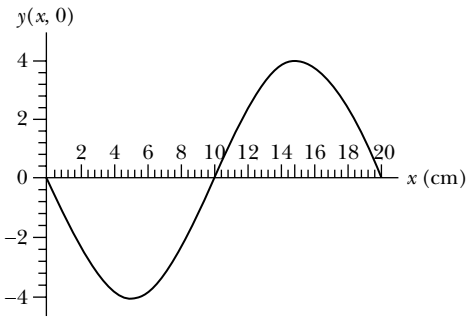
14. (a) 0.040 m; (b) 0; (c) 0 m/s; (d) 1.0 s;

(e)



15. (a);



(c) 4.0 cm; (c) 0.35 rad/cm; (d) 0.63 rad/s; (e) pi; (f) The sign is minus since the wave is traveling in the + x direction; (g) 1.8 cm/s; (h) –2.5 cm/s

16. (a) pi rad; (b) 6.50 mm; (c) 0 rad; (d) 17.5 mm; (e) 13.2 mm

17. (a) 158 m/s ≈ 1.58×10^2 m/s; (b) 60.0 cm; (c) 264 Hz

18. (a) 14.2 W ≈ 14 W; (b) 28.4 W ≈ 28 W; (c) 56.8 W 56.8 W ≈ 57 W; (d) 37.2 W; (e) 0 W

19. (a) 1.50 Hz; (b) 2.00 m; (c) 3.00 m/s; (d) $x = 0.500$ m = 50.0 cm ($n = 0$; (e) $x = 1.50$ m = 150 cm ($n = 1$); (f) $x = 2.50$ m = 250 cm ($n = 2$); (g) $x = 0$ ($n = 0$); (h) $x = 1.00$ m = 100 cm ($n = 1$); (i) $x = 2.00$ m = 200 cm ($n = 2$)

20. (a) under root 8; f_3; (b) λ_3

21. (a) 289 Hz; (b) eight nodes

22. (a) 17 seats/s; (b) 30 seats

23. (a) 15 m/s; (b) 0.043N

24. 6.0 cm

25. (a) 142 Hz; (b) 213 m/s

26. (a) 8.36 Hz; (b) 8.36 Hz

27. (a) 0.635kg; (b) the mass cannot set up a standing wave on the string

28. (a) 3.0 mm; (b) 16 rad/m; (c) 3.2×10^2 rad/s; (d) minus sign

29. 0.845 g/m

30. (a) 107 mm; (b) *pi* rad

31. 2.67 m/s

32. (a) 4.11 m; (b) 14.6 Hz

33. 940 Hz

34. (a) 36.2m/s; (b) 28.0m/s; (c) 300 g; (d) 500 g

35. 4.5 cm

36. ±0.6435 rad

37. (a) 6.25 mm; (b) 70.54° (or 1.23 rad); (c) 70.54° (or 1.23 rad)

38. (a) $x = 0$; (b) $x = 0.25$ m; (c) $x = 0.50$ m; (d) 50 ms; (e) 10 m/s; (f) 2.0 cm; (g) $t = 0$ s; (h) 25 ms; (i) 50 ms

39. $y_m = 0.0020$ m 2.00 mm

40. (a) 3.0 cm; (b) 1.0×10^2 cm; (c) 2.0 Hz; (d) 2.0×10^2 cm/s; (e) +x direction; (f) 38 cm/s; (g) –0.28 cm

41. 121 m/s

42. 6.48 m/s

43. (a) 8.29 Hz; (b) 16.6 Hz; (c) 24.9 Hz

44. 5.0 ms

45. 1.19 rads, or 68°

46. No resultant wave

47. 0.0014 s or 1.4 ms

48. 332 Hz

50. 752 N

51. (a) 5.0 cm; (b) 0.40; (c) 11.3 m/s; (d) 0.035 s; (e) 889 cm/s ≈ 8.9 m/s; (f) 16 m^{-1}; (g) 1.8×10^2 rad/s; (h) 0.93 rad; (i) plus sign

52. 0.364 m/s

53. (a) 64 Hz; (b) 1.1 m; (c) 4.0×10^{-2} m; (d) 5.7 rad/m; (e) 4.0×10^2 rad/s; (f) pi/2; (g) minus sign

54. 0.00637 m ≈ 6.4 mm

55. (a) 0.40 cm; (b) 1.2×10^2 cm/s; (c) 3.0 cm; (d) 0

56. (a) 3.0 mm; (b) 10.47 m^{-1} ≈ 10 m^{-1}; (c) 5.2×10^2 rad/s; (d) minus sign

57. 1.98×10^{11} N/m^2

58. 8.0 cm

59. (a) 2 s; (b) 3 s; (c) 2×10^{-3} N

60. 180 Hz

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Practice Questions

Single Correct Choice Type

1. (a)	2. (a)	3. (b)	4. (c)	5. (d)
6. (b)	7. (b)	8. (c)	9. (c)	10. (b)
11. (b)	12. (b)	13. (a)	14. (b)	15. (d)
16. (d)	17. (c)	18. (a)	19. (c)	20. (b)
21. (d)	22. (a)	23. (d)	24. (d)	25. (d)
26. (d)	27. (c)	28. (d)	29. (b)	30. (b)
31. (b)	32. (c)	33. (a)		

More than One Correct Choice Type

34. (a), (d)	35. (c), (d)	36. (a), (d)	37. (a), (b), (c), (d)
38. (a), (b)	39. (a), (c), (d)	40. (a), (b), (c), (d)	41. (a), (b), (d)
42. (a), (c)	43. (b), (c)		

Linked Comprehension

44. (b)	45. (b)	46. (c)	47. (c)
48. (c)	49. (a)	50. (b)	51. (b)

Matrix-Match

52. (a) → (q); (b) → (p); (c) → (s); (d) → (r)

53. (1) → (a); (2) → (a); (3) → (c)

54. (1) → (d); (2) → (b); (3) → (a)

Integer Type

55. 6	56. 10	57. 50	58. 10

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CHAPTER 17

Waves – II

17.1 | WHAT IS PHYSICS?

The physics of sound waves is the basis of countless studies in the research journals of many fields. Here are just a few examples. Some physiologists are concerned with how speech is produced, how speech impairment might be corrected, how hearing loss can be alleviated, and even how snoring is produced. Some acoustic engineers are concerned with improving the acoustics of concert halls, with reducing noise near freeways and road construction, and with reproducing music by speaker systems. Some aviation engineers are concerned with the shock waves produced by supersonic aircraft and the aircraft noise produced in communities near an airport. Some medical researchers are concerned with how noises produced by the heart and lungs can signal a medical problem in a patient. Some military engineers are concerned with how the sounds of sniper fire might allow a soldier to pinpoint the sniper's location, and, on the gentler side, some biologists are concerned with how a cat purrs.

To begin our discussion of the physics of sound, we must first answer the question "What *are* sound waves?"

17.2 | SOUND WAVES

Key Concept

- Sound waves are longitudinal mechanical waves that can travel through solids, liquids, or gases.

We know that, mechanical waves are waves that require a material medium to exist. There are two types of mechanical waves: *Transverse waves* involve oscillations perpendicular to the direction in which the wave travels; *longitudinal waves* involve oscillations parallel to the direction of wave travel.

In this book, a **sound wave** is defined roughly as any longitudinal wave. Seismic prospecting teams use such waves to probe Earth's crust for oil.

Contents

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17.2 Sound Waves
17.3 The Speed of Sound
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17.5 Intensity and Sound Characteristics
17.6 Three-Dimensional Propagation of Waves
17.7 Standing Waves
17.8 Interference
17.9 Beats
17.10 The Doppler Effect
17.11 Supersonic Speeds, Shock Waves

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Ships carry sound-ranging gear (sonar) to detect underwater obstacles. Submarines use sound waves to stalk other submarines, largely by listening for the characteristic noises produced by the propulsion system. Figure 17-1 suggests how sound waves can be used to explore the soft tissues of an animal or human body. In this chapter we shall focus on sound waves that travel through the air and that are audible to people.

Figure 17-2 illustrates several ideas that we shall use in our discussions. Point S represents a tiny sound source, called a *point source,* that emits sound waves in all directions. The *wavefronts* and *rays* indicate the direction of travel and the spread of the sound waves. **Wavefronts** are surfaces over which the oscillations due to the sound wave have the same value; such surfaces are represented by whole or partial circles in a two-dimensional drawing for a point source. **Rays** are directed lines perpendicular to the wavefronts that indicate the direction of travel of the wavefronts. The short double arrows superimposed on the rays of Fig. 17-2 indicate that the longitudinal oscillations of the air are parallel to the rays.

Near a point source like that of Fig. 17-2, the wavefronts are spherical and spread out in three dimensions, and there the waves are said to be *spherical.* As the wavefronts move outward and their radii become larger, their curvature decreases. Far from the source, we approximate the wavefronts as planes (or lines on two-dimensional drawings), and the waves are said to be *planar.*

Mauro Fermariello/SPL/Photo Researchers, Inc.

Figure 17-1 A loggerhead turtle is being checked with ultrasound (which has a frequency above your hearing range); an image of its interior is being produced on a monitor off to the right.

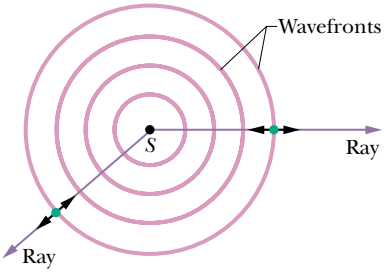

Figure 17-2 A sound wave travels from a point source S through a three-dimensional medium. The wavefronts form spheres centered on S; the rays are radial to S. The short, double-headed arrows indicate that elements of the medium oscillate parallel to the rays.

17.3 | THE SPEED OF SOUND

Key Concept

- The speed v of a sound wave in a medium having bulk modulus B and density ρ is

$$v = \sqrt{\frac{B}{\rho}} \quad \text{(speed of sound).}$$

In air at 20°C, the speed of sound is 343 m/s.

The speed of any mechanical wave, transverse or longitudinal, depends on both an inertial property of the medium (to store kinetic energy) and an elastic property of the medium (to store potential energy). Thus, we can generalize Eq. 16-29, which gives the speed of a transverse wave along a stretched string, by writing

$$v = \sqrt{\frac{\tau}{\mu}} = \sqrt{\frac{\text{elastic property}}{\text{inertial property}}}, \tag{17-1}$$

where (for transverse waves) τ is the tension in the string and μ is the string's linear density. If the medium is air and the wave is longitudinal, we can guess that the inertial property, corresponding to μ, is the volume density ρ of air. What shall we put for the elastic property?

In a stretched string, potential energy is associated with the periodic stretching of the string elements as the wave passes through them. As a sound wave passes through air, potential energy is associated with periodic compressions and expansions of small volume elements of the air. The property that determines the extent to

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which an element of a medium changes in volume when the pressure (force per unit area) on it changes is the **bulk modulus** B, defined as

$$B = -\frac{\Delta p}{\Delta V/V} \quad \text{(definition of bulk modulus).} \tag{17-2}$$

Here $\Delta V/V$ is the fractional change in volume produced by a change in pressure Δp. As explained earlier, the SI unit for pressure is the newton per square meter, which is given a special name, the *pascal* (Pa). From Eq. 17-2 we see that the unit for B is also the pascal. The signs of Δp and ΔV are always opposite: When we increase the pressure on an element (Δp is positive), its volume decreases (ΔV is negative). We include a minus sign in Eq. 17-2 so that B is always a positive quantity. Now substituting B for τ and ρ for μ in Eq. 17-1 yields as the speed of sound in a medium with bulk modulus B and density ρ. Table 17-1 lists the speed of sound in various media.

$$v = \sqrt{\frac{B}{\rho}} \quad \text{(speed of sound)} \tag{17-3}$$

Table 17-1 The Speed of Sound[a]

Medium	Speed (m/s)
Gases	
Air (0°C)	331
Air (20°C)	343
Helium	965
Hydrogen	1284
Liquids	
Water (0°C)	1402
Water (20°C)	1482
Seawater[b]	1522
Solids	
Aluminum	6420
Steel	5941
Granite	6000

[a] At 0°C and 1 atm pressure, except where noted.
[b] At 20°C and 3.5% salinity.

The density of water is almost 1000 times greater than the density of air. If this were the only relevant factor, we would expect from Eq. 17-3 that the speed of sound in water would be considerably less than the speed of sound in air. However, Table 17-1 shows us that the reverse is true. We conclude (again from Eq. 17-3) that the bulk modulus of water must be more than 1000 times greater than that of air. This is indeed the case. Water is much more incompressible than air, which (see Eq. 17-2) is another way of saying that its bulk modulus is much greater.

Formal Derivation of Eq. 17-3

We now derive Eq. 17-3 by direct application of Newton's laws. Let a single pulse in which air is compressed travel (from right to left) with speed v through the air in a long tube, like that in Fig. 16-2. Let us run along with the pulse at that speed, so that the pulse appears to stand still in our reference frame. Figure 17-3*a* shows the situation as it is viewed from that frame. The pulse is standing still, and air is moving at speed v through it from left to right.

Let the pressure of the undisturbed air be p and the pressure inside the pulse be $p + \Delta p$, where Δp is positive due to the compression. Consider an element of air of thickness Δx and face area A, moving toward the pulse at speed v. As this element enters the pulse, the leading face of the element encounters a region of higher pressure, which slows the element to speed $v + \Delta v$, in which Δv is negative. This slowing is complete when the rear face of the element reaches the pulse, which requires time interval

$$\Delta t = \frac{\Delta x}{v}. \tag{17-4}$$

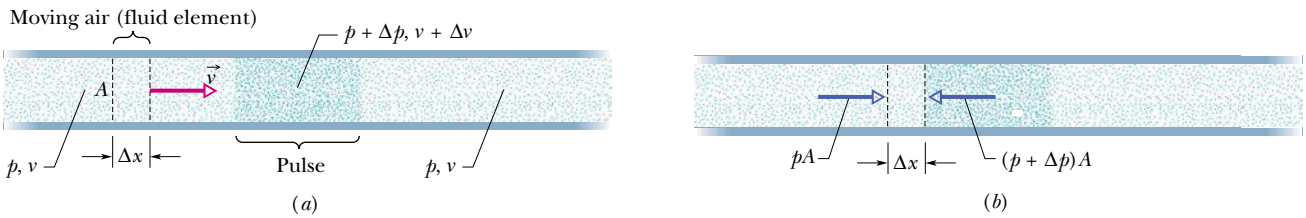

Figure 17-3 A compression pulse is sent from right to left down a long air-filled tube. The reference frame of the figure is chosen so that the pulse is at rest and the air moves from left to right. (*a*) An element of air of width Δx moves toward the pulse with speed v. (*b*) The leading face of the element enters the pulse. The forces acting on the leading and trailing faces (due to air pressure) are shown.

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Let us apply Newton's second law to the element. During Δt, the average force on the element's trailing face is pA toward the right, and the average force on the leading face is $(p + \Delta p)A$ toward the left (Fig. 17-3*b*). Therefore, the average net force on the element during Δt is

$$F = pA - (p + \Delta p)A$$
$$= -\Delta pA \quad \text{(net force)}. \tag{17-5}$$

The minus sign indicates that the net force on the air element is directed to the left in Fig. 17-3*b*. The volume of the element is $A\,\Delta x$, so with the aid of Eq. 17-4, we can write its mass as

$$\Delta m = \rho \Delta V = \rho A\,\Delta x = \rho A v\,\Delta t \quad \text{(mass)}. \tag{17-6}$$

The average acceleration of the element during Δt is

$$a = \frac{\Delta v}{\Delta t} \quad \text{(acceleration)}. \tag{17-7}$$

Thus, from Newton's second law ($F = ma$), we have, from Eqs. 17-5, 17-6, and 17-7,

$$-\Delta p\,A = (\rho A v\,\Delta t)\frac{\Delta v}{\Delta t}, \tag{17-8}$$

which we can write as

$$\rho v^2 = -\frac{\Delta p}{\Delta v/v}. \tag{17-9}$$

The air that occupies a volume $V\,(= Av\,\Delta t)$ outside the pulse is compressed by an amount $\Delta V\,(= A\,\Delta v\,\Delta t)$ as it enters the pulse. Thus,

$$\frac{\Delta V}{V} = \frac{A\,\Delta v\,\Delta t}{Av\,\Delta t} = \frac{\Delta v}{v}. \tag{17-10}$$

Substituting Eq. 17-10 and then Eq. 17-2 into Eq. 17-9 leads to

$$\rho v^2 = -\frac{\Delta p}{\Delta v/v} = -\frac{\Delta p}{\Delta V/V} = B. \tag{17-11}$$

Solving for v yields Eq. 17-3 for the speed of the air toward the right in Fig. 17-3, and thus for the actual speed of the pulse toward the left.

SAMPLE PROBLEM 17.01

Speed of sound

A handclap on stage in an amphitheater sends out sound waves that scatter from terraces of width $w = 0.65$ m (Fig. 17-4). The sound returns to the stage as a periodic series of pulses, one from each terrace; the parade of pulses sounds like a played note. (a) Assuming that all the rays in Fig. 17-4 are horizontal, find the frequency at which the pulses return (that is, the frequency of the perceived note).

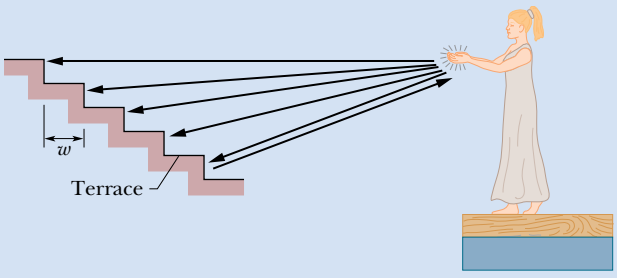

Figure 17-4 Sound waves from handclaps on the stage of an amphitheater.

KEY IDEA

Consider a string of pulses returning to the stage. A pulse that came back just before the previous one has traveled an extra distance of $2w$, taking an extra amount of time $\Delta t = 2w/v$.

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Calculations: The frequency of the pulse is therefore

$$f = \frac{1}{\Delta t} = \frac{v}{2w} = \frac{343 \text{ m/s}}{2(0.65 \text{ m})} = 2.6 \times 10^2 \text{ Hz}$$

(b) If the width w of the terraces were smaller, would the frequency be higher or lower?

Reasoning: Since $f \propto 1/w$, the frequency would be higher if w were smaller.

17.4 | TRAVELING SOUND WAVES

Key Concepts

- A sound wave causes a longitudinal displacement s of a mass element in a medium as given by

$$s = s_m \cos(kx - \omega t),$$

where s_m is the displacement amplitude (maximum displacement) from equilibrium, $k = 2\pi/\lambda$, and $\omega = 2\pi f$, λ and f being the wavelength and frequency, respectively, of the sound wave.

- The sound wave also causes a pressure change Δp of the medium from the equilibrium pressure:

$$\Delta p = \Delta p_m \sin(kx - \omega t),$$

where the pressure amplitude is

$$\Delta p_m = (v\rho\omega)s_m.$$

Here we examine the displacements and pressure variations associated with a sinusoidal sound wave traveling through air. Figure 17-5*a* displays such a wave traveling rightward through a long air-filled tube. Recall from Chapter 16 that we can produce such a wave by sinusoidally moving a piston at the left end of the tube (as in Fig. 16-2). The piston's rightward motion moves the element of air next to the piston face and compresses that air; the piston's leftward motion allows the element of air to move back to the left and the pressure to decrease. As each element of air pushes on the next element in turn, the right–left motion of the air and the change in its pressure travel along the tube as a sound wave.

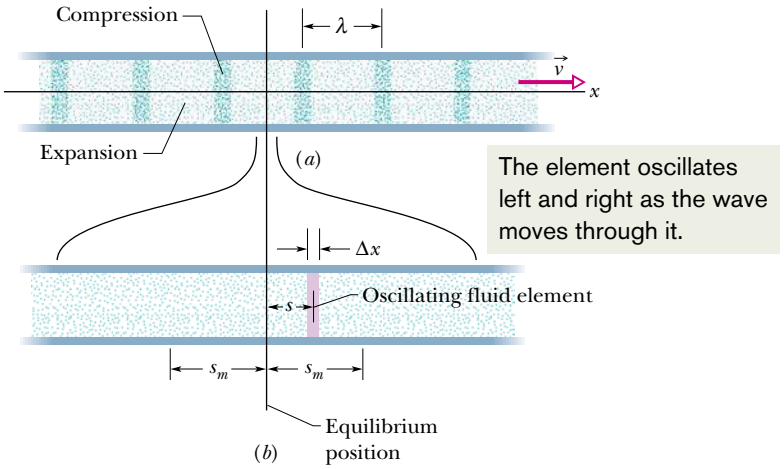

Figure 17-5 (*a*) A sound wave, traveling through a long air-filled tube with speed v, consists of a moving, periodic pattern of expansions and compressions of the air. The wave is shown at an arbitrary instant. (*b*) A horizontally expanded view of a short piece of the tube. As the wave passes, an air element of thickness Δx oscillates left and right in simple harmonic motion about its equilibrium position. At the instant shown in (*b*), the element happens to be displaced a distance s to the right of its equilibrium position. Its maximum displacement, either right or left, is s_m.

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Consider the thin element of air of thickness Δx shown in Fig. 17-5*b*. As the wave travels through this portion of the tube, the element of air oscillates left and right in simple harmonic motion about its equilibrium position. Thus, the oscillations of each air element due to the traveling sound wave are like those of a string element due to a transverse wave, except that the air element oscillates *longitudinally* rather than *transversely*. Because string elements oscillate parallel to the y axis, we write their displacements in the form $y(x, t)$. Similarly, because air elements oscillate parallel to the x axis, we could write their displacements in the confusing form $x(x, t)$, but we shall use $s(x, t)$ instead.

Displacement. To show that the displacements $s(x, t)$ are sinusoidal functions of x and t, we can use either a sine function or a cosine function. In this chapter we use a cosine function, writing

$$s(x, t) = s_m \cos(kx - \omega t). \tag{17-12}$$

Figure 17-6*a* labels the various parts of this equation. In it, s_m is the **displacement amplitude**—that is, the maximum displacement of the air element to either side of its equilibrium position (see Fig. 17-5*b*). The angular wave number k, angular frequency ω, frequency f, wavelength λ, speed v, and period T for a sound (longitudinal) wave are defined and interrelated exactly as for a transverse wave, except that λ is now the distance (again along the direction of travel) in which the pattern of compression and expansion due to the wave begins to repeat itself (see Fig. 17-5*a*). (We assume s_m is much less than λ.)

Pressure. As the wave moves, the air pressure at any position x in Fig. 17-5*a* varies sinusoidally, as we prove next. To describe this variation we write

$$\Delta p(x, t) = \Delta p_m \sin(kx - \omega t). \tag{17-13}$$

Figure 17-6*b* labels the various parts of this equation. A negative value of Δp in Eq. 17-13 corresponds to an expansion of the air, and a positive value to a compression. Here Δp_m is the **pressure amplitude**, which is the maximum increase or decrease in pressure due to the wave; Δp_m is normally very much less than the pressure p present when there is no wave. As we shall prove, the pressure amplitude Δp_m is related to the displacement amplitude s_m in Eq. 17-12 by

$$\Delta p_m = (v\rho\omega)s_m. \tag{17-14}$$

Figure 17-7 shows plots of Eqs. 17-12 and 17-13 at $t = 0$; with time, the two curves would move rightward along the horizontal axes. Note that the displacement and pressure variation are $\pi/2$ rad (or 90°) out of phase. Thus, for example, the pressure variation Δp at any point along the wave is zero when the displacement there is a maximum.

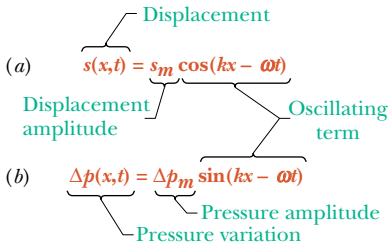

Figure 17-6 (*a*) The displacement function and (*b*) the pressure-variation function of a traveling sound wave consist of an amplitude and an oscillating term.

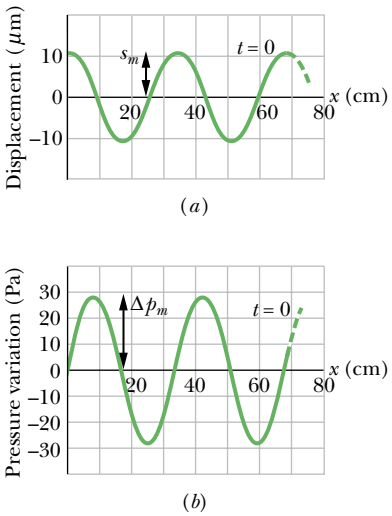

Figure 17-7 (*a*) A plot of the displacement function (Eq. 17-12) for $t = 0$. (*b*) A similar plot of the pressure-variation function (Eq. 17-13). Both plots are for a 1000 Hz sound wave whose pressure amplitude is at the threshold of pain.

CHECKPOINT 1

When the oscillating air element in Fig. 17-5*b* is moving rightward through the point of zero displacement, is the pressure in the element at its equilibrium value, just beginning to increase, or just beginning to decrease?

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Derivation of Eqs. 17-13 and 17-14

Figure 17-5*b* shows an oscillating element of air of cross-sectional area A and thickness Δx, with its center displaced from its equilibrium position by distance s. From Eq. 17-2 we can write, for the pressure variation in the displaced element,

$$\Delta p = -B\frac{\Delta V}{V}. \tag{17-15}$$

The quantity V in Eq. 17-15 is the volume of the element, given by

$$V = A\ \Delta x. \tag{17-16}$$

The quantity ΔV in Eq. 17-15 is the change in volume that occurs when the element is displaced. This volume change comes about because the displacements of the two faces of the element are not quite the same, differing by some amount Δs. Thus, we can write the change in volume as

$$\Delta V = A\ \Delta s. \tag{17-17}$$

Substituting Eqs. 17-16 and 17-17 into Eq. 17-15 and passing to the differential limit yield

$$\Delta p = -B\frac{\Delta s}{\Delta x} = -B\frac{\partial s}{\partial x}. \tag{17-18}$$

The symbols ∂ indicate that the derivative in Eq. 17-18 is a *partial derivative*, which tells us how s changes with x when the time t is fixed. From Eq. 17-12 we then have, treating t as a constant,

$$\frac{\partial s}{\partial x} = \frac{\partial}{\partial x}[s_m \cos(kx - \omega t)] = -ks_m \sin(kx - \omega t).$$

Substituting this quantity for the partial derivative in Eq. 17-18 yields

$$\Delta p = Bks_m \sin(kx - \omega t).$$

This tells us that the pressure varies as a sinusoidal function of time and that the amplitude of the variation is equal to the terms in front of the sine function. Setting $\Delta p_m = Bks_m$, this yields Eq. 17-13, which we set out to prove.

Using Eq. 17-3, we can now write

$$\Delta p_m = (Bk)s_m = (v^2\rho k)s_m.$$

Equation 17-14, which we also wanted to prove, follows at once if we substitute ω/v for k from Eq. 16-14.

SAMPLE PROBLEM 17.02

Pressure amplitude, displacement amplitude

The maximum pressure amplitude Δp_m that the human ear can tolerate in loud sounds is about 28 Pa (which is very much less than the normal air pressure of about 10^5 Pa). What is the displacement amplitude s_m for such a sound in air of density $\rho = 1.21$ kg/m^3, at a frequency of 1000 Hz and a speed of 343 m/s?

KEY IDEA

The displacement amplitude s_m of a sound wave is related to the pressure amplitude Δp_m of the wave according to Eq. 17-14.

Calculations: Solving that equation for s_m yields

$$s_m = \frac{\Delta p_m}{v\rho\omega} = \frac{\Delta p_m}{v\rho(2\pi f)}.$$

Substituting known data then gives us

$$s_m = \frac{28\text{ Pa}}{(343\text{ m/s})(1.21\text{ kg/m}^3)(2\pi)(1000\text{ Hz})}$$
$$= 1.1\times10^{-5}\text{ m} = 11\ \mu\text{m}. \qquad \text{(Answer)}$$

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That is only about one-seventh the thickness of a book page. Obviously, the displacement amplitude of even the loudest sound that the ear can tolerate is very small. Temporary exposure to such loud sound produces temporary hearing loss, probably due to a decrease in blood supply to the inner ear. Prolonged exposure produces permanent damage.

The pressure amplitude Δp_m for the *faintest* detectable sound at 1000 Hz is 2.8×10^{-5} Pa. Proceeding as above leads to $s_m = 1.1 \times 10^{-11}$ m or 11 pm, which is about one-tenth the radius of a typical atom. The ear is indeed a sensitive detector of sound waves.

17.5 | INTENSITY AND SOUND CHARACTERISTICS

Key Concepts

- The intensity I of a sound wave at a surface is the average rate per unit area at which energy is transferred by the wave through or onto the surface:

$$I = \frac{P}{A},$$

where P is the time rate of energy transfer (power) of the sound wave and A is the area of the surface intercepting the sound. The intensity I is related to the displacement amplitude s_m of the sound wave by

$$I = \frac{1}{2}\rho v \omega^2 s_m^2.$$

- The intensity at a distance r from a point source that emits sound waves of power P_s equally in all directions (isotropically) is

$$I = \frac{P_s}{4\pi r^2}.$$

- The sound level β in decibels (dB) is defined as

$$\beta = (10 \text{ dB}) \log \frac{I}{I_0},$$

where I_0 ($= 10^{-12}$ W/m^2) is a reference intensity level to which all intensities are compared. For every factor-of-10 increase in intensity, 10 dB is added to the sound level.

If you have ever tried to sleep while someone played loud music nearby, you are well aware that there is more to sound than frequency, wavelength, and speed. There is also intensity. The **intensity** I of a sound wave at a surface is the average rate per unit area at which energy is transferred by the wave through or onto the surface. We can write this as

$$I = \frac{P}{A}, \tag{17-19}$$

where P is the time rate of energy transfer (the power) of the sound wave and A is the area of the surface intercepting the sound. As we shall derive shortly, the intensity I is related to the displacement amplitude s_m of the sound wave by

$$I = \frac{1}{2}\rho v \omega^2 s_m^2. \tag{17-20}$$

Intensity can be measured on a detector. *Loudness* is a perception, something that you sense. The two can differ because your perception depends on factors such as the sensitivity of your hearing mechanism to various frequencies.

Variation of Intensity with Distance

How intensity varies with distance from a real sound source is often complex. Some real sources (like loudspeakers) may transmit sound only in particular directions, and the environment usually produces echoes (reflected sound waves) that overlap the direct sound waves. In some situations, however, we can ignore echoes and assume that the sound

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source is a point source that emits the sound *isotropically*—that is, with equal intensity in all directions. The wavefronts spreading from such an isotropic point source S at a particular instant are shown in Fig. 17-8.

Let us assume that the mechanical energy of the sound waves is conserved as they spread from this source. Let us also center an imaginary sphere of radius r on the source, as shown in Fig. 17-8. All the energy emitted by the source must pass through the surface of the sphere. Thus, the time rate at which energy is transferred through the surface by the sound waves must equal the time rate at which energy is emitted by the source (that is, the power P_S of the source). From Eq. 17-19, the intensity I at the sphere must then be

$$I = \frac{P_S}{4\pi r^2}, \tag{17-21}$$

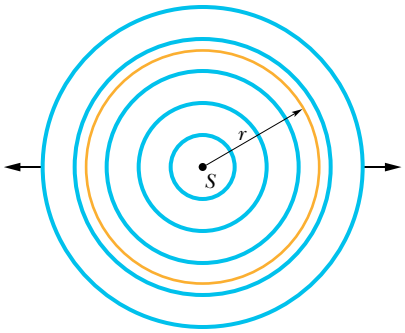
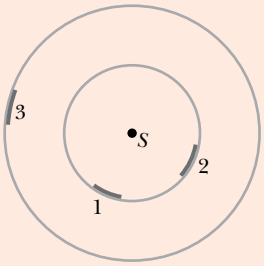


Figure 17-8 A point source S emits sound waves uniformly in all directions. The waves pass through an imaginary sphere of radius r that is centered on S.

where $4\pi r^2$ is the area of the sphere. Equation 17-21 tells us that the intensity of sound from an isotropic point source decreases with the square of the distance r from the source.

CHECKPOINT 2

The figure indicates three small patches 1, 2, and 3 that lie on the surfaces of two imaginary spheres; the spheres are centered on an isotropic point source S of sound. The rates at which energy is transmitted through the three patches by the sound waves are equal. Rank the patches according to (a) the intensity of the sound on them and (b) their area, greatest first.



The Decibel Scale

The displacement amplitude at the human ear ranges from about 10^{-5} m for the loudest tolerable sound to about 10^{-11} m for the faintest detectable sound, a ratio of 10^6. From Eq. 17-20 we see that the intensity of a sound varies as the *square* of its amplitude, so the ratio of intensities at these two limits of the human auditory system is 10^{12}. Humans can hear over an enormous range of intensities.

We deal with such an enormous range of values by using logarithms. Consider the relation

$$y = \log x,$$

in which x and y are variables. It is a property of this equation that if we *multiply* x by 10, then y increases by 1. To see this, we write

$$y' = \log(10x) = \log 10 + \log x = 1 + y.$$

Similarly, if we multiply x by 10^{12}, y increases by only 12.

Thus, instead of speaking of the intensity I of a sound wave, it is much more convenient to speak of its **sound level** β, defined as

$$\beta = (10\ \text{dB}) \log \frac{I}{I_0}. \tag{17-22}$$

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Table 17-2 Some Sound Levels (dB)

Hearing threshold	0
Rustle of leaves	10
Conversation	60
Rock concert	110
Pain threshold	120
Jet engine	130

Here dB is the abbreviation for **decibel**, the unit of sound level, a name that was chosen to recognize the work of Alexander Graham Bell. I_0 in Eq. 17-22 is a standard reference intensity ($=10^{-12}$ W/m^2), chosen because it is near the lower limit of the human range of hearing. For $I = I_0$, Eq. 17-22 gives $\beta = 10 \log 1 = 0$, so our standard reference level corresponds to zero decibels. Then β increases by 10 dB every time the sound intensity increases by an order of magnitude (a factor of 10). Thus, $\beta = 40$ corresponds to an intensity that is 10^4 times the standard reference level. Table 17-2 lists the sound levels for a variety of environments.

Derivation of Eq. 17-20

Consider, in Fig. 17-5*a*, a thin slice of air of thickness dx, area A, and mass dm, oscillating back and forth as the sound wave of Eq. 17-12 passes through it. The kinetic energy dK of the slice of air is

$$dK = \frac{1}{2} dm\, v_s^2. \tag{17-23}$$

Here v_s is not the speed of the wave but the speed of the oscillating element of air, obtained from Eq. 17-12 as

$$v_s = \frac{\partial s}{\partial t} = -\omega s_m \sin(kx - \omega t).$$

Using this relation and putting $dm = \rho A\, dx$ allow us to rewrite Eq. 17-23 as

$$dK = \frac{1}{2}(\rho A\, dx)(-\omega s_m)^2 \sin^2(kx - \omega t). \tag{17-24}$$

Dividing Eq. 17-24 by dt gives the rate at which kinetic energy moves along with the wave. As we saw in Chapter 16 for transverse waves, dx/dt is the wave speed v, so we have

$$\frac{dK}{dt} = \frac{1}{2} \rho A v \omega^2 s_m^2 \sin^2(kx - \omega t). \tag{17-25}$$

The *average* rate at which kinetic energy is transported is

$$\begin{aligned} \left(\frac{dK}{dt}\right)_{\text{avg}} &= \frac{1}{2} \rho A v \omega^2 s_m^2 [\sin^2(kx - \omega t)]_{\text{avg}} \\ &= \frac{1}{4} \rho A v \omega^2 s_m^2. \end{aligned} \tag{17-26}$$

To obtain this equation, we have used the fact that the average value of the square of a sine (or a cosine) function over one full oscillation is 1/2.

We assume that *potential* energy is carried along with the wave at this same average rate. The wave intensity I, which is the average rate per unit area at which energy of both kinds is transmitted by the wave, is then, from Eq. 17-26,

$$I = \frac{2(dK/dt)_{\text{avg}}}{A} = \frac{1}{2} \rho v \omega^2 s_m^2,$$

which is Eq. 17-20, the equation we set out to derive.

Pitch

Pitch is also a sensation experienced by a listener. It depends mainly on the frequency of vibration of the air, which of course is the same as that of the source of sound. Frequency is a physical quantity and is measured in hertz (Hz), one vibration per second is known as one hertz. A high frequency produces a high-pitched note and a low frequency gives a low-pitched note.

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Quality or Timber

The same note played on two different instruments does not sound the same and different waveforms are obtained on a cathode ray oscillator. The notes are said to have different quality or timber. This is due to the fact that with one exception (a tuning fork), sounds are never "pure notes", that is, of one frequency; their waveforms are not sinusoidal. They consist of a main or fundamental note which usually predominates and others, generally with smaller intensities, called overtones. The fundamental is the component of lowest frequency and the overtones have frequencies that are multiples of the fundamental frequency. The number and intensities of the overtones determine the quality of a sound.

17.6 | THREE-DIMENSIONAL PROPAGATION OF WAVES

Key Concepts

- If $\vec{r}$ is the position vector of any point in the wavefront, the wave motion in three dimensions can be described by
$$\xi = f(\hat{a}\cdot\vec{r} - vt)$$
- A harmonic traveling plane wave propagating in a homogeneous dispersive medium in the direction of the unit vector $\hat{c}$, then at a plane at some distance c from the origin the disturbance is given by
$$\psi(c,t) = y_m \sin(\omega t - kc)$$
The quantity kc is
$$kc = k(\hat{c}\cdot\vec{r}) = (k\hat{c})\cdot\vec{r} \equiv \vec{k}\cdot\vec{r}$$
where $k\hat{c}$ is the propagation vector $\vec{k}$; the magnitude of $\vec{k}$ is $(2\pi/\lambda)$.

Waves in Two and Three Dimensions

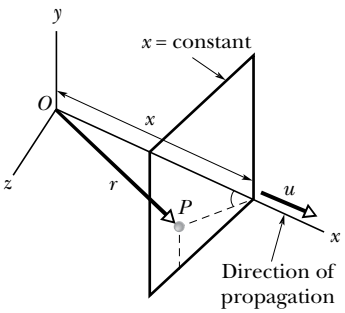
Although $\xi = f(x - vt)$ represents a wave motion propagating along the x axis, we do not necessarily have to interpret it as meaning a wave concentrated on the x axis. If the physical disturbance described by ξ is present over all space, then at a given time t, the function $\xi = f(x - vt)$ gives the same value at all points having the same x. Here x = constant represents a plane perpendicular to the x axis (Fig. 17-9), which contains all the points in the plane perpendicular to the x axis. This plane is called a wavefront. Therefore, in three dimensions, the expression $\xi = f(x - vt)$ describes a plane wave propagating parallel to the x axis.

We must note that for a plane wave, the direction of propagation is important and is represented by unit vector of the wave velocity vector $\hat{a}$, perpendicular to the plane of the wave. This vector may not lie along any of the three coordinate axes. It is therefore convenient to express the plane wave equation in a form that does not require the $\hat{a}$ to be along any of the axes. In the case of Fig. 17-9*a*, the unit vector $\hat{a}$ is parallel to the x axis. If $\vec{r}$ is the position vector of any point in the wavefront, we can see that

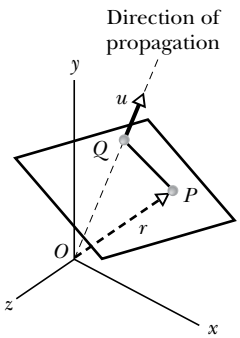
$$x = \hat{a}\cdot\vec{r}$$

and therefore, we may write equation

$$\xi = f(\hat{a}\cdot\vec{r} - vt)$$



(*a*)



(*b*)

Figure 17-9 (*a*) Plane wave propagating along the x axis (*b*) Plane wave propagating in an arbitrary direction.

The Propagation Vector

Suppose we have a harmonic traveling plane wave propagating in a homogeneous dispersive medium in the direction of the unit vector $\hat{c}$, (along the direction of propagation of wave). Suppose that at the origin the wave function has the time dependence

$$\Psi(0,t) = \sin\omega t \qquad (17\text{-}27)$$

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Then at a plane at some distance c from the origin the disturbance is given by

$$\Psi(c,t) = y_m \sin(\omega t - kc) \tag{17-28}$$

We need to write this wave function in terms of general coordinates x, y, z, instead by the distance c along the propagation direction. Let $\vec{r} = x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}}$ denote a point in space as measured from the origin of the x, y, z system. Thus, the quantity kc in Eq. 17-28 can be written

$$kc = k(\hat{c}\cdot\vec{r}) = (k\hat{c})\cdot\vec{r} \equiv \vec{k}\cdot\vec{r} \tag{17-29}$$

The quantity $k\hat{c}$ is called the propagation vector $\vec{k}$;

$$\vec{k} = k\hat{c} \tag{17-30}$$

The magnitude of $\vec{k}$ is $(2\pi/\lambda)$ the direction of $\vec{k}$ is $\hat{c}$, along the wave propagation direction. Equation 17-29 becomes for a wave having three-dimensional wave vector,

$$kc = \vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z$$

The physical meaning of the wave number number k is the increase in phase per unit displacement along the propagation direction $\hat{c}$. So that kc is the phase increased in distance c. The meaning of k_x is the increased phase per unit displacement along the $+x$ axis, that is, along $\hat{\mathrm{i}}$ makes an angle θ with $\hat{c}$. Suppose that the wavelength is λ. Then if one moves along the direction $\hat{c}$ by a distance λ (at fixed time), the phase increases by 2π If instead one advances along $\hat{\mathrm{i}}$, one must travel a distance $\lambda/\cos\theta$ to increase phase by 2π. Then the phase has increased by 2π in a distance along $\hat{\mathrm{i}}$ that is larger than λ-by a factor $(\cos\theta)^{-1}$. In the case of a harmonic or sinusoidal plane wave propagating in the direction $\hat{u}$, we write

$$\psi = y_m \sin[k(\hat{u}\cdot r - vt)] = y_m \sin(k_x x + k_y y + k_z z - \omega t)$$

where k_x, k_y, and k_z, are the components of $\vec{k}$ satisfying the relation

$$k_x^2 + k_y^2 + k_z^2 = k^2 = \frac{\omega^2}{v^2}$$

The plane waves in Fig. 17-10*a*, although contain the three coordinates x, y, z, are really one dimensional propagation, since the propagation is along one particular direction and the physical situation is the same in all planes perpendicular to the direction of propagation (Fig. 17-10*a*). But in nature there are other kinds of waves that propagate in several directions as shown in Fig. 17-10*b*.

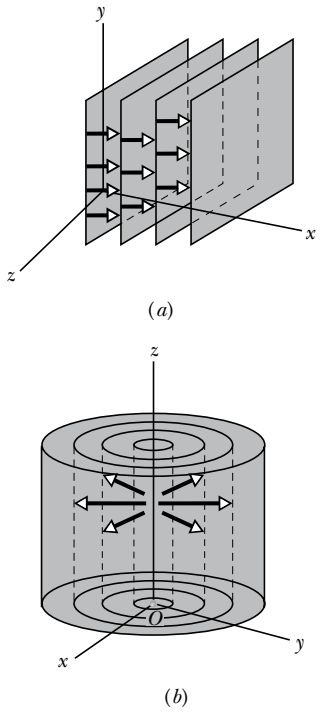

Figure 17-10 Various wavefronts.

Spherical Waves

Let's say point source (Fig. 17-11) is emanating power P isotropically.

All particles at distance r from O are in same phase. Wavefronts are sphere with center at O. Intensity at distance R is I, where

$$I = \frac{P_s}{4\pi R^2}$$

Equating with the formula for intensity derived earlier in Eqs. 17-20 and 17-21

$$\frac{P_s}{4\pi r^2} = \frac{P_0^2 v}{2B} \tag{17-31}$$

Thus, excess pressure amplitude becomes, $p_0 = D/r$, where

$$D = \sqrt{\frac{P_s B}{2\pi v}}$$

Therefore,
$$p = \frac{D}{r}\sin(\omega t - \vec{k}\cdot\vec{r} + \phi) \tag{17-32}$$

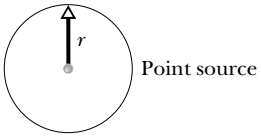

Figure 17-11 Point source emanating power P.

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SAMPLE PROBLEM 17.03

Intensity change with distance, cylindrical sound wave

An electric spark jumps along a straight line of length $L = 10$ m, emitting a pulse of sound that travels radially outward from the spark. (The spark is said to be a *line source* of sound.) The power of this acoustic emission is $P_s = 1.6 \times 10^4$ W.

(a) What is the intensity I of the sound when it reaches a distance $r = 12$ m from the spark?

KEY IDEAS

(1) Let us center an imaginary cylinder of radius $r = 12$ m and length $L = 10$ m (open at both ends) on the spark, as shown in Fig. 17-12. Then the intensity I at the cylindrical surface is the ratio P/A, where P is the time rate at which sound energy passes through the surface and A is the surface area. (2) We assume that the principle of conservation of energy applies to the sound energy. This means that the rate P at which energy is transferred through the cylinder must equal the rate P_s at which energy is emitted by the source.

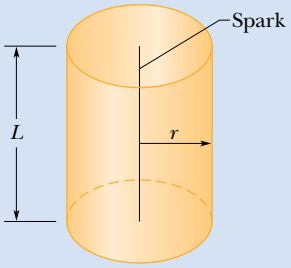

Figure 17-12 A spark along a straight line of length L emits sound waves radially outward. The waves pass through an imaginary cylinder of radius r and length L that is centered on the spark.

Calculations: Putting these ideas together and noting that the area of the cylindrical surface is $A = 2\pi rL$, we have

$$I = \frac{P}{A} = \frac{P_s}{2\pi rL}. \tag{17-33}$$

This tells us that the intensity of the sound from a line source decreases with distance r (and not with the square of distance r as for a point source). Substituting the given data, we find

$$I = \frac{1.6\times10^4 \text{ W}}{2\pi(12 \text{ m})(10 \text{ m})}$$
$$= 21.2 \text{ W/m}^2 \approx 21 \text{ W/m}^2. \quad \text{(Answer)}$$

(b) At what time rate P_d is sound energy intercepted by an acoustic detector of area $A_d = 2.0$ cm^2, aimed at the spark and located a distance $r = 12$ m from the spark?

Calculations: We know that the intensity of sound at the detector is the ratio of the energy transfer rate P_d there to the detector's area A_d:

$$I = \frac{P_d}{A_d}. \tag{17-34}$$

We can imagine that the detector lies on the cylindrical surface of (a). Then the sound intensity at the detector is the intensity I (= 21.2 W/m^2) at the cylindrical surface. Solving Eq. 17-34 for P_d gives us

$$P_d = (21.2 \text{ W/m}^2)(2.0 \times 10^{-4} \text{ m}^2) = 4.2 \text{ mW}. \quad \text{(Answer)}$$

SAMPLE PROBLEM 17.04

Decibels, sound level, change in intensity

Many veteran rockers suffer from acute hearing damage because of the high sound levels they endured for years. Many rockers now wear special earplugs to protect their hearing during performances (Fig. 17-13). If an earplug decreases the sound level of the sound waves by 20 dB, what is the ratio of the final intensity I_f of the waves to their initial intensity I_i?

KEY IDEA

For both the final and initial waves, the sound level β is related to the intensity by the definition of sound level in Eq. 17-22.

Calculations: For the final waves we have

$$\beta_f = (10 \text{ dB}) \log \frac{I_f}{I_0},$$

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Tim Mosenfelder/Getty Images, Inc.

Figure 17-13 Lars Ulrich of Metallica is an advocate for the organization HEAR (Hearing Education and Awareness for Rockers), which warns about the damage high sound levels can have on hearing.

and for the initial waves we have

$$\beta_i = (10\ \text{dB}) \log \frac{I_i}{I_0}.$$

The difference in the sound levels is

$$\beta_f - \beta_i = (10\ \text{dB}) \left(\log \frac{I_f}{I_0} - \log \frac{I_i}{I_0} \right). \qquad (17\text{-}35)$$

Using the identity

$$\log \frac{a}{b} - \log \frac{c}{d} = \log \frac{ad}{bc},$$

we can rewrite Eq. 17-36 as

$$\beta_f - \beta_i = (10\ \text{dB}) \log \frac{I_f}{I_i}. \qquad (17\text{-}36)$$

Rearranging and then substituting the given decrease in sound level as $\beta_f - \beta_i = -20$ dB, we find

$$\log \frac{I_f}{I_i} = \frac{\beta_f - \beta_i}{10\ \text{dB}} = \frac{-20\ \text{dB}}{10\ \text{dB}} = -2.0.$$

We next take the antilog of the far left and far right sides of this equation. We find

$$\frac{I_f}{I_i} = \log^{-1}(-2.0) = 0.010. \qquad \text{(Answer)}$$

Thus, the earplug reduces the intensity of the sound waves to 0.010 of their initial intensity (two orders of magnitude).

17.7 | STANDING WAVES

Key Concepts

- Standing sound wave patterns can be set up in pipes (that is, resonance can be set up) if sound of the proper wavelength is introduced in the pipe.
- A pipe open at both ends will resonate at frequencies

$$f = \frac{v}{\lambda} = \frac{nv}{2L}, \quad n = 1, 2, 3, \ldots,$$

where v is the speed of sound in the air in the pipe.

- For a pipe closed at one end and open at the other, the resonant frequencies are

$$f = \frac{v}{\lambda} = \frac{nv}{4L}, \quad n = 1, 3, 5, \ldots,$$

Sources of Musical Sound

Musical sounds can be set up by oscillating strings (guitar, piano, violin), membranes (kettledrum, snare drum), air columns (flute, oboe, pipe organ, and the didgeridoo of Fig. 17-14), wooden blocks or steel bars (marimba, xylophone), and many other oscillating bodies. Most common instruments involve more than a single oscillating part.

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Standing waves can be set up on a stretched string that is fixed at both ends. They arise because waves traveling along the string are reflected back onto the string at each end. If the wavelength of the waves is suitably matched to the length of the string, the superposition of waves traveling in opposite directions produces a standing wave pattern (or oscillation mode). The wavelength required of the waves for such a match is one that corresponds to a *resonant frequency* of the string. The advantage of setting up standing waves is that the string then oscillates with a large, sustained amplitude, pushing back and forth against the surrounding air and thus generating a noticeable sound wave with the same frequency as the oscillations of the string. This production of sound is of obvious importance to, say, a guitarist.

Sound Waves. We can set up standing waves of sound in an air-filled pipe in a similar way. As sound waves travel through the air in the pipe, they are reflected at each end and travel back through the pipe. (The reflection occurs even if an end is open, but the reflection is not as complete as when the end is closed.) If the wavelength of the sound waves is suitably matched to the length of the pipe, the superposition of waves traveling in opposite directions through the pipe sets up a standing wave pattern. The wavelength required of the sound waves for such a match is one that corresponds to a resonant frequency of the pipe. The advantage of such a standing wave is that the air in the pipe oscillates with a large, sustained amplitude, emitting at any open end a sound wave that has the same frequency as the oscillations in the pipe. This emission of sound is of obvious importance to, say, an organist.

Many other aspects of standing sound wave patterns are similar to those of string waves: The closed end of a pipe is like the fixed end of a string in that there must be a node (zero displacement) there, and the open end of a pipe is like the end of a string attached to a freely moving ring, as in Fig. 16-27*b*, in that there must be an antinode there. (Actually, the antinode for the open end of a pipe is located slightly beyond the end, but we shall not dwell on that detail.)

Two Open Ends. The simplest standing wave pattern that can be set up in a pipe with two open ends is shown in Fig. 17-15*a*. There is an antinode across each open end, as required. There is also a node across the middle of the pipe. An easier way of representing this standing longitudinal sound wave is shown in Fig. 17-15*b*—by drawing it as a standing transverse string wave.

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Figure 17-14 The air column within a didgeridoo ("a pipe") oscillates when the instrument is played.

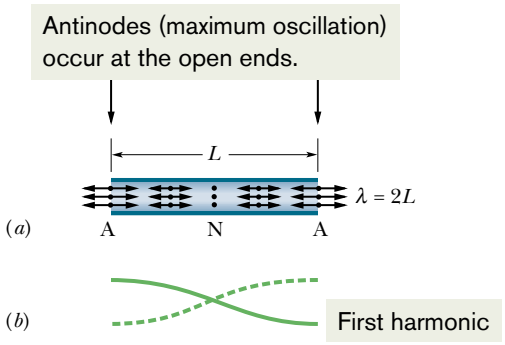

Figure 17-15 (*a*) The simplest standing wave pattern of displacement for (longitudinal) sound waves in a pipe with both ends open has an antinode (A) across each end and a node (N) across the middle. (The longitudinal displacements represented by the double arrows are greatly exaggerated.) (*b*) The corresponding standing wave pattern for (transverse) string waves.

The standing wave pattern of Fig. 17-15*a* is called the *fundamental mode* or *first harmonic*. For it to be set up, the sound waves in a pipe of length L must have a wavelength given by $L = \lambda/2$, so that $\lambda = 2L$. Several more standing sound wave patterns for a pipe with two open ends are shown in Fig. 17-16*a* using string wave representations. The *second harmonic* requires sound waves of wavelength $\lambda = L$, the *third harmonic* requires wavelength $\lambda = 2L/3$, and so on.

More generally, the resonant frequencies for a pipe of length L with two open ends correspond to the wavelengths

$$\lambda = \frac{2L}{n}, \quad \text{for } n = 1, 2, 3, \ldots, \tag{17-37}$$

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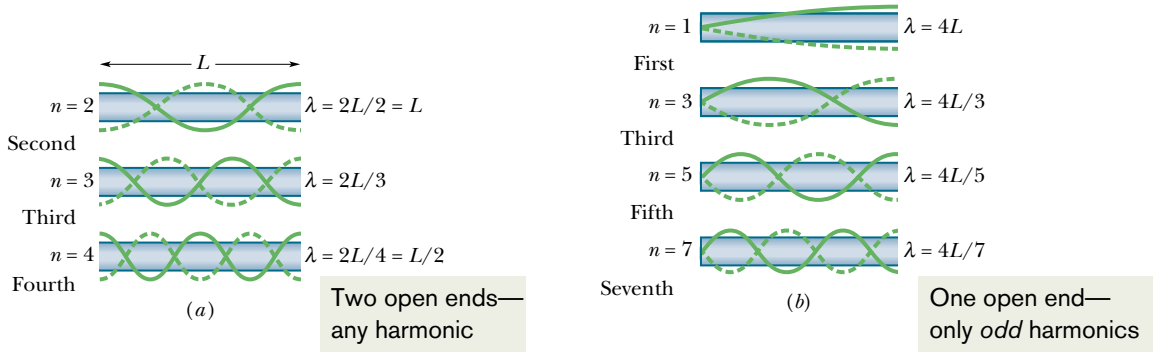

Figure 17-16 Standing wave patterns for string waves superimposed on pipes to represent standing sound wave patterns in the pipes. (*a*) With *both* ends of the pipe open, any harmonic can be set up in the pipe. (*b*) With only *one* end open, only odd harmonics can be set up.

where n is called the *harmonic number*. Letting v be the speed of sound, we write the resonant frequencies for a pipe with two open ends as

$$f = \frac{v}{\lambda} = \frac{nv}{2L}, \quad \text{for } n = 1, 2, 3, \ldots \quad \text{(pipe, two open ends)}. \tag{17-38}$$

One Open End. Figure 17-16*b* shows (using string wave representations) some of the standing sound wave patterns that can be set up in a pipe with only one open end. As required, across the open end there is an antinode and across the closed end there is a node. The simplest pattern requires sound waves having a wavelength given by $L = \lambda/4$, so that $\lambda = 4L$. The next simplest pattern requires a wavelength given by $L = 3\lambda/4$, so that $\lambda = 4L/3$, and so on.

More generally, the resonant frequencies for a pipe of length L with only one open end correspond to the wavelengths

$$\lambda = \frac{4L}{n}, \quad \text{for } n = 1, 3, 5, \ldots, \tag{17-39}$$

in which the harmonic number n *must be an odd number*. The resonant frequencies are then given by

$$f = \frac{v}{\lambda} = \frac{nv}{4L}, \quad \text{for } n = 1, 3, 5, \ldots \quad \text{(pipe, one open end)}. \tag{17-40}$$

Note again that only odd harmonics can exist in a pipe with one open end. For example, the second harmonic, with $n = 2$, cannot be set up in such a pipe. Note also that for such a pipe the adjective in a phrase such as "the third harmonic" still refers to the harmonic number n (and not to, say, the third possible harmonic). Finally note that Eqs. 17-37 and 17-38 for two open ends contain the number 2 and any integer value of n, but Eqs. 17-39 and 17-40 for one open end contain the number 4 and only odd values of n.

Length. The length of a musical instrument reflects the range of frequencies over which the instrument is designed to function, and smaller length implies higher frequencies, as we can tell from Eq. 16-75 for string instruments and Eqs. 17-38 and 17-40 for instruments with air columns. Figure 17-17, for example, shows the saxophone and violin families, with their frequency ranges suggested by the piano keyboard. Note that, for every instrument, there is overlap with its higher- and lower-frequency neighbors.

Net Wave. In any oscillating system that gives rise to a musical sound, whether it is a violin string or the air in an organ pipe, the fundamental and one or more of the higher harmonics are usually generated simultaneously. Thus, you hear them together—that is, superimposed as a net wave. When different instruments are played at the same note, they produce the same fundamental frequency but different intensities for the higher harmonics.

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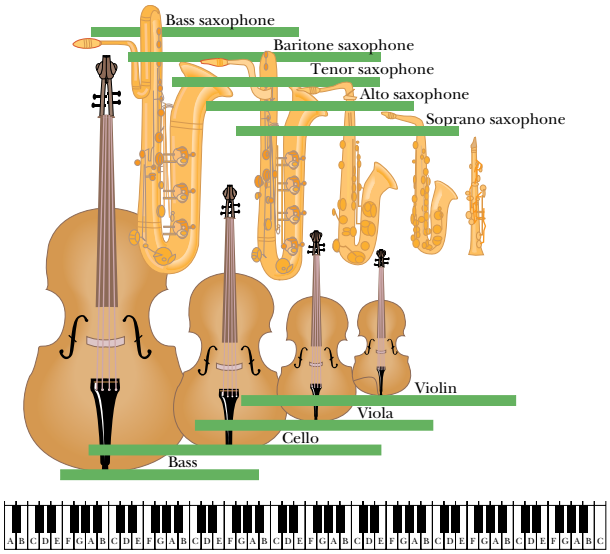

Figure 17-17 The saxophone and violin families, showing the relations between instrument length and frequency range. The frequency range of each instrument is indicated by a horizontal bar along a frequency scale suggested by the keyboard at the bottom; the frequency increases toward the right.

For example, the fourth harmonic of middle C might be relatively loud on one instrument and relatively quiet or even missing on another. Thus, because different instruments produce different net waves, they sound different to you even when they are played at the same note. That would be the case for the two net waves shown in Fig. 17-18, which were produced at the same note by different instruments. If you heard only the fundamentals, the music would not be musical.

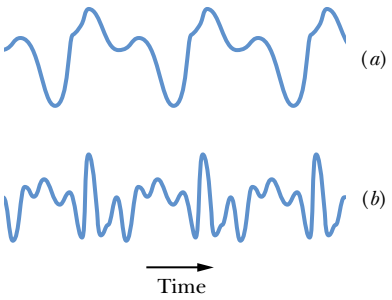

Figure 17-18 The wave forms produced by (*a*) a flute and (*b*) an oboe when played at the same note, with the same first harmonic frequency.

CHECKPOINT 3

Pipe A, with length L, and pipe B, with length $2L$, both have two open ends. Which harmonic of pipe B has the same frequency as the fundamental of pipe A?

SAMPLE PROBLEM 17.05

Standing wave pattern in two ends open pipe

Weak background noises from a room set up the fundamental standing wave in a cardboard tube of length $L = 67.0$ cm with two open ends. Assume that the speed of sound in the air within the tube is 343 m/s.

(a) What frequency do you hear from the tube?

KEY IDEA

With both pipe ends open, we have a symmetric situation in which the standing wave has an antinode at each end of the tube. The standing wave pattern (in string wave style) is that of Fig. 17-15*b*.

Calculation: The frequency is given by Eq. 17-32 with $n = 1$ for the fundamental mode:

$$f = \frac{nv}{2L} = \frac{(1)(343 \text{ m/s})}{2(0.670 \text{ m})} = 256 \text{ Hz} \qquad \text{(Answer)}$$

If the background noises set up any higher harmonics, such as the second harmonic, you also hear frequencies that are *integer* multiples of 256 Hz.

(b) If you jam your ear against one end of the tube, what fundamental frequency do you hear from the tube?

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KEY IDEA

With your ear effectively closing one end of the tube, we have an asymmetric situation – an antinode still exists at the open end, but a node is now at the other (closed) end. The standing wave pattern is the top one in Fig. 17-16*b*.

Calculation: The frequency is given by Eq. 17-40 with $n = 1$ for the fundamental mode:

$$f = \frac{nv}{4L} = \frac{(1)(343 \text{ m/s})}{4(0.670 \text{ m})} = 128 \text{ Hz} \quad \text{(Answer)}$$

If the background noises set up any higher harmonics, they will be *odd* multiples of 128 Hz. That means that the frequency of 256 Hz (which is an even multiple) cannot now occur.

End-Correction of Pipes

The air at the open end of a pipe is free to move. So, the vibrations at this end of a sounding pipe extend a little into the air outside the pipe. The antinode of the stationary wave due to any note is then a small distance e away from the open end in practice and this is known as the end-correction. So, the fundamental wavelength λ in the case of a pipe with one end open is given by $\lambda/4 = L + e$, where L is the length of the pipe, (Fig. 17-19*a*). In the case of an open pipe sounding its fundamental note, the wavelength λ is given by $\lambda/2 = L + e + e$, since two end-corrections are required (Fig. 17-19*b*). So $\lambda = 2(L + 2e)$.

Calculation shows that $e = 0.6r$, where r is the radius of the pipe. So, the wider the pipe, the greater is the end-correction.

Resonance Tube Experiment, Measurement of Velocity of Sound and "End-Correction" of Tube

If a tuning-fork is held over the open end of a tube T filled with water, resonance is obtained at some position as the level of water is gradually lowered as shown in Fig. 17-19*c*. We recognize the resonance when the sound heard from the vibrating air in the tube becomes distinctly louder. The stationary wave which sets up then is shown in Fig. 17-19*c*. If e is the end-correction of the tube and L is the length from the water level to the top of the tube, then

$$L + e = \frac{\lambda}{4}$$

But

$$\lambda = \frac{v}{f}$$

where f is the frequency of the fork and v is the velocity of sound in air.

$$L + e = \frac{v}{4f} \tag{17-41}$$

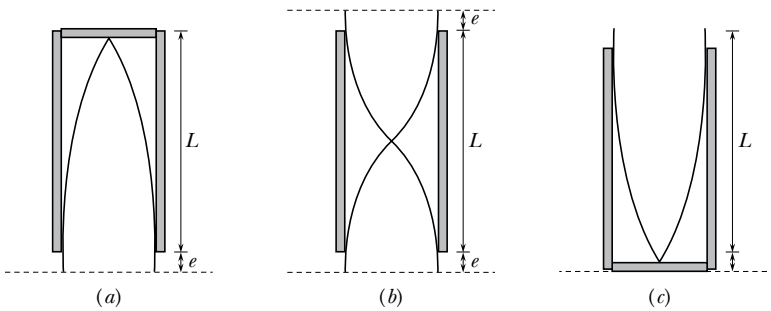

Figure 17-19 (*a*) and (*b*) End-correction of pipes. (*c*) Resonance tube experiment.

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If different tuning-forks of known frequency are used, and the corresponding values of L are measured when resonance occurs, we can plot a graph from Eq. 17-41 of $1/f$ against L (Fig. 17-20). From Eq. 17-41, the slope of the line is $4/v$. So we can find speed of sound v. Also, the intercept of the line on the x axis of L is e, so the end-correction can also be found.

The experiment can also be performed by one fork only if the tube is sufficiently long. In this case the level of the water is lowered even after the first resonance, until resonance is again obtained at a lower water level w_1, as shown in Fig. 17-21. The stationary wave that sets up is shown in Fig. 17-21 and the new length to the top from w_1 is L_1.

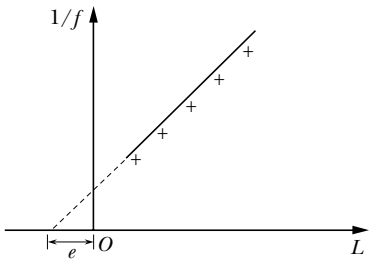

Figure 17-20 Plot of $1/f$ vs l.

Considering the end correction,

$$L_1 + e = \frac{3}{4}\lambda \qquad (17\text{-}42)$$

Subtracting Eq. 17-41 from Eq. 17-42, we get

$$L_1 - L = \frac{\lambda}{2}$$

Therefore, $$\lambda = 2(L_1 - L)$$

and $$v = f\lambda = 2f(L_1 - L)$$

By this manipulation, the end-correction e is eliminated. The magnitude of e can be found from Eqs. 17-41 and 17-43. Thus, from Eqs. 17-41 and 17-43,

$$3L + 3e = \frac{3}{4}\lambda = L_1 + e \qquad (17\text{-}43)$$

Therefore, $$2e = L_1 - 3L \quad \text{or} \quad e = \frac{1}{2}(L_1 - 3L)$$

So, e can be found from measurements of L_1 and L.

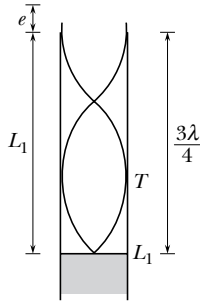

Figure 17-21 Resonance tube.

Standing Waves in Rods

A rod clamped in the middle and disturbed parallel to the length at one end will vibrate as shown in Fig. 17-22*a*. For clarity, the displacements have been drawn in the transverse direction although they are actually parallel to the rod. The midpoint of the rod will be a displacement node as it is restricted by the clamp and the ends of the rod are displacement antinodes as they are free to vibrate. The vibrations in the set-up are similar to those in a pipe open at both ends. The fundamental mode is shown in Fig. 17-22 having wavelength $2L$. Other normal modes can be produced by clamping the rod at different points. For example, when the rod is clamped at a distance $L/4$ from one end, the second normal mode (Fig. 17-22*b*) is observed.

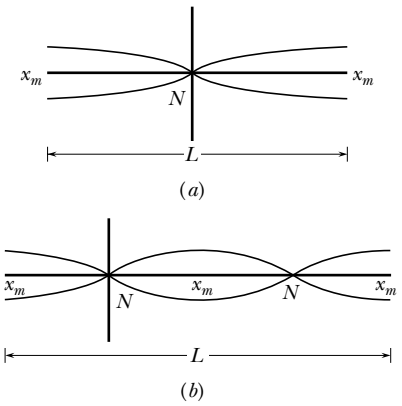

Figure 17-22 Vibrations in a rod clamped in the middle and disturbed parallel to it's length.

SAMPLE PROBLEM 17.06

Resonance in pipes

A cylindrical pipe of length 28 cm closed at one end is found to be at resonance when a tuning fork of frequency 850 Hz is sounded near the open end. Find the mode of vibration of the air in the pipe and calculate the value of the end-correction. (Take the speed of sound in air as 340 m/s).

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KEY IDEA

We can draw the first three possibilities for resonance remembering that open end is a displacement node. This problem involves using Eq. 17-40 with possible end correction.

Calculation: Let λ be the wavelength of the sound in the pipe.

Then $$\lambda = \frac{v}{f} = \frac{340}{850} = 0.4 \text{ m} = 40 \text{ cm}$$

If the pipe is resonating at its fundamental frequency f_0, the stationary wave in the pipe is shown in Fig. 17-23*a* and the wavelength λ_0 is given by $\lambda_0/4 = 28$ cm. So $\lambda_0 = 112$ cm. As the wavelength of the sound is just 40 cm, the pipe cannot be sounding at its fundamental frequency. The first overtone of the pipe is $3f_0$, which corresponds to a wavelength λ_1 given by $3\lambda_1/4 = 28$ cm as shown in Fig. 17-23*b*.

Therefore, $\lambda_1 = 37.33$ cm which is close to the actual wavelength. Consequently, allowing for the effect of an end correction, the pipe can be considered sounding at its first overtone. Let e = the end-correction in centimeters.

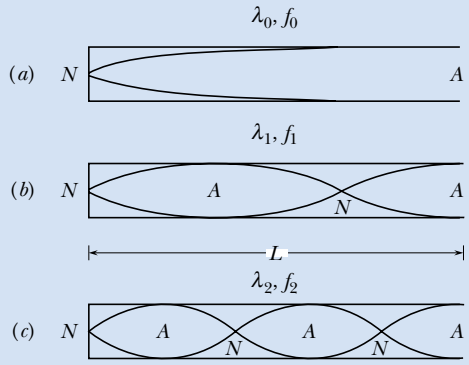

Figure 17-23 Organ pipe with various modes of vibrations.

Then $$28 + e = \frac{3}{4}\lambda_1$$

Thus, $$28 + e = \frac{3}{4} \times 40$$

so, $$e = 20 \text{ cm} \qquad \text{(Answer)}$$

This answer seems to be of the reasonable order of magnitude.

SAMPLE PROBLEM 17.07

Resonance between pipes of different lengths

Pipe A is open at both ends and has length $L_A = 0.343$ m. We want to place it near three other pipes in which standing waves have been set up, so that the sound can set up a standing wave in pipe A. Those other three pipes are each closed at one end and have lengths $L_B = 0.500L_A$, $L_C = 0.250L_A$, and $L_D = 2.00L_A$. For each of these three pipes, which of their harmonics can excite a harmonic in pipe A?

KEY IDEAS

(1) The sound from one pipe can set up a standing wave in another pipe only if the harmonic frequencies match. (2) Equation 17-38 gives the harmonic frequencies in a pipe with two open ends (a symmetric pipe) as $f = nv/2L$, for $n = 1, 2, 3, \ldots$, that is, for any positive integer. (3) Equation 17-40 gives the harmonic frequencies in a pipe with only one open end (an asymmetric pipe) as $f = nv/4L$, for $n = 1, 3, 5, \ldots$, that is, for only odd positive integers.

Pipe A: Let's first find the resonant frequencies of symmetric pipe A (with two open ends) by evaluating Eq. 17-38:

$$f_A = \frac{n_A v}{2L_A} = \frac{n_A(343 \text{ m/s})}{2(0.343 \text{ m})}$$
$$= n_A(500 \text{ Hz}) = n_A(0.50 \text{ kHz}), \quad \text{for } n_A = 1, 2, 3, \ldots.$$

The first six harmonic frequencies are shown in the top plot in Fig. 17-24.

Pipe B: Next let's find the resonant frequencies of asymmetric pipe B (with only one open end) by evaluating Eq. 17-40, being careful to use only odd integers for the harmonic numbers:

$$f_B = \frac{n_B v}{4L_B} = \frac{n_B v}{4(0.500L_A)} = \frac{n_B(343 \text{ m/s})}{2(0.343 \text{ m})}$$
$$= n_B(500 \text{ Hz}) = n_B(0.500 \text{ kHz}), \quad \text{for } n_B = 1, 3, 5, \ldots.$$

Comparing our two results, we see that we get a match for each choice of n_B:

$$f_A = f_B \quad \text{for} \quad n_A = n_B \quad \text{with } n_B = 1, 3, 5, \ldots. \qquad \text{(Answer)}$$

For example, as shown in Fig. 17-24, if we set up the fifth harmonic in pipe B and bring the pipe close to pipe A,

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the fifth harmonic will then be set up in pipe A. However, no harmonic in B can set up an even harmonic in A.

Pipe C: Let's continue with pipe C (with only one end) by writing Eq. 17-40 as

$$f_C = \frac{n_C v}{4L_C} = \frac{n_C v}{4(0.250L_A)} = \frac{n_C(343\text{ m/s})}{0.343\text{ m/s}}$$
$$= n_C(1000\text{ Hz}) = n_C(1.00\text{ kHz}), \quad \text{for } n_C = 1, 3, 5, \ldots.$$

From this we see that C can excite some of the harmonics of A but only those with harmonic numbers n_A that are twice an odd integer:

$$f_A = f_C \quad \text{for } n_A = 2n_C, \quad \text{with } n_C = 1, 3, 5, \ldots. \quad \text{(Answer)}$$

Pipe D: Finally, let's check D with our same procedure:

$$f_D = \frac{n_D v}{4L_D} = \frac{n_D v}{4(2L_A)} = \frac{n_D(343\text{ m/s})}{8(0.343\text{ m/s})}$$
$$= n_D(125\text{ Hz}) = n_D(0.125\text{ kHz}), \quad \text{for } n_D = 1, 3, 5, \ldots.$$

As shown in Fig. 17-24, none of these frequencies match a harmonic frequency of A. (Can you see that we would get a match if $n_D = 4n_A$? But that is impossible because $4n_A$ cannot yield an odd integer, as required of n_D.) Thus D cannot set up a standing wave in A.

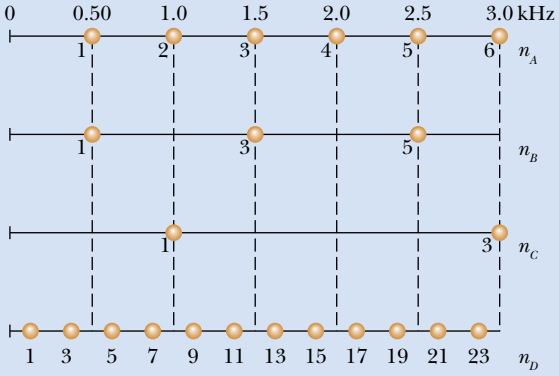

Figure 17-24 Harmonic frequencies of four pipes.

SAMPLE PROBLEM 17.08

End correction of pipes

An open pipe 30 cm long and a closed pipe 23 cm long both of the same diameter, are each sounding their first overtone, and these are in unison. What is the end correction of these pipes?

KEY IDEA

This problem involves using Eqs. 17-38 and 17-40 with possible end correction.

Calculations: Suppose v is the speed of sound in air, and f is the frequency of the note. The wavelength, λ, is then v/f. When the open pipe is sounding its first overtone, the length of the pipe plus end-corrections is equal to λ. Therefore,

$$\lambda = \frac{v}{f} = 30 + 2e \qquad (17\text{-}44)$$

since there are *two end corrections*.

When the closed pipe is sounding its first overtone,

$$\frac{3}{4}\lambda = 23 + e \qquad (17\text{-}45)$$

From Eqs. 17-44 and 17-45, it follows that

$$23 + e = \frac{3}{4}(30 + 2e)$$

Therefore, $92 + 4e = 90 + 6e$

so, $e = 1\text{ cm}$ (Answer)

SAMPLE PROBLEM 17.09

Resonance in pipes submerged in seawater

The two pipes in Fig. 17-25 are submerged in seawater at 20°C. Pipe A, with length $L_A = 1.50$ m and one open end, contains a small sound source that is to set up the standing wave with the second lowest resonant frequency of that pipe.

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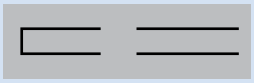

Figure 17-25 Two pipes submerged in seawater. A sound source in pipe A sets up acoustic resonance in that pipe. Sound then leaks from pipe A into pipe B.

(a) What is resonant frequency?

KEY IDEA

Only odd harmonics can exist in a pipe with one open end. The resonant frequencies are given by $f = nv/4L$ for harmonic numbers $n = 1, 3, 5, \ldots$.

Calculation: Because resonant frequency increases with increasing n, the second lowest resonant frequency corresponds to $n = 3$, the second lowest choice of n.

Caution: The frequency is the *second* lowest but, because $n = 3$, it is for the *third* harmonic of pipe A.

Using the speed of sound in seawater from Table 17-1, we then have

$$f = \frac{nv}{4L_A} = \frac{(3)(1522\text{ m/s})}{(4)(1.50\text{ m})} = 761\text{ Hz} \qquad \text{(Answer)}$$

(b) Sound leaking from pipe A is to set up resonance in pipe B, which has two open ends. That resonance is to be at the second lowest resonant frequency of pipe B. What is the required length L_B of pipe B?

KEY IDEA

Any harmonic can exist in a pipe with two open ends. The resonant frequencies are given by $f = nv/2L$ for harmonic numbers $n = 1, 2, 3, \ldots$.

Calculations: Because resonant frequency increases with increasing n, the second lowest resonant frequency corresponds to $n = 2$, the second lowest choice of n. Therefore, we can write

$$f = \frac{2v}{2L_B}$$

Solving for L_B and substituting known data yield

$$L_B = \frac{v}{f} = \frac{1522\text{ m/s}}{761\text{ Hz}} = 2.00\text{ m} \qquad \text{(Answer)}$$

17.8 | INTERFERENCE

Key Concepts

- The interference of two sound waves with identical wavelengths passing through a common point depends on their phase difference ϕ there. If the sound waves were emitted in phase and are traveling in approximately the same direction, ϕ is given by

$$\phi = \frac{\Delta L}{\lambda} 2\pi,$$

where ΔL is their path length difference.

- Fully constructive interference occurs when ϕ is an integer multiple of 2π,

$$\phi = m(2\pi), \quad \text{for } m = 0, 1, 2, \ldots,$$

and, equivalently, when ΔL is related to wavelength λ by

$$\frac{\Delta L}{\lambda} = 0, 1, 2, \ldots.$$

- Fully destructive interference occurs when ϕ is an odd multiple of π,

$$\phi = (2m + 1)\,\pi, \quad \text{for } m = 0, 1, 2, \ldots,$$

and

$$\frac{\Delta L}{\lambda} = 0.5, 1.5, 2.5, \ldots.$$

Like transverse waves, sound waves can undergo interference. In fact, we can write equations for the interference as we did in Section 16-7 for transverse waves. Suppose two sound waves with the same amplitude and wavelength are traveling in the positive direction of an x axis with a phase difference of ϕ. We can express the waves in the form of Eqs. 16-50 and 16-51 but, to be consistent with Eq. 17-12, we use cosine functions instead of sine functions:

$$s_1(x, t) = s_m \cos(kx - \omega t)$$

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and

$$s_2(x, t) = s_m \cos(kx - \omega t + \phi).$$

These waves overlap and interfere. From Eq. 16-54, we can write the resultant wave as

$$s' = \left[2s_m \cos\frac{1}{2}\phi\right]\cos\left(kx - \omega t + \frac{1}{2}\phi\right).$$

As we saw with transverse waves, the resultant wave is itself a traveling wave. Its amplitude is the magnitude

$$s'_m = \left|2s_m \cos\frac{1}{2}\phi\right|. \tag{17-46}$$

As with transverse waves, the value of ϕ determines what type of interference the individual waves undergo.

One way to control ϕ is to send the waves along paths with different lengths. Figure 17-26*a* shows how we can set up such a situation: Two point sources S_1 and S_2 emit sound waves that are in phase and of identical wavelength λ. Thus, the *sources* themselves are said to be in phase; that is, as the waves emerge from the sources, their displacements are always identical. We are interested in the waves that then travel through point P in Fig. 17-26*a*. We assume that the distance to P is much greater than the distance between the sources so that we can approximate the waves as traveling in the same direction at P.

If the waves traveled along paths with identical lengths to reach point P, they would be in phase there. As with transverse waves, this means that they would undergo fully constructive interference there. However, in Fig. 17-26*a*, path L_2 traveled by the wave from S_2 is longer than path L_1 traveled by the wave from S_1. The difference in path lengths means that the waves may not be in phase at point P. In other words, their phase difference ϕ at P depends on their **path length difference** $\Delta L = |L_2 - L_1|$.

To relate phase difference ϕ to path length difference ΔL, we recall (from Section 16-3) that a phase difference of 2π rad corresponds to one wavelength. Thus, we can write the proportion

$$\frac{\phi}{2\pi} = \frac{\Delta L}{\lambda}, \tag{17-47}$$

from which

$$\phi = \frac{\Delta L}{\lambda}2\pi. \tag{17-48}$$

Fully constructive interference occurs when ϕ is zero, 2π, or any integer multiple of 2π. We can write this condition as

$$\phi = m(2\pi), \quad \text{for } m = 0, 1, 2, \ldots \quad \text{(fully constructive interference)}. \tag{17-49}$$

From Eq. 17-48, this occurs when the ratio $\Delta L/\lambda$ is

$$\frac{\Delta L}{\lambda} = 0, 1, 2, \ldots. \quad \text{(fully constructive interference)}. \tag{17-50}$$

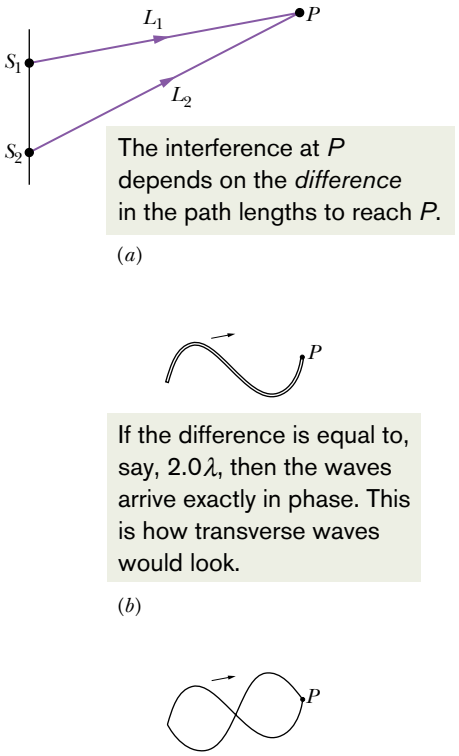
L_1
P
S_1
L_2
S_2
The interference at P depends on the *difference* in the path lengths to reach P.
(*a*)
P
If the difference is equal to, say, 2.0λ, then the waves arrive exactly in phase. This is how transverse waves would look.
(*b*)
P
If the difference is equal to, say, 2.5λ, then the waves arrive exactly out of phase. This is how transverse waves would look.
(*c*)

Figure 17-26 (*a*) Two point sources S_1 and S_2 emit spherical sound waves in phase. The rays indicate that the waves pass through a common point P. The waves (represented with *transverse* waves) arrive at P (*b*) exactly in phase and (*c*) exactly out of phase.

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For example, if the path length difference $\Delta L = |L_2 - L_1|$ in Fig. 17-26*a* is equal to 2λ, then $\Delta L/\lambda = 2$ and the waves undergo fully constructive interference at point P (Fig. 17-26*b*). The interference is fully constructive because the wave from S_2 is phase-shifted relative to the wave from S_1 by 2λ, putting the two waves *exactly in phase* at P.

Fully destructive interference occurs when ϕ is an odd multiple of π:

$$\phi = (2m + 1)\pi, \quad \text{for } m = 0, 1, 2, \ldots \quad \text{(fully destructive interference).} \tag{17-51}$$

From Eq. 17-48, this occurs when the ratio $\Delta L/\lambda$ is

$$\frac{\Delta L}{\lambda} = 0.5, 1.5, 2.5, \ldots. \quad \text{(fully destructive interference).} \tag{17-52}$$

For example, if the path length difference $\Delta L = |L_2 - L_1|$ in Fig. 17-26*a* is equal to 2.5λ, then $\Delta L/\lambda = 2.5$ and the waves undergo fully destructive interference at point P (Fig. 17-26*c*). The interference is fully destructive because the wave from S_2 is phase-shifted relative to the wave from S_1 by 2.5 wavelengths, which puts the two waves *exactly out of phase* at P.

Of course, two waves could produce intermediate interference as, say, when $\Delta L/\lambda = 1.2$. This would be closer to fully constructive interference ($\Delta L/\lambda = 1.0$) than to fully destructive interference ($\Delta L/\lambda = 1.5$).

SAMPLE PROBLEM 17.10

Interference points along a big circle

In Fig. 17-27, two point sources S_1 and S_2, which are in phase and separated by distance $D = 1.5\lambda$, emit identical sound waves of wavelength λ.

(a) What is the path length difference of the waves from S_1 and S_2 at point P_1, which lies on the perpendicular bisector of distance D, at a distance greater than D from the sources (Fig. 17-27*b*)? (That is, what is the difference in the distance from source S_1 to point P_1 and the distance from source S_2 to P_1?) What type of interference occurs at P_1?

Reasoning: Because the waves travel identical distances to reach P_1, their path length difference is

$$\Delta L = 0. \quad \text{(Answer)}$$

From Eq. 17-50, this means that the waves undergo fully constructive interference at P_1 because they start in phase at the sources and reach P_1 in phase.

(b) What are the path length difference and type of interference at point P_2 in Fig. 17-27*c*?

Reasoning: The wave from S_1 travels the extra distance D ($= 1.5\lambda$) to reach P_2. Thus, the path length difference is

$$\Delta L = 1.5\lambda. \quad \text{(Answer)}$$

From Eq. 17-51, this means that the waves are exactly out of phase at P_2 and undergo fully destructive interference there.

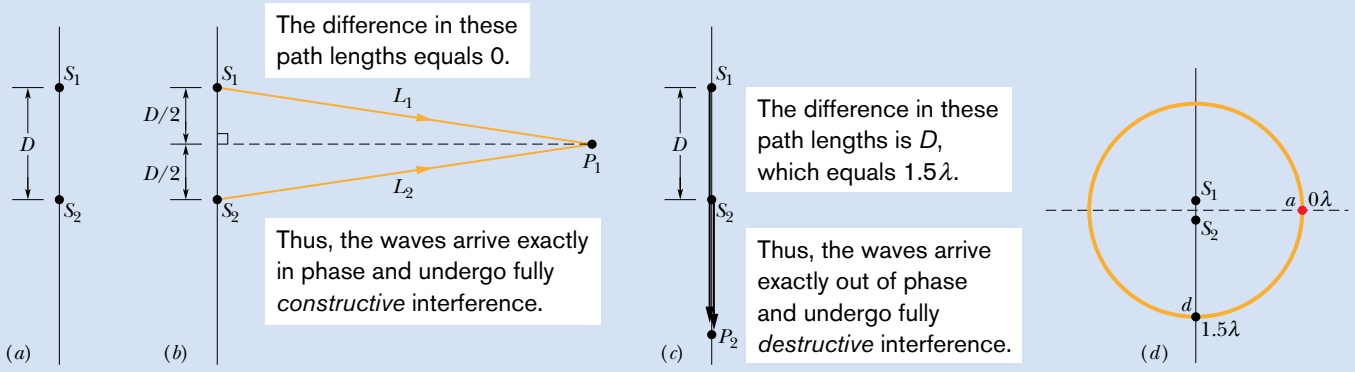

Figure 17-27 (*a*) Two point sources S_1 and S_2, separated by distance D, emit spherical sound waves in phase. (*b*) The waves travel equal distances to reach point P_1. (*c*) Point P_2 is on the line extending through S_1 and S_2. (*d*) We move around a large circle. (*Figure continues*)

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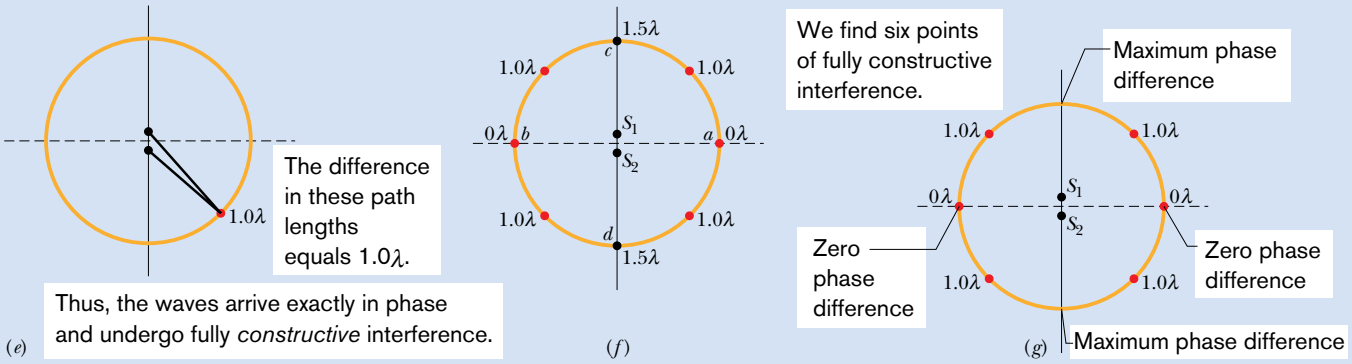

Figure 17-27 *(continued)* (*e*) Another point of fully constructive interference. (*f*) Using symmetry to determine other points. (*g*) The six points of fully constructive interference.

(c) Figure 17-27*d* shows a circle with a radius much greater than D, centered on the midpoint between sources S_1 and S_2. What is the number of points N around this circle at which the interference is fully constructive? (That is, at how many points do the waves arrive exactly in phase?)

Reasoning: Starting at point a, let's move clockwise along the circle to point d. As we move, path length difference ΔL increases and so the type of interference changes. From (a), we know that is $\Delta L = 0\lambda$ at point a. From (b), we know that $\Delta L = 1.5\lambda$ at point d. Thus, there must be one point between a and d at which $\Delta L = \lambda$ (Fig. 17-27*e*). From Eq. 17-50, fully constructive interference occurs at that point. Also, there can be no other point along the way from point a to point d at which fully constructive interference occurs, because there is no other integer than 1 between 0 at point a and 1.5 at point d.

We can now use symmetry to locate other points of fully constructive or destructive interference (Fig. 17-27*f*). Symmetry about line cd gives us point b, at which $\Delta L = 0\lambda$. Also, there are three more points at which $\Delta L = \lambda$. In all (Fig. 17-27*g*) we have

$$N = 6. \qquad \text{(Answer)}$$

SAMPLE PROBLEM 17.11

Interference points of two sound waves reflected from five surfaces

A sound source emits two sinusoidal sound waves, both of wavelength A, along paths A and B in Fig. 17-28. The sound traveling along path B is reflected from five surfaces as shown and then merges at point Q with the sound that traveled along path A. For what values of d (in wavelengths) are the waves exactly out of phase when they merge at point Q? For what values of d are the waves exactly in phase at point Q?

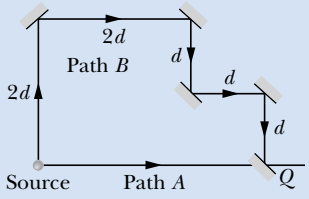

Figure 17-28 A sound source sends sound waves along paths A and B. The wave moving along path B is reflected five times and then travels with the wave moving along path A.

KEY IDEA

Because the two waves emerge from the same source, they must emerge in phase with each other. When they merge at point Q their phase difference can depend only on the path length difference ΔL between the two paths they followed.

Calculations: To reach Q, the wave following path A travels distance $3d$, and the wave following path B travels distance $7d$. Thus, their path length difference ΔL is $4d$.

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If the waves are to be exactly out of phase at Q, we must have

$$\frac{\Delta L}{\lambda} = 0.5, 1.5, 2.5, \ldots$$

Substituting $\Delta L = 4d$ and solving for d then give us

$$d = \frac{1}{8}\lambda, \frac{3}{8}\lambda, \frac{5}{8}\lambda, \ldots \qquad \text{(Answer)}$$

Similarly, if the waves are to be exactly in phase at point Q, we must have

$$\frac{\Delta L}{\lambda} = 0, 1, 2, 3, \ldots$$

We can eliminate 0 from the list (which implies that there is no path difference in Fig. 17-28). Then solving for d, we find

$$d = \frac{1}{4}\lambda, \frac{1}{2}\lambda, \frac{3}{4}\lambda, \ldots. \qquad \text{(Answer)}$$

17.9 | BEATS

Key Concept

- Beats arise when two waves having slightly different frequencies, f_1 and f_2, are detected together. The beat frequency is

$$f_{\text{beat}} = f_1 - f_2.$$

If we listen, a few minutes apart, to two sounds whose frequencies are, say, 552 and 564 Hz, most of us cannot tell one from the other because the frequencies are so close to each other. However, if the sounds reach our ears simultaneously, what we hear is a sound whose frequency is 558 Hz, the *average* of the two combining frequencies. We also hear a striking variation in the intensity of this sound—it increases and decreases in slow, wavering **beats** that repeat at a frequency of 12 Hz, the *difference* between the two combining frequencies. Figure 17-29 shows this beat phenomenon.

Let the time-dependent variations of the displacements due to two sound waves of equal amplitude s_m be

$$s_1 = s_m \cos \omega_1 t \quad \text{and} \quad s_2 = s_m \cos \omega_2 t, \tag{17-53}$$

where $\omega_1 > \omega_2$. From the superposition principle, the resultant displacement is the sum of the individual displacements:

$$s = s_1 + s_2 = s_m(\cos \omega_1 t + \cos \omega_2 t).$$

Using the trigonometric identity (see Appendix E)

$$\cos \alpha + \cos \beta = 2 \cos\left[\frac{1}{2}(\alpha - \beta)\right] \cos\left[\frac{1}{2}(\alpha + \beta)\right]$$

allows us to write the resultant displacement as

$$s = 2s_m \cos\left[\frac{1}{2}(\omega_1 - \omega_2)t\right] \cos\left[\frac{1}{2}(\omega_1 + \omega_2)t\right]. \tag{17-54}$$

If we write

$$\omega' = \frac{1}{2}(\omega_1 - \omega_2) \quad \text{and} \quad \omega = \frac{1}{2}(\omega_1 + \omega_2), \tag{17-55}$$

we can then write Eq. 17-54 as

$$s(t) = [2s_m \cos \omega' t] \cos \omega t. \tag{17-56}$$

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We now assume that the angular frequencies ω_1 and ω_2 of the combining waves are almost equal, which means that $\omega \gg \omega'$ in Eq. 17-55. We can then regard Eq. 17-56 as a cosine function whose angular frequency is ω and whose amplitude (which is not constant but varies with angular frequency ω') is the absolute value of the quantity in the brackets.

A maximum amplitude will occur whenever cos $\omega' t$ in Eq. 17-56 has the value +1 or −1, which happens twice in each repetition of the cosine function. Because cos $\omega' t$ has angular frequency ω', the angular frequency ω_{beat} at which beats occur is $\omega_{beat} = 2\omega'$. Then, with the aid of Eq. 17-55, we can write the beat angular frequency as

$$\omega_{beat} = 2\omega' = (2)\left(\frac{1}{2}\right)(\omega_1 - \omega_2) = \omega_1 - \omega_2.$$

Because $\omega = 2\pi f$, we can recast this as

$$f_{beat} = f_1 - f_2 \quad \text{(beat frequency)}. \tag{17-57}$$

Musicians use the beat phenomenon in tuning instruments. If an instrument is sounded against a standard frequency (for example, the note called "concert A" played on an orchestra's first oboe) and tuned until the beat disappears, the instrument is in tune with that standard. In musical Vienna, concert A (440 Hz) is available as a convenient telephone service for the city's many musicians.

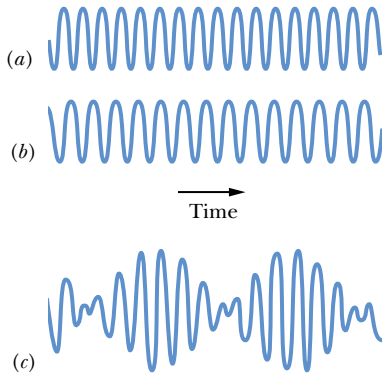

Figure 17-29 (a, b) The pressure variations Δp of two sound waves as they would be detected separately. The frequencies of the waves are nearly equal. (c) The resultant pressure variation if the two waves are detected simultaneously.

SAMPLE PROBLEM 17.12

Beat frequencies and penguins finding one another

When an emperor penguin returns from a search for food, how can it find its mate among the thousands of penguins huddled together for warmth in the harsh Antarctic weather? It is not by sight, because penguins all look alike, even to a penguin.

The answer lies in the way penguins vocalize. Most birds vocalize by using only one side of their two-sided vocal organ, called the *syrinx*. Emperor penguins, however, vocalize by using both sides simultaneously. Each side sets up acoustic standing waves in the bird's throat and mouth, much like in a pipe with two open ends. Suppose that the frequency of the first harmonic produced by side A is $f_{A1} = 432$ Hz and the frequency of the first harmonic produced by side B is $f_{B1} = 371$ Hz. What is the beat frequency between those two first-harmonic frequencies and between the two second-harmonic frequencies?

KEY IDEA

The beat frequency between two frequencies is their difference, as given by Eq. 17-52 ($f_{beat} = f_1 - f_2$).

Calculations: For the two first-harmonic frequencies f_{A1} and f_{B1}, the beat frequency is

$$f_{beat,1} = f_{A1} - f_{B1} = 432 \text{ Hz} - 371 \text{ Hz}$$
$$= 61 \text{ Hz}. \quad \text{(Answer)}$$

Because the standing waves in the penguin are effectively in a pipe with two open ends, the resonant frequencies are given by Eq. 17-38 ($f = nv/2L$), in which L is the (unknown) length of the effective pipe. The first-harmonic frequency is $f_1 = v/2L$, and the second-harmonic frequency is $f_2 = 2v/2L$. Comparing these two frequencies, we see that, in general,

$$f_2 = 2f_1.$$

For the penguin, the second harmonic of side A has frequency $f_{A2} = 2f_{A1}$ and the second harmonic of side B has frequency $f_{B2} = 2f_{B1}$. Using Eq. 17-52 with frequencies f_{A2} and f_{B2}, we find that the corresponding beat frequency associated with the second harmonics is

$$f_{beat,2} = f_{A2} - f_{B2} = 2f_{A1} - 2f_{B1}$$
$$= 2(432 \text{ Hz}) - 2(371 \text{ Hz})$$
$$= 122 \text{ Hz}. \quad \text{(Answer)}$$

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Experiments indicate that penguins can perceive such large beat frequencies. (Humans cannot hear a beat frequency any higher than about 12 Hz — we perceive the two separate frequencies.) Thus, a penguin's cry can be rich with different harmonics and different beat frequencies, allowing the voice to be recognized even among the voices of thousands of other, closely huddled penguins.

17.10 | THE DOPPLER EFFECT

Key Concepts

- The Doppler effect is a change in the observed frequency of a wave when the source or the detector moves relative to the transmitting medium (such as air). For sound the observed frequency f' is given in terms of the source frequency f by

$$f' = f\frac{v \pm v_D}{v \pm v_S} \quad \text{(general Doppler effect)},$$

where v_D is the speed of the detector relative to the medium, v_S is that of the source, and v is the speed of sound in the medium.

- The signs are chosen such that f' tends to be *greater* for relative motion toward (one of the objects moves toward the other) and *less* for motion away.

A police car is parked by the side of the highway, sounding its 1000 Hz siren. If you are also parked by the highway, you will hear that same frequency. However, if there is relative motion between you and the police car, either toward or away from each other, you will hear a different frequency. For example, if you are driving *toward* the police car at 120 km/h (about 75 mi/h), you will hear a *higher* frequency (1096 Hz, an *increase* of 96 Hz). If you are driving *away from* the police car at that same speed, you will hear a *lower* frequency (904 Hz, a *decrease* of 96 Hz).

These motion-related frequency changes are examples of the **Doppler effect**. The effect was proposed (although not fully worked out) in 1842 by Austrian physicist Johann Christian Doppler. It was tested experimentally in 1845 by Buys Ballot in Holland, "using a locomotive drawing an open car with several trumpeters."

The Doppler effect holds not only for sound waves but also for electromagnetic waves, including microwaves, radio waves, and visible light. Here, however, we shall consider only sound waves, and we shall take as a reference frame the body of air through which these waves travel. This means that we shall measure the speeds of a source S of sound waves and a detector D of those waves *relative to that body of air.* (Unless otherwise stated, the body of air is stationary relative to the ground, so the speeds can also be measured relative to the ground.) We shall assume that S and D move either directly toward or directly away from each other, at speeds less than the speed of sound.

General Equation. If either the detector or the source is moving, or both are moving, the emitted frequency f and the detected frequency f' are related by

$$f' = f\frac{v \pm v_D}{v \pm v_S} \quad \text{(general Doppler effect)}, \tag{17-58}$$

where v is the speed of sound through the air, v_D is the detector's speed relative to the air, and v_S is the source's speed relative to the air. The choice of plus or minus signs is set by this rule:

When the motion of detector or source is toward the other, the sign on its speed must give an upward shift in frequency. When the motion of detector or source is away from the other, the sign on its speed must give a downward shift in frequency.

In short, *toward* means *shift up,* and *away* means *shift down.*

Here are some examples of the rule. If the detector moves toward the source, use the plus sign in the numerator of Eq. 17-58 to get a shift up in the frequency. If it moves away, use the minus sign in the numerator to get a

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shift down. If it is stationary, substitute 0 for v_D. If the source moves toward the detector, use the minus sign in the denominator of Eq. 17-58 to get a shift up in the frequency. If it moves away, use the plus sign in the denominator to get a shift down. If the source is stationary, substitute 0 for v_S.

Next, we derive equations for the Doppler effect for the following two specific situations and then derive Eq. 17-58 for the general situation.

1. When the detector moves relative to the air and the source is stationary relative to the air, the motion changes the frequency at which the detector intercepts wavefronts and thus changes the detected frequency of the sound wave.
2. When the source moves relative to the air and the detector is stationary relative to the air, the motion changes the wavelength of the sound wave and thus changes the detected frequency (recall that frequency is related to wavelength).

Moving Detector and Stationary Source

In Fig. 17-30, a detector D (represented by an ear) is moving at speed v_D toward a stationary source S that emits spherical wavefronts, of wavelength λ and frequency f, moving at the speed v of sound in air. The wavefronts are drawn one wavelength apart. The frequency detected by detector D is the rate at which D intercepts wavefronts (or individual wavelengths). If D were stationary, that rate would be f, but since D is moving into the wavefronts, the rate of interception is greater, and thus the detected frequency f' is greater than f.

Let us for the moment consider the situation in which D is stationary (Fig. 17-31). In time t, the wavefronts move to the right a distance vt. The number of wavelengths in that distance vt is the number of wavelengths intercepted by D in time t, and that number is vt/λ. The rate at which D intercepts wavelengths, which is the frequency f detected by D, is

$$f = \frac{vt/\lambda}{t} = \frac{v}{\lambda}. \tag{17-59}$$

In this situation, with D stationary, there is no Doppler effect—the frequency detected by D is the frequency emitted by S.

Now let us again consider the situation in which D moves in the direction opposite the wavefront velocity (Fig. 17-32). In time t, the wavefronts move to the right a distance vt as previously, but now D moves to the left a distance $v_D t$. Thus, in this time t, the distance moved by the wavefronts relative to D is $vt + v_D t$. The number of wavelengths in this relative distance $vt + v_D t$ is the number of wavelengths intercepted by D in time t and is $(vt + v_D t)/\lambda$. The *rate* at which D intercepts wavelengths in this situation is the frequency f', given by

$$f' = \frac{(vt + v_D t)/\lambda}{t} = \frac{v + v_D}{\lambda}. \tag{17-60}$$

From Eq. 17-59, we have $\lambda = v/f$. Then Eq. 17-60 becomes

$$f' = \frac{v + v_D}{v/f} = f\frac{v + v_D}{v}. \tag{17-61}$$

Note that in Eq. 17-61, $f' > f$ unless $v_D = 0$ (the detector is stationary).

Shift up: The detector moves *toward* the source.

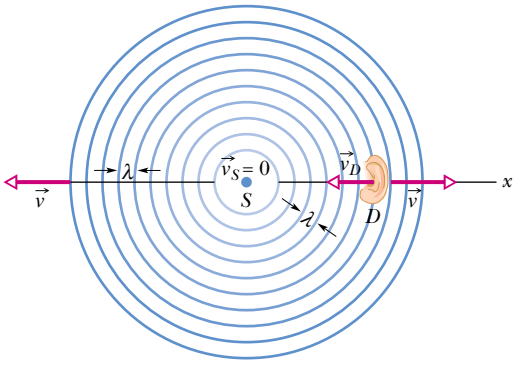

Figure 17-30 A stationary source of sound S emits spherical wavefronts, shown one wavelength apart, that expand outward at speed v. A sound detector D, represented by an ear, moves with velocity $\vec{v}_D$ toward the source. The detector senses a higher frequency because of its motion.

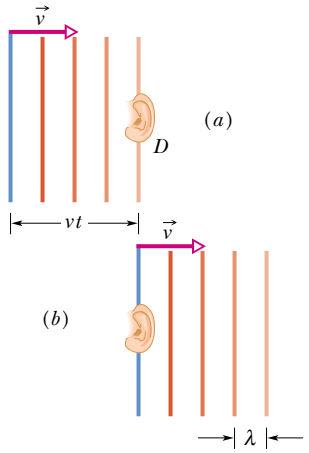

Figure 17-31 The wavefronts of Fig. 17-30, assumed planar, (*a*) reach and (*b*) pass a stationary detector D; they move a distance vt to the right in time t.

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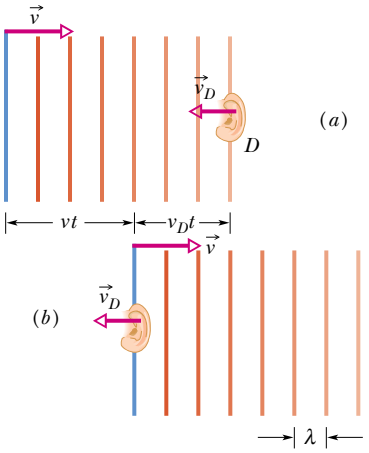

Figure 17-32 Wavefronts traveling to the right (*a*) reach and (*b*) pass detector *D*, which moves in the opposite direction. In time *t*, the wavefronts move a distance vt to the right and *D* moves a distance $v_D t$ to the left.

Similarly, we can find the frequency detected by *D* if *D* moves away from the source. In this situation, the wavefronts move a distance $vt - v_D t$ relative to *D* in time *t*, and f' is given by

$$f' = f\frac{v - v_D}{v}. \tag{17-62}$$

In Eq. 17-62, $f' < f$ unless $v_D = 0$. We can summarize Eqs. 17-61 and 17-62 with

$$f' = f\frac{v \pm v_D}{v} \quad \text{(detector moving, source stationary)}. \tag{17-63}$$

Moving Source and Detector Stationary

Let detector *D* be stationary with respect to the body of air, and let source *S* move toward *D* at speed v_S (Fig. 17-33). The motion of *S* changes the wavelength of the sound waves it emits and thus the frequency detected by *D*.

To see this change, let T (= $1/f$) be the time between the emission of any pair of successive wavefronts W_1 and W_2. During T, wavefront W_1 moves a distance vT and the source moves a distance $v_S T$. At the end of T, wavefront W_2 is emitted. In the direction in which *S* moves, the distance between W_1 and W_2, which is the wavelength λ' of the waves moving in that direction, is $vT - v_S T$. If *D* detects those waves, it detects frequency f' given by

$$f' = \frac{v}{\lambda'} = \frac{v}{vT - v_S T} = \frac{v}{v/f - v_S/f}$$

$$= f\frac{v}{v - v_S}. \tag{17-64}$$

Note that f' must be greater than f unless $v_S = 0$.

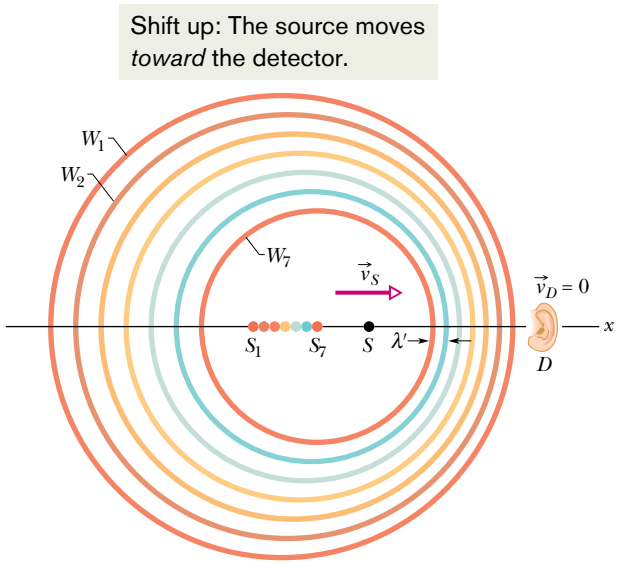

Figure 17-33 A detector *D* is stationary, and a source *S* is moving toward it at speed v_S. Wavefront W_1 was emitted when the source was at S_1, wavefront W_7 when it was at S_7. At the moment depicted, the source is at *S*. The detector senses a higher frequency because the moving source, chasing its own wavefronts, emits a reduced wavelength λ' in the direction of its motion.

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In the direction opposite that taken by S, the wavelength λ' of the waves is again the distance between successive waves but now that distance is $vT + v_S T$. If D detects those waves, it detects frequency f' given by

$$f' = f\frac{v}{v + v_S}. \tag{17-65}$$

Now f' must be less than f unless $v_S = 0$.

We can summarize Eqs. 17-64 and 17-65 with

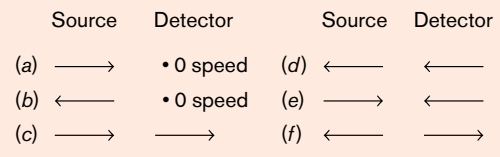
$$f' = f\frac{v}{v \pm v_S} \quad \text{(source moving, detector stationary).} \tag{17-66}$$

General Doppler Effect Equation

We can now derive the general Doppler effect equation by replacing f in Eq. 17-66 (the source frequency) with f' of Eq. 17-63 (the frequency associated with motion of the detector). That simple replacement gives us Eq. 17-58 for the general Doppler effect. That general equation holds not only when both detector and source are moving but also in the two specific situations we just discussed. For the situation in which the detector is moving and the source is stationary, substitution of $v_S = 0$ into Eq. 17-58 gives us Eq. 17-63, which we previously found. For the situation in which the source is moving and the detector is stationary, substitution of $v_D = 0$ into Eq. 17-58 gives us Eq. 17-66, which we previously found. Thus, Eq. 17-58 is the equation to remember.

CHECKPOINT 4

The figure indicates the directions of motion of a sound source and a detector for six situations in stationary air. For each situation, is the detected frequency greater than or less than the emitted frequency, or can not we tell without more information about the actual speeds?



SAMPLE PROBLEM 17.13

Double Doppler shift in the echoes used by bats

Bats navigate and search out prey by emitting, and then detecting reflections of, ultrasonic waves, which are sound waves with frequencies greater than can be heard by a human. Suppose a bat emits ultrasound at frequency $f_{be} = 82.52$ kHz while flying with velocity $\vec{v}_b = (9.00 \text{ m/s})\hat{i}$ as it chases a moth that flies with velocity $\vec{v}_m = (8.00 \text{ m/s})\hat{i}$. What frequency f_{md} does the moth detect? What frequency f_{bd} does the bat detect in the returning echo from the moth?

KEY IDEAS

The frequency is shifted by the relative motion of the bat and moth. Because they move along a single axis, the shifted frequency is given by Eq. 17-58. Motion *toward* tends to shift the frequency *up*, and motion *away* tends to shift it *down*.

Detection by moth: The general Doppler equation is

$$f' = f\frac{v \pm v_D}{v \pm v_S}. \tag{17-67}$$

Here, the detected frequency f' that we want to find is the frequency f_{md} detected by the moth. On the right side, the emitted frequency f is the bat's emission frequency $f_{be} = 82.52$ kHz, the speed of sound is $v = 343$ m/s, the speed v_D of the detector is the moth's speed $v_m = 8.00$ m/s, and the speed v_S of the source is the bat's speed $v_b = 9.00$ m/s.

The decisions about the plus and minus signs can be tricky. Think in terms of *toward* and *away*. We have the speed of the moth (the detector) in the numerator of Eq. 17-62. The moth moves *away* from the bat, which tends to lower the detected frequency. Because the speed is in the numerator, we choose

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the minus sign to meet that tendency (the numerator becomes smaller). These reasoning steps are shown in Table 17-3.

We have the speed of the bat in the denominator of Eq. 17-67. The bat moves *toward* the moth, which tends to increase the detected frequency. Because the speed is in the denominator, we choose the minus sign to meet that tendency (the denominator becomes smaller).

With these substitutions and decisions, we have

$$f_{md} = f_{be}\frac{v - v_m}{v - v_b}$$
$$= (82.52\text{ kHz})\frac{343\text{ m/s} - 8.00\text{ m/s}}{343\text{ m/s} - 9.00\text{ m/s}}$$
$$= 82.767\text{ kHz} \approx 82.8\text{ kHz}. \qquad \text{(Answer)}$$

Detection of echo by bat: In the echo back to the bat, the moth acts as a source of sound, emitting at the frequency f_{md} we just calculated. So now the moth is the source (moving *away*) and the bat is the detector (moving *toward*). The reasoning steps are shown in Table 17-3.

To find the frequency f_{bd} detected by the bat, we write Eq. 17-62 as

$$f_{bd} = f_{md}\frac{v + v_b}{v + v_m}$$
$$= (82.767\text{ kHz})\frac{343\text{ m/s} + 9.00\text{ m/s}}{343\text{ m/s} + 8.00\text{ m/s}}$$
$$= 83.00\text{ kHz} \approx 83.0\text{ kHz}. \qquad \text{(Answer)}$$

Some moths evade bats by "jamming" the detection system with ultrasonic clicks.

Table 17-3

Bat to Moth		Echo Back to Bat	
Detector	Source	Detector	Source
moth	bat	bat	moth
speed $v_D = v_m$	speed $v_S = v_b$	speed $v_D = v_b$	speed $v_S = v_m$
away	toward	toward	away
shift down	shift up	shift up	shift down
numerator	denominator	numerator	denominator
minus	minus	plus	plus

SAMPLE PROBLEM 17.14

Finding frequency when the source is moving, and detector is stationary

An engine is continuously emitting sound of frequency 1.2 kHz moving uniformly along a straight track. A stationary listener is situated off the track at a perpendicular distance of $l = 350$ m. The velocity of engine is equal to 50 m/s and the velocity of sound is 350 m/s. Find the frequency of sound received by the listener at the moment when the engine gets closest to him.

KEY IDEA

This problem is based on the fact that sound takes finite time to reach from the engine to the listener. You can imagine it like this, suppose you are running along a straight path and you are throwing balls of different colors toward your friend standing off the path at some distance. By the time the first ball reaches your friend you will be at some other position throwing some other ball.

Similarly, the frequency emitted by the engine when it is closest to the listener does not reach instantaneously. Sound from engine takes finite time to reach the listener. The frequency heard when the engine is closest, was emitted from some previous position of the engine.

Calculations: Let the velocity of engine make an angle θ with the line joining engine and listener, and the sound emitted by the engine at this moment, reaches the listener exactly at the same instant, when the engine is closest to the listener. Thus, the time in which sound reaches the listener along SL is equal to the time when engine reaches point O from point S.

From Fig. 17-34,

$$\cos\theta = \frac{SO}{SL} = \frac{v_s t}{vt} = \frac{v_s}{v} \qquad (17\text{-}68)$$

The component of velocity, along the direction of sound propagation will affect the wavelength. Thus $v_s \cos\theta$ should be used in the equation of doppler effect. Frequency heard by the listener is

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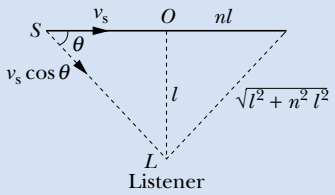

Figure 17-34 An engine moving on a railway track and sound is heard by the listener, L.

$$f_L = f_S\left(\frac{v}{v - v_s\cos\theta}\right) \qquad (17\text{-}69)$$

$$= f_S\left(\frac{v}{v - v_s^2/v}\right)$$

$$= f_S\left(\frac{1}{1 - v_s^2/v^2}\right)$$

On substituting numerical values, we get

$$f_L = 1.225 \text{ kHz}$$

Since the sound heard by the listener was emitted by the engine while approaching the listener, the frequency observed should have been higher than the source frequency.

In Case of: Find the location of the engine at the moment when the listener receives actual frequency of the engine.

Calculation: When the source is at O, the velocity component of source toward listener is $v_s \cos 90° = 0$. Hence from Eq. 17-69

$$f_L = f_S\left(\frac{v}{v - v_s\cos 90°}\right) = f_S$$

Time taken by sound to reach the listener is l/v. Distance moved by engine in this time

$$= \frac{l}{v} \times v_s = \frac{350}{350} \times 50 = 50 \text{ m} \qquad \text{(Answer)}$$

Thus, engine is located at distance of 50 m from the nearest point when the listener hears the actual frequency.

17.11 | SUPERSONIC SPEEDS, SHOCK WAVES

Key Concept

- If the speed of a source relative to the medium exceeds the speed of sound in the medium, the Doppler equation no longer applies. In such a case, shock waves result. The half-angle θ of the Mach cone is given by

$$\sin\theta = \frac{v}{v_S} \quad \text{(Mach cone angle).}$$

If a source is moving toward a stationary detector at a speed v_S equal to the speed of sound v, Eqs. 17-58 and 17-66 predict that the detected frequency f' will be infinitely great. This means that the source is moving so fast that it keeps pace with its own spherical wavefronts (Fig. 17-35*a*). What happens when $v_S > v$? For such *supersonic* speeds, Eqs. 17-58 and 17-66 no longer apply. Figure 17-35*b* depicts the spherical wavefronts that originated at various positions of the source. The radius of any wavefront is vt, where t is the time that has elapsed since the source emitted that wavefront. Note that all the wavefronts bunch along a V-shaped envelope in this two-dimensional drawing. The wavefronts actually extend in three dimensions, and the bunching actually forms a cone called the *Mach cone*. A *shock wave* exists along the surface of this cone, because the bunching of wavefronts causes an abrupt rise and fall of air pressure as the surface passes through any point. From Fig. 17-35*b*, we see that the half-angle θ of the cone (the *Mach cone angle*) is given by

$$\sin\theta = \frac{vt}{v_S t} = \frac{v}{v_S} \quad \text{(Mach cone angle).} \qquad (17\text{-}70)$$

The ratio v_S/v is the *Mach number*. If a plane flies at Mach 2.3, its speed is 2.3 times the speed of sound in the air through which the plane is flying. The shock wave generated by a supersonic aircraft (Fig. 17-36) or projectile

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produces a burst of sound, called a *sonic boom,* in which the air pressure first suddenly increases and then suddenly decreases below normal before returning to normal. Part of the sound that is heard when a rifle is fired is the sonic boom produced by the bullet. When a long bull whip is snapped, its tip is moving faster than sound and produces a small sonic boom—the *crack* of the whip.

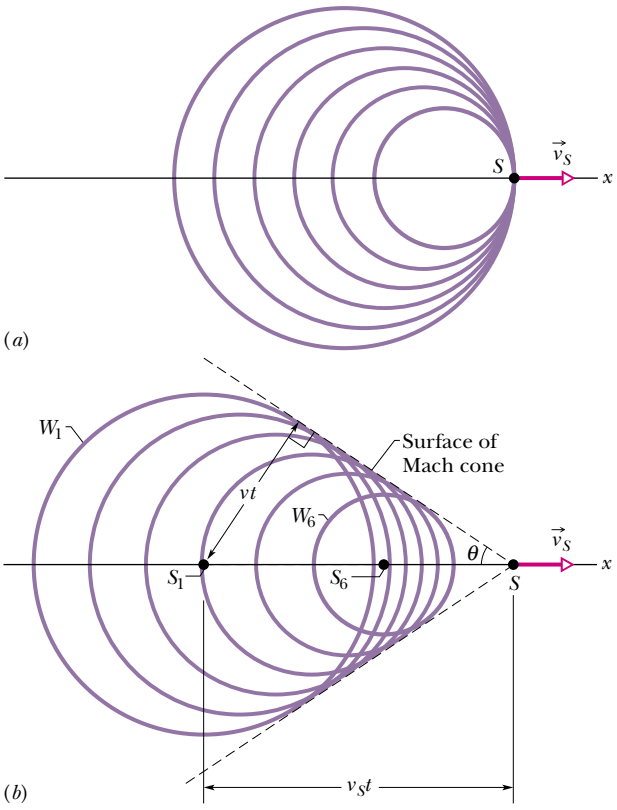

Figure 17-35 (*a*) A source of sound *S* moves at speed v_S equal to the speed of sound and thus as fast as the wavefronts it generates. (*b*) A source *S* moves at speed v_S faster than the speed of sound and thus faster than the wavefronts. When the source was at position S_1 it generated wavefront W_1, and at position S_6 it generated W_6. All the spherical wavefronts expand at the speed of sound *v* and bunch along the surface of a cone called the Mach cone, forming a shock wave. The surface of the cone has half-angle θ and is tangent to all the wavefronts.

U.S. Navy photo by Ensign John Gay

Figure 17-36 Shock waves produced by the wings of a Navy FA 18 jet. The shock waves are visible because the sudden decrease in air pressure in them caused water molecules in the air to condense, forming a fog.

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REVIEW AND SUMMARY

Sound Waves Sound waves are longitudinal mechanical waves that can travel through solids, liquids, or gases. The speed v of a sound wave in a medium having **bulk modulus** B and density ρ is

$$v = \sqrt{\frac{B}{\rho}} \quad \text{(speed of sound)}. \tag{17-3}$$

In air at 20°C, the speed of sound is 343 m/s.

A sound wave causes a longitudinal displacement s of a mass element in a medium as given by

$$s = s_m \cos(kx - \omega t), \tag{17-12}$$

where s_m is the **displacement amplitude** (maximum displacement) from equilibrium, $k = 2\pi/\lambda$, and $\omega = 2\pi f$, λ and f being the wavelength and frequency of the sound wave. The wave also causes a pressure change Δp from the equilibrium pressure:

$$\Delta p = \Delta p_m \sin(kx - \omega t), \tag{17-13}$$

where the **pressure amplitude** is

$$\Delta p_m = (v\rho\omega)s_m. \tag{17-14}$$

Sound Intensity The **intensity** I of a sound wave at a surface is the average rate per unit area at which energy is transferred by the wave through or onto the surface:

$$I = \frac{P}{A}, \tag{17-19}$$

where P is the time rate of energy transfer (power) of the sound wave and A is the area of the surface intercepting the sound. The intensity I is related to the displacement amplitude s_m of the sound wave by

$$I = \frac{1}{2}\rho v\omega^2 s_m^2. \tag{17-20}$$

The intensity at a distance r from a point source that emits sound waves of power P_S is

$$I = \frac{P_S}{4\pi r^2}, \tag{17-21}$$

Sound Level in Decibels The *sound level* β in *decibels* (dB) is defined as

$$\beta = (10\text{ dB})\log\frac{I}{I_0}, \tag{17-22}$$

where I_0 ($= 10^{-12}$ W/m^2) is a reference intensity level to which all intensities are compared. For every factor-of-10 increase in intensity, 10 dB is added to the sound level.

The propagation Vector A harmonic traveling plane wave propagating in a homogeneous dispersive medium in the direction of the unit vector $\hat{c}$, then at a plane at some distance c from the origin the disturbance is given by

$$\Psi(c,t) = y_m \sin(\omega t - kc) \tag{17-28}$$

The quantity kc is

$$kc = k(\hat{c}\cdot\vec{r}) = (k\hat{c})\cdot\vec{r} \equiv \vec{k}\cdot\vec{r} \tag{17-29}$$

where $k\hat{c}$ is the propagation vector $\vec{k}$; the magnitude of $\vec{k}$ is $(2\pi/\lambda)$. The direction of $\vec{k}$ is $\hat{c}$, along the wave propagation direction. For a wave having three-dimensional wave vector

$$kc = \vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z$$

The physical meaning of the wave number number k is the increase in phase per unit displacement along the propagation direction $\hat{c}$. So that kc is the phase increased in distance c.

Standing Wave Patterns in Pipes Standing sound wave patterns can be set up in pipes. A pipe open at both ends will resonate at frequencies

$$f = \frac{v}{\lambda} = \frac{nv}{2L}, \quad n = 1, 2, 3, \ldots, \tag{17-38}$$

where v is the speed of sound in the air in the pipe. For a pipe closed at one end and open at the other, the resonant frequencies are

$$f = \frac{v}{\lambda} = \frac{nv}{4L}, \quad n = 1, 3, 5, \ldots. \tag{17-40}$$

Interference The interference of two sound waves with identical wavelengths passing through a common point depends on their phase difference ϕ there. If the sound waves were emitted in phase and are traveling in approximately the same direction, ϕ is given by

$$\phi = \frac{\Delta L}{\lambda}2\pi, \tag{17-48}$$

where ΔL is their **path length difference** (the difference in the distances traveled by the waves to reach the common point). Fully constructive interference occurs when ϕ is an integer multiple of 2π,

$$\phi = m(2\pi), \qquad \text{for } m = 0, 1, 2, \ldots, \tag{17-49}$$

and, equivalently, when ΔL is related to wavelength λ by

$$\frac{\Delta L}{\lambda} = 0, 1, 2, \ldots. \tag{17-50}$$

Fully destructive interference occurs when ϕ is an odd multiple of π,

$$\phi = (2m + 1)\,\pi, \quad \text{for } m = 0, 1, 2, \ldots, \tag{17-51}$$

and, equivalently, when ΔL is related to λ by

$$\frac{\Delta L}{\lambda} = 0.5, 1.5, 2.5, \ldots. \tag{17-52}$$

Beats *Beats* arise when two waves having slightly different frequencies, f_1 and f_2, are detected together. The beat frequency is

$$f_{\text{beat}} = f_1 - f_2. \tag{17-57}$$

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The Doppler Effect The *Doppler effect* is a change in the observed frequency of a wave when the source or the detector moves relative to the transmitting medium (such as air). For sound the observed frequency f' is given in terms of the source frequency f by

$$f' = f\frac{v \pm v_D}{v \pm v_S} \quad \text{(general Doppler effect)}, \qquad (17\text{-}58)$$

where v_D is the speed of the detector relative to the medium, v_S is that of the source, and v is the speed of sound in the medium. The signs are chosen such that f' tends to be *greater* for motion toward and *less* for motion away.

Shock Wave If the speed of a source relative to the medium exceeds the speed of sound in the medium, the Doppler equation no longer applies. In such a case, shock waves result. The half-angle θ of the Mach cone is given by

$$\sin\theta = \frac{v}{v_S} \quad \text{(Mach cone angle)}. \qquad (17\text{-}70)$$

PROBLEMS

Where needed in the problems, use

speed of sound in air = 343 m/s

and density of air = 1.21 kg/m^3

unless otherwise specified.

1. Diagnostic ultrasound of frequency 3.80 MHz is used to examine tumors in soft tissue. (a) What is the wavelength in air of such a sound wave? (b) If the speed of sound in tissue is 1500 m/s, what is the wavelength of this wave in tissue?
2. What is the intensity at radial distances (a) 2.50 m and (b) 6.00 m from an isotropic point source of sound that emits energy at the rate 12.0 W, assuming no energy absorption by the surrounding air?
3. When you "crack" a knuckle, you suddenly widen the knuckle cavity, allowing more volume for the synovial fluid inside it and causing a gas bubble suddenly to appear in the fluid. The sudden production of the bubble, called "cavitation," produces a sound pulse—the cracking sound. Assume that the sound is transmitted uniformly in all directions and that it fully passes from the knuckle interior to the outside. If the pulse has a sound level of 50 dB at your ear, estimate the rate at which energy is produced by the cavitation.
4. A tuning fork of unknown frequency makes 4.00 beats per second with a standard fork of frequency 384 Hz. The beat frequency decreases when a small piece of wax is put on a prong of the first fork. What is the frequency of this fork?
5. A sound wave of frequency 280 Hz has an intensity of 1.00 μW/m^2. What is the amplitude of the air oscillations caused by this wave?
6. *Party hearing*. As the number of people at a party increases, you must raise your voice for a listener to hear you against the *background noise* of the other partygoers. However, once you reach the level of yelling, the only way you can be heard is if you move closer to your listener, into the listener's "personal space." Model the situation by replacing you with an isotropic point source of fixed power P and replacing your listener with a point that absorbs part of your sound waves. These points are initially separated by r_1 = 1.20 m. If the background noise increases by $\Delta\beta$ = 9.0 dB, the sound level at your listener must also increase. What separation r_f is then required?
7. Two identical piano wires have a fundamental frequency of 600 Hz when kept under the same tension. What fractional increase in the tension of one wire will lead to the occurrence of 8.0 beats/s when both wires oscillate simultaneously?
8. A sound source A and a reflecting surface B move directly toward each other. Relative to the air, the speed of source A is 20.0 m/s, the speed of surface B is 80.0 m/s, and the speed of sound is 329 m/s. The source emits waves at frequency 2000 Hz as measured in the source frame. In the reflector frame, what are the (a) frequency and (b) wavelength of the arriving sound waves? In the source frame, what are the (c) frequency and (d) wavelength of the sound waves reflected back to the source?
9. A whistle of frequency 540 Hz moves in a circle of radius 60.0 cm at an angular speed of 20.0 rad/s. What are the (a) lowest and (b) highest frequencies heard by a listener a long distance away, at rest with respect to the center of the circle?
10. The shock wave off the cockpit of the FA 18 in Fig. 17-36 has an angle of about 60°. The airplane was traveling at about 1350 km/h when the photograph was taken. Approximately what was the speed of sound at the airplane's altitude?
11. A bat is flitting about in a cave, navigating via ultrasonic bleeps. Assume that the sound emission frequency of the bat is 39 000 Hz. During one fast swoop directly toward a flat wall surface, the bat is moving at 0.020 times the speed of sound in air. What frequency does the bat hear reflected off the wall?
12. The source of a sound wave has a power of 3.00 μW. If it is a point source, (a) what is the intensity 4.20 m away and (b) what is the sound level in decibels at that distance?
13. A sound source sends a sinusoidal sound wave of angular frequency 3000 rad/s and amplitude 10.0 nm through a tube of air. The internal radius of the tube is 2.00 cm. (a) What is the average rate at which energy (the sum of the kinetic and potential energies) is transported to the opposite end of the tube? (b) If, simultaneously, an

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identical wave travels along an adjacent, identical tube, what is the total average rate at which energy is transported to the opposite ends of the two tubes by the waves? If, instead, those two waves are sent along the *same* tube simultaneously, what is the total average rate at which they transport energy when their phase difference is (c) 0, (d) 0.40π rad, and (e) π rad?

14. Two atmospheric sound sources A and B emit isotropically at constant power. The sound levels β of their emissions are plotted in Fig. 17-37 versus the radial distance r from the sources. The vertical axis scale is set by $\beta_1 = 85.0$ dB and $\beta_2 = 65.0$ dB. What are (a) the ratio of the larger power to the smaller power and (b) the sound level difference at $r = 23$ m?

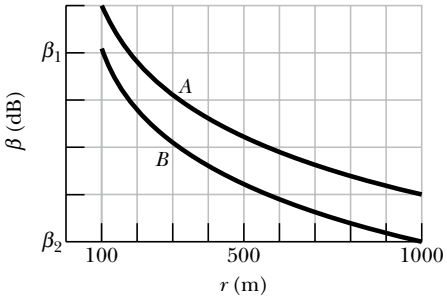

Figure 17-37 Problem 14.

15. A jet plane passes over you at a height of 4800 m and a speed of Mach 1.5. (a) Find the Mach cone angle (the sound speed is 331 m/s). (b) How long after the jet passes directly overhead does the shock wave reach you?

16. Organ pipe A, with both ends open, has a fundamental frequency of 425 Hz. The fifth harmonic of organ pipe B, with one end open, has the same frequency as the second harmonic of pipe A. How long are (a) pipe A and (b) pipe B?

17. A certain sound source is increased in sound level by 40.0 dB. By what multiple is (a) its intensity increased and (b) its pressure amplitude increased?

18. Figure 17-38 shows four tubes with lengths 1.0 m or 2.0 m, with one or two open ends as drawn. The fifth harmonic is set up in each tube, and some of the sound that escapes from them is detected by detector D, which moves directly away from the tubes. In terms of the speed of sound v, what speed must the detector have such that the detected frequency of the sound from (a) tube 1, (b) tube 2, (c) tube 3, and (d) tube 4 is equal to the tube's fundamental frequency?

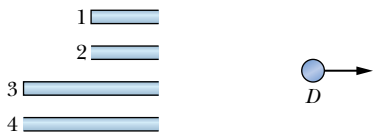

Figure 17-38 Problem 18.

19. A point source emits 30.0 W of sound isotropically. A small microphone intercepts the sound in an area of 0.750 cm^2, 180 m from the source. Calculate (a) the sound intensity there and (b) the power intercepted by the microphone.

20. A plane flies at 2.00 times the speed of sound. Its sonic boom reaches a man on the ground 35.4 s after the plane passes directly overhead. What is the altitude of the plane? Assume the speed of sound to be 330 m/s.

21. A point source emits sound waves isotropically. The intensity of the waves 6.00 m from the source is 4.50×10^{-4} W/m^2. Assuming that the energy of the waves is conserved, find the power of the source.

22. A tube 0.900 m long is closed at one end. A stretched wire is placed near the open end. The wire is 0.330 m long and has a mass of 9.60 g. It is fixed at both ends and oscillates in its fundamental mode. By resonance, it sets the air column in the tube into oscillation at that column's second lowest harmonic frequency. Find (a) that frequency and (b) the tension in the wire.

23. Earthquakes generate sound waves inside Earth. Unlike a gas, Earth can experience both transverse (S) and longitudinal (P) sound waves. Typically, the speed of S waves is about 4.5 km/s, and that of P waves 8.0 km/s. A seismograph records P and S waves from an earthquake. The first P waves arrive 3.5 min before the first S waves. If the waves travel in a straight line, how far away did the earthquake occur?

24. Pipe A, which is 1.80 m long and open at both ends, oscillates at its third lowest harmonic frequency. It is filled with air for which the speed of sound is 343 m/s. Pipe B, which is closed at one end, oscillates at its second lowest harmonic frequency. This frequency of B happens to match the frequency of A. An x axis extends along the interior of B, with $x = 0$ at the closed end. (a) How many nodes are along that axis? What are the (b) smallest and (c) second smallest value of x locating those nodes? (d) What is the fundamental frequency of B?

25. (a) Find the speed of waves on a violin string of mass 860 mg and length 22.0 cm if the fundamental frequency is 920 Hz. (b) What is the tension in the string? For the fundamental, what is the wavelength of (c) the waves on the string and (d) the sound waves emitted by the string?

26. Figure 17-39 shows the output from a pressure monitor mounted at a point along the path taken by a sound wave of a single frequency traveling at 343 m/s through air with a uniform density of 1.21 kg/m^3. The vertical axis scale is set by $\Delta p_s = 5.0$ mPa. If the *displacement* function of the wave is $s(x, t) = s_m \cos(kx - \omega t)$, what are (a) s_m, (b) k, and (c) ω? The air is then cooled so that its density is 1.35 kg/m^3 and the speed of a sound wave through it is 320 m/s. The sound source again emits the sound wave at the same frequency and same pressure amplitude. What now are (d) s_m, (e) k, and (f) ω?

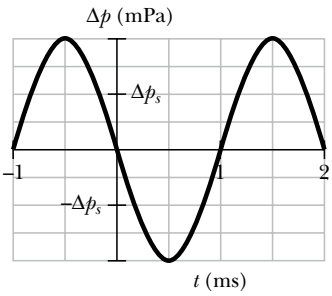

Figure 17-39 Problem 26.

27. A violin string 15.0 cm long and fixed at both ends oscillates in its $n = 1$ mode. The speed of waves on the string is 280 m/s, and the speed of sound in air is 348 m/s. What are

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the (a) frequency and (b) wavelength of the emitted sound wave?

28. Figure 17-40 shows four isotropic point sources of sound that are uniformly spaced on an x axis. The sources emit sound at the same wavelength λ and same amplitude s_m, and they emit in phase. A point P is shown on the x axis. Assume that as the sound waves travel to P, the decrease in their amplitude is negligible. What multiple of s_m is the amplitude of the net wave at P if distance d in the figure is (a) $\lambda/4$, (b) $\lambda/2$, and (c) λ?

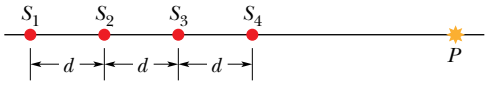

Figure 17-40 Problem 28.

29. The water level in a vertical glass tube 1.00 m long can be adjusted to any position in the tube. A tuning fork vibrating at 900 Hz is held just over the open top end of the tube, to set up a standing wave of sound in the air-filled top portion of the tube. (That airfilled top portion acts as a tube with one end closed and the other end open.) (a) For how many different positions of the water level will sound from the fork set up resonance in the tube's air-filled portion? What are the (b) least and (c) second least water heights in the tube for resonance to occur?

30. In Fig. 17-41, a French submarine and a U.S. submarine move toward each other during maneuvers in motionless water in the North Atlantic. The French sub moves at the speed $v_F = 48.00$ km/h, and the U.S. sub at $v_{US} = 72.00$ km/h. The French sub sends out a sonar signal (sound wave in water) at 1.560×10^3 Hz. Sonar waves travel at 5470 km/h. (a) What is the signal's frequency as detected by the U.S. sub? (b) What frequency is detected by the French sub in the signal reflected back to it by the U.S. sub?

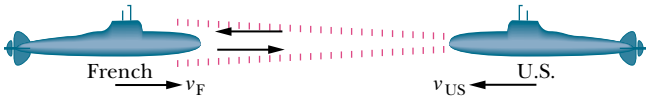

Figure 17-41 Problem 30.

31. Two sounds differ in sound level by 3.00 dB. What is the ratio of the greater intensity to the smaller intensity?

32. The A string of a violin is a little too tightly stretched. Beats at 4.50 per second are heard when the string is sounded together with a tuning fork that is oscillating accurately at concert A (440 Hz). What is the period of the violin string oscillation?

33. A man strikes one end of a thin rod with a hammer. The speed of sound in the rod is 15 times the speed of sound in air. A woman, at the other end with her ear close to the rod, hears the sound of the blow twice with a 60 ms interval between; one sound comes through the rod and the other comes through the air alongside the rod. If the speed of sound in air is 343 m/s, what is the length of the rod?

34. An acoustic burglar alarm consists of a source emitting waves of frequency 30.0 kHz. What is the beat frequency between the source waves and the waves reflected from an intruder walking at an average speed of 0.950 m/s directly away from the alarm?

35. *Hot chocolate effect*. Tap a metal spoon inside a mug of water and note the frequency f_i you hear. Then add a spoonful of powder (say, chocolate mix or instant coffee) and tap again as you stir the powder. The frequency you hear has a lower value f_s because the tiny air bubbles released by the powder change the water's bulk modulus. As the bubbles reach the water surface and disappear, the frequency gradually shifts back to its initial value. During the effect, the bubbles do not appreciably change the water's density or volume or the sound's wavelength. Rather, they change the value of dV/dp—that is, the differential change in volume due to the differential change in the pressure caused by the sound wave in the water. If $f_s/f_i = 0.500$, what is the ratio $(dV/dp)_s/(dV/dp)_i$?

36. A 2000 Hz siren and a civil defense official are both at rest with respect to the ground. What frequency does the official hear if the wind is blowing at 15 m/s (a) from source to official and (b) from official to source?

37. One of the harmonic frequencies of tube A with two open ends is 400 Hz. The next-highest harmonic frequency is 480 Hz. (a) What harmonic frequency is next highest after the harmonic frequency 160 Hz? (b) What is the number of this next-highest harmonic? One of the harmonic frequencies of tube B with only one open end is 1080 Hz. The next-highest harmonic frequency is 1320 Hz. (c) What harmonic frequency is next highest after the harmonic frequency 600 Hz? (d) What is the number of this next-highest harmonic?

38. A violin string 30.0 cm long with linear density 0.650 g/m is placed near a loudspeaker that is fed by an audio oscillator of variable frequency. It is found that the string is set into oscillation only at the frequencies 1040 and 1560 Hz as the frequency of the oscillator is varied over the range 600–1600 Hz. What is the tension in the string?

39. A stationary motion detector sends sound waves of frequency 3.00 MHz toward a truck approaching at a speed of 61.0 m/s. What is the frequency of the waves reflected back to the detector?

40. A stone is dropped into a well. The splash is heard 3.35 s later. What is the depth of the well?

41. Two trains are traveling toward each other at 35.7 m/s relative to the ground. One train is blowing a whistle at 850 Hz. (a) What frequency is heard on the other train in still air? (b) What frequency is heard on the other train if the wind is blowing at 35.7 m/s toward the whistle and away from the listener? (c) What frequency is heard if the wind direction is reversed?

42. A sound wave of the form $s = s_m \cos(kx - \omega t + \phi)$ travels at 343 m/s through air in a long horizontal tube. At one instant, air molecule A at $x = 2.000$ m is at its maximum positive displacement of 5.00 nm and air molecule B at $x = 2.070$ m is at a positive displacement of 2.00 nm. All the molecules between A and B are at intermediate displacements. What is the frequency of the wave?

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43. You have five tuning forks that oscillate at close but different resonant frequencies. What are the (a) maximum and (b) minimum number of different beat frequencies you can produce by sounding the forks two at a time, depending on how the resonant frequencies differ?

44. Figure 17-42 shows two point sources S_1 and S_2 that emit sound of wavelength $\lambda = 3.00$ m. The emissions are isotropic and in phase, and the separation between the sources is $d = 16.0$ m. At any point P on the x axis, the wave from S_1 and the wave from S_2 interfere. When P is very far away ($x \approx \infty$), what are (a) the phase difference between the arriving waves from S_1 and S_2 and (b) the type of interference they produce? Now move point P along the x axis toward S_1. (c) Does the phase difference between the waves increase or decrease? At what distance x do the waves have a phase difference of (d) 0.50λ, (e) 1.00λ, and (f) 1.50λ?

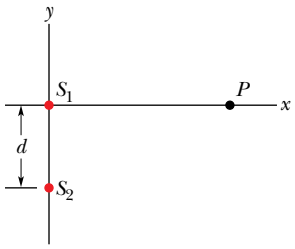

Figure 17-42 Problem 44.

45. An ambulance with a siren emitting a whine at 1620 Hz overtakes and passes a cyclist pedaling a bike at 2.44 m/s. After being passed, the cyclist hears a frequency of 1590 Hz. How fast is the ambulance moving?

46. When the door of the Chapel of the Mausoleum in Hamilton, Scotland, is slammed shut, the last echo heard by someone standing just inside the door reportedly comes 15 s later. (a) If that echo were due to a single reflection off a wall opposite the door, how far from the door is the wall? (b) If, instead, the wall is 32.0 m away, how many reflections (back and forth) occur?

47. A stationary detector measures the frequency of a sound source that first moves at constant velocity directly toward the detector and then (after passing the detector) directly away from it. The emitted frequency is f. During the approach the detected frequency is f'_{app} and during the recession it is f'_{rec}. If $(f'_{app} - f'_{rec})/f = 0.200$, what is the ratio v_s/v of the speed of the source to the speed of sound?

48. Figure 17-43 shows two isotropic point sources of sound, S_1 and S_2. The sources emit waves in phase at wavelength 0.50 m; they are separated by $D = 2.20$ m. If we move a sound detector along a large circle centered at the midpoint between the sources, at how many points do waves arrive at the detector (a) exactly in phase and (b) exactly out of phase?

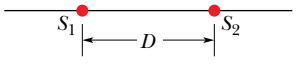

Figure 17-43 Problem 48.

49. Approximately a third of people with normal hearing have ears that continuously emit a low-intensity sound outward through the ear canal. A person with such *spontaneous otoacoustic emission* is rarely aware of the sound, except perhaps in a noise-free environment, but occasionally the emission is loud enough to be heard by someone else nearby. In one observation, the sound wave had a frequency of 1200 Hz and a pressure amplitude of 2.50×10^{-3} Pa. What were (a) the displacement amplitude and (b) the intensity of the wave emitted by the ear?

50. Two loud speakers are located 3.35 m apart on an outdoor stage. A listener is 17.5 m from one and 19.5 m from the other. During the sound check, a signal generator drives the two speakers in phase with the same amplitude and frequency. The transmitted frequency is swept through the audible range (20 Hz to 20 kHz). (a) What is the lowest frequency $f_{min,1}$ that gives minimum signal (destructive interference) at the listener's location? By what number must $f_{min,1}$ be multiplied to get (b) the second lowest frequency $f_{min,2}$ that gives minimum signal and (c) the third lowest frequency $f_{min,3}$ that gives minimum signal? (d) What is the lowest frequency $f_{max,1}$ that gives maximum signal (constructive interference) at the listener's location? By what number must $f_{max,1}$ be multiplied to get (e) the second lowest frequency $f_{max,2}$ that gives maximum signal and (f) the third lowest frequency $f_{max,3}$ that gives maximum signal?

51. In Fig. 17-44, sound with a 65.0 cm wavelength travels rightward from a source and through a tube that consists of a straight portion and a half-circle. Part of the sound wave travels through the half-circle and then rejoins the rest of the wave, which goes directly through the straight portion. This rejoining results in interference. What is the smallest radius r that results in an intensity minimum at the detector?

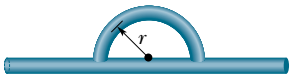

Figure 17-44 Problem 51.

52. What is the speed of sound in oxygen if 28.0 g of the gas occupies 22.4 L and the bulk modulus is 0.144 MPa?

53. In Fig. 17-45, two speakers separated by distance $d_1 = 2.00$ m are in phase. Assume the amplitudes of the sound waves from the speakers are approximately the same at the listener's ear at distance $d_2 = 4.00$ m directly in front of one speaker. Consider the full audible range for normal hearing, 20 Hz to 20 kHz. (a) What is the lowest frequency $f_{min,1}$ that gives minimum signal (destructive interference) at the listener's ear? By what number must $f_{min,1}$ be multiplied to get (b) the second lowest frequency $f_{min,2}$ that gives minimum signal and (c) the third lowest frequency $f_{min,3}$ that gives minimum signal? (d) What is the lowest frequency $f_{max,1}$ that gives maximum signal (constructive interference) at the listener's ear? By what number must $f_{max,1}$ be multiplied to get (e) the second lowest frequency $f_{max,2}$ that gives maximum signal and (f) the third lowest frequency $f_{max,3}$ that gives maximum signal?

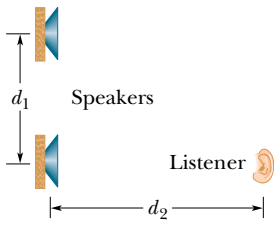

Figure 17-45 Problem 53.

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54. In Fig. 17-46, sound waves A and B, both of wavelength λ, are initially in phase and traveling rightward, as indicated by the two rays. Wave A is reflected from four surfaces but ends up traveling in its original direction. Wave B ends in that direction after reflecting from two surfaces. Let distance L in the figure be expressed as a multiple q of λ: $L = q\lambda$. What are the (a) smallest and (b) third smallest values of q that put A and B exactly out of phase with each other after the reflections?

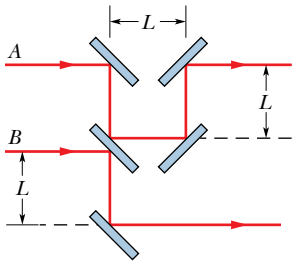

Figure 17-46 Problem 54.

55. A girl is sitting near the open window of a train that is moving at a velocity of 10.00 m/s to the east. The girl's uncle stands near the tracks and watches the train move away. The locomotive whistle emits sound at frequency 520.0 Hz. The air is still. (a) What frequency does the uncle hear? (b) What frequency does the girl hear? A wind begins to blow from the east at 10.00 m/s. (c) What frequency does the uncle now hear? (d) What frequency does the girl now hear?

56. In Fig. 17-47, S is a small loudspeaker driven by an audio oscillator with a frequency that is varied from 1000 Hz to 2000 Hz, and D is a cylindrical pipe with two open ends and a length of 48.9 cm. The speed of sound in the air-filled pipe is 344 m/s. (a) At how many frequencies does the sound from the loudspeaker set up resonance in the pipe? What are the (b) lowest and (c) second lowest frequencies at which resonance occurs?

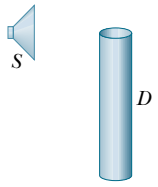

Figure 17-47 Problem 56.

57. The pressure in a traveling sound wave is given by the equation

$$\Delta p = (2.00\ \text{Pa}) \sin \pi[(0.900\ \text{m}^{-1})\, x - (450\ \text{s}^{-1})t].$$

Find the (a) pressure amplitude, (b) frequency, (c) wavelength, and (d) speed of the wave.

58. A well with vertical sides and water at the bottom resonates at 9.00 Hz and at no lower frequency. The air-filled portion of the well acts as a tube with one closed end (at the bottom) and one open end (at the top). The air in the well has a density of 1.10 kg/m^3 and a bulk modulus of 1.33×10^5 Pa. How far down in the well is the water surface?

59. A column of soldiers, marching at 100 paces per minute, keep in step with the beat of a drummer at the head of the column. The soldiers in the rear end of the column are striding forward with the left foot when the drummer is advancing with the right foot. What is the approximate length of the column?

60. From two sources, sound waves of frequency 270 Hz are emitted in phase in the positive direction of an x axis. At a detector that is on the axis and 5.00 m from one source and 4.00 m from the other source, what is the phase difference between the waves (a) in radians and (b) as a multiple of the wavelength?

61. If the form of a sound wave traveling through air is

$$s(x, t) = (6.0\ \text{nm}) \cos(kx + (3000\ \text{rad/s})t + \phi),$$

how much time does any given air molecule along the path take to move between displacements $s = +4.0$ nm and $s = -4.0$ nm?

62. Suppose that the sound level of a conversation is initially at an angry 75 dB and then drops to a soothing 55 dB. Assuming that the frequency of the sound is 500 Hz, determine the (a) initial and (b) final sound intensities and the (c) initial and (d) final sound wave amplitudes.

63. A sound wave in a fluid medium is reflected at a barrier so that a standing wave is formed. The distance between nodes is 4.9 cm, and the speed of propagation is 1250 m/s. Find the frequency of the sound wave.

64. A train approaching a railway crossing at a speed of 120 km/h sounds a short whistle at frequency 640 Hz when it is 300 m away from the crossing. The speed of sound in air is 340 m/s. What will be the frequency heard by a person standing on a road perpendicular to the track through the crossing at a distance of 400 m from the crossing?

65. A source emitting sound of frequency 180 Hz is placed in front of a wall at a distance of 2 m from it. A detector is also placed in front of the wall at the same distance from it. Find the minimum distance between the source and the detector for which the detector detects a maximum of sound. Speed of sound in air is 360 m/s.

66. Figure 17-48 shows a source of sound moving along the x-axis at a speed of 22 m/s continuously emitting a sound of frequency 2.0 kHz, which travels in air at a speed of 330 m/s. A listener Q stands on the y-axis at a distance of 330 m from the origin. At $t = 0$, the source crosses the origin P. (a) When does the sound emitted from the source at P reach the listener Q? (b) What will be the frequency heard by the listener at this instant? (c) Where will the source be at this instant?

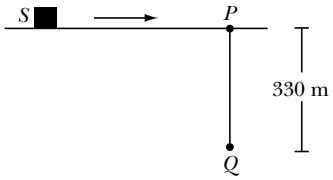

Figure 17-48 Problem 66.

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PRACTICE QUESTIONS

Single Correct Choice Type

1. A bell is ringing inside of a sealed glass jar that is connected to a vacuum pump. Initially, the jar is filled with air. What does one hear as the air is slowly removed from the jar by the pump?
(a) The sound intensity from the bell gradually decreases.
(b) The frequency of the sound from the bell gradually increases.
(c) The frequency of the sound from the bell gradually decreases.
(d) The speed of the sound from the bell gradually increases.

2. Take the speed of sound to be 340 m/s. A thunder clap is heard about 3 s after the lightning is seen. The source of both light and sound is
(a) moving overhead faster than the speed of sound.
(b) emitting a much higher frequency than is heard.
(c) emitting a much lower frequency than is heard.
(d) about 1000 m away.

3. Two closed pipes, A and B have the same length. A is wider than B. They resonate in the fundamental mode at frequencies n_A and n_B, respectively,
(a) $n_A = n_B$.
(b) $n_A > n_B$.
(c) $n_A < n_B$.
(d) Either (b) or (c) depending on the ratio of their diameters.

4. A car moving at 35 m/s approaches a stationary whistle that emits a 220 Hz sound. The speed of sound is 343 m/s. What is the speed of the sound relative to the driver of the car?
(a) 300 m/s **(b)** 340 m/s
(c) 378 m/s **(d)** 305 m/s

5. The speed of sound in fresh water at 20°C is 1482 m/s. At what temperature is the speed of sound in helium gas the same as that of fresh water at 20°C? Helium is considered a monatomic ideal gas ($\gamma = 1.67$ and atomic mass = 4.003 u).
(a) 313 K **(b)** 442 K
(c) 633 K **(d)** 377 K

6. Ethanol has a density of 659 kg/m^3. If the speed of sound in ethanol is 1162 m/s, what is its adiabatic bulk modulus?
(a) 1.7×10^8 N/m^2 **(b)** 7.7×10^8 N/m^2
(c) 2.2×10^8 N/m^2 **(d)** 8.9×10^8 N/m^2

7. A fire whistle emits a tone of 170 Hz. Take the speed of sound in air to be 340 m/s. The wavelength of this sound is about
(a) 0.5 m **(b)** 1.0 m
(c) 2.0 m **(d)** 3.0 m

8. An aircraft carrier has a speed of 13.0 m/s relative to the water. A jet is catapulted from the deck and has a speed of 67.0 m/s relative to the water. The engines produce a 1550-Hz whine, and the speed of sound is 343 m/s. What is the frequency of the sound heard by the crew on the ship?
(a) 1200 Hz **(b)** 1450 Hz
(c) 1180 Hz **(d)** 1350 Hz

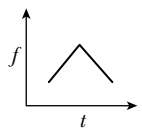
9. Two golf carts have horns that emit sound with a frequency of 390 Hz. The golf carts are traveling toward one another, each traveling with a speed of 9.0 m/s with respect to the ground. What frequency do the drivers of the carts hear? The speed of sound at the golf course is 343 m/s.
(a) 390 Hz **(b)** 400 Hz
(c) 410 Hz **(d)** 420 Hz

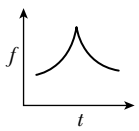
10. At a distance of 5.0 m from a point sound source, the sound intensity level is 110 dB. At what distance is the intensity level 95 dB?
(a) 5.0 m **(b)** 14 m
(c) 7.1 m **(d)** 28 m

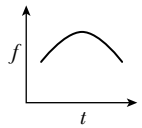
11. When we clap our hands, the sound produced is best described by
(a) $p = p_0 \sin(kx - \omega t)$ **(b)** $p = p_0 \sin kx \cos \omega t$
(c) $p = p_0 \cos kx \sin \omega t$ **(d)** $p = \Sigma\, p_0 \sin(k_n x - \omega_n t)$
(Here p denotes the change in pressure from the equilibrium value).

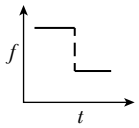
12. The sound level at a point P is 14 dB below the sound level at a point 1.0 m from a point source. The distance from the source to point P is
(a) 4.0 cm **(b)** 20.2 m
(c) 2.0 m **(d)** 5.0 m

13. Two pianos each sound the same note simultaneously, but they are both out of tune. On a day when the speed of sound is 343 m/s, piano A produces a wavelength of 0.769 m, while piano B produces a wavelength of 0.776 m. How much time separates successive beats?
(a) 1.4 s **(b)** 0.76 s
(c) 0.25 s **(d)** 0.07 s

14. A railway engine whistling at a constant frequency moves with a constant speed. It goes past a stationary observer standing beside the railway track. The frequency f of the sound heard by the observer is plotted against time t. Which of the following best represents the resulting curve?
(a) 

(b) 

(c) 

(d) 

15. How far must one stand from a 5 mW point sound source if the intensity is at the hearing threshold? Assume the sound waves travel to the listener without being disturbed.
(a) 500 m **(b)** 20 km
(c) 2 km **(d)** 1 km

16. A vibrating tuning fork is held over a water column with one end closed and the other open. As the water level

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is allowed to fall, a loud sound is heard for water levels separated by 17 cm. If the speed of sound in air is 340 m/s, the frequency of the tuning fork is:
(a) 500 Hz (b) 1000 Hz
(c) 2000 Hz (d) 5780 Hz

17. Two identical tuning forks vibrate at 256 Hz. One of them is then loaded with a drop of wax, after which 6 beats per second are heard. The period of the loaded tuning fork is
(a) 0.006 s (b) 0.005 s
(c) 0.004 s (d) 0.003 s

18. A pebble is dropped in a lake; and it produces ripples with a frequency of 0.25 Hz. When should a second pebble be dropped at the same place to produce destructive interference?
(a) 0.50 s after the first (b) 1.0 s after the first
(c) 2.0 s after the first (d) 0.75 s after the first

19. A tube, open at only one end, is cut into two shorter (nonequal) lengths. The piece that is open at both ends has a fundamental frequency of 425 Hz, while the piece open only at one end has a fundamental frequency of 675 Hz. What is the fundamental frequency of the original tube?
(a) 127 Hz (b) 162 Hz
(c) 209 Hz (d) 148 Hz

20. A stationary source generates 5.0 Hz water waves whose speed is 2.0 m/s. A boat is approaching the source at 10 m/s. The frequency of these waves, as observed by a person in the boat, is
(a) 25 Hz (b) 15 Hz
(c) 20 Hz (d) 30 Hz

21. When two waves with same frequency and constant phase difference interface,
(a) there is a gain of energy.
(b) there is a loss of energy.
(c) the energy is redistributed and the distribution changes with time.
(d) the energy is redistributed and the distribution remains constant in time.

22. A rocket in a fireworks display explodes high in the air. The sound spreads out uniformly in all directions. The intensity of the sound is 2.0×10^{-6} W/m^2 at a distance of 120 m from the explosion. Find the distance from the source at which the intensity is 0.80×10^{-6} W/m^2.
(a) 690 m (b) 220 m
(c) 110 m (d) 190 m

23. A source emits sound with a frequency of 1000 Hz. It is moving at 20 m/s toward a stationary reflecting wall. If the speed of sound is 340 m/s an observer at rest directly behind the source hears a beat frequency of
(a) 11 Hz (b) 86 Hz
(c) 97 Hz (d) 118 Hz

24. Pipe A is 0.50 m long and open at both ends. Pipe B is open at one end and closed at the other end. Determine the length of pipe B so that it has the same fundamental frequency as A.
(a) 0.25 m (b) 0.75 m
(c) 0.50 m (d) 1.0 m

25. A piano wire has a length of 81 cm and a mass of 2.0 g. If its fundamental frequency is to be 394 Hz, it tension must be
(a) 0.32 N (b) 63 N
(c) 130 N (d) 250 N

26. An organ pipe is open at both ends. It is producing sound at its third harmonic, the frequency of which is 262 Hz. The speed of sound is 343 m/s. What is the length of the pipe?
(a) 1.96 m (b) 1.31 m
(c) 1.85 m (d) 0.926 m

27. The security alarm on a parked car goes off and produces a frequency of 960 Hz. The speed of sound is 343 m/s. As you drive toward this parked car, pass it, and drive away, you observe the frequency to change by 95 Hz. At what speed are you driving?
(a) 9.2 m/s (b) 17 m/s
(c) 12 m/s (d) 25 m/s

28. A bird is flying directly toward a stationary bird-watcher and emits a frequency of 1250 Hz. The bird-watcher, however, hears a frequency of 1290 Hz. What is the speed of the bird, expressed as a percentage of the speed of sound?
(a) 2.0% (b) 0.97%
(c) 3.1% (d) 1.4%

29. Two timpani (tunable drums) are played at the same time. One is correctly tuned so that when it is struck, sound is produced that has a wavelength of 2.20 m. The second produces sound with a wavelength of 2.08 m. If the speed of sound is 343 m/s, what beat frequency is heard?
(a) 7 beats/s (b) 9 beats/s
(c) 11 beats/s (d) 13 beats/s

30. Two identical tuning forks vibrate at 587 Hz. After a small piece of clay is placed on one of them, eight beats per second are heard. What is the period of the tuning fork that holds the clay?
(a) 1.68×10^{-3} s (b) 1.70×10^{-3} s
(c) 1.73×10^{-3} s (d) 1.76×10^{-3} s

31. Four standing wave segments, or loops, are observed on a string fixed at both ends as it vibrates at a frequency of 140 Hz. What is the fundamental frequency of the string?
(a) 23 Hz (b) 28 Hz
(c) 35 Hz (d) 47 Hz

32. A 4.00-m long string, clamped at both ends, vibrates at 2.00×10^2 Hz. If the string resonates in six segments, what is the speed of transverse waves on the string?
(a) 100 m/s (b) 133 m/s
(c) 267 m/s (d) 328 m/s

33. A string with a linear density of 0.035 kg/m and a mass of 0.014 kg is clamped at both ends. Under what tension in the string will it have a fundamental frequency of 110 Hz?
(a) 270 N (b) 410 N
(c) 550 N (d) 680 N

34. A bat emits a sound whose frequency is 91 kHz. The speed of sound in air at 20.0°C is 343 m/s. However, the air temperature is 35°C, so the speed of sound is not 343 m/s. Find the wavelength of the sound.
(a) 5.0×10^{-2} s (b) 3.9×10^{-3} s
(c) 1.1×10^{-2} s (d) 5.5×10^{-3} s

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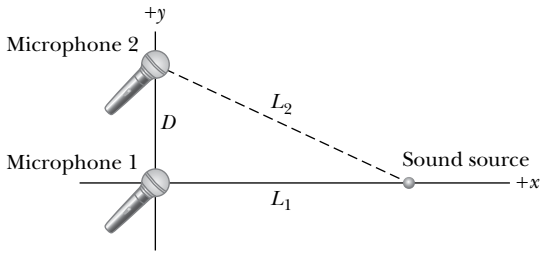
35. As the drawing shows, one microphone is located at the origin, and a second microphone is located on the $+y$ axis. The microphones are separated by a distance of $D = 1.50$ m. A source of sound is located on the $+x$ axis, its distances from microphones 1 and 2 being L_1 and L_2, respectively. The speed of sound is 343 m/s. The sound reaches microphone 1 first, and then, 1.46 ms later, it reaches microphone 2. Find the distances L_1 and L_2.



	L_1	L_2
(a)	1.00 m	1.80 m
(b)	1.25 m	1.95 m
(c)	1.50 m	2.12 m
(d)	2.00 m	2.50 m

More than One Correct Choice Type

36. Coherent sources are characterized by the same
- **(a)** phase and phase velocity.
- **(b)** wavelength, amplitude, and phase velocity.
- **(c)** wavelength, amplitude, and frequency.
- **(d)** wavelength and phase.

37. When an organ pipe resonates in its fundamental mode then at the center of the pipe,
- **(a)** the gas molecules undergo vibrations of maximum amplitude.
- **(b)** the gas molecules are at rest.
- **(c)** the pressure of the gas is constant.
- **(d)** the pressure of the gas undergo maximum variation.

38. A listener is at rest with respect to the source of sound. A wind starts blowing along the line joining the source and the observer. Which of the following quantities do not change?
- **(a)** Frequency
- **(b)** Velocity of sound
- **(c)** Wavelength
- **(d)** Time period

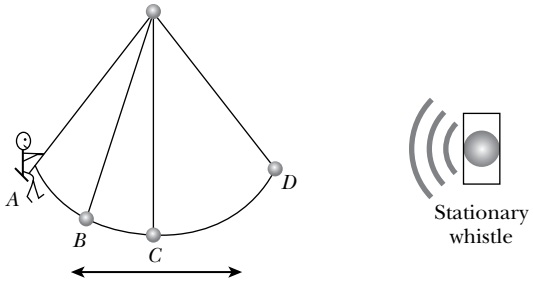
39. To raise the pitch of a stringed musical instrument, the player can
- **(a)** loosen the string.
- **(b)** tighten the string.
- **(c)** shorten the string.
- **(d)** lengthen the string.

40. A sound wave passes from a medium A to a medium B. The velocity of sound B is greater than that in A. Assume that there is no absorption or reflection at the boundary. As the wave moves across the boundary
- **(a)** the frequency of sound will not change.
- **(b)** the wavelength will increase.
- **(c)** the wavelength will decrease.
- **(d)** the intensity of sound will not change.

41. Two identical straight wires are stretched so as to produce 6 beats per second. When vibrating simultaneously, on changing the tension slightly in one of them, the beat frequency remains unchanged. Denoting the higher and lower initial tensions in the string by T_1 and T_2, respectively, it could be said that while making the above changes in tension,
- **(a)** T_2 was decreased.
- **(b)** T_2 was increased.
- **(c)** T_1 was increased.
- **(d)** T_1 was decreased.

Linked Comprehension

Paragraph for Questions 42–44: The diagram shows the various positions of a child in motion on a swing. Somewhere in front of the child a stationary whistle is blowing.

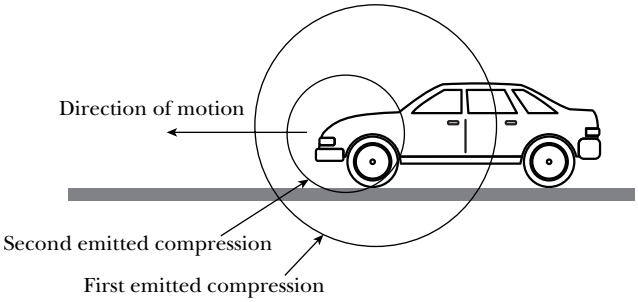


42. At which position(s) will the child hear the highest frequency for the sound from the whistle?
- **(a)** At both A and D.
- **(b)** At C when moving toward B.
- **(c)** At B when moving toward A.
- **(d)** At C when moving toward D.

43. At which position(s) will the child hear the lowest frequency for the sound from the whistle?
- **(a)** At both A and D.
- **(b)** At C when moving toward B.
- **(c)** At B when moving toward A.
- **(d)** At C when moving toward D.

44. At which position(s) will the child hear the same frequency as that heard by a stationary observer standing next to the whistle?
- **(a)** At both A and D.
- **(b)** At C when moving toward B.
- **(c)** At B when moving toward A.
- **(d)** At C when moving toward D.

Paragraph for Questions 45–48: The car in the figure is moving to the left at 35 m/s. The car's horn continuously emits a 2.20×10^2 Hz sound. The figure also shows the first two regions of compression of the emitted sound waves. The speed of sound is 343 m/s.



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45. How far does the car move in one period of the sound emitted from the horn?
(a) 0.08 m **(b)** 0.16 m
(c) 8 m **(d)** 16 m

46. How far has the initial compression traveled when the second compression is emitted?
(a) 0.77 m **(b)** 7.7 m
(c) 1.56 m **(d)** 15.5 m

47. What is the wavelength of the sound in the direction of motion of the car?
(a) 1.40 m **(b)** 1.70 m
(c) 1.56 m **(d)** 1.93 m

48. What is the frequency heard by a stationary observer standing in front of the car?
(a) 9.7 Hz **(b)** 245 Hz
(c) 176 Hz **(d)** 219 Hz

Paragraph for Questions 49 and 50: Two loudspeakers are located 3 m apart on the stage of an auditorium. A listener at point P is seated 29.0 m from one speaker and 25.0 m from the other. A signal generator drives the speakers in phase with the same amplitude and frequency. The wave amplitude at P due to each speaker alone is A. The frequency is then varied between 20 Hz and 300 Hz. The speed of sound is 343 m/s.

49. At what frequency or frequencies will the listener at P hear a maximum intensity?
(a) 170 Hz only
(b) 57 Hz, 113 Hz, 170 Hz, 227 Hz, and 284 Hz
(c) 113 Hz and 226 Hz
(d) 86 Hz, 170 Hz, 257 Hz

50. Determine the value of the maximum amplitude in terms of A.
(a) 2.0 A **(b)** 3.0 A
(c) 2.5 A **(d)** 4.0 A

Paragraph for Questions 51 and 52: Vibrations with frequency 6.00×10^2 Hz are established on a 1.33 m length of string that is clamped at both ends. The speed of waves on the string is 4.00×10^2 m/s.

51. How many antinodes are contained in the resulting standing wave pattern?
(a) 2 **(b)** 4
(c) 3 **(d)** 5

52. How far from either end of the string does the first node occur?
(a) 0.17 m **(b)** 0.49 m
(c) 0.33 m **(d)** 0.66 m

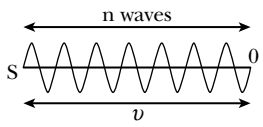
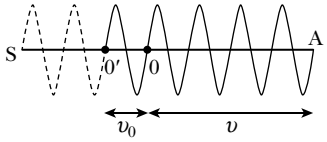
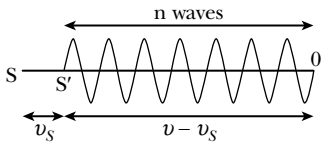
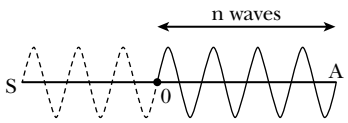
Matrix-Match

53. A source of sound of frequency 1000 Hz moves to the right with a speed of 32 m/s relative to the ground. To its right is a reflecting surface moving to the left with a speed of 64 m/s relative to the ground. Take speed of sound in air to be 332 m/s.

Column I	Column II
(a) Wavelength of sound emitted in air by source in the right side of source (in mm)	**(p)** 1320
(b) Number of waves arriving per second at the reflecting surface	**(q)** 364
(c) Speed of reflected wave (in m/s)	**(r)** 332
(d) Wavelength of reflected wave (in mm)	**(s)** 200

Directions for Questions 54: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

54. Doppler's effect

Column I	Column II	Column III
(I) Sound waves propagate in the form of spherical wavefronts	**(i)** the distance between two successive circles is equal to wavelength λ	**(J)** 
(II) No change in wavelength received by either observer X or Y	**(ii)** sound waves propagate in the form of spherical wavefronts	**(K)** 
(III) Source at three different positions	**(iii)** waves are represented by concentric circles	**(L)** 
(IV) Change in wavelength received by either observer X or Y	**(iv)** waves are represented by non-concentric circles	**(M)** 

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(1) What happens when the observer and source are stationary?
(a) (I) (i) (J) (b) (III) (ii) (K)
(c) (II) (ii) (L) (d) (I) (iii) (J)

(2) What happens when the source is moving but observer is at rest?
(a) (I) (ii) (M) (b) (III) (iv) (L)
(c) (II) (ii) (K) (d) (II) (iv) (M)

(3) What happens when the source is at rest, but observer is moving?
(a) (III) (i) (L) (b) (IV) (i) (J)
(c) (III) (iv) (J) (d) (II) (iii) (M)

Integer Type

55. Two tuning forks when sounded together give 8 beats per second. When tuning fork A is sounded with air column of length 37.5 cm closed at one end, resonance occurs in its fundamental mode. Tuning fork B gives resonance with air column of length 38.5 cm and is closed at one end in its fundamental mode. Find the frequencies of tuning forks.

56. The speed of a sound in a container of hydrogen at 201 K is 1220 m/s. What would be the speed of sound if the temperature were raised to 405 K? Assume that hydrogen behaves like an ideal gas.

57. A hunter is standing on flat ground between two vertical cliffs that are directly opposite one another. He is closer to one cliff than to the other. He fires a gun and, after a while, hears three echoes. The second echo arrives 1.6 s after the first, and the third echo arrives 1.1 s after the second. Assuming that the speed of sound is 343 m/s and that there are no reflections of sound from the ground, find the distance between the cliffs.

58. Two submarines are underwater and approaching each other head-on. Sub A has a speed of 12 m/s and sub B has a speed of 8 m/s. Sub A sends out a 1550-Hz sonar wave that travels at a speed of 1522 m/s. What is the frequency detected by sub B?

59. A tube is open only at one end. A certain harmonic produced by the tube has a frequency of 450 Hz. The next higher harmonic has a frequency of 750 Hz. The speed of sound in air is 343 m/s. What is the integer n that describes the harmonic whose frequency is 450 Hz?

ANSWER KEY

Checkpoints

1. Beginning to decrease (example: mentally move the curves of Fig. 17-7 rightward past the point at $x = 42$ m)
2. 1 and 2 tie, then 3 (see Eq. 17-21)
3. Second (see Eqs. 17-32 and 17-34)
4. (a) greater; (b) less; (c) can't tell; (d) can't tell; (e) greater; (f) less

Problems

1. (a) 9.03×10^{-5} m or 90.3 μm; (b) 3.95×10^{-4} m or 0.395 mm
2. (a) 0.153W/m^2; (b) 0.0265 W/m^2
3. 1.13×10^{-7} W ≈ 0.1 μW.
4. 388 Hz
5. 3.95×10^{-8} m
6. 0.43 m
7. 0.0268
8. (a) 2.65×10^3 Hz; (b) 0.124 m; (c) 3.71×10^3 Hz; (d) 0.0887 m
9. (a) 522 Hz; (b) 560 Hz
10. 3.3×10^2 m/s
11. 4.1 10 Hz
12. (a) 1.35×10^{-8} W/m^2; (b) 41.3 dB
13. (a) 2.347×10^{-10} W or 0.23 nW; (b) 4.7×10^{-10} W; (c) 9.4×10^{-10} W; (d) 6.1×10^{-10} W; (e) 0
14. (a) ≈ 3.2; (b) 23 m
15. (a) 42°; (b) 11 s
16. (a) 40.4 cm; (b) 50.4 cm
17. (a) 1.0×10^4; (b) 100
18. (a) 0.80v; (b) 0.80v; (c) 0.80v; (d) $2v/3$
19. (a) 7.37×10^{-5} W/m^2 ; (b) 5.53×10^{-9} W
20. 1.35×10^4 m
21. 0.204 W
22. (a) 286 Hz; (b) 1.04×10^3 N
23. 2.2×10^3 km
24. (a) 2 nodes; (b) $x = 0$; (c) 0.60 m; (d) 95.3 Hz
25. (a) 405 m/s; (b) 641 N; (c) 0.440 m; (d) 0.373 m
26. (a) 343 m/s; (b); 9.2 rad/m (c) 3142 rad/s $\approx 3.1 \times 10^3$ rad/s; (d) 7.4 nm; (e) 9.8 rad/m; (f) 3142 rad/s $\approx 3.1 \times 10^3$ rad/s
27. (a) 933 Hz; (b) 0.373 m
28. (a) 0; (b) 0; (c) $4S_m$
29. (a) 5 values of n ($n = 1, 3, 5, 7, 9$); (b) $n = 9$ with $h = 0.143$ m; (c) $n = 7$ with $h = 0.333$ m
30. (a) 1.595×10^3 Hz; (b) 1.630×10^3 Hz
31. ≈ 2.00
32. 2.25×10^{-3} s
33. 22 m
34. 166 Hz
35. 4.00
36. (a) 2.0×10^3 Hz (no Doppler shift); (b) 2.0×10^3 Hz (no Doppler shift)
37. (a) 240 Hz; (b) 3; (c) 840 Hz; (d) 7
38. 63.3 N
39. 4.30 MHz
40. 50.3 m
41. (a) 1.05×10^3 Hz; (b) 1.07×10^3 Hz; (c) 1.03×10^3 Hz
42. 905 Hz
43. (a) 10; (b) 4
44. (a) no phase difference; (b) constructive interference; (c) increase; (d) 84.6 m; (e) 41.2 m; (f) 26.2 m
45. 8.96 m/s
46. (a) 2.6 km; (b) $159.8 \approx 1.6 \times 10^2$
47. 9.90×10^{-2}
48. (a) 18; (b)16

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49. (a) 7.99×10^{-10} m ≈ 0.80 nm; (b) 7.5 nW/m^2
50. (a) 2.0 m; (b) factor of 3; (c) factor of 5; (d) 172 Hz; (e) factor of 2; (f) factor of 3
51. 28.5 cm **52.** 339 m/s
53. (a) 363 Hz; (b) factor of 3; (c) factor of 5; (d) 726 Hz; (e) factor of 2; (f) factor of 3
54. (a) 0.5; (b) 2.5 **55.** (a) 505.3 Hz; (b) 520.0 Hz; (c) 505.7 Hz; (d) 520.0 Hz
56. (a) 3 frequencies; (b) 1055 Hz; (c) 1407 Hz **57.** (a) 2.00 Pa; (b) 225 Hz; (c) 2.22 m; (d) 500 m/s **58.** 9.66 m
59. 2.1×10^2 m **60.** (a) 4.95 rad; (b) 0.787
61. 0.49 ms **62.** (a) 32 μW/m^2; (b) 0.32 μW/m^2; (c) 1.24×10^{-7} m ≈ 0.12 μm; (d) 1.24×10^{-8} m ≈ 0.012 μm
63. 1.27×10^4 Hz ≈ 13 kHz
64. 60 Hz **65.** 3 m **66.** (a) $t = 1$ s; (b) 2.0 Hz; (c) at $x = 22$ m

Practice Questions

Single Correct Choice Type

1. (a)	**2.** (d)	**3.** (c)	**4.** (c)	**5.** (c)
6. (d)	**7.** (c)	**8.** (d)	**9.** (c)	**10.** (d)
11. (d)	**12.** (d)	**13.** (c)	**14.** (d)	**15.** (b)
16. (b)	**17.** (c)	**18.** (c)	**19.** (b)	**20.** (d)
21. (d)	**22.** (d)	**23.** (d)	**24.** (a)	**25.** (b)
26. (a)	**27.** (b)	**28.** (c)	**29.** (b)	**30.** (c)
31. (c)	**32.** (c)	**33.** (a)	**34.** (b)	**35.** (d)

More than One Correct Choice Type

36. (b), (c)	**37.** (b), (d)	**38.** (a), (d)	**39.** (b), (c)	**40.** (a), (b), (d)
41. (b), (d)				

Linked Comprehension

42. (d)	**43.** (b)	**44.** (a)	**45.** (b)	**46.** (c)
47. (a)	**48.** (b)	**49.** (d)	**50.** (a)	**51.** (b)
52. (c)				

Matrix-Match

53. (a) $\rightarrow$ (q); (b) $\rightarrow$ (p); (c) $\rightarrow$ (r); (d) $\rightarrow$ (s)
54. (1) $\rightarrow$ (a); (2) $\rightarrow$ (b); (3) $\rightarrow$ (d)

Integer Type

55. 300	**56.** 1730	**57.** 650	**58.** 1570	**59.** 3

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Temperature, Zeroth Law of Thermodynamics and Thermal Expansion

CHAPTER 18

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18.1 What is Physics?
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18.3 The Zeroth Law of Thermodynamics
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18.5 The Celsius and Fahrenheit Scales
18.6 Thermal Expansion
18.7 Expansion of Solids
18.8 Expansion of Liquids

18.1 | WHAT IS PHYSICS?

One of the principal branches of physics and engineering is **thermodynamics**, which is the study and application of the *thermal energy* (often called the *internal energy*) of systems. One of the central concepts of thermodynamics is temperature. Since childhood, you have been developing a working knowledge of thermal energy and temperature. For example, you know to be cautious with hot foods and hot stoves and to store perishable foods in cool or cold compartments. You also know how to control the temperature inside home and car, and how to protect yourself from wind chill and heat stroke.

The starting point in our discussion of thermodynamics is the concept of temperature and how it is measured.

18.2 | TEMPERATURE

Key Concept

- Temperature is an SI base quantity related to our sense of hot and cold. It is measured with a thermometer, which contains a working substance with a measurable property, such as length or pressure, that changes in a regular way as the substance becomes hotter or colder.

Temperature is one of the seven SI base quantities. Physicists measure temperature on the **Kelvin scale**, which is marked in units called *kelvins*. Although the temperature of a body apparently has no upper limit, it does have a lower limit; this limiting low temperature is taken as the zero of the Kelvin temperature scale. Room temperature is about 290 kelvins, or 290 K as we write it, above this *absolute zero*. Figure 18-1 shows a wide range of temperatures.

When the universe began 13.7 billion years ago, its temperature was about 10^{39} K. As the universe expanded it cooled, and it has now reached an average temperature of about 3 K. We on Earth are a little warmer than that because we happen to live near a star. Without our Sun, we too would be at 3 K (or, rather, we could not exist).

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18.3 | THE ZEROTH LAW OF THERMODYNAMICS

Key Concept

- When a thermometer and some other object are placed in contact with each other, they eventually reach thermal equilibrium. The reading of the thermometer is then taken to be the temperature of the other object. The process provides consistent and useful temperature measurements because of the zeroth law of thermodynamics: If bodies A and B are each in thermal equilibrium with a third body C (the thermometer), then A and B are in thermal equilibrium with each other.

The properties of many bodies change as we alter their temperature, perhaps by moving them from a refrigerator to a warm oven. To give a few examples: As their temperature increases, the volume of a liquid increases, a metal rod grows a little longer, and the electrical resistance of a wire increases, as does the pressure exerted by a confined gas. We can use any one of these properties as the basis of an instrument that will help us pin down the concept of temperature.

Figure 18-2 shows such an instrument. Any resourceful engineer could design and construct it, using any one of the properties listed above. The instrument is fitted with a digital readout display and has the following properties: If you heat it (say, with a Bunsen burner), the displayed number starts to increase; if you then put it into a refrigerator, the displayed number starts to decrease. The instrument is not calibrated in any way, and the numbers have (as yet) no physical meaning. The device is a *thermoscope* but not (as yet) a *thermometer*.

Suppose that, as in Fig. 18-3*a*, we put the thermoscope (which we shall call body T) into intimate contact with another body (body A). The entire system is confined within a thick-walled insulating box. The numbers displayed by the thermoscope roll by until, eventually, they come to rest (let us say the reading is "137.04") and no further change takes place. In fact, we suppose that every measurable property of body T and of body A has assumed a stable, unchanging value. Then we say that the two bodies are in *thermal equilibrium* with each other. Even though the displayed readings for body T have not been calibrated, we conclude that bodies T and A must be at the same (unknown) temperature.

Suppose that we next put body T into intimate contact with body B (Fig. 18-3*b*) and find that the two bodies come to thermal equilibrium *at the same reading* of *the thermoscope*. Then bodies T and B must be at the same (still unknown) temperature. If we now put bodies A and B into intimate contact (Fig. 18-3*c*), are they immediately in thermal equilibrium with each other? Experimentally, we find that they are.

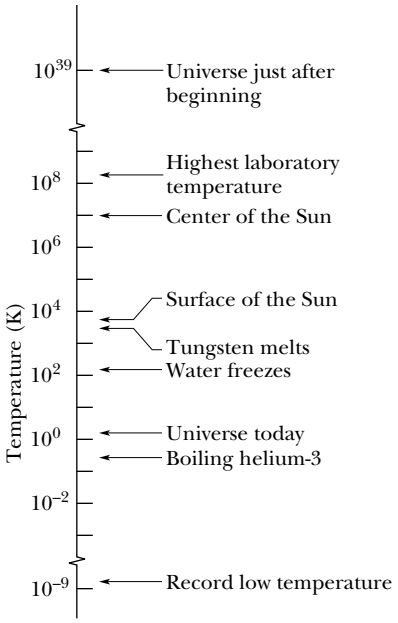

Figure 18-1 Some temperatures on the Kelvin scale. Temperature $T = 0$ corresponds to $10^{-\infty}$ and cannot be plotted on this logarithmic scale.

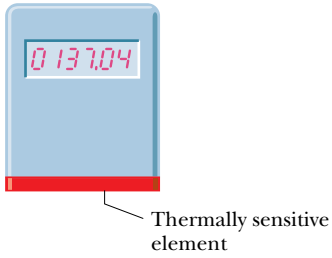

Figure 18-2 A thermoscope. The numbers increase when the device is heated and decrease when it is cooled. The thermally sensitive element could be—among many possibilities—a coil of wire whose electrical resistance is measured and displayed.

The experimental fact shown in Fig. 18-3 is summed up in the **zeroth law of thermodynamics**:

If bodies A and B are each in thermal equilibrium with a third body T, then A and B are in thermal equilibrium with each other.

In less formal language, the message of the zeroth law is: "Every body has a property called **temperature**. When two bodies are in thermal equilibrium, their temperatures are equal. And vice versa." We can now make our thermoscope (the third body T) into a thermometer, confident that its readings will have physical meaning. All we have to do is calibrate it.

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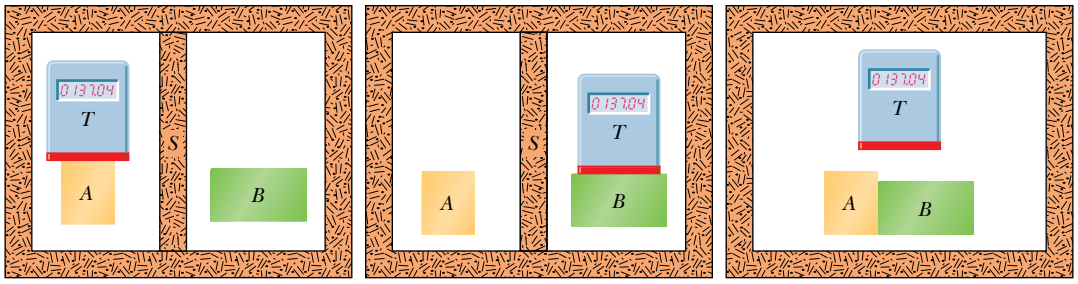

Figure 18-3 (*a*) Body *T* (a thermoscope) and body *A* are in thermal equilibrium. (Body *S* is a thermally insulating screen.) (*b*) Body *T* and body *B* are also in thermal equilibrium, at the same reading of the thermoscope. (*c*) If (*a*) and (*b*) are true, the zeroth law of thermodynamics states that body *A* and body *B* are also in thermal equilibrium.

We use the zeroth law constantly in the laboratory. If we want to know whether the liquids in two beakers are at the same temperature, we measure the temperature of each with a thermometer. We do not need to bring the two liquids into intimate contact and observe whether they are or are not in thermal equilibrium.

The zeroth law, which has been called a logical afterthought, came to light only in the 1930s, long after the first and second laws of thermodynamics had been discovered and numbered. Because the concept of temperature is fundamental to those two laws, the law that establishes temperature as a valid concept should have the lowest number—hence the zero.

18.4 | MEASURING TEMPERATURE

Key Concept

- In the SI system, temperature is measured on the Kelvin scale, which is based on the triple point of water (273.16 K). Other temperatures are then defined by use of a constant-volume gas thermometer, in which a sample of gas is maintained at constant volume so its pressure is proportional to its temperature. We define the temperature T as measured with a gas thermometer to be

$$T = (273.16\ \text{K})\left(\lim_{\text{gas}\to 0}\frac{p}{p_3}\right).$$

Here T is in kelvins, and p_3 and p are the pressures of the gas at 273.16 K and the measured temperature, respectively.

Here we first define and measure temperatures on the Kelvin scale. Then we calibrate a thermoscope so as to make it a thermometer.

The Triple Point of Water

To set up a temperature scale, we pick some reproducible thermal phenomenon and, quite arbitrarily, assign a certain Kelvin temperature to its environment; that is, we *select a standard fixed point* and give it a standard fixed-point *temperature*. We could, for example, select the freezing point or the boiling point of water but, for technical reasons, we select instead the **triple point of water**.

Liquid water, solid ice, and water vapor (gaseous water) can coexist, in thermal equilibrium, at only one set of values of pressure and temperature. Figure 18-4 shows a triple-point cell, in which this so-called triple point of water can be achieved in the laboratory. By international agreement, the triple point of water has been assigned a value of 273.16 K as the standard fixed-point temperature for the calibration of thermometers; that is,

$$T_3 = 273.16\ \text{K} \quad \text{(triple-point temperature)}, \tag{18-1}$$

in which the subscript 3 means "triple point." This agreement also sets the size of the kelvin as 1/273.16 of the difference between the triple-point temperature of water and absolute zero.

Note that we do not use a degree mark in reporting Kelvin temperatures. It is 300 K (not 300°K), and it is read "300 kelvins" (not "300 degrees Kelvin"). The usual SI prefixes apply. Thus, 0.0035 K is 3.5 mK. No distinction in

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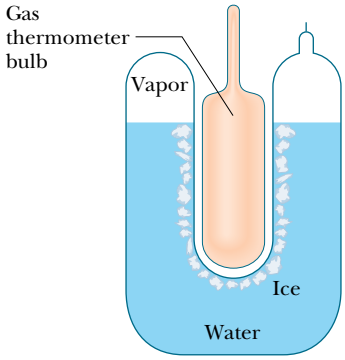

Figure 18-4 A triple-point cell, in which solid ice, liquid water, and water vapor coexist in thermal equilibrium. By international agreement, the temperature of this mixture has been defined to be 273.16 K. The bulb of a constant-volume gas thermometer is shown inserted into the well of the cell.

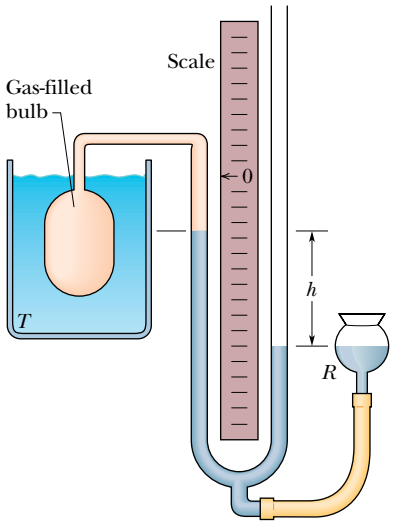

Figure 18-5 A constant-volume gas thermometer, its bulb immersed in a liquid whose temperature T is to be measured.

nomenclature is made between Kelvin temperatures and temperature differences, so we can write, "the boiling point of sulfur is 717.8 K" and "the temperature of this water bath was raised by 8.5 K."

The Constant-Volume Gas Thermometer

The standard thermometer, against which all other thermometers are calibrated, is based on the pressure of a gas in a fixed volume. Figure 18-5 shows such a **constant-volume gas thermometer**; it consists of a gas-filled bulb connected by a tube to a mercury manometer. By raising and lowering reservoir R, the mercury level in the left arm of the U-tube can always be brought to the zero of the scale to keep the gas volume constant (variations in the gas volume can affect temperature measurements).

The temperature of any body in thermal contact with the bulb (such as the liquid surrounding the bulb in Fig. 18-5) is then defined to be

$$T = Cp, \tag{18-2}$$

in which p is the pressure exerted by the gas and C is a constant. From Eq. 14-10, the pressure p is

$$p = p_0 - \rho g h, \tag{18-3}$$

in which p_0 is the atmospheric pressure, ρ is the density of the mercury in the manometer, and h is the measured difference between the mercury levels in the two arms of the tube.* (The minus sign is used in Eq. 18-3 because pressure p is measured *above* the level at which the pressure is p_0.)

If we next put the bulb in a triple-point cell (Fig. 18-4), the temperature now being measured is

$$T_3 = Cp_3, \tag{18-4}$$

in which p_3 is the gas pressure now. Eliminating C between Eqs. 18-2 and 18-4 gives us the temperature as

$$T = T_3\left(\frac{p}{p_3}\right) = (273.16\,\text{K})\left(\frac{p}{p_3}\right) \quad \text{(provisional)}. \tag{18-5}$$

We still have a problem with this thermometer. If we use it to measure, say, the boiling point of water, we find that different gases in the bulb give slightly different results. However, as we use smaller and smaller amounts of gas to fill the bulb, the readings converge nicely to a single temperature, no matter what gas we use. Figure 18-6 shows this convergence for three gases.

Thus the recipe for measuring a temperature with a gas thermometer is

$$T = (273.16\ \text{K})\left(\lim_{\text{gas}\to 0}\frac{p}{p_3}\right). \tag{18-6}$$

The recipe instructs us to measure an unknown temperature T as follows: Fill the thermometer bulb with an arbitrary amount of *any* gas (for example, nitrogen) and measure p_3 (using a triple-point cell) and p, the gas pressure at the

*For pressure units, we shall use units introduced in Section 14.2. The SI unit for pressure is the newton per square meter, which is called the pascal (Pa). The pascal is related to other common pressure units by

$$1\ \text{atm} = 1.01 \times 10^5\ \text{Pa} = 760\ \text{torr} = 14.7\ \text{lb/in}^2.$$

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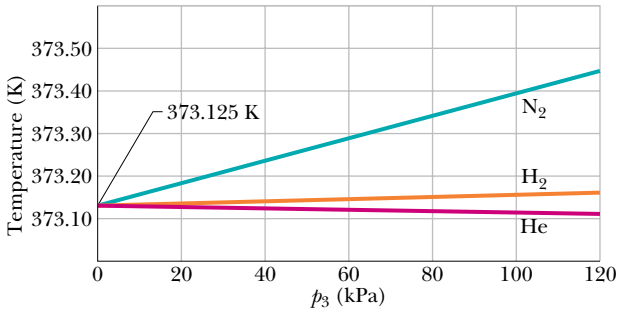

Figure 18-6 Temperatures measured by a constant-volume gas thermometer, with its bulb immersed in boiling water. For temperature calculations using Eq. 18-5, pressure p_3 was measured at the triple point of water. Three different gases in the thermometer bulb gave generally different results at different gas pressures, but as the amount of gas was decreased (decreasing p_3), all three curves converged to 373.125 K.

temperature being measured. (Keep the gas volume the same.) Calculate the ratio p/p_3. Then repeat both measurements with a smaller amount of gas in the bulb, and again calculate this ratio. Continue this way, using smaller and smaller amounts of gas, until you can extrapolate to the ratio p/p_3 that you would find if there were approximately no gas in the bulb. Calculate the temperature T by substituting that extrapolated ratio into Eq. 18-6. (The temperature is called the *ideal gas temperature*.)

18.5 | THE CELSIUS AND FAHRENHEIT SCALES

Key Concept

- The Celsius temperature scale is defined by

$$T_C = T - 273.15°.$$

with T in kelvins. The Fahrenheit temperature scale is defined by

$$T_F = \frac{9}{5}T_C + 32°.$$

So far, we have discussed only the Kelvin scale, used in basic scientific work. In nearly all countries of the world, the Celsius scale (formerly called the centigrade scale) is the scale of choice for popular and commercial use and much scientific use. Celsius temperatures are measured in degrees, and the Celsius degree has the same size as the kelvin. However, the zero of the Celsius scale is shifted to a more convenient value than absolute zero. If T_C represents a Celsius temperature and T a Kelvin temperature, then

$$T_C = T - 273.15°, \tag{18-7}$$

In expressing temperatures on the Celsius scale, the degree symbol is commonly used. Thus, we write 20.00°C for a Celsius reading but 293.15 K for a Kelvin reading.

The Fahrenheit scale, used in the United States, employs a smaller degree than the Celsius scale and a different zero of temperature. You can easily verify both these differences by examining an ordinary room thermometer on which both scales are marked. The relation between the Celsius and Fahrenheit scales is

$$T_F = \frac{9}{5}T_C + 32°, \tag{18-8}$$

where T_F is Fahrenheit temperature. Converting between these two scales can be done easily by remembering a few corresponding points, such as the freezing and boiling points of water (Table 18-1). Figure 18-7 compares the Kelvin, Celsius, and Fahrenheit scales.

Table 18-1 Some Corresponding Temperatures

Temperature	°C	°F
Boiling point of water[a]	100	212
Normal body temperature	37.0	98.6
Accepted comfort level	20	68
Freezing point of water[a]	0	32
Zero of Fahrenheit scale	≈ –18	0
Scales coincide	–40	–40

[a]Strictly, the boiling point of water on the Celsius scale is 99.975°C, and the freezing point is 0.00°C. Thus, there is slightly less than 100°C between those two points.

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We use the letters C and F to distinguish measurements and degrees on the two scales. Thus,

$$0°C = 32°F$$

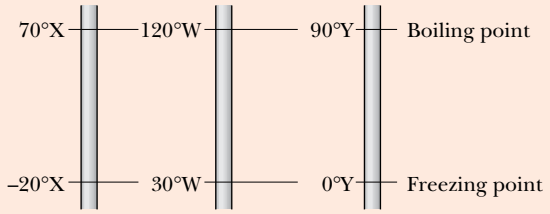
means that 0° on the Celsius scale measures the same temperature as 32°on the Fahrenheit scale, whereas

$$5°C = 9°F$$

means that a temperature difference of 5 Celsius degrees (note the degree symbol appears *after* C) is equivalent to a temperature difference of 9 Fahrenheit degrees.

CHECKPOINT 1

The figure here shows three linear temperature scales with the freezing and boiling points of water indicated. (a) Rank the degrees on these scales by size, greatest first. (b) Rank the following temperatures, highest first: 50°X, 50°W, and 50°Y.



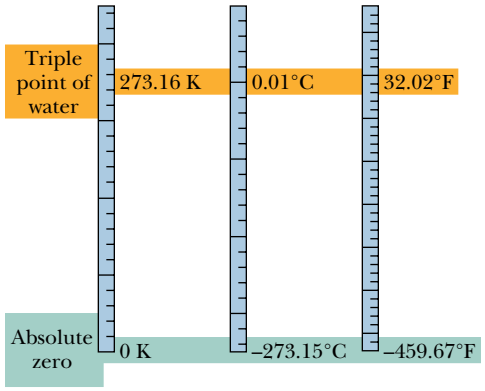


Figure 18-7 The Kelvin, Celsius, and Fahrenheit temperature scales compared.

SAMPLE PROBLEM 18.01

Conversion between two temperature scales

Suppose you come across old scientific notes that describe a temperature scale called Z on which the boiling point of water is 65.0°Z and the freezing point is –14.0°Z. To what temperature on the Fahrenheit scale would a temperature of T = –98.0°Z correspond? Assume that the Z scale is linear; that is, the size of a Z degree is the same everywhere on the Z scale.

KEY IDEA

A conversion factor between two (linear) temperature scales can be calculated by using two known (benchmark) temperatures, such as the boiling and freezing points of water. The number of degrees between the known temperatures on one scale is equivalent to the number of degrees between them on the other scale.

Calculations: We begin by relating the given temperature T to *either* known temperature on the Z scale. Since T = –98.0°Z is closer to the freezing point (–14.0°Z) than to the boiling point (65.0°Z), we use the freezing point.

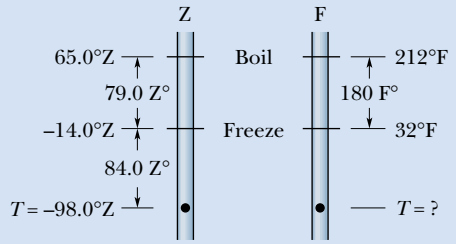


Figure 18-8 An unknown temperature scale compared with the Fahrenheit temperature scale.

Then we note that the T we seek is *below this point* by –14.0°Z – (–98.0°Z) = 84.0°Z (Fig. 18-8). (Read this difference as "84.0 Z degrees.")

Next, we set up a conversion factor between the Z and Fahrenheit scales to convert this difference. To do so, we use *both* known temperatures on the Z scale and the corresponding temperatures on the Fahrenheit scale. On the Z scale, the difference between the boiling and freezing points is 65.0°Z – (–14.0°Z) = 79.0°Z. On the Fahrenheit scale, it is 212°F – 32.0°F = 180°F.

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Thus, a temperature difference of 79.0°Z is equivalent to a temperature difference of 180°F (Fig. 18-8), and we can use the ratio (180°F)/(79.0°Z) as our conversion factor.

Now, since T is below the freezing point by 84.0°Z, it must also be below the freezing point by

$$(84.0°\text{Z})\frac{180°\text{F}}{79.0°\text{Z}} = 191°\text{F}.$$

Because the freezing point is at 32.0°F, this means that

$$T = 32.0°\text{F} - 191°\text{F} = -159°\text{F}. \qquad \text{(Answer)}$$

18.6 | THERMAL EXPANSION

You can often loosen a tight metal jar lid by holding it under a stream of hot water. Both the metal of the lid and the glass of the jar expand as the hot water adds energy to their atoms. (With the added energy, the atoms can move a bit farther from one another than usual, against the spring-like interatomic forces that hold every solid together.) However, because the atoms in the metal move farther apart than those in the glass, the lid expands more than the jar and thus is loosened.

Such **thermal expansion** of materials with an increase in temperature must be anticipated in many common situations. When a bridge is subject to large seasonal changes in temperature, for example, sections of the bridge are separated by *expansion slots* so that the sections have room to expand on hot days without the bridge buckling. When a dental cavity is filled, the filling material must have the same thermal expansion properties as the surrounding tooth; otherwise, consuming cold ice cream and then hot coffee would be very painful. When the Concorde aircraft (Fig. 18-9) was built, the design had to allow for the thermal expansion of the fuselage during supersonic flight because of frictional heating by the passing air.

Hugh Thomas/BWP Media/Getty Images, Inc.

Figure 18-9 When a Concorde flew faster than the speed of sound, thermal expansion due to the rubbing by passing air increased the aircraft's length by about 12.5 cm. (The temperature increased to about 128°C at the aircraft nose and about 90°C at the tail, and cabin windows were noticeably warm to the touch.)

Cause of Thermal expansion

All molecules undergo oscillations due to kinetic energy associated with temperature and the oscillations in a solid are not simple harmonic. Thus, the potential energy curve between neighboring atoms, as shown in Fig. 18-10, is not simple harmonic. The energy increases faster when the atoms move close to each other than when they move apart. The horizontal lines show the range of motion of atoms for different total energy. As the energy increases, the mean position of the atoms moves apart from each other because of the asymmetric nature of the vibrations. The thermal energy increases with temperature, increasing the average separation of the atoms and thus the size of a solid sample.

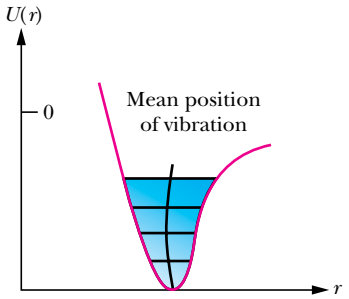

Figure 18-10 Potential energy due to interatomic forces plotted against the distance. Mean position of vibration is also shown.

18.7 | EXPANSION OF SOLIDS

Key Concept

- All objects change size with changes in temperature. For a temperature change ΔT, a change ΔL in any linear dimension L is given by

$$\Delta L = L\alpha\,\Delta T,$$

in which a α is the coefficient of linear expansion.

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Table 18-2 Some Coefficients of Linear Expansion[a]

Substance	α (10^{-6}/°C)
Ice (at 0°C)	51
Lead	29
Aluminum	23
Brass	19
Copper	17
Concrete	12
Steel	11
Glass (ordinary)	9
Glass (Pyrex)	3.2
Diamond	1.2
Invar[b]	0.7
Fused quartz	0.5

[a] Room temperature values except for the listing for ice.
[b] This alloy was designed to have a low coefficient of expansion. The word is a shortened form of "invariable."

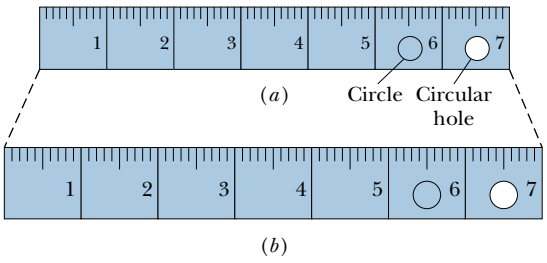

Figure 18-11 The same steel ruler at two different temperatures. When it expands, the scale, the numbers, the thickness, and the diameters of the circle and circular hole are all increased by the same factor. (The expansion has been exaggerated for clarity.)

Linear Expansion

If the temperature of a metal rod of length L is raised by an amount ΔT, its length is found to increase by an amount

$$\Delta L = L\alpha\,\Delta T, \qquad (18\text{-}9)$$

in which α is a constant called the **coefficient of linear expansion**. The coefficient α has the unit "per degree" or "per kelvin" and depends on the material. Although α varies somewhat with temperature, for most practical purposes it can be taken as constant for a particular material. Table 18-2 shows some coefficients of linear expansion. Note that the unit °C there could be replaced with the unit K.

The thermal expansion of a solid is like photographic enlargement except it is in three dimensions. Figure 18-11*b* shows the (exaggerated) thermal expansion of a steel ruler. Equation 18-9 applies to every linear dimension of the ruler, including its edge, thickness, diagonals, and the diameters of the circle etched on it and the circular hole cut in it. If the disk cut from that hole originally fits snugly in the hole, it will continue to fit snugly if it undergoes the same temperature increase as the ruler.

Bimetallic Strip

The thermal expansion properties of some materials can be put to common use. Thermometers and thermostats may be based on the differences in expansion between the components of a *bimetal strip* (Fig. 18-12). Also, the familiar liquid-in-glass thermometers are based on the fact that liquids such as mercury and alcohol expand to a different (greater) extent than their glass containers.

Calculation of Radius of Curvature

Consider a bimetallic strip as shown in Fig. 18-13*a*. On heating it will bend as shown in Fig. 18-13*b* if $\alpha_1 > \alpha_2$ where, α_1 and α_2 are the thermal expansion coefficients of outer and inner regions and d is the diameter of the rod.

Length of the outer portion is l_1 where, $l_1 = (R + d/2)\,\theta$, as the rod will be subtending an angle θ at the center.

$$l_1 = l(1 + \alpha_1 \Delta t)$$

Comparing the two values of l_1 we have,

$$(R + d/2)\theta = l(1 + \alpha_1 \Delta t)$$

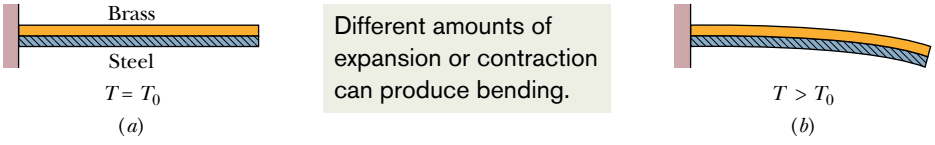

Figure 18-12 (*a*) A bimetal strip, consisting of a strip of brass and a strip of steel welded together, at temperature T_0. (*b*) The strip bends as shown at temperatures above this reference temperature. Below the reference temperature the strip bends the other way. Many thermostats operate on this principle, making and breaking an electrical contact as the temperature rises and falls.

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Length of the inner portion is l_2, where $l_2 = (R - d/2)\theta$, as the rod will be subtending angle θ at the center.

Also, $$l_2 = l(1 + \alpha_2 \Delta t)$$

Comparing the two values of l_2, we have

$$(R - d/2)\theta = l(1 + \alpha_2 \Delta t)$$

Solving the above equations, we get the radius of curvature

$$R = \left(\frac{d(2 + (\alpha_1 + \alpha_2)\Delta t}{2((\alpha_1 - \alpha_2)\Delta t} \right) \qquad (18\text{-}10)$$

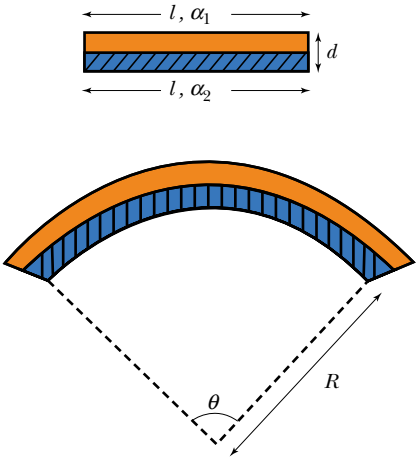
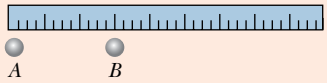


Figure 18-13 A bimetallic strip (*a*) Before thermal expansion and (*b*) after thermal expansion.

CHECKPOINT 2

A meter scale was calibrated at 0°C. If at 40°C it is giving the distance between two fixed points *A* and *B* as 12 mm (see figure), then what is the actual distance between them, given that coefficient of thermal expansion $\alpha = 10^{-5}$/°C.

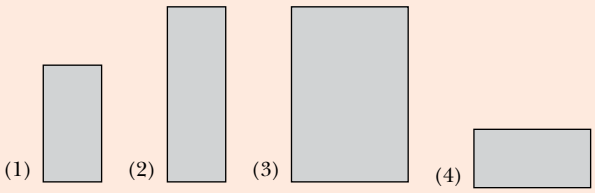


The Coefficient of Superficial Expansion

The coefficient of superficial expansion of a substance represents the alteration in area of a sheet of the substance initially possessing unit area, when it is heated through unit difference of temperature. Its numerical value, from the above, is equal to twice the coefficient of linear expansion of the substance in question. The coefficient of superficial expansion of an isotropic solid is equal to twice the coefficient of linear expansion of the solid.

CHECKPOINT 3

The figure here shows four rectangular metal plates, with sides of L, $2L$, or $3L$. They are all made of the same material, and their temperature is to be increased by the same amount. Rank the plates according to the expected increase in (a) their vertical heights and (b) their areas, greatest first.



CHECKPOINT 4

Why is only iron or steel used as reinforcement in concrete structures, while other metals, duralumin for example, are never employed?

CHECKPOINT 5

A metallic sheet is 50 cm long and 20 cm wide at 0°C. If it is heated to 60°C, what will be its area? Given that $\alpha = 10^{-4}$/°C.

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SAMPLE PROBLEM 18.02

Thermal stress

A steel wire of 2.0 mm^2 cross-section is held straight (but under no tension) by attaching it firmly to two rigid walls at a distance 1.50 m apart, at 30°C. If the temperature now decreases to –10°C, and if the end points remain fixed, what will be the tension in the wire? For steel, Y = 2 00 000 MPa.

KEY IDEA

If free to do so, the wire would contract but since we have tied its ends, it will not contract and maintain the original length.

Till now we have seen that when the length of a wire is changed it produces strain and hence stress. This situation is different as strain will be produced because of wire maintaining its length. At a lower temperature, the wire would have a unstrained length smaller than the original length. However, since its ends are tied, it will maintain its length but develop strain. Or in other words it has longer length than it would have at this temperature if not tied at its ends.

Calculations: If free to do so, the wire would contract a distance ΔL as it cooled, where

$$\begin{aligned}\Delta L &= \alpha L_0 \Delta T \\ &= (1.2 \times 10^{-5}/°C)\,(1.5\text{ m})\,(40°C) \\ &= 7.2 \times 10^{-4}\text{ m}\end{aligned}$$

But the ends are fixed. As a result, forces at the ends must, in effect, stretch the wire this same length, ΔL. Therefore, from $Y = (F/A)/(\Delta L/L)$, we have tension

$$\begin{aligned}F &= \frac{YA\,\Delta L}{L} \\ &= \frac{(2\times10^{11}\text{ N/m}^2)(2\times10^{-6}\text{ m}^2)(7.2\times10^{-4}\text{ m})}{1.5\text{ m}} \\ &= 192\text{ N} \qquad \text{(Answer)}\end{aligned}$$

Note: Strictly, we should have substituted $(1.5 - 7.2 \times 10^{-4})$ m for L in the expression for tension. However, the error incurred in not doing so, is negligible.

SAMPLE PROBLEM 18.03

Pendulum with differential expansion

A second's pendulum clock is shown in Fig. 18-14. It is compensated for temperature change by using differential expansion of the light rod and the heavy bob. This keeps the center of oscillation of the pendulum at a fixed distance below the point of suspension. Find the length of the light rod, h_0 and radius r_0 of the heavy bob of this pendulum. Given that $\alpha'' \gg \alpha'$.

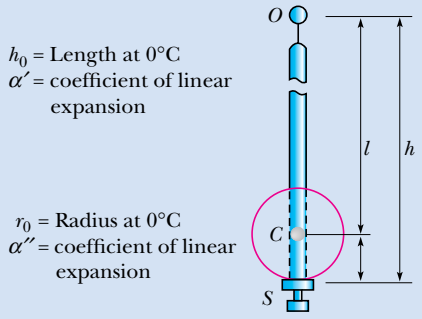

Figure 18-14 A pendulum with differential expansion.

KEY IDEA

A second's pendulum clock has the period 2 s. We can consider the center of the bob C as approximately the center of oscillation of the pendulum, the effective length of the pendulum is $l = h - r$. If expansion coefficients are so chosen that the difference of the lengths remain constant, then we will have constant, time period irrespective of the temperature change.

Calculations: Now,

$$h = h_0(1 + \alpha'\Delta T)$$

$$r = r_0(1 + \alpha''\Delta T)$$

where $\Delta\alpha'' \gg \alpha'$. Then, $l = h_0 - r_0 + (h_0\alpha' - r_0\alpha'')\Delta T$, which will be independent of temperature if

$$(h_0\alpha' - r_0\alpha'') = 0.$$

Also, $$l_0 = h_0 - r_0,$$

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where l_0 is the length that must be maintained. Solving these last two equations for h_0 and r_0, we get

$$h_0 = \frac{\alpha'' l_0}{\alpha'' - \alpha'}, \quad r_0 = \frac{\alpha' l_0}{\alpha'' - \alpha'}$$

The period (2 s) of the pendulum is

$$\tau = 2\pi\sqrt{\frac{l_0}{g}} \Rightarrow 2 = 2\pi\sqrt{\frac{l_0}{g}} \Rightarrow l_0 = \frac{g}{\pi^2}$$

Finally, the dimension (h_0, r_0) of the pendulum becomes

$$h_0 = \frac{\alpha'' l_0}{\alpha'' - \alpha'} = \frac{\alpha''(g/\pi^2)}{\alpha'' - \alpha'} \qquad \text{(Answer)}$$

And

$$r_0 = \frac{\alpha'(g/\pi^2)}{\alpha'' - \alpha'} \qquad \text{(Answer)}$$

Note: If we would have used same material, that is, $\alpha'' - \alpha' = 0$, the required dimensions would have become infinite. Or in other words, using same material will not ensure the temperature independence.

SAMPLE PROBLEM 18.04

Force exerted by rods of differential materials on thermal expansion

Two rods made of different materials are placed between massive walls (Fig. 18-15). The cross-section of the rods is A and their respective lengths are l_1 and l_2. The temperature increases by ΔT.

Find the force which the rods exert on each other if their coefficients of linear thermal expansion α_1 and α_2 and the moduli of elasticity of their materials E_1 and E_2 are known. Disregard the deformation of the walls.

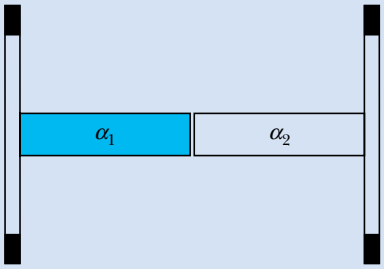

Figure 18-15 Two composite rods kept between two fixed walls.

KEY IDEAS

When the rods are heated in a free state, their total length will increase by

$$\Delta l = \Delta l_1 + \Delta l_2 = (\alpha_1 l_1 + \alpha_2 l_2)\Delta T$$

Compression by the same amount Δl will reduce the lengths of the rods by $\Delta l_1'$ and $\Delta l_2'$, where

$$\Delta l_1' + \Delta l_2' = \Delta l$$

This requires the force

$$F = \frac{E_1 A}{l_1}\Delta l_1' = \frac{E_2 A \Delta l_1'}{l_2}$$

Calculation: Upon solving this system of equations, we find that

$$F = \frac{a_1 l_1 + a_2 l_2}{\frac{l_1}{E_1} + \frac{l_2}{E_2}} A\Delta T$$

The rods will act upon each other with this force.

SAMPLE PROBLEM 18.05

Tension in plates due to thermal expansion

A copper plate is soldered (rigidly connected) to two steel plates as shown in Fig. 18-16. What tensions will arise in the plates if the temperature is increased by ΔT? Given coefficient of linear thermal expansion of copper is α_c and of steel is α_s, and α_c is greater than α_s. Also, the coefficients of elasticity for steel and copper are Y_s and Y_c, respectively. Cross-section area of the all three plates is A.

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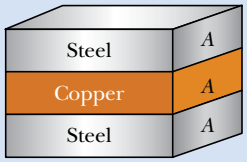

Figure 18-16 Three plates, a copper plate between two steel plates, having cross-section area A soldered together.

KEY IDEAS

Imagine if we have kept the three plates separately, the two steel plates would have expanded equally, but more than the copper plate. But since they are rigidly connected, the steel plates' expansion will be hindered by the copper plate that was unable to expand equally and similarly the copper plate will expand more because of stretching by the steel plates.

Since the coefficient of linear thermal expansion of copper α_c is greater than that of steel α_s, the increase in temperature will lead to compression of the copper plate and developmental of tension in steel plate. Thus, to conclude the steel plates will be experiencing compressive stress and the copper plates will be experiencing tensile stress. Because they are rigidly joined, the relative elongations of all the three plates are the same.

Calculations: Denoting the compressive force acting on the copper plate from the sides of the steel plates by F, we shall have for the relative elongation of the copper plate:

$$\frac{\Delta l}{l} = \alpha_c t - \frac{F}{AY_c}$$

Both steel plates are subjected to the tensile force $F/2$ from the side of the copper plate. Upon equating the relative elongation of the plates, we obtain.

$$\alpha_c t - \frac{F}{AY_c} = \alpha_s t + \frac{F}{2AY_s}$$

Hence, $$F = \frac{2AY_sY_c(\alpha_c - \alpha_s)t}{2Y_s + Y_c}$$

Note: If $\alpha_c = \alpha_s$ then we get force as zero, that is what we should expect, as both plates will have same expansion and they will not be pulling each other.

18.8 | EXPANSION OF LIQUIDS

Key Concepts

- The change ΔV in the volume V of a solid or liquid is

$$\Delta V = V\beta\,\Delta T.$$

Here $\beta = 3\alpha$ is the material's coefficient of volume expansion.

Volume Expansion

If all dimensions of a solid expand with temperature, the volume of that solid must also expand. For liquids, volume expansion is the only meaningful expansion parameter. If the temperature of a solid or liquid whose volume is V is increased by an amount ΔT, the increase in volume is found to be

$$\Delta V = V\beta\,\Delta T, \tag{18-11}$$

where β is the **coefficient of volume expansion** of the solid or liquid. The coefficients of volume expansion and linear expansion for a solid are related by

$$\beta = 3\alpha. \tag{18-12}$$

Like solids, different liquids expand by different amounts for the same temperature rise. Fig. 18-17 is a simple set up to demonstrate the expansion in different liquids. Three flasks are filled to the brim with different (preferably colored) liquids, and identical capillary tubes attached as shown in the figure. The flasks are placed in a large trough of water. When the trough is heated, the liquids expand by different amounts, thereby rising to different heights in their respective tubes. Liquids show greater expansion than solids. The cubical expansivities of liquids are about ten

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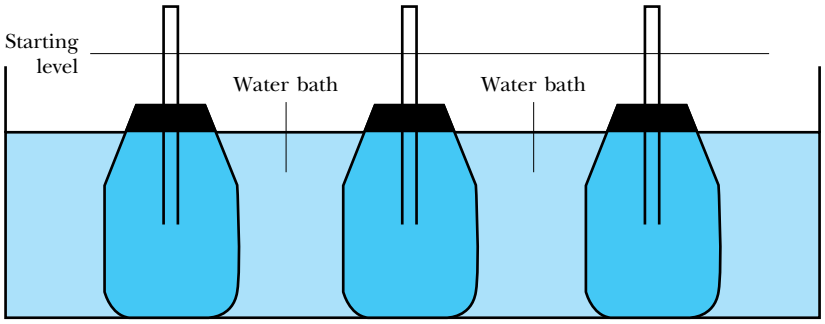

Figure 18-17 Set-up for demonstration of expansion in different liquids filled to same level in identical flasks.

times those of solids. Table 18-3 gives coefficients of cubical expansion of some common liquids.

Table 18-3 Coefficient of volume expansion at 20°C for some common liquids

Liquid	$\gamma(\times 10^{-4})/K$
Gasoline	9.5
Mercury	1.8
Ethyl alcohol	11.0
Glycerine	5.0
Water	2.1

One of the difficulties posed when measurements are to be made on the expansion of liquids is because liquids always need a containing vessel. When a liquid sample is heated, the container also expands. Thus, measurements of the volume increase on heating must also take into account the fact that the container has also expanded, meaning the liquid now occupies a slightly larger volume than it did before heating.

Consider the case of a uniform capillary tube of length L, as measured by a ruler, that contains some liquid, where γ is the coefficient of volume expansion of the liquid and α is the coefficient of linear expansion of the tube material. (Note that the specification that a ruler is used for the measurement is to avoid the use of scale that expands with the tube.) When the temperature of the tube and the liquid is raised by ΔT, an increase in height of liquid column is observed. We know that the vessel and the liquid both expand so the observed increase in the height of liquid column is not representative of the increase in volume of liquid only. If the radius of the capillary is r and its cross-sectional area is A, the increase in radius of the capillary tube due to thermal expansion is given by $dr = r\alpha\, dT$, where α is coefficient of linear expansion of the tube material, and dT is the temperature increase. We know that $dr/r \ll 1$.

Since $A = \pi r^2$, we find that $dA/A = 2dr/r$, or that $dA \equiv A\beta dT = A(2\alpha)dT$, where β is used as the symbol for the coefficient of area expansion.

If the temperature is changed from T to $T' \equiv T + dT$, the cross-sectional area changes from A to $A' \equiv A + dA = A(1 + 2\alpha dT)$.

The change in length from L to $L' \equiv L + dL$ of the liquid thread is governed by the volume expansion of the liquid.

Since $V' = V(1 + \gamma dT)$, where γ is the coefficient of volume expansion for the liquid, we have

$$L'A' = V' = V(1 + \gamma dT) = LA(1 + \gamma dT).$$

But $A'/A = 1 + 2\alpha dT$, so

$$L' = L\frac{(1+\gamma dT)}{(1+2\alpha dT)}$$

$$L\{[1 + (\gamma - 2\alpha)]\, dT - 2\gamma\alpha\,(dT)^2\}$$

For a temperature increase ΔT, the length change is $\Delta L \equiv L' - L \approx L(\gamma - 2\alpha)\Delta T$, and the relative change in length is

$$\frac{\Delta L}{L} \approx (\gamma - 2\alpha)\Delta T$$

Here we have assumed that $\alpha\Delta T = 1$.

Density Variation on Heating

Since the volume of a liquid increases when heated, without change in mass, it follows that the density of the liquid will change on heating. How do we explain this? From the previous section, it is seen that expansion is a result of

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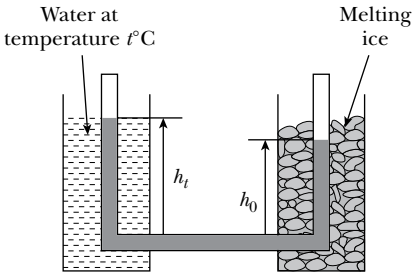

Figure 18-18 A U tube with both arms in two different baths.

the molecules moving farther apart on the application of heat. It therefore follows that the density of a liquid will be less at a higher current created in liquids when heated at the bottom. To calculate the density at the new temperature, we need to know the original density of the liquid. If we let ρ_0 represent the original density of the liquid, then the new density ρ for a temperature change ΔT is given by

$$\rho = \frac{\rho_0}{1+\gamma\Delta T}$$

If value of $\gamma\Delta T$ is negligible, then from binomial theorem: $\rho = \rho_0 (1 - \gamma\Delta T)$.

When the expansion of a liquid in a vessel is measured to obtain the coefficient γ, what is actually obtained directly is γ relative to the material of which the container is made. Figure 18-18 illustrates an apparatus from which the correct value of γ can be found without any knowledge of γ for the container.

Hydrostatic equilibrium of the liquid in the horizontal section of the tube implies that $\rho_t g h_t = \rho_0 g h_0$, where ρ_t is the density of the liquid in the warm column and ρ_0 is the density of the liquid in the cold column. Using the definition of the coefficient of volume expansion γ, the volumes of a given mass M of liquid at temperatures t and 0°C are related by $V_t = V_0 (1 + \gamma t)$. Since $\rho_t V_t = \rho_0 V_0$, the densities are related by $\rho_0 = \rho_t(1 + \gamma t)$. But then, since $\rho_t h_t = \rho_0 h_0$, we have $h_t = h_0 (1 + \gamma t)$. Solving this for γ, we find $\gamma = (h_t - h_0)/(h_0 t)$, as desired.

CHECKPOINT 6

In Fig. 18-18, if $h_1 - h_0 = 1.0$ cm, $h_0 = 100$ cm, and $t = 20$°C, what is the value of γ for the liquid?

CHECKPOINT 7

A sinker of weight W_0 has an apparent weight W_1 when weighed in a liquid at a temperature T_1 and W_2 when weighed in the same liquid at a temperature T_2. The coefficient of cubical expansion of material of sinker is γ_s. What is the coefficient of volume expansion of the liquid?

CHECKPOINT 8

A glass cylinder can contain $m_0 = 100$ g of mercury at a temperature of $t_0 = 0$°C. When $t_1 = 20$°C, the cylinder can contain $m_1 = 99.7$ g of mercury. In both cases the temperature of the mercury is assumed to be equal to that of the cylinder. Use this data to find the coefficient of linear expansion of glass α, bearing in mind that the coefficient of volume expansion of mercury $\gamma_1 = 18 \times 10^{-5}$/deg.

SAMPLE PROBLEM 18.06

Thermal expansion of a volume

On a hot day in Las Vegas, an oil trucker loaded 37000 L of diesel fuel. He encountered cold weather on the way to New York, where the temperature was 23.0 K lower than in Las Vegas, and where he delivered his entire load. How many liters did he deliver? The coefficient of volume expansion for diesel fuel is 9.50×10^{-4}/°C, and the coefficient of linear expansion for his steel truck tank is 11×10^{-6}/°C.

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KEY IDEA

The volume of the diesel fuel depends directly on the temperature. Thus, because the temperature decreased, the volume of the fuel did also, as given by Eq. 18-11 ($\Delta V = V\beta\Delta T$).

Calculations: We find

$$\Delta V = (37\,000\text{ L})(9.50 \times 10^{-4}/°\text{C})(-23.0\text{ K}) = -808\text{ L}.$$

Thus, the amount delivered was

$$V_{del} = V + \Delta V = 37\,000\text{ L} - 808\text{ L}$$
$$= 36\,190\text{ L}. \qquad \text{(Answer)}$$

Note that the thermal expansion of the steel tank has nothing to do with the problem.

SAMPLE PROBLEM 18.07

Simultaneous thermal expansion of liquid and container

A glass flask whose volume is exactly 1000 cm^3 at 0°C is completely filled with mercury at this temperature. When the flask and mercury are heated to 100°C, it is found that 15.4 cm^3 of mercury overflows. If the coefficient of volume expansion of mercury is 1.8×10^{-4}/K, calculate the coefficient of volume expansion of glass.

KEY IDEA

This sample problem represents the normal situation in which both liquid and container expand simultaneously when heated. The higher volume expansion of the liquid over the material of the container causes the overflow. We equate the overflow to the difference in the final volumes of the liquid and the container.

Calculations:

Given: Original volume of mercury, $V_0 = 1000$ cm^3
Original temperature of mercury,

$$T_0 = 0°\text{C} = 273\text{ K}$$

Final temperature of flask and mercury,

$$T = 100°\text{C} = 373\text{ K}$$

Volume of mercury overflow, $\Delta V = 15.4$ cm^3
Volume expansivity of mercury, $\gamma_{Hg} = 1.8 \times 10^{-4}$/K

Let volume expansivity of glass be γ_g. Let V_1 represent the final volume of the glass container, and V_2 that of the mercury. Then,

$$\Delta V = V_2 - V_1$$

where the new volumes are

$$V_2 = V_0(1 + \gamma_{Hg}\Delta T) \quad \text{and} \quad V_1 = V_0(1 + \gamma_g\Delta T)$$

Thus, $$V_0(1 + \gamma_{Hg}\Delta T) - V_0(1 + \gamma_g\Delta T) = \Delta V$$

Substituting values, we get

$$1000 \times (1 + 1.8 \times 10^{-4} \times 100) - 1000 \times (1 + \gamma_g \times 100)$$
$$= 15.4\text{ cm}^3$$

from which, the expansivity of glass,

$$\gamma_g = 2.6 \times 10^{-5}/\text{K} \qquad \text{(Answer)}$$

SAMPLE PROBLEM 18.08

Density variation on heating

A glass cylinder can contain $m_0 = 100$ g of mercury at a temperature of $T_0 = 0°$C. When $T_1 = 20°$C, the cylinder can contain $m_1 = 99.7$ g of mercury. In both cases the temperature of the mercury is assumed to be equal to that of the cylinder. Use this data to find the coefficient of linear expansion of glass α, bearing in mind that the coefficient of volume expansion of mercury $\gamma_1 = 18 \times 10^{-5}$/deg.

KEY IDEA

When the cylinder is heated, its volume increases according to the same law as that of the glass:

$V_1 = V_0(1 + \gamma T_1)$, where γ is the coefficient of volume expansion of glass. If the densities of mercury at temperatures T_0 and T_1 are denoted by ρ_0 and ρ_1, we can

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write that $m_0 = V_0 \rho_0$ and $m_1 = V_1 \rho_1$, where

$$\rho_1 = \frac{\rho_0}{1+\gamma_1 T_1}$$

Calculations: This system of equations will give the following expression for γ:

$$\gamma = \frac{m_1(1+\gamma_1 T_1)}{m_0 T_1} \approx 3\times 10^{-5}/\text{deg}$$

The coefficient of linear expansion,

$$\alpha = \frac{\gamma}{3} = 10^{-5}/\text{deg} \qquad \text{(Answer)}$$

SAMPLE PROBLEM 18.09

Calculating volume of liquid from coefficient of linear and volume expansion

A one-liter glass flask contains some mercury. It is found that at different temperatures the volume of air inside the flask remains the same. What is the volume of mercury in this flask if coefficient of linear expansion of glass is $9 \times 10^{-6}/°C$ while volume expansion of mercury is $1.8 \times 10^{-4}/°C$?

KEY IDEA

We know that both the vessel and the liquid expand, and they will expand in such a manner that the difference of their volumes will remain unchanged or in other words the expansion of mercury will be equal to that of the whole flask.

Calculations: If V is the volume of flask, V_L is volume of mercury and V_A is volume of air in it, then

$$V = V_L + V_A$$

As with the change in temperature, volume of air remains constant, the expansion of mercury will be equal to that of the whole flask, that is

$$\Delta V = \Delta V_L$$

or $$V\gamma_G \Delta\theta = V_L \gamma_L \Delta\theta \quad (\text{as } \Delta V = V\gamma\Delta\theta)$$

Here $V = 1$, $L = 1000$ cc and $\gamma_G = 3\alpha_G = 27 \times 10^{-6}/°C$. So,

$$V_L = (1000 \times 27 \times 10^{-6}/1.8 \times 10^{-4})$$
$$= 150 \text{ cc} \qquad \text{(Answer)}$$

SAMPLE PROBLEM 18.10

Effect of thermal expansion on floating block

An aluminum cube of side 20 cm floats on mercury (Fig. 18-19). How much farther will the block sink when the temperature rises from 300 to 350 K? Given that the densities of aluminum and mercury at 300 K are 2.7 and 13.6 g/cc while the coefficient of volume expansion of mercury and linear expansion of aluminum are $1.8 \times 10^{-4}/°C$ and $23 \times 10^{-6}/°C$, respectively

KEY IDEA

As the temperature rises, the block expands and the density of the liquid decreases, so it is like we are putting a bigger block of same weight in a lighter liquid. So, to apply the same buoyant force, a larger fraction of the volume should sink into the liquid.

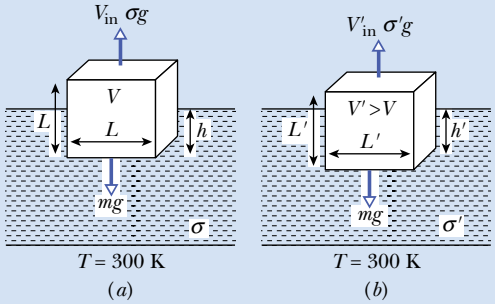

Figure 18-19 An aluminum cube floating in mercury. The temperature of mercury is (*a*) 300 K and (*b*) 350 K.

Calculations: If side of cube, density of liquid (mercury) and length of the floating cube inside the liquid are L,

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σ, h, and L', σ', h' at 300 K and 350 K, respectively, then by principle of floatation

$$m = V_{in}\sigma = V'_{in}\sigma' \quad (\text{with } m = V\rho = \rho L^3)$$

This is, as $V_{in} = hA = hL^2$,

$$hL^2\sigma = h'(L')^2\sigma'$$

or

$$\left[\frac{h'}{h}\right] = \left[\frac{L}{L'}\right]^2 \times \left[\frac{\sigma}{\sigma'}\right]$$

Now as $L' = L(1 + \alpha_s \Delta T)$ and $\sigma' = \sigma/(1 + \gamma_L \Delta T)$, therefore,

$$\left[\frac{h'}{h}\right] = \frac{(1+\gamma_L \Delta T)}{(1+\alpha_s \Delta T)} = \frac{(1+\gamma \Delta T)}{(1+2\alpha_s \Delta T)} = 1 + (\gamma_L - 2\alpha_s)\Delta T$$

So,

$$\Delta h = (h' - h) = h(\gamma_L - 2\alpha_s) \times \Delta T$$

Also, from the equation $m = V_{in}\sigma$, we have

$$\rho L^3 = hL^2\sigma$$

That is,

$$h = \left[\frac{\rho}{\sigma}\right] L = \left[\frac{2.7}{13.6}\right] \times 20 = 3.970 \text{ cm}$$

So,

$$\Delta h = 3.970\ (1.8 \times 10^{-4} - 2 \times 23 \times 10^{-6})\ (350 - 300)$$
$$= 2.66 \times 10^{-2} \text{ cm} \quad \text{(Answer)}$$

As expected the block sinks deeper.

Anomalous Expansion of Water

The most common liquid, water, does not behave like other liquids. Above about 4°C, water expands as the temperature rises, as we would expect. Between 0 and about 4°C, however, water *contracts* with increasing temperature. Thus, at about 4°C, the density of water passes through a maximum. At all other temperatures, the density of water is less than this maximum value.

This behavior of water is the reason lakes freeze from the top down rather than from the bottom up. As water on the surface is cooled from, say, 10°C toward the freezing point, it becomes denser ("heavier") than lower water and sinks to the bottom. Below 4°C, however, further cooling makes the water then on the surface *less* dense ("lighter") than the lower water, so it stays on the surface until it freezes. Thus the surface freezes while the lower water is still liquid. If lakes froze from the bottom up, the ice so formed would tend not to melt completely during the summer, because it would be insulated by the water above. After a few years, many bodies of open water in the temperate zones of Earth would be frozen solid all year round—and aquatic life could not exist.

REVIEW AND SUMMARY

Temperature; Thermometers Temperature is an SI base quantity related to our sense of hot and cold. It is measured with a thermometer, which contains a working substance with a measurable property, such as length or pressure, that changes in a regular way as the substance becomes hotter or colder.

Zeroth Law of Thermodynamics When a thermometer and some other object are placed in contact with each other, they eventually reach thermal equilibrium. The reading of the thermometer is then taken to be the temperature of the other object. The process provides consistent and useful temperature measurements because of the **zeroth law of thermodynamics**: If bodies A and B are each in thermal equilibrium with a third body C (the thermometer), then A and B are in thermal equilibrium with each other.

The Kelvin Temperature Scale In the SI system, temperature is measured on the **Kelvin scale**, which is based on the *triple point* of water (273.16 K). Other temperatures are then defined by use of a *constant-volume gas thermometer*, in which a sample of gas is maintained at constant volume so its pressure is proportional to its temperature. We define the *temperature* T as measured with a gas thermometer to be

$$T = (273.16 \text{ K}) \left(\lim_{\text{gas} \to 0} \frac{p}{p_3} \right). \quad (18\text{-}6)$$

Here T is in kelvins, and p_3 and p are the pressures of the gas at 273.16 K and the measured temperature, respectively.

Celsius and Fahrenheit Scales The Celsius temperature scale is defined by

$$T_C = T - 273.15°, \quad (18\text{-}7)$$

with T in kelvins. The Fahrenheit temperature scale is defined by

$$T_F = \frac{9}{5} T_C + 32°. \quad (18\text{-}8)$$

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Linear Expansion All objects change size with changes in temperature. For a temperature change ΔT, a change ΔL in any linear dimension L is given by

$$\Delta L = L\alpha\,\Delta T, \tag{18-9}$$

in which α is the **coefficient of linear expansion**.

Volume Expansion The change ΔV in the volume V of a solid or liquid is

$$\Delta V = V\beta\,\Delta T. \tag{18-11}$$

Here $\beta = 3\alpha$ is the material's **coefficient of volume expansion**.

PROBLEMS

1. As a result of a temperature rise of 64°C, a bar with a crack at its center buckles upward (Fig. 18-20). The fixed distance L_0 is 3.77 m and the coefficient of linear expansion of the bar is 25×10^{-6}/°C. Find the rise x of the center.

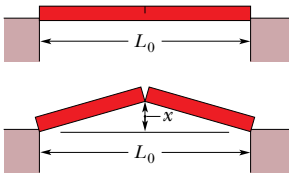

Figure 18-20 Problem 1.

2. On a linear X temperature scale, water freezes at –125.0°X and boils at 360.0°X. On a linear Y temperature scale, water freezes at –70.00°Y and boils at –30.00°Y. A temperature of 50.00°Y corresponds to what temperature on the X scale?
3. The volume of a solid aluminum ball with initial radius 20 cm increases by 347 cm^3 when the ball is heated. What is the temperature change?
4. (a) In 1983, the temperature at the Soviet Vostok Station in Antarctica reached a record low of –89.2°C. What temperature is this on the Fahrenheit scale? (b) The highest officially recorded temperature in the continental United States was 134°F in Death Valley, California. What is this temperature on the Celsius scale?
5. A vertical glass tube of length $L = 1.280\,000$ m is half filled with a liquid at 20.000 000°C. How much will the height of the liquid column change when the tube and liquid are heated to 30.000 000°C? Use coefficients $\alpha_{glass} = 2.000\,000 \times 10^{-5}$/K and $\beta_{liquid} = 4.000\,000 \times 10^{-5}$/K.
6. An aluminum cup of 200 cm^3 capacity is completely filled with glycerine at 22°C. How much glycerin, if any, will spill out of the cup if the temperature of both the cup and the glycerin is increased to 28°C? (The coefficient of volume expansion of glycerin is 5.1×10^{-4}/°C)
7. At what temperature is the Fahrenheit scale reading equal to (a) three times that of the Celsius scale and (b) one-third that of the Celsius scale?
8. At 20°C, a brass cube has edge length 25 cm. What is the increase in the surface area when it is heated from 20°C to 75°C?
9. At 20°C, a rod is exactly 20.05 cm long on a steel ruler. Both are placed in an oven at 250°C, where the rod now measures 20.11 cm on the same ruler. What is the coefficient of linear expansion for the material of which the rod is made?
10. An aluminum-alloy rod has a length of 10.000 cm at 20.000°C and a length of 10.020 cm at the boiling point of water. (a) What is the length of the rod at the freezing point of water? (b) What is the temperature if the length of the rod is 10.009 cm?
11. An aluminum flagpole is 30 m high. By how much does its length increase as the temperature increases by 15°C?
12. A steel rod is 3.000 cm in diameter at –10.00°C. A brass ring has an interior diameter of 2.992 cm at –10.00°C. At what common temperature will the ring just slide onto the rod?
13. Suppose that on a linear temperature scale X, water boils at –72.0°X and freezes at –123.0°X. What is a temperature of 59.0 K on the X scale? (Approximate water's boiling point as 373 K.)
14. What is the volume of a lead ball at 20.00°C if the ball's volume at 60.00°C is 33.58 cm^3?
15. A circular hole in an aluminum plate is 3.115 cm in diameter at 0.000°C. What is its diameter when the temperature of the plate is raised to 180.0°C?
16. When the temperature of a copper coin is raised by 100°C, its diameter increases by 0.20%. To two significant figures, give the percent increase in (a) the area of a face, (b) the thickness, (c) the volume, and (d) the mass of the coin. (e) Calculate the coefficient of linear expansion of the coin.
17. When the temperature of a metal cylinder is raised from 47.4°C to 100°C, its length increases by 0.10%. (a) Find the percent change in density. (b) What is the metal? Use Table 18-2.
18. In a certain experiment, a small radioactive source must move at selected, extremely slow speeds. This motion is accomplished by fastening the source to one end of an aluminum rod and heating the central section of the rod in a controlled way. If the effective heated section of the rod in Fig. 18-21 has length $d = 2.00$ cm, at what constant rate must the temperature of the rod be changed if the source is to move at a constant speed of 150 nm/s?

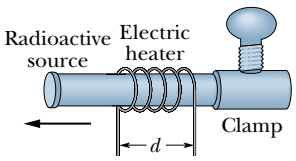

Figure 18-21 Problem 18.

19. A rectangular plate of glass initially has the dimensions 0.200 m by 0.300 m. The coefficient of linear expansion for the glass is 9.00×10^{-6}/K. What is the change in the plate's area if its temperature is increased by 20.0 K?

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20. A pendulum clock gives correct time at 20°C at a place where $g = 9.800\ m/s^2$. The pendulum consists of a light steel rod connected to a heavy ball. It is taken to a different place where $g = 9.788\ m/s^2$. At what temperature will it give correct time? Coefficient of linear expansion of steel = 12×10^{-6}/°C.

21. The densities of wood and benzene at 0°C are 880 kg/m³ and 900 kg/m³, respectively. The coefficients of volume expansion are 1.2×10^{-3}/°C for wood and 1.5×10^{-3}/°C for benzene. At what temperature will a piece of wood just sink in benzene?

22. A glass window pane is exactly 20 cm by 30 cm at 10°C. By how much has its area increased when its temperature is 40°C, assuming that it can expand freely?

23. A pendulum clock with a pendulum made of Invar ($a = 0.7 \times 10^{-6}$/°C) has a period of 0.5 s and is accurate at 25°C. If the clock is used in a country where the temperature averages 35°C, what correction is necessary at the end of a month (30 days) to the time given by the clock?

24. Two constant-volume gas thermometers are assembled, one with nitrogen and the other with hydrogen. Both contain enough gas so that $p_3 = 78$ kPa. (a) What is the difference between the pressures in the two thermometers if both bulbs are in boiling water? (*Hint:* See Fig. 18-5.) (b) Which gas is at higher pressure?

PRACTICE QUESTIONS

Single Correct Choice Type

1. In constructing a thermometer it is necessary to use a substance that
- **(a)** expands with rising temperature.
- **(b)** expands linearly with rising temperature.
- **(c)** will not freeze.
- **(d)** undergoes some change when heated or cooled.

2. What is the limiting low temperature of a physical object?
- **(a)** There is no limiting low temperature
- **(b)** 0 K
- **(c)** 0°C
- **(d)** 0°F

3. A balloon is filled with cold air and placed in a warm room. It is NOT in thermal equilibrium with the air of the room until
- **(a)** it rises to the ceiling.
- **(b)** it sinks to the floor.
- **(c)** it stops expanding.
- **(d)** it starts to contract.

4. Three thermometers are placed in a closed, insulated box and are allowed to reach thermal equilibrium. One is calibrated in Fahrenheit degrees, one in Celsius degrees, and one in Kelvins. The Celsius thermometer reads –40°C and the Kelvin thermometer reads 233 K. Which one of the following statements is necessarily true?
- **(a)** The Kelvin thermometer should read –233 K.
- **(b)** The Kelvin thermometer should read –313 K.
- **(c)** The Fahrenheit thermometer must read –40°F.
- **(d)** If water were found within the box, it must be in the liquid state.

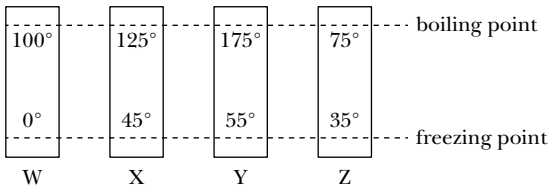
5. Suppose object C is in thermal equilibrium with object A and with object B. The zeroth law of thermodynamics states
- **(a)** that C will always be in thermal equilibrium with both A and B.
- **(b)** that C must transfer energy to both A and B.
- **(c)** that A is in thermal equilibrium with B.
- **(d)** that A cannot be in thermal equilibrium with B.

6. The "triple point" of a substance is that point for which the temperature and pressure are such that:
- **(a)** only solid and liquid are in equilibrium.
- **(b)** only liquid and vapor are in equilibrium.
- **(c)** only solid and vapor are in equilibrium.
- **(d)** solid, liquid and vapor are all in equilibrium.

7. Which one of the following statements is the best explanation for the fact that metal pipes that carry water often burst during cold winter months?
- **(a)** Water contracts upon freezing while the metal expands at lower temperatures.
- **(b)** The metal contracts to a greater extent than the water.
- **(c)** The interior of the pipe contracts less than the outside of the pipe.
- **(d)** Water expands upon freezing while the metal contracts at lower temperatures.

8. A constant-volume gas thermometer is used to measure the temperature of an object. When the thermometer is in contact with water at its triple point (273 K) the pressure in the thermometer is 8.50×10^4 Pa. When it is in contact with the object the pressure is 9.65×10^4 Pa. The temperature of the object is
- **(a)** 41.0 K
- **(b)** 114 K
- **(c)** 241 K
- **(d)** 310 K

9. The diagram shows four thermometers, labeled W, X, Y, and Z. The freezing and boiling points of water are indicated. Rank the thermometers according to the size of a degree on their scales, smallest to largest.



- **(a)** W, X, Y, Z
- **(b)** Y, W, X, Z
- **(c)** Z, Y, W, X
- **(d)** Z, X, W, Y

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10. Which one of the following properties could *not* be used as a temperature sensitive property in the construction of a thermometer?
(a) The change in mass of a solid.
(b) The change in volume of a liquid.
(c) The change in length of a metal rod.
(d) The change in electrical resistance of a wire.

11. There is a temperature at which the reading on the Kelvin scale is numerically
(a) equal to that on the Celsius scale.
(b) lower than that on the Celsius scale.
(c) equal to that on the Fahrenheit scale.
(d) less than zero.

12. The coefficient of linear expansion of aluminum is 23×10^{-6}/°C. A circular hole in an aluminum plate is 2.725 cm in diameter at 0°C. What is the diameter of the hole if the temperature of the plate is raised to 100°C?
(a) 0.0063 cm **(b)** 2.728 cm
(c) 2.731 cm **(d)** 2.757 cm

13. A thermometer indicates 98.6°C. It may be
(a) outdoors on a cold day.
(b) in a comfortable room.
(c) in a cup of hot tea.
(d) in a normal person's mouth.

14. A copper plate has a length of 0.12 m and a width of 0.10 m at 25°C. The plate is uniformly heated to 175°C. If the linear expansion coefficient for copper is 1.7×10^{-5}/°C, what is the *change* in the area of the plate as a result of the increase in temperature?
(a) 2.6×10^{-5} m^2 **(b)** 6.1×10^{-5} m^2
(c) 3.2×10^{-6} m^2 **(d)** 4.9×10^{-7} m^2

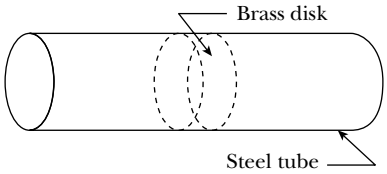
15. It is more difficult to measure the coefficient of volume expansion of a liquid than that of a solid because
(a) no relation exists between linear and volume expansion coefficients.
(b) a liquid tends to evaporate.
(c) a liquid expands too much when heated.
(d) the containing vessel also expands.

16. A steel string guitar is strung so that there is negligible tension in the strings at a temperature of 24.9°C. The guitar is taken to an outdoor winter concert where the temperature of the strings decreases to –15.1°C. The cross-sectional area of a particular string is 5.5×10^{-6} m^2. The distance between the points where the string is attached does not change. For steel, Young's modulus is 2.0×10^{11} N/m^2; and the coefficient of linear expansion is 1.2×10^{-5}/°C. Use your knowledge of linear thermal expansion and stress to calculate the tension in the string at the concert.
(a) 530 N **(b)** 240 N
(c) 120 N **(d)** 60 N

17. Thin strips of iron and zinc are riveted together to form a bimetallic strip which bends when heated. The iron is on the inside of the bend because
(a) it has a higher coefficient of linear expansion.
(b) it has a lower coefficient of linear expansion.
(c) it has a higher specific heat.
(d) it has a lower specific heat.

18. The coefficient of linear expansion of a certain solid is 9×10^{-6}/°C. Assuming this solid behaves like most solids, what is its coefficient of volume expansion?
(a) 9.1×10^{-6}/°C **(b)** 2.7×10^{-6}/°C
(c) 27×10^{-6}/°C **(d)** 729×10^{-6}/°C

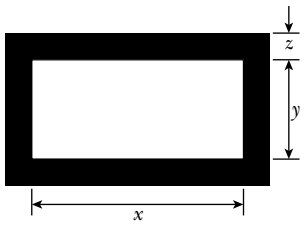
19. The given figure shows a brass disk tightly filled into a steel tube. The coefficient of linear expansion of the brass is 2.00×10^{-5} per °C. The system was assembled by cooling the disks in dry ice (–57°C) to enable them to just slide into the close-fitting tube. If the diameter of a disk is 80.00 mm at 43°C, what is its diameter in the dry ice?



(a) 78.400 mm **(b)** 79.998 mm
(c) 80.160 mm **(d)** 79.840 mm

20. A tanker ship is filled with 2.25×10^5 m^3 of gasoline at a refinery in southern Texas when the temperature is 17.2°C. When the ship arrives in New York City, the temperature is 1.3°C. If the coefficient of volume expansion for gasoline is 9.50×10^{-4}/°C, how much has the volume of the gasoline decreased when it is unloaded in New York?
(a) 1.50×10^{-2} m^3 **(b)** 66.2 m^3
(c) 1290 m^3 **(d)** 3400 m^3

21. The figure shows a rectangular brass plate at 0°C in which there is cut a rectangular hole of dimensions indicated. If the temperature of the plate is raised to 150°C, then



(a) x will increase and y will decrease.
(b) both x and y will decrease.
(c) x will decrease and y will increase.
(d) both x and y will increase.

22. The coefficient of volume expansion for gold is 4.20×10^{-5}/°C. The density of gold is 19 300 kg/m^3 at 0.0°C. What is the density of gold at 1050°C?
(a) 20 200 kg/m^3 **(b)** 18 500 kg/m^3
(c) 19 300 kg/m^3 **(d)** 18 800 kg/m^3

23. Dermatologists often remove small precancerous skin lesions by freezing them quickly with liquid nitrogen, which has a temperature of 77 K. What is this temperature on the (a) Celsius?
(a) –196°C **(b)** –321°C
(c) –164°C **(d)** –242°C

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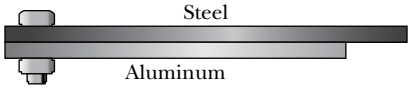
24. An aluminum baseball bat has a length of 0.86 m at a temperature of 17°C. When the temperature of the bat is raised, the bat lengthens by 0.000 16 m. Determine the final temperature of the bat.

(a) 19°C **(b)** 25°C
(c) 34°C **(d)** 29°C

25. When the temperature of a coin is raised by 75°C, the coin's diameter increases by 2.3×10^{-5} m. If the original diameter of the coin is 1.8×10^{-2} m, find the coefficient of linear expansion.

(a) 5.4×10^{-5}/°C
(b) 1.7×10^{-5}/°C
(c) 9.8×10^{-6}/°C
(d) 3.1×10^{-6}/°C

26. The given figure shows that, two thin strips of metal are bolted together at one end and have the same temperature. One is steel, and the other is aluminum. The steel strip is 0.10% longer than the aluminum strip. By how much should the temperature of the strips be increased, so that the strips have the same length?



(a) 91°C **(b)** 99°C
(c) 77°C **(d)** 85°C

27. At a temperature of 0°C, the mass and volume of a fluid are 825 kg and 1.17 m³. The coefficient of volume expansion is 1.26×10^{-3}/°C. What is the density of the fluid when the temperature has risen to 20.0°C?

(a) 688 kg/m³
(b) 562 kg/m³
(c) 705 kg/m³
(d) 641 kg/m³

28. A steel rod of length L_0 has a cross-sectional area A. The force required to stretch this rod by the same amount as the expansion produced by heating it through ΔT is (coefficient of linear expansion of steel is α and young's modulus for steel is Y)

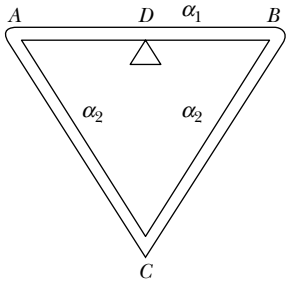
(a) $YA\alpha\Delta T$
(b) $1/2YA\alpha\Delta T$
(c) $2YA\alpha\Delta T$
(d) $3YA\alpha\Delta T$

29. A piece of metal floats on mercury. The coefficients of volume expansion of the metal and mercury are γ_1 and γ_2, respectively. If the temperatures of both mercury and the metal are increased by ΔT, the fraction of the volume of the metal submerged in mercury changes by the factor of

(a) $\dfrac{1+\gamma_2\Delta T}{1+\gamma_1\Delta T}$ **(b)** $1+\gamma_1\Delta T$

(c) $1+\gamma_2\Delta T$ **(d)** $\dfrac{1-\gamma_2\Delta T}{1-\gamma_1\Delta T}$

30. Three rods of equal length are joined to form an equilateral triangle ABC (see figure). D is the midpoint of AB. The coefficient of linear expansion is α_1 for AB, and α_2 for AC and BC. If the distances DC remains constant, for small changes in temperature



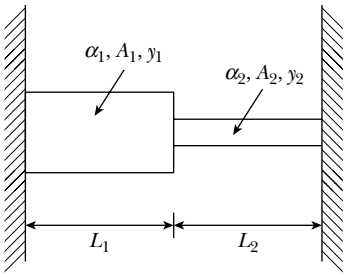
(a) $\alpha_1 = \alpha_2$ **(b)** $\alpha_1 = 2\alpha_2$

(c) $\alpha_1 = 4\alpha_2$ **(d)** $\alpha_1 = \dfrac{1}{2}\alpha_2$

31. A iron rod of length 50 cm is joined at an end to an aluminium rod of length 100 cm. All measurements refer to 20°C. The coefficients of linear expansion of iron and aluminum are 12×10^{-6}/°C and 24×10^{-6}/°C, respectively. The average coefficient of composite system is

(a) 36×10^{-6}/°C
(b) 12×10^{-6}/°C
(c) 20×10^{-6}/°C
(d) 48×10^{-6}/°C

32. Two elastic rods are joined between fixed supports as shown in figure. Condition for no change in the lengths of individual rods with the increase of temperature (α_1, α_2 = linear expansion coefficient, A_1 and A_2 = area of rods, y_1 and y_2 = young modulus)



(a) $\dfrac{A_1}{A_2} = \dfrac{\alpha_1 y_1}{\alpha_2 y_2}$ **(b)** $\dfrac{A_1}{A_2} = \dfrac{L_1\alpha_1 y_1}{L_2\alpha_2 y_2}$

(c) $\dfrac{A_1}{A_2} = \dfrac{L_2\alpha_1 y_1}{L_1\alpha_2 y_2}$ **(d)** $\dfrac{A_1}{A_2} = \dfrac{\alpha_2 y_2}{\alpha_1 y_1}$

33. An iron ball (coefficient of linear expansion = 12×10^{-5}/°C) has a diameter of 6 cm and is 0.010 mm too large to pass through a hole in a brass plate (coefficient of linear expansion = 1.9×10^{-5}/°C) when the ball and the plate are both at a temperature of 30°C. At what common temperature of the ball and the plate will the ball just pass through the hole in the plate?

(a) 23.8°C **(b)** 53.8°C
(c) 42.5°C **(d)** 63.5°C

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More than One Correct Choice Type

34. A vessel is partly filled with liquid. When the vessel is cooled to a lower temperature, the space in the vessel, unoccupied by the liquid, remains constant. Then, the volume of the liquid (V_L), volume of the vessel (V_V), the coefficient of cubical expansion of the material of the vessel (γ_V), and of the liquid (γ_L), are related as

(a) $\gamma_L > \gamma_V$ **(b)** $\gamma_L < \gamma_V$

(c) $\gamma_V/\gamma_L = V_V/V_L$ **(d)** $\gamma_V/\gamma_L = V_L/V_V$

35. A metal rod is shaped into a ring with a small gap. If this is heated, then

(a) the length of the rod will increase.

(b) the gap will decrease.

(c) the gap will increase.

(d) the diameter of the ring will increase in the same ratio as the length of the rod.

36. When the temperature of a copper coin is raised by 80°C, its diameter increases by 0.2%. Then

(a) Percentage rise in the area of a face is 0.4%.

(b) Percentage rise in the thickness is 0.4%.

(c) Percentage rise in the volume is 0.6%.

(d) Coefficient of linear expansion of copper is 0.25×10^{-4}/°C.

37. Two rods of length L_1 and L_2 are made of materials of coefficient of linear expansion α_1 and α_2, respectively, such that $L_1\alpha_1 = L_2\alpha_2$. The temperature of the rods is increased by ΔT and correspondingly the change in their respective lengths is ΔL_1 and ΔL_2.

(a) $\Delta L_1 \neq \Delta L_2$

(b) $\Delta L_1 = \Delta L_2$

(c) The difference in the length $(L_1 - L_2)$ is a constant and is independent of the rise of temperature.

(d) Data insufficient to arrive at a conclusion.

Linked Comprehension

Paragraph for Questions 38–40: When a solid is heated, its length changes according to the relation $l = l_0(1 + \alpha\Delta T)$, where l is the final length, l_0 is the initial length, ΔT is the change in temperature, and α is the coefficient of linear expansion. The change in area of a solid upon heating is called superficial expansion. The area changes according to the relation $A = A_0(1 + \beta\Delta T)$, where A is the final area, A_0 is the initial area, and β is the coefficient of areal expansion.

38. A metal disc having a circular hole at its center is heated. If the metal expands on heating, the diameter of the hole

(a) increases.

(b) decreases.

(c) remains unchanged.

(d) increases or decreases depending on the metal.

39. On heating a liquid of coefficient of cubical or volume expansion γ in a container having coefficient of linear expansion $\gamma/3$, the level of liquid in the container

(a) rises.

(b) falls.

(c) remain almost stationary.

(d) none of these.

40. The coefficients of linear expansion of brass and steel are α_1 and α_2. If we take a brass rod of length l_1 and a steel rod of length l_2 at 0°C, their difference in length remains the same at any temperature if

(a) $\alpha_1 l_2 = \alpha_2 l_1$

(b) $\alpha_1 l_2^2 = \alpha_2 l_1^2$

(c) $\alpha_1^2 l_2 = \alpha_2^2 l_1$

(d) $\alpha_1 l_1 = \alpha_2 l_2$

Matrix-Match

Directions for Question 41: In the question below, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

41. In the given table, the first column shows the lower fixed point (LFP); Column II shows the upper fixed point (UFP) and the Column III gives corresponding divisions on the scale for three different scales for temperature measurement.

Column I	Column II	Column III
(I) Lower fixed point: 460	**(i)** Upper fixed point: 80	**(J)** 212
(II) Lower fixed point: 32	**(ii)** Upper fixed point: 100	**(K)** 80
(III) Lower fixed point: 0	**(iii)** Upper fixed point: 672	**(L)** 100
(IV) Lower fixed point: 273.15	**(iv)** Upper fixed point: 212	**(M)** 180

(1) Determine the features of Fahrenheit scale.

(a) (III) (ii) (L) **(b)** (II) (iv) (M)

(c) (I) (i) (M) **(d)** (II) (i) (K)

(2) Determine the features of Rankine scale.

(a) (I) (ii) (K) **(b)** (IV) (iii) (J)

(c) (II) (iii) (M) **(d)** (I) (iii) (J)

(3) Determine the features of Reaumer scale.

(a) (I) (i) (K) **(b)** (I) (i) (L)

(c) (III) (i) (K) **(d)** (I) (iv) (L)

Integer Type

42. A pendulum clock loses 12 s/day if the temperature is 40°C and goes fast by 4 s/day if the temperature is 20°C. Find the temperature at which the clock shows the correct time.

43. Find the length of the steel rod that would have the same difference in length with a copper rod of length 24 cm at all temperatures ($\alpha_{copper} = 18 \times 10^{-6}$/K, $\alpha_{steel} = 12 \times 10^{-6}$/K).

44. Find the approximate length of the Golden Gate bridge if it is known that the steel in the roadbed expands by 0.53 m when the temperature changes from +2 to +32°C.

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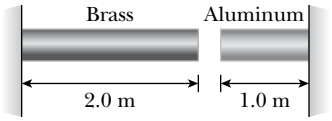
45. The brass bar and the aluminum bar in the drawing are each attached to an immovable wall. At 28°C the air gap between the rods is 1.3×10^{-3} m. At what temperature will the gap be closed?



ANSWER KEY

Checkpoints

1. (a) all tie; (b) 50°X, 50°Y, 50°W **2.** 12.0048 mm

3. (a) 2 and 3 tie, then 1, then 4; (b) 3, 2 then 1 and 4 tie (from Eq. 18-9, assume that change in area is proportional to initial area)

4. Reinforced concrete structures are very strong because the expansion coefficient is very close to that of iron and steel.

5. 1012 cm^2 **6.** $\gamma = 1.0/[(100)(20) = 500 \times 10^{-6}/°C$ **7.** $\frac{W_0 - W_1}{W_0 - W_2}\left[\frac{1+(T_2 - T_1)\gamma_{1s}}{T_2 - T_1}\right] - \frac{1}{T_2 - T_1}$

8. 10^{-5}/deg

Problems

1. 0.1066 m ≈ 0.11 m **2.** 1330°X **3.** 150°K **4.** (a) –129°F; (b) 56.7 on the Celsius scale

5. 0 **6.** 0.53 cm^3 **7.** (a) 80.0°F; (b) –7.27°F **8.** 7.8 cm^2

9. $24 \times 10^{-6}/°C$ **10.** (a) 9.995 cm; (b) 56°C **11.** 0.010 m **12.** 325.0°C

13. –232°X **14.** 33.46 cm^3 **15.** 3.128 cm

16. (a) 0.40%; (b) 0.20%; (c) 0.60%; (d) mass does not change; (e) $2.0 \times 10^{-5}/°C$

17. (a) –0.30%; (b) the metal is brass **18.** 0.326 K/s **19.** 2.16×10^5 m^2 **20.** 82°C

21. 83°C **22.** 0.32 cm^2 **23.** 9.1 s

24. 0.054 kPa ≈ 0.05 kPa; (b) pressure in the nitrogen thermometer is higher than the pressure in the hydrogen thermometer

Practice Questions

Single Correct Choice Type

1. (d) **2.** (b) **3.** (c) **4.** (c) **5.** (c)

6. (d) **7.** (d) **8.** (d) **9.** (d) **10.** (a)

11. (c) **12.** (c) **13.** (c) **14.** (b) **15.** (d)

16. (a) **17.** (b) **18.** (c) **19.** (d) **20.** (d)

21. (d) **22.** (b) **23.** (a) **24.** (b) **25.** (b)

26. (a) **27.** (a) **28.** (a) **29.** (a) **30.** (c)

31. (c) **32.** (d) **33.** (b)

More than One Correct Choice Type

34. (a), (d) **35.** (a), (b), (d) **36.** (a), (c), (d) **37.** (b), (c)

Linked Comprehension

38. (a) **39.** (c) **40.** (d)

Matrix-Match

41. (1) → (b); (2) → (d); (3) → (c)

Integer Type

42. 22 **43.** 36 **44.** 1500 **45.** 49

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CHAPTER 19

Heat – Measurement and Transfer

19.1 | WHAT IS PHYSICS?

If you take a can of cola from the refrigerator and leave it on the kitchen table, its temperature will rise—rapidly at first but then more slowly—until the temperature of the cola equals that of the room (the two are then in thermal equilibrium). In the same way, the temperature of a cup of hot coffee, left sitting on the table, will fall until it also reaches room temperature.

In generalizing this situation, we describe the cola or the coffee as a *system* (with temperature T_S) and the relevant parts of the kitchen as the *environment* (with temperature T_E) of that system. Our observation is that if T_S is not equal to T_E, then T_S will change (T_E can also change some) until the two temperatures are equal and thus thermal equilibrium is reached.

19.2 | TEMPERATURE AND HEAT

Key Concept

- Heat Q is energy that is transferred between a system and its environment because of a temperature difference between them. It can be measured in joules (J), calories (cal), kilocalories (Cal or kcal), or British thermal units (Btu), with

$$1 \text{ cal} = 3.968 \times 10^{-3} \text{ Btu} = 4.1868 \text{ J}.$$

The change in temperature discussed above is due to a change in the thermal energy of the system because of a transfer of energy between the system and the system's environment. (Recall that *thermal energy* is an internal energy that consists of the kinetic and potential energies associated with the random motions of the atoms, molecules, and other microscopic bodies within an object.) The transferred energy is called **heat** and is symbolized Q. Heat is *positive* when energy is transferred to a system's thermal energy from its environment (we say that heat is absorbed by the system). Heat is *negative* when energy is transferred from a system's thermal energy to its environment (we say that heat is released or lost by the system).

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19.1 What is Physics?
19.2 Temperature and Heat
19.3 The Absorption of Heat by Solids and Liquids
19.4 Calorimetry
19.5 Heat Transfer Mechanisms
19.6 Applications of Stefan's Law
19.7 Wien's Displacement Law
19.8 Newton's Law of Cooling

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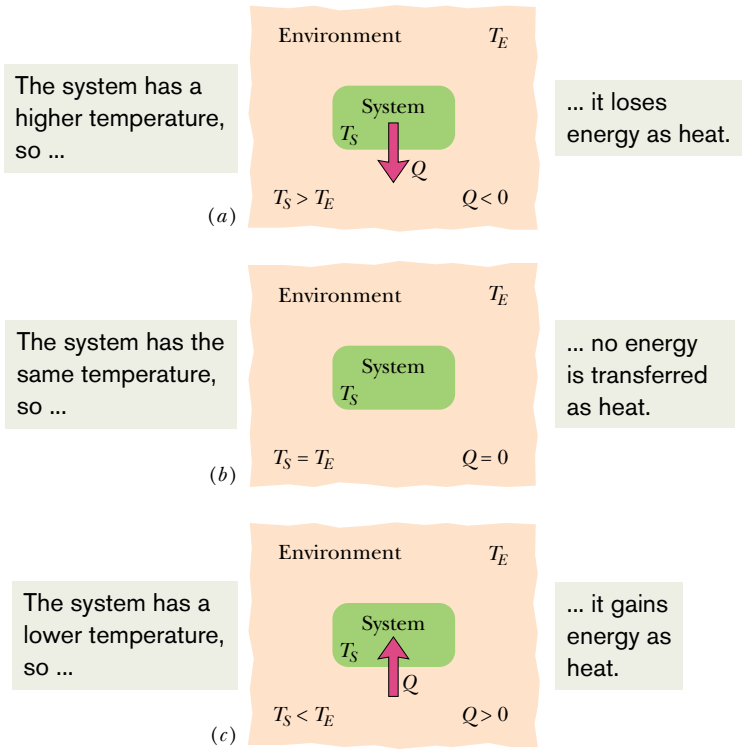

Figure 19-1 If the temperature of a system exceeds that of its environment as in (*a*), heat Q is lost by the system to the environment until thermal equilibrium (*b*) is established. (*c*) If the temperature of the system is below that of the environment, heat is absorbed by the system until thermal equilibrium is established.

This transfer of energy is shown in Fig. 19-1. In the situation of Fig. 19-1*a*, in which $T_S > T_E$, energy is transferred from the system to the environment, so Q is negative. In Fig. 19-1*b*, in which $T_S = T_E$, there is no such transfer, Q is zero, and heat is neither released nor absorbed. In Fig. 19-1*c*, in which $T_S < T_E$, the transfer is to the system from the environment; so Q is positive.

We are led then to this definition of heat:

Heat is the energy transferred between a system and its environment because of a temperature difference that exists between them.

Language. Recall that energy can also be transferred between a system and its environment as *work* W via a force acting on a system. Heat and work, unlike temperature, pressure, and volume, are not intrinsic properties of a system. They have meaning only as they describe the transfer of energy into or out of a system. Similarly, the phrase "a $600 transfer" has meaning if it describes the transfer to or from an account, not what is in the account, because the account holds money, not a transfer.

Units. Before scientists realized that heat is transferred energy, heat was measured in terms of its ability to raise the temperature of water. Thus, the **calorie** (cal) was defined as the amount of heat that would raise the temperature of 1 g of water from 14.5°C to 15.5°C. In the British system, the corresponding unit of heat was the **British thermal unit** (Btu), defined as the amount of heat that would raise the temperature of 1 lb of water from 63°F to 64°F.

In 1948, the scientific community decided that since heat (like work) is transferred energy, the SI unit for heat should be the one we use for energy—namely, the **joule**. The calorie is now defined to be 4.1868 J (exactly), with no reference to the heating of water. (The "calorie" used in nutrition, sometimes called the Calorie (Cal), is really a kilocalorie.) The relations among the various heat units are

$$1 \text{ cal} = 3.968 \times 10^{-3} \text{ Btu} = 4.1868 \text{ J}. \tag{19-1}$$

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19.3 | THE ABSORPTION OF HEAT BY SOLIDS AND LIQUIDS

Key Concepts

- If heat Q is absorbed by an object, the object's temperature change $T_f - T_i$ is related to Q by
$$Q = C(T_f - T_i),$$
in which C is the heat capacity of the object. If the object has mass m, then
$$Q = cm(T_f - T_i),$$
where c is the specific heat of the material making up the object.
- The molar specific heat of a material is the heat capacity per mole, which means per 6.02×10^{23} elementary units of the material.
- Heat absorbed by a material may change the material's physical state—for example, from solid to liquid or from liquid to gas. The amount of energy required per unit mass to change the state (but not the temperature) of a particular material is its heat of transformation L. Thus,
$$Q = Lm.$$
- The heat of vaporization L_V is the amount of energy per unit mass that must be added to vaporize a liquid or that must be removed to condense a gas.
- The heat of fusion L_F is the amount of energy per unit mass that must be added to melt a solid or that must be removed to freeze a liquid.

Thermal Parameters

Heat Capacity

The **heat capacity** C of an object is the proportionality constant between the heat Q that the object absorbs or loses and the resulting temperature change ΔT of the object; that is,

$$Q = C\Delta T = C(T_f - T_i), \tag{19-2}$$

in which T_i and T_f are the initial and final temperatures of the object. Heat capacity C has the unit of energy per degree or energy per kelvin. The heat capacity C of, say, a marble slab used in a bun warmer might be 179 cal/°C, which we can also write as 179 cal/K or as 749 J/K.

The word "capacity" in this context is really misleading in that it suggests analogy with the capacity of a bucket to hold water. *That analogy is false*, and you should not think of the object as "containing" heat or being limited in its ability to absorb heat. Heat transfer can proceed without limit as long as the necessary temperature difference is maintained. The object may, of course, melt or vaporize during the process.

Specific Heat

Two objects made of the same material—say, marble—will have heat capacities proportional to their masses. It is therefore convenient to define a "heat capacity per unit mass" or **specific heat** c that refers not to an object but to a unit mass of the material of which the object is made. Equation 19-2 then becomes

$$Q = cm\Delta T = cm(T_f - T_i). \tag{19-3}$$

Through experiment we would find that although the heat capacity of a particular marble slab might be 179 cal/°C (or 749 J/K), the specific heat of marble itself (in that slab or in any other marble object) is 0.21 cal/g · °C (or 880 J/kg · K).

From the way the calorie and the British thermal unit were initially defined, the specific heat of water is

$$c = 1 \text{ cal/g} \cdot \text{°C} = 1 \text{ Btu/lb} \cdot \text{°F} = 4186.8 \text{ J/kg} \cdot \text{K}. \tag{19-4}$$

Table 19-1 shows the specific heats of some substances at room temperature. Note that the value for water is relatively high. The specific heat of any substance actually depends somewhat on temperature, but the values in Table 19-1 apply reasonably well in a range of temperatures near room temperature.

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CHECKPOINT 1

A certain amount of heat Q will warm 1 g of material A by 3°C and 1 g of material B by 4°C. Which material has the greater specific heat?

Molar Specific Heat

In many instances the most convenient unit for specifying the amount of a substance is the mole (mol), where

$$1 \text{ mol} = 6.02 \times 10^{23} \text{ elementary units}$$

of *any* substance. Thus 1 mol of aluminum means 6.02×10^{23} atoms (the atom is the elementary unit), and 1 mol of aluminum oxide means 6.02×10^{23} molecules (the molecule is the elementary unit of the compound).

When quantities are expressed in moles, specific heats must also involve moles (rather than a mass unit); they are then called **molar specific heats**. Table 19-1 shows the values for some elemental solids (each consisting of a single element) at room temperature.

AN IMPORTANT POINT

In determining and then using the specific heat of any substance, we need to know the conditions under which energy is transferred as heat. For solids and liquids, we usually assume that the sample is under constant pressure (usually atmospheric) during the transfer. It is also conceivable that the sample is held at constant volume while the heat is absorbed. This means that thermal expansion of the sample is prevented by applying external pressure. For solids and liquids, this is very hard to arrange experimentally, but the effect can be calculated, and it turns out that the specific heats under constant pressure and constant volume for any solid or liquid differ usually by no more than a few percent. Gases, as you will see, have quite different values for their specific heats under constant-pressure conditions and under constant-volume conditions.

Table 19-1 Some Specific Heats and Molar Specific Heats at Room Temperature

Substance	Specific Heat $\frac{\text{cal}}{\text{g}\cdot\text{K}}$	Specific Heat $\frac{\text{J}}{\text{kg}\cdot\text{K}}$	Molar Specific Heat $\frac{\text{J}}{\text{mol}\cdot\text{K}}$
Elemental Solids			
Lead	0.0305	128	26.5
Tungsten	0.0321	134	24.8
Silver	0.0564	236	25.5
Copper	0.0923	386	24.5
Aluminum	0.215	900	24.4
Other Solids			
Brass	0.092	380	
Granite	0.19	790	
Glass	0.20	840	
Ice (–10°C)	0.530	2220	
Liquids			
Mercury	0.033	140	
Ethyl alcohol	0.58	2430	
Seawater	0.93	3900	
Water	1.00	4187	

Heats of Transformation

When energy is absorbed as heat by a solid or liquid, the temperature of the sample does not necessarily rise. Instead, the sample may change from one *phase*, or *state*, to another. Matter can exist in three common states: In the *solid state*, the molecules of a sample are locked into a fairly rigid structure by their mutual attraction. In the *liquid state*, the molecules have more energy and move about more. They may form brief clusters, but the sample does not have a rigid structure and can flow or settle into a container. In the *gas*, or *vapor*, *state*, the molecules have even more energy, are free of one another, and can fill up the full volume of a container.

Melting. To *melt* a solid means to change it from the solid state to the liquid state. The process requires energy because the molecules of the solid must be freed from their rigid structure. Melting an ice cube to form liquid water is a common example. To *freeze* a liquid to form a solid is the reverse of melting and requires that energy be removed from the liquid, so that the molecules can settle into a rigid structure.

Vaporizing. To *vaporize* a liquid means to change it from the liquid state to the vapor (gas) state. This process, like melting, requires energy because the molecules must be freed from their clusters. Boiling liquid water to transfer to water vapor (or steam—a gas of individual water molecules) is a common example. *Condensing* a gas to form a liquid is the reverse of vaporizing; it requires that energy be removed from the gas, so that the molecules can cluster instead of flying away from one another.

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Table 19-2 Some Heats of Transformation

	Melting		Boiling	
Substance	**Melting Point (K)**	**Heat of Fusion L_F (kJ/kg)**	**Boiling Point (K)**	**Heat of Vaporization L_V (kJ/kg)**
Hydrogen	14.0	58.0	20.3	455
Oxygen	54.8	13.9	90.2	213
Mercury	234	11.4	630	296
Water	273	333	373	2256
Lead	601	23.2	2017	858
Silver	1235	105	2323	2336
Copper	1356	207	2868	4730

The amount of energy per unit mass that must be transferred as heat when a sample completely undergoes a phase change is called the **heat of transformation** L. Thus, when a sample of mass m completely undergoes a phase change, the total energy transferred is

$$Q = Lm. \tag{19-5}$$

When the phase change is from liquid to gas (then the sample must absorb heat) or from gas to liquid (then the sample must release heat), the heat of transformation is called the **heat of vaporization** L_V. For water at its normal boiling or condensation temperature,

$$L_V = 539 \text{ cal/g} = 40.7 \text{ kJ/mol} = 2256 \text{ kJ/kg}. \tag{19-6}$$

When the phase change is from solid to liquid (then the sample must absorb heat) or from liquid to solid (then the sample must release heat), the heat of transformation is called the **heat of fusion** L_F. For water at its normal freezing or melting temperature,

$$L_F = 79.5 \text{ cal/g} = 6.01 \text{ kJ/mol} = 333 \text{ kJ/kg}. \tag{19-7}$$

Table 19-2 shows the heats of transformation for some substances.

SAMPLE PROBLEM 19.01

Heat to change temperature and state

(a) How much heat must be absorbed by ice of mass m = 720 g at –10°C to take it to the liquid state at 15°C?

KEY IDEAS

The heating process is accomplished in three steps: (1) The ice cannot melt at a temperature below the freezing point—so initially, any energy transferred to the ice as heat can only increase the temperature of the ice, until 0°C is reached. (2) The temperature then cannot increase until all the ice melts—so any energy transferred to the ice as heat now can only change ice to liquid water, until all the ice melts. (3) Now the energy transferred to the liquid water as heat can only increase the temperature of the liquid water.

Calculations:

Warming the ice: The heat Q_1 needed to take the ice from the initial T_i = –10°C to the final T_f = –0°C (so that the ice can then melt) is given by Eq. 19-3 ($Q = cm\,\Delta T$). Using the specific heat of ice c_{ice} in Table 19-1 gives us

$$\begin{aligned} Q_1 &= c_{ice} m(T_f - T_i) \\ &= (2220 \text{ J/kg} \cdot \text{K})(0.720 \text{ kg})[0°\text{C} - (-10°\text{C})] \\ &= 15\,984 \text{ J} \approx 15.98 \text{ kJ}. \end{aligned}$$

Melting the ice: The heat Q_2 needed to melt all the ice is given by Eq. 19-5 ($Q = Lm$). Here L is the heat of fusion L_F, with the value given in Eq. 19-7 and Table 19-2. We find

$$Q_2 = L_F m = (333 \text{ kJ/kg})(0.720 \text{ kg}) \approx 239.8 \text{ kJ}.$$

Warming the liquid: The heat Q_3 needed to increase the temperature of the water from the initial value T_i = 0°C to the final value T_f = 15°C is given by Eq. 19-3 (with the specific heat of liquid water c_{liq}):

$$\begin{aligned} Q_3 &= c_{liq} m(T_f - T_i) \\ &= (4186.8 \text{ J/kg} \cdot \text{K})(0.720 \text{ kg})(15°\text{C} - 0°\text{C}) \\ &= 45\,217 \text{ J} \approx 45.22 \text{ kJ}. \end{aligned}$$

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Total: The total required heat Q_{tot} is the sum of the amounts required in the three steps:

$$\begin{aligned} Q_{tot} &= Q_1 + Q_2 + Q_3 \\ &= 15.98 \text{ kJ} + 239.8 \text{ kJ} + 45.22 \text{ kJ} \\ &\approx 300 \text{ kJ}. \end{aligned} \quad \text{(Answer)}$$

Note that most of the energy goes into melting the ice rather than raising the temperature.

(b) If we supply the ice with a total energy of only 210 kJ (as heat), what are the final state and temperature of the water?

KEY IDEA

From step 1, we know that 15.98 kJ is needed to raise the temperature of the ice to the melting point. The remaining heat Q_{rem} is then 210 kJ – 15.98 kJ, or about 194 kJ. From step 2, we can see that this amount of heat is insufficient to melt all the ice. Because the melting of the ice is incomplete, we must end up with a mixture of ice and liquid; the temperature of the mixture must be the freezing point, 0°C.

Calculations: We can find the mass m of ice that is melted by the available energy Q_{rem} by using Eq. 19-5 with L_F:

$$m = \frac{Q_{rem}}{L_F} = \frac{194 \text{ kJ}}{333 \text{ kJ/kg}} = 0.583 \text{ kg} \approx 580 \text{ g}.$$

Thus, the mass of the ice that remains is 720 g – 580 g, or 140 g, and we have

580 g water and 140 g ice, at 0°C. (Answer)

19.4 | CALORIMETRY

Key Concept

- The amount of heat energy transfer (absorbed or released) of a material can be measured by a technique called calorimetry. The heat associated with the process is determined by measuring the change in temperature it causes in its surroundings by using an apparatus called a calorimeter.

To measure specific heat of a material, we heat a sample to some known temperature T_m, and keep it in a vessel containing water of known mass and temperature T_w, ($T_w < T$) and measuring the temperature of the water after equilibrium has been reached. This method is called *calorimetry*, and vessel in which this energy transfer occurs is called *calorimeter*. If the system of the sample, water, and calorimeter is isolated, and we assume that heat capacity of the calorimeter is negligible, then the amount of energy that leaves the sample (of unknown specific heat) must be equal the amount of energy that enters the water.

$$Q_{cold} = -Q_{hot} \quad (19\text{-}8)$$

Suppose m_m is the mass of sample 2 composed of a substance of unknown specific heat, c_m and initial temperature T_m. Likewise, let m_w, c_w, and T_w represent corresponding values for water. If T_f is the final equilibrium temperature after everything is mixed, Eq. 19-8 shows that the energy transfer for the water is $m_w c_w (T_f - T_w)$, which is positive because $T_f > T_w$, and that the energy transfer for the sample of unknown specific heat is $m_m c_m (T_f - T_m)$ which is negative. Substituting these expressions into Eq. 19-8 gives

$$m_w c_w (T_f - T_w) = -m_m c_m (T_f - T_m)$$

Solving for c_m gives

$$c_m = \frac{m_w c_w (T_f - T_w)}{m_m (T_m - T_f)}$$

SAMPLE PROBLEM 19.02

Hot slug in water, coming to equilibrium

A copper slug whose mass m_c is 75 g is heated in a laboratory oven to a temperature T of 312°C. The slug is then dropped into a glass beaker containing a mass $m_w = 220$ g of water. The heat capacity C_b of the beaker is 45 cal/K. The initial temperature T_i of the water and the beaker is 12°C. Assuming that the slug, beaker, and water are an isolated system and the water does not vaporize, find the final temperature T_f of the system at thermal equilibrium.

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KEY IDEAS

(1) Because the system is isolated, the system's total energy cannot change and only internal transfers of thermal energy can occur. (2) Because nothing in the system undergoes a phase change, the thermal energy transfers can only change the temperatures.

Calculations: To relate the transfers to the temperature changes, we can use Eqs. 19-2 and 19-3 to write

$$\text{for the water: } Q_w = c_w m_w (T_f - T_i); \quad (19\text{-}9)$$

$$\text{for the beaker: } Q_b = C_b (T_f - T_i); \quad (19\text{-}10)$$

$$\text{for the copper: } Q_c = c_c m_c (T_f - T). \quad (19\text{-}11)$$

Because the total energy of the system cannot change, the sum of these three energy transfers is zero:

$$Q_w + Q_b + Q_c = 0. \quad (19\text{-}12)$$

Substituting Eqs. 19-9 through 19-11 into Eq. 19-12 yields

$$c_w m_w (T_f - T_i) + C_b (T_f - T_i) + c_c m_c (T_f - T) = 0. \quad (19\text{-}13)$$

Temperatures are contained in Eq. 19-13 only as differences. Thus, because the differences on the Celsius and Kelvin scales are identical, we can use either of those scales in this equation. Solving it for T_f, we obtain

$$T_f = \frac{c_c m_c T + C_b T_i + c_w m_w T_i}{c_w m_w + C_b + c_c m_c}.$$

Using Celsius temperatures and taking values for c_c and c_w from Table 19-1, we find the numerator to be

$$(0.0923 \text{ cal/g} \cdot \text{K})(75 \text{ g})(312°\text{C}) + (45 \text{ cal/K})(12°\text{C}) + (1.00 \text{ cal/g} \cdot \text{K})(220 \text{ g})(12°\text{C}) = 5339.8 \text{ cal},$$

and the denominator to be

$$(1.00 \text{ cal/g} \cdot \text{K})(220 \text{ g}) + 45 \text{ cal/K} + (0.0923 \text{ cal/g} \cdot \text{K})(75 \text{ g}) = 271.9 \text{ cal/°C}.$$

We then have

$$T_f = \frac{5339.8 \text{ cal}}{271.9 \text{ cal/°C}} = 19.6°\text{C} \approx 20°\text{C}. \quad \text{(Answer)}$$

From the given data you can show that

$$Q_w \approx 1670 \text{ cal}, \quad Q_b \approx 342 \text{ cal}, \quad Q_c \approx -2020 \text{ cal}.$$

Apart from rounding errors, the algebraic sum of these three heat transfers is indeed zero, as required by the conservation of energy (Eq. 19-12).

19.5 | HEAT TRANSFER MECHANISMS

Key Concepts

- The rate P_{cond} at which energy is conducted through a slab for which one face is maintained at the higher temperature T_H and the other face is maintained at the lower temperature T_C is

$$P_{cond} = \frac{Q}{t} = kA\frac{T_H - T_C}{L}.$$

  Here each face of the slab has area A, the length of the slab (the distance between the faces) is L, and k is the thermal conductivity of the material.
- Convection occurs when temperature differences cause an energy transfer by motion within a fluid.
- Radiation is an energy transfer via the emission of electromagnetic energy. The rate P_{rad} at which an object emits energy via thermal radiation is

$$P_{rad} = \sigma \varepsilon A T^4,$$

  where σ (= 5.6704×10^{-8} W/m$^2 \cdot$K^4) is the Stefan–Boltzmann constant, ε is the emissivity of the object's surface, A is its surface area, and T is its surface temperature (in kelvins). The rate P_{abs} at which an object absorbs energy via thermal radiation from its environment, which is at the uniform temperature T_{env} (in kelvins), is

$$P_{abs} = \sigma \varepsilon A T_{env}^4.$$

There are three heat transfer mechanisms: conduction, convection, and radiation. Let's next examine these mechanisms in turn.

Conduction

If you leave the end of a metal poker in a fire for enough time, its handle will get hot. Energy is transferred from the fire to the handle by (thermal) **conduction** along the length of the poker. The vibration amplitudes of the atoms and

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electrons of the metal at the fire end of the poker become relatively large because of the high temperature of their environment. These increased vibrational amplitudes, and thus the associated energy, are passed along the poker, from atom to atom, during collisions between adjacent atoms. In this way, a region of rising temperature extends itself along the poker to the handle.

Consider a slab of face area A and thickness L, whose faces are maintained at temperatures T_H and T_C by a hot reservoir and a cold reservoir, as in Fig. 19-2. Let Q be the energy that is transferred as heat through the slab, from its hot face to its cold face, in time t. Experiment shows that the *conduction rate* P_{cond} (the amount of energy transferred per unit time) is

$$P_{cond} = \frac{Q}{t} = kA\frac{T_H - T_C}{L}, \tag{19-14}$$

in which k, called the *thermal conductivity*, is a constant that depends on the material of which the slab is made. A material that readily transfers energy by conduction is a *good thermal conductor* and has a high value of k. Table 19-3 gives the thermal conductivities of some common metals, gases, and building materials.

We assume a steady transfer of energy as heat.

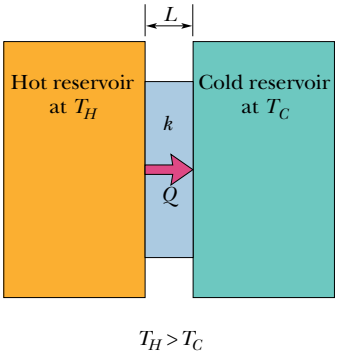

Figure 19-2 Thermal conduction. Energy is transferred as heat from a reservoir at temperature T_H to a cooler reservoir at temperature T_C through a conducting slab of thickness L and thermal conductivity k.

Thermal Resistance to Conduction (R-Value)

If you are interested in insulating your house or in keeping cola cans cold on a picnic, you are more concerned with poor heat conductors than with good ones. For this reason, the concept of *thermal resistance* R has been introduced into engineering practice. The R-value of a slab of thickness L is defined as

$$R = \frac{L}{k}. \tag{19-15}$$

The lower the thermal conductivity of the material of which a slab is made, the higher the R-value of the slab; so something that has a high R-value is a *poor thermal conductor* and thus a *good thermal insulator*.

Note that R is a property attributed to a slab of a specified thickness, not to a material. The commonly used unit for R (which, in the United States at least, is almost never stated) is the square foot – Fahrenheit degree – hour per British thermal unit ($ft^2 \cdot °F \cdot h/Btu$). (Now you know why the unit is rarely stated.)

Table 19-3 Some Thermal Conductivities

Substance	k (W/m·K)
Metals	
Stainless steel	14
Lead	35
Iron	67
Brass	109
Aluminum	235
Copper	401
Silver	428
Gases	
Air (dry)	0.026
Helium	0.15
Hydrogen	0.18
Building Materials	
Polyurethane foam	0.024
Rock wool	0.043
Fiberglass	0.048
White pine	0.11
Window glass	1.0

Conduction Through a Composite Slab

Figure 19-3 shows a composite slab, consisting of two materials having different thicknesses L_1 and L_2 and different thermal conductivities k_1 and k_2. The temperatures

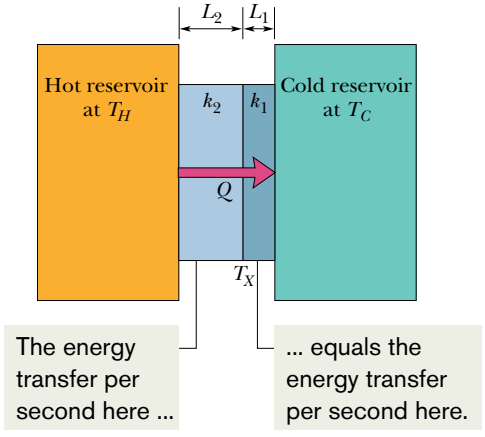

Figure 19-3 Heat is transferred at a steady rate through a composite slab made up of two different materials with different thicknesses and different thermal conductivities. The steady-state temperature at the interface of the two materials is T_X.

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of the outer surfaces of the slab are T_H and T_C. Each face of the slab has area A. Let us derive an expression for the conduction rate through the slab under the assumption that the transfer is a *steady-state* process; that is, the temperatures everywhere in the slab and the rate of energy transfer do not change with time.

In the steady state, the conduction rates through the two materials must be equal. This is the same as saying that the energy transferred through one material in a certain time must be equal to that transferred through the other material in the same time. If this were not true, temperatures in the slab would be changing and we would not have a steady-state situation. Letting T_X be the temperature of the interface between the two materials, we can now use Eq. 19-14 to write

$$P_{\text{cond}} = \frac{k_2 A(T_H - T_X)}{L_2} = \frac{k_1 A(T_X - T_C)}{L_1}. \tag{19-16}$$

Solving Eq. 19-16 for T_X yields, after a little algebra,

$$T_X = \frac{k_1 L_2 T_C + k_2 L_1 T_H}{k_1 L_2 + k_2 L_1}. \tag{19-17}$$

Substituting this expression for T_X into either equality of Eq. 19-16 yields

$$P_{\text{cond}} = \frac{A(T_H - T_C)}{L_1/k_1 + L_2/k_2}. \tag{19-18}$$

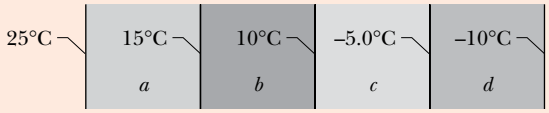
We can extend Eq. 19-18 to apply to any number n of materials making up a slab:

$$P_{\text{cond}} = \frac{A(T_H - T_C)}{\sum(L/k)}. \tag{19-19}$$

The summation sign in the denominator tells us to add the values of L/k for all the materials.

CHECKPOINT 2

The figure shows the face and interface temperatures of a composite slab consisting of four materials, of identical thicknesses, through which the heat transfer is steady. Rank the materials according to their thermal conductivities, greatest first.



Conduction through a Cylinder

Consider a long cylinder of inside radius r_i, outside radius r_o, and length L, such as the one shown in Fig. 19-4. A hot fluid is flowing through the cylinder and its environment is at constant temperature T_{env}. Let's find rate of heat loss through the lateral surface. For a cylinder with length very much larger than diameter, it may be assumed that the heat flows only in a radial direction. Let's consider an elemental cylinder of radius r and thickness dr and length L, then the area for heat flow in the cylindrical system is

$$A_r = 2\pi rL$$

So that substituting in Eq. 19-14, $P = -kA_r(\Delta T/\Delta x)$ becomes

$$P = -2\pi krL\frac{dT}{dr}$$

P is going to be same for all elements once steady state reaches

$$\frac{2\pi kLdT}{P} = -\frac{dr}{r}$$

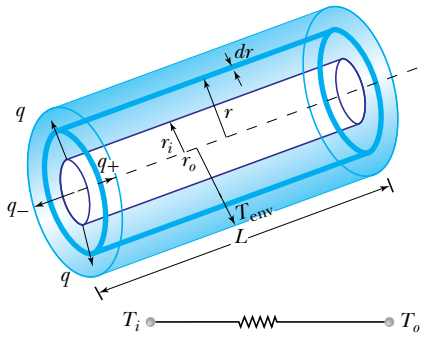


Figure 19-4 A long cylinder with hot fluid flowing through it, acts like a resistor.

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Integrating with the limits

$$T = T_i \text{ at } r = r_i$$
$$T = T_{\text{env}} \text{ at } r = r_o$$

We get,
$$P = \frac{2\pi kL(T_i - T_{\text{env}})}{\ln(r_o/r_i)}$$

and the thermal resistance in this case can be said to be

$$R_{\text{th}} = \frac{\ln(r_o/r_i)}{2\pi kL}$$

The thermal resistance concept may be used for multiple-layer cylindrical walls just as it was used for plane walls. For the three-layer system shown in Fig. 19-5*a* the solution is

$$P = \frac{2\pi L(T_1 - T_4)}{\ln(r_2/r_1)/k_A + \ln(r_3/r_2)/k_B + \ln(r_4/r_3)/k_C}$$

The thermal circuit is shown in Fig. 19-5*b*.

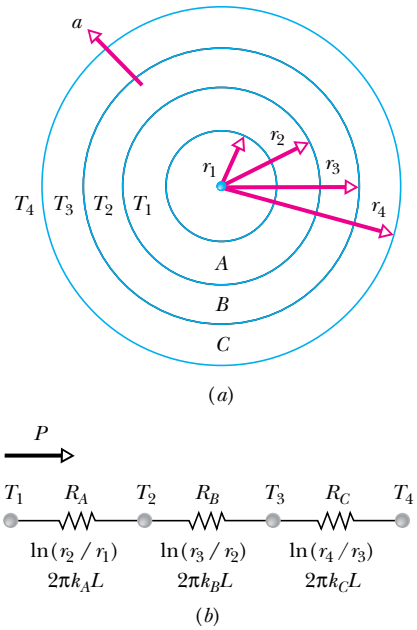

Figure 19-5 Multiple layer cylinders with hot liquid flowing through act like a resistors in series.

CHECKPOINT 3

Derive conduction rate of spherical system similar to the cylindrical system discussed above.

SAMPLE PROBLEM 19.03

Thermal conduction through a layered wall

Figure 19-6 shows the cross section of a wall made of white pine of thickness L_a and brick of thickness L_d (= $2.0L_a$), sandwiching two layers of unknown material with identical thicknesses and thermal conductivities. The thermal conductivity of the pine is k_a and that of the brick is k_d(= $5.0k_a$). The face area A of the wall is unknown. Thermal conduction through the wall has reached the steady state; the only known interface temperatures are $T_1 = 25°C$, $T_2 = 20°C$, and $T_5 = -10°C$. What is interface temperature T_4?

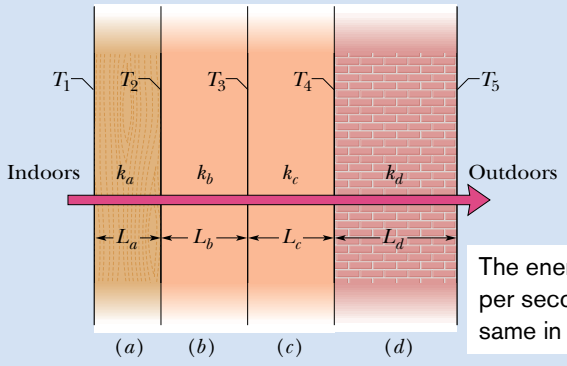

Figure 19-6 Steady-state heat transfer through a wall.

KEY IDEAS

(1) Temperature T_4 helps determine the rate P_d at which energy is conducted through the brick, as given by Eq. 19-14. However, we lack enough data to solve Eq. 19-14 for T_4. (2) Because the conduction is steady, the conduction rate P_d through the brick must equal the conduction rate P_a through the pine. That gets us going.

Calculations: From Eq. 19-14 and Fig. 19-6, we can write

$$p_a = k_a A\frac{T_1 - T_2}{L_a} \quad \text{and} \quad P_d = k_d A\frac{T_4 - T_5}{L_d}.$$

Setting $P_a = P_d$ and solving for T_4 yield

$$T_4 = \frac{k_a L_d}{k_d L_a}(T_1 - T_2) + T_5.$$

Letting $L_d = 2.0L_a$ and $k_d = 5.0k_a$, and inserting the known temperatures, we find

$$T_4 = \frac{k_a(2.0L_a)}{(5.0k_a)L_a}(25°C - 20°C) + (-10°C)$$
$$= -8.0°C. \qquad \text{(Answer)}$$

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Convection

When you look at the flame of a candle or a match, you are watching thermal energy being transported upward by **convection**. Such energy transfer occurs when a fluid, such as air or water, comes in contact with an object whose temperature is higher than that of the fluid. The temperature of the part of the fluid that is in contact with the hot object increases, and (in most cases) that fluid expands and thus becomes less dense. Because this expanded fluid is now lighter than the surrounding cooler fluid, buoyant forces cause it to rise. Some of the surrounding cooler fluid then flows so as to take the place of the rising warmer fluid, and the process can then continue.

Convection is part of many natural processes. Atmospheric convection plays a fundamental role in determining global climate patterns and daily weather variations. Glider pilots and birds alike seek rising thermals (convection currents of warm air) that keep them aloft. Huge energy transfers take place within the oceans by the same process. Finally, energy is transported to the surface of the Sun from the nuclear furnace at its core by enormous cells of convection, in which hot gas rises to the surface along the cell core and cooler gas around the core descends below the surface.

Radiation

The third method by which an object and its environment can exchange energy as heat is via electromagnetic waves (visible light is one kind of electromagnetic wave). Energy transferred in this way is often called **thermal radiation** to distinguish it from electromagnetic *signals* (as in, say, television broadcasts) and from nuclear radiation (energy and particles emitted by nuclei). (To "radiate" generally means to emit.) When you stand in front of a big fire, you are warmed by absorbing thermal radiation from the fire; that is, your thermal energy increases as the fire's thermal energy decreases. No medium is required for heat transfer via radiation—the radiation can travel through vacuum from, say, the Sun to you.

The rate P_{rad} at which an object emits energy via electromagnetic radiation depends on the object's surface area A and the temperature T of that area in kelvins and is given by

$$P_{rad} = \sigma \varepsilon A T^4. \tag{19-20}$$

Here $\sigma = 5.6704 \times 10^{-8}\ \mathrm{W/m^2 \cdot K^4}$ is called the *Stefan–Boltzmann constant* after Josef Stefan (who discovered Eq. 19-20 experimentally in 1879) and Ludwig Boltzmann (who derived it theoretically soon after). The symbol ε represents the *emissivity* of the object's surface, which has a value between 0 and 1, depending on the composition of the surface. A surface with the maximum emissivity of 1.0 is said to be a *blackbody radiator*, but such a surface is an ideal limit and does not occur in nature. Note again that the temperature in Eq. 19-20 must be in kelvins so that a temperature of absolute zero corresponds to no radiation. Note also that every object whose temperature is above 0 K—including you—emits thermal radiation. (See Fig. 19-7.)

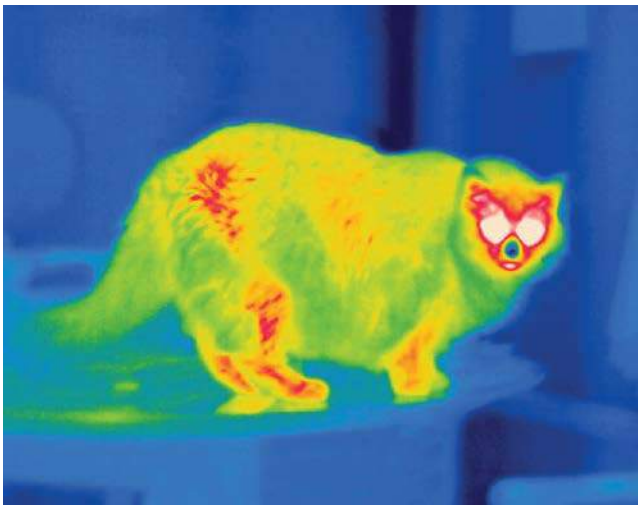
The rate P_{abs} at which an object absorbs energy via thermal radiation from its environment, which we take to be at uniform temperature T_{env} (in kelvins), is

$$P_{abs} = \sigma \varepsilon A T_{env}^4. \tag{19-21}$$

The emissivity ε in Eq. 19-21 is the same as that in Eq. 19-20. An idealized blackbody radiator, with $\varepsilon = 1$, will absorb all the radiated energy it intercepts (rather than sending a portion back away from itself through reflection or scattering).

Because an object both emits and absorbs thermal radiation, its net rate P_{net} of energy exchange due to thermal radiation is

$$P_{net} = P_{abs} - P_{rad} = \sigma \varepsilon A (T_{env}^4 - T^4). \tag{19-22}$$



Edward Kinsman/Photo Researchers, Inc.

Figure 19-7 A false-color thermogram reveals the rate at which energy is radiated by a cat. The rate is color-coded, with white and red indicating the greatest radiation rate. The nose is cool.

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P_{net} is positive if net energy is being absorbed via radiation and negative if it is being lost via radiation.

Thermal radiation is involved in the numerous medical cases of a *dead* rattlesnake striking a hand reaching toward it. Pits between each eye and nostril of a rattlesnake (Fig. 19-8) serve as sensors of thermal radiation. When, say, a mouse moves close to a rattlesnake's head, the thermal radiation from the mouse triggers these sensors, causing a reflex action in which the snake strikes the mouse with its fangs and injects its venom. The thermal radiation from a reaching hand can cause the same reflex action even if the snake has been dead for as long as 30 min because the snake's nervous system continues to function. As one snake expert advised, if you must remove a recently killed rattlesnake, use a long stick rather than your hand.

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Figure 19-8 A rattlesnake's face has thermal radiation detectors, allowing the snake to strike at an animal even in complete darkness.

Behavior of Bodies During Heat Radiation

When radiant energy falls on a body, a part of the energy is absorbed, a part is reflected, and remaining part is transmitted. If we denote by a, r, and t the fractions absorbed, reflected, and transmitted, respectively, then

$$a + r + t = 1$$

We can use the parameters a, r, and t to classify substances in terms of their behavior toward radiant energy.

For a good absorber, $a > (r + t)$, and for a near perfect absorber, $a \gg (r + t)$,

For a good reflector, $r > (a + t)$, and for a near perfect reflector, $r \gg (a + t)$,

For a good absorber, $t > (a + r)$, and for a near perfect transmitter, $t \gg (a + r)$.

The nature of surfaces of bodies decides the behavior of bodies toward radiation. It turns out that shiny surfaces are good reflectors, while black surfaces are good absorbers of heat. Surfaces which are transparent to light are also transparent to heat, and are hence good transmitters of heat.

Prevost's Theory of Exchanges

Let's imagine a small body A, at temperature T_A being suspended by a nonconducting thread inside the evacuated box B whose walls are maintained at a constant, different temperature T_B as shown in Fig. 19-9. It is very clear that energy exchange between A and B can occur only by radiation. Also, if $T_A > T_B$, A's temperature falls until it is equal to T_B, but if $T_A < T_B$, it rises to T_B. Thus, finally in any case A acquires B's temperature and it may appear that energy exchange then stops.

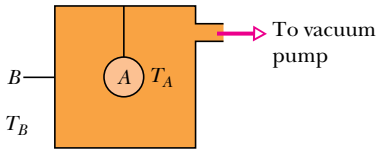

Figure 19-9 Small body maintained at constant temperature.

Prevost suggested in 1792 that when a body is at the same temperature as its surroundings its rate of emission of radiation to the surroundings equals its rate of absorption of radiation from the surroundings. That is, there is dynamic equilibrium and energy exchange continues, at a rate depending on the temperature. Thus, a body which is a good absorber of radiation must also be a good emitter of radiation otherwise its temperature would rise above that of its surroundings. Conversely, a good emitter must be a good absorber. Experiments confirm these conclusions and indicate that a dull, black surface is the best absorber and also the best emitter.

Simultaneous Emission and Absorption of Thermal Radiation

Any object located in environment is simultaneously emitting and absorbing thermal radiation and net rate of heat flow due to thermal radiation is given by $P_{net} = P_{emit} - P_{abs}$. Suppose an object with surface area A and temperature T

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is exposed to thermal radiation coming from its surroundings in all directions that are at a uniform temperature T_{env}. Then the net rate of heat flow due to thermal radiation is

$$P_{net} = \varepsilon\sigma A T^4 - \varepsilon\sigma A T_{evn}^4 = \varepsilon\sigma A(T^4 - T_{evn}^4) \tag{19-23}$$

A body emits energy even if it is at the same temperature as its surroundings; it just emits at the same rate at which it absorbs, so $P_{net} = 0$. If $T > T_{env}$, the object emits more thermal radiation that it absorbs. If $T < T_{env}$, the object absorbs more thermal radiation than it emits.

The rate of absorption is proportional to the emissivity because a good emitter is also a good absorber. The emissivity ε measures not only how much the object emits compared to a blackbody but it also measures how much the object absorbs compared to a blackbody. A blackbody at the same temperature as its surroundings would have to absorb radiation at the rate $P_{abs} = \sigma A T_{env}^4$ to exactly balance the rate of emission. However, emissivity does depend on temperature. Equation 19-23 assumes the emissivity at temperature T is the same as the emissivity at temperature T_{env}. If T and T_{env} are very different, we would have to modify Eq. 19-23 to use two different emissivities.

(*Caution*: Do not substitute temperature in Celsius degrees into Eq. 19-23)

SAMPLE PROBLEM 19.04

Rate of fall in temperature of a black body

A copper cube with sides of length 20 mm had its surface blackened. The cube is heated up to 500 K and placed inside a room at a constant temperature of 300 K. Calculate the initial rate of temperature fall of the cube, assuming it as a perfect black-body. (Stefan's constant = 5.7×10^{-8} W/m^2/K^4 specific heat capacity of aluminum = 400 J/kg/K; density of aluminum = 7800 kg/m^3) where A is the surface area of the cube, T_1 is the temperature of the body, and T_2 is the temperature of the enclosure.

KEY IDEA

By the Prevost's theory of heat exchange, when a body is placed in an environment, the body absorbs heat from the environment and also radiates heat to it. The net heat radiated per second is given by

$$P = \sigma A(T_1^4 - T_2^4) \tag{19-24}$$

Calculations: For the cube of side a, the area emitting radiation is, $A = 6a^2$. Thus, rate of energy radiated per second $\Delta Q/\Delta t$ is

$$\frac{\Delta Q}{\Delta t} = mc\frac{\Delta T}{\Delta t} = \sigma 6a^2(T_1^4 - T_2^4) \tag{19-25}$$

where $\Delta T/\Delta t$ is the rate of fall of temperature, m is the mass of the cube ($m = \rho V$), ρ the density of aluminum, V the volume of the cube, and c is the specific heat capacity of the metal.

We set $V = a^3$. From Eq. 19-25, we have

$$\frac{\Delta T}{\Delta t} = \frac{\sigma 6a^2(T_1^4 - T_2^4)}{mc} = \frac{\sigma 6a^2(T_1^4 - T_2^4)}{\rho a^3 c}$$

which gives

$$\frac{\Delta T}{\Delta t} = \frac{\sigma 6(T_1^4 - T_2^4)}{\rho a c} \tag{19-26}$$

Substituting the given data into Eq. 19-26 we get the initial rate of fall of temperature as

$$\frac{\Delta T}{\Delta t} = \frac{6 \times 5.7 \times 10^{-8}\,\text{W/m}^2/\text{K}^4 \times (500^4 - 300^4)\,\text{K}^4}{7.8 \times 10^3\,\text{kg/m}^3 \times 0.020\,\text{m} \times 400\,\text{J/kgK}}$$

$$= 0.3\,\text{K/s}$$

The rate of heat loss after this will be decreasing with time as temperature decreases.

SAMPLE PROBLEM 19.05

Thermal radiation by a skunk cabbage can melt surrounding snow

Unlike most other plants, a special type of cabbage called skunk cabbage can regulate its internal temperature (set at $T = 22°\text{C}$) by altering the rate at which it produces energy. If it becomes covered with snow, it can increase that production so that its thermal radiation melts the snow in order to re-expose the plant to sunlight. Let's model a skunk cabbage with a cylinder of height $h = 5.0$ cm and radius $R = 1.5$ cm and assume it is surrounded by a

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snow wall at temperature $T_{env} = -3.0°C$ (Fig. 19-10). If the emissivity ε is 0.80, what is the net rate of energy exchange via thermal radiation between the plant's curved side and the snow?

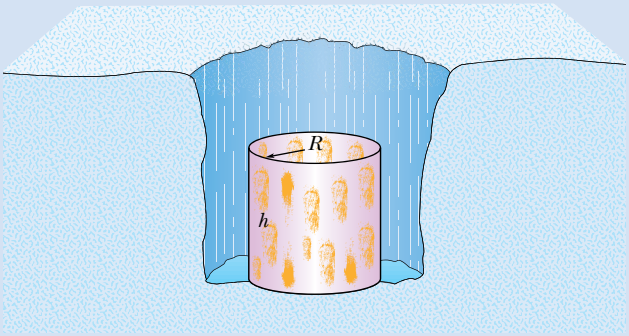

Figure 19-10 Model of skunk cabbage that has melted snow to uncover itself.

KEY IDEAS

(1) In a steady-state situation, a surface with area A, emissivity ε, and temperature T loses energy to thermal radiation at the rate given by Eq. 19-20 ($P_{rad} = \sigma\varepsilon A T^4$). (2) Simultaneously, it gains energy by thermal radiation from its environment at temperature T_{env} at the rate given by Eq. 19-21 ($P_{abs} = \sigma\varepsilon A T_{env}^4$).

Calculations: To find the net rate of energy exchange, we subtract Eq. 19-20 from Eq. 19-21 to write

$$P_{net} = P_{abs} - P_{rad}$$
$$= \sigma\varepsilon A(T_{env}^4 - T^4). \qquad (19\text{-}27)$$

We need the area of the curved surface of the cylinder, which is $A = h(2\pi R)$. We also need the temperatures in kelvins: $T_{env} = 273\ K - 3\ K = 270\ K$ and $T = 273\ K + 22\ K = 295\ K$. Substituting in Eq. 19-27 for A and then substituting known values in SI units (which are not displayed here), we find

$$P_{net} = (5.67 \times 10^{-8})(0.80)(0.050)(2\pi)(0.015)(270^4 - 295^4)$$
$$= -0.48\ W. \qquad \text{(Answer)}$$

Thus, the plant has a net loss of energy via thermal radiation of 0.48 W. The plant's energy production rate is comparable to that of a hummingbird in flight.

19.6 | APPLICATIONS OF STEFAN'S LAW

Key Concepts

- For an object having small temperature difference with surroundings the rate of fall is directly proportional to its temperature difference with surrounding.

$$-\frac{dT}{dt} = k(T - T_{env})$$

- Total energy emitted by the Earth as a sphere is $\sigma T_{env}^4 \left(4\pi R_E^2\right)$, as heat emitted will be from the entire surface of the Earth.

Rate of Fall in Temperature for Small Temperature Difference with Surroundings

For a given object rate of heat loss due to radiation only can be written as

$$-mc\frac{dT}{dt} = \sigma A(T^4 - T_{env}^4)$$
$$-\frac{dT}{dt} = \frac{\sigma A}{mc}(T^4 - T_{env}^4)$$

We can write, $T = T_{env} + \Delta T$

$$-\frac{dT}{dt} = \frac{\sigma A}{mc}\left[(T_{env} + \Delta T)^4 - T_{env}^4\right]$$
$$= \frac{\sigma A T_{env}^4}{mc}\left[\left(1 + \frac{\Delta T}{T_{env}}\right)^4 - 1\right]$$

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If $\Delta T \ll T_{env}$ (in kelvin)

$$-\frac{dT}{dt} = \frac{\sigma A T_{env}^4}{mc}\left[1+\frac{4\Delta T_{env}^3}{T_{env}}-1\right]$$

$$= \frac{4\sigma A T_{env}^3}{mc}(T - T_{env})$$

$$= k(T - T_{env}), \text{ where } k = \frac{4TAT_{env}^3}{mc}$$

For an object losing heat only due to radiation, the rate of fall in temperature is directly proportional to its temperature difference with surrounding provided temperature difference is small as compared to surrounding temperature which is expressed in kelvin.

Approximate Surface Temperature of the Sun

Sun supplies the radiant energy to our solar system. Using Stefan's law, and considering a planet in thermal equilibrium, it is possible to estimate the temperature of the Sun. We define a quantity known as the solar constant (SC) as the radiant energy falling per unit time on unit area at right angles to a sunbeam.

Let's find temperature of the Sun assuming we know:

1. The surface area of the Sun $(4\pi R_S^2)$ is $6.07\times10^{18}\,m^2$.
2. The radius of the Earth's orbit round the Sun (assumed circular) is $R_O = 1.1488\times10^{11}\,m$.
3. Earth's ambient temperature is 300 K.

The total power P (energy per second) radiated by the Sun is

$$P_{\text{emit by Sun}} = \sigma T_S^4 \times 4\pi R_S^2$$

$$= 3.763 \times 10^{26}\ W$$

Heat available per unit area of the Earth's orbit with radius R_O is given by

$$\frac{P_{\text{emit by Sun}}}{4\pi R_O^2} = SC_E$$

On calculating solar constant for Earth, SC_E, we get the value as 1350 W/m^2.

$$P_{\text{abs by Earth}} = \left[\frac{P_{\text{emit by Sun}}}{4\pi R_O^2}\right]\times \pi R_E^2$$

Exposed area is πr^2 and not $2\pi r^2$. Imagine if a small sphere Earth is put in the imaginary sphere of radius R_O then only πR_E^2 area becomes occupied (Fig. 19-11).

Total energy emitted by the Earth as a sphere is $\sigma T_{env}^4(4\pi R_E^2)$, as heat emitted will be from the entire surface of the Earth.

For thermal equilibrium energy received per unit time must be equal to the energy emitted per unit time.

So,

$$\frac{\sigma T_S^4 \times 4\pi R_S^2}{4\pi R_O^2}\times \pi R_E^2 = \sigma T_{env}^4(4\pi R_E^2)$$

Solving for T_S we get $T_S = 5743$ K.

This value is only approximate, as the distance cannot be determined accurately.

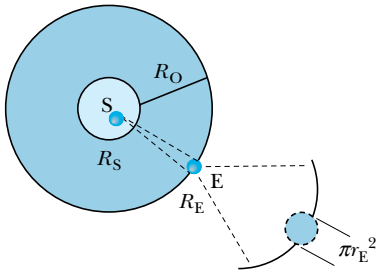

Figure 19-11 Earth receiving small portion of the energy radiated by the Sun.

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SAMPLE PROBLEM 19.06

Finding surface temperature

A space station is covered by an envelope which is a blackened shell. The temperature of the envelope is $T = 500$ K that is constant due to the operation of appliances of the station. Determine the temperature of the shell if the station is enveloped by a thin spherical black screen of nearly the same radius as the radius of the shell. Assume that there are no radiations falling on this space station.

KEY IDEAS

The total amount of heat Q emitted in space per unit time remains unchanged since it is determined by the energy liberated during the operation of the appliances of the station. As given in the question, the appliances produce heat at constant rate.

Since only the outer surface of the screen emits into space (this radiation depends only on its temperature), the temperature of the screen must be equal to the initial temperature $T = 500$ K of the station. Also, it is important to remember that the screen emits the same amount of heat Q inward. This radiation reaches the envelope of the station and is absorbed by it.

Calculations: Therefore, the total amount of heat supplied to the station per unit time is the sum of the heat Q liberated during the operation of the appliances and the amount of heat Q absorbed by the inner surface of the screen, that is, equal to $2Q$. According to the heat balance condition, the same amount of heat must be emitted,

$$2Q = A\sigma T_x^4$$

and hence

$$\frac{Q}{2Q} = \frac{T^4}{T_x^4}$$

where T_x is the required temperature of the envelope of the station. Finally, we obtain

$$T_x = \sqrt[4]{2}T \equiv 600 \text{ K}$$

The temperature of the envelope has increased because to gain in thermal equilibrium it must emit more energy than before as it is receiving more energy than before.

19.7 | WIEN'S DISPLACEMENT LAW

Key Concept

- According to Wien's displacement law, the peak of the wavelength distribution shifts to shorter wavelengths as the temperature increases. This relationship is described by the following equation

$$\lambda_{max} T = 2.88 \times 10^{-3} \text{ m} \cdot \text{K}$$

where λ_{max} max is the wavelength at which the curve peaks and T is the absolute temperature of the surface of the object emitting the radiation.

The heat radiation emitted by a body consists of different wavelengths. But some wavelengths contribute much more than the others in total radiation energy. These wavelengths are distributed continuously over a range. In Fig. 19-12*a* graph of intensity versus wavelength has been plotted using experimental data for a body at three different temperatures. We can see that intensity is not same for all wavelengths. These graphs at a given temperature are same for all black bodies. As the temperature of the body increases, the wavelength of dominant region decreases. At around 1100 K, the radiation has a good contribution from red region of wavelengths and the object appears red. At temperatures around 3000 K, the radiation contains enough shorter wavelengths and the object appears white.

The peak of the wavelength distribution shifts to shorter wavelengths as the temperature increases. This behavior is described by the following relationship, called Wien's displacement law:

$$\lambda_{max} T = 2.88 \times 10^{-3} \text{ m} \cdot \text{K}$$

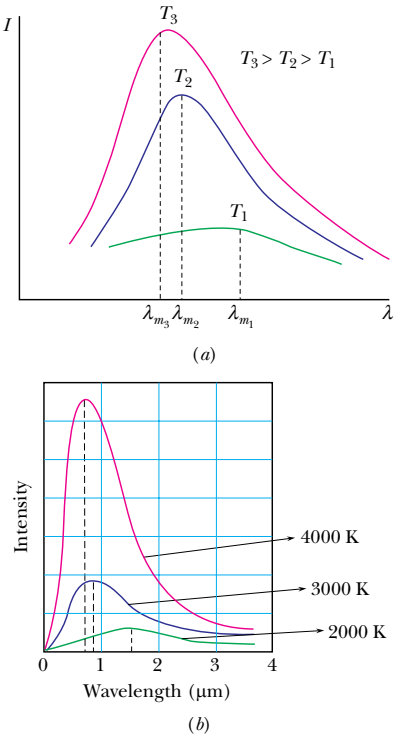

Figure 19-12 (*a*) General variation of intensity with wavelength at different temperatures. (*b*) Variation of intensity with wavelength at different temperatures for black body.

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where λ_{max} max is the wavelength at which the curve peaks and T is the absolute temperature of the surface of the object emitting the radiation. The wavelength at the curve's peak is inversely proportional to the absolute temperature; that is, as the temperature increases, the peak is "displaced" to shorter wavelengths (Fig. 19-12*b*).

Wien's displacement law is consistent with the behavior of the object mentioned at the beginning of this section. At room temperature, the object does not appear to glow because the peak is in the infrared region of the electromagnetic spectrum. At higher temperatures, it glows red because the peak is in the near infrared with some radiation at the red end of the visible spectrum, and at still higher temperatures, it glows white because the peak is in the visible region so that all colors are emitted.

Thus, we can summarize

1. For all black bodies, radiation spectrum at a given temperature is same.
2. Black body emits all wavelength from 0 to ∞.
3. Area bound by $E_\lambda - \lambda$ curve and λ axis is the emissive power.

$$\int_0^\infty E_\lambda d\lambda = E$$

So, this area is directly proportional to the fourth power of absolute temperature.

4. With increase in temperature, energy associated with each wavelength increases.
5. Wavelength that corresponds to maximum spectral emissive power (λ_{max}) decreases with increase in temperature. This decrease is according to Wien's law.

$$\lambda_{max} \propto \frac{1}{T}$$

SAMPLE PROBLEM 19.07

Visible range

The maximum rate of emissive power from the Sun occurs in the middle of the visible range at about λ = 0.5 μm. Estimate the temperature of the Sun's surface. The evolutionists suggest this as the reason of eyes being sensitive to a particular band of the spectrum known as visible range.

KEY IDEA

We assume the Sun to be a blackbody. Then the wavelength of maximum emission and the surface temperature are related by Wien's law.

Calculation: Given: $\lambda_{max} = 0.5\ \mu m = 5 \times 10^{-3}\ mK$

Then from Wien's law, we know that the product of the wavelength for maximum power emission is

$$\lambda_{max} T = 2.898 \times 10^{-3}\ mK$$

We can solve for the temperature since we know λ_{max}

$$T = \frac{2.898 \times 10^{-3}\ mK}{5 \times 10^{-7}\ m}$$

$$= 6000\ K \qquad \text{(Answer)}$$

19.8 | NEWTON'S LAW OF COOLING

Key Concepts

- According to Newton's law of cooling the rate of heat loss from a body is directly proportional to the excess temperature of the body over the surroundings. It can be written as

$$-\frac{\Delta Q}{\Delta t} = \text{constant} \times (T - T_{env})$$

where T_{env} is the temperature of the surroundings, and T is the temperature of the body.

- The negative slope obtained by the cooling curve is proportional to the rate of heat loss, all other quantities remain unaltered.

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Newton did an extensive study into cooling of hot bodies by convection and radiation, he formulated Newton's law of cooling, as the conclusion of his research. It states that the rate of heat loss from a body is directly proportional to the excess temperature of the body over the surroundings.

If T_{env} is the temperature of the surroundings, and T is the temperature of the body, and we represent the rate of heat loss by $-\Delta Q/\Delta t$, Newton's law of cooling can written as

$$-\frac{\Delta Q}{\Delta t} = \text{constant} \times (T - T_{env})$$

The value of the constant of proportionality depends on the area of surface exposed to the surrounding air. The larger the exposed area, the faster the body lose heat. The negative sign expresses the fact that the temperature decreases with time. Let's call the constant of proportionality as D. The result looks same as the Eq. 19-23 but the value of constants are different as the Newton's law of cooling takes both radiation and convection into account.

$$-mc\frac{dT}{dt} = D(T - T_{env})$$

$$-\frac{dT}{dt} = \frac{D}{mc}(T - T_{env})$$

Let's say

$$k = \frac{D}{mc}$$

Then,

$$\int \frac{dT}{T - T_{env}} = \int k\,dt$$

$$\ln(T - T_{env}) = -kt + C$$

$$\ln\left(\frac{T - T_{env}}{T_{in} - T_{env}}\right) = -kt$$

$$T - T_{env} = (T_{in} - T_{env})e^{-kt}$$

$$T = T_{env} + (T_{in} - T_{env})e^{-kt}$$

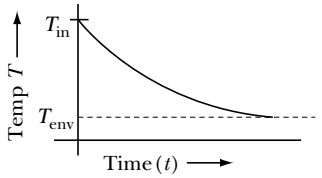

Figure 19-13 Cooling curve based on Newton's law of cooling.

Plotting temperature of the body versus time we get, what is known as cooling curve (Fig. 19-13).

Cooling Curves

A laboratory experiment to verify Newton's law of cooling was carried out by allowing a hot solid to cool in a current of air. Temperatures of the solid were taken at regular intervals of one minute until the solid cooled to about 10°C above room temperature. A graph of temperature against time was plotted. This plot is called a cooling curve. A typical cooling curve is shown in Fig. 19-14*a*.

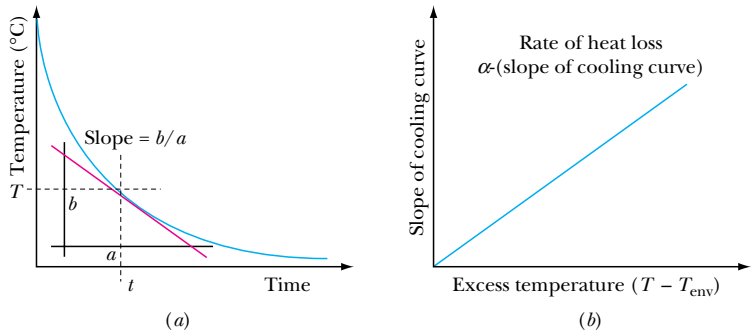

Figure 19-14 (a) Cooling curve. (b) Plot to verify Newton's law of cooling.

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From points along the cooling curve, tangents were drawn an, and the slope of each was calculated from the ratio b/a, as shown in Fig. 19-14*a*. The negative of this slope is proportional to the rate of heat loss, all other quantities (like mass, surface area, etc.) remaining unaltered. A plot of the negative slope against the temperature excess, as shown in Fig. 19-13*b*, is a straight line passing through the origin, thus verifying Newton's law of cooling.

A simple cooling curve can be obtained experimentally by heating a beaker of water to 80°C, say, and allowing it to cool to room temperature while taking readings of its temperature at regular intervals.

SAMPLE PROBLEM 19.08

Newton's law of cooling

An object at temperature 70°C is placed in a surrounding medium of temperature 30°C. Its temperature decreases by 10°C in first 5 min. Find the time interval from starting in which its temperature reduces to 40°C. Assume Newton's law to be valid.

$$t = 0 \quad T = 70°C \quad T_{env} = 30°C$$
$$t = 5 \quad T = 60°C \quad T_{env} = 30°C$$
$$\text{time} = t \quad T = 40°C \quad T_{env} = 30°C$$

Calculations:

Method 1 (approximate): From Newton's law of cooling

$$-\frac{dT}{dt} = k(T - T_{env})$$

This equation can be interpreted approximately such that

$$-\frac{dT}{dt} = -\frac{(T_f - T_i)}{t_f - t_i}$$

On substituting values, we get

$$-\frac{dT}{dt} = -\frac{(60°C - 70°C)}{5\ \text{min}}$$

In place of instantaneous temperature T we can write T_{avg}.

Thus, $$T = \left(\frac{70°C + 60°C}{2\ \text{min}}\right)$$

Therefore, $$k(T - T_{env}) = k\left(\frac{70°C + 60°C}{2\ \text{min}} - 30°C\right)$$

or

$$-\left(\frac{60°C - 70°C}{5\ \text{min}}\right) = k\left(\frac{132°C}{2\ \text{min}} - 30°C\right)$$

Solving we get, $$k = \frac{2}{35}$$

Now applying same ideas for the change from 60°C to 40°C, we get

$$-\left(\frac{40°C - 60°C}{t_1}\right) = \frac{2}{35}\left(\frac{40°C + 60°C}{2} - 30°C\right)$$

Solving we get, $$\frac{20}{t_1} = \frac{2}{35} \times 20$$

or $$t_1 = 17.5\ \text{min}.$$

Thus, answer is 17.5 + 5 = 22.5.

Method 2 (exact): From Newton's law of cooling.

$$-\frac{dT}{dt} = k(T - T_{env})$$

Integrating we get,

$$\ln(T - T_{env}) = -kt + C$$

Using $t = 0, T = 70°C, T_{env} = 30°C$, we get

$$\ln(70 - 30) = C \Rightarrow C = \ln 40$$
$$\ln(T - T_{env}) = -kt + \ln 40$$

Similarly using $t = 5, T = 60°C, T_{env} = 30°C$, we get

$$\ln 30 = -5k + \ln 40$$
$$5k = \ln\frac{4}{3}$$
$$k = \frac{1}{5}\ln\frac{4}{3}$$

Using $t = t_1, T = 40°C, T_{env} = 30°C$, we get

$$\ln(40 - 30) = -\frac{1}{5}\ln\frac{4}{3}t_1 = \ln 40$$
$$\frac{\ln(4/3)}{5}t_1 = \ln\left(\frac{40}{10}\right) = \ln 4$$
$$t_1 = \frac{5\ln 4}{\ln 4 - \ln 3} = \frac{10\ln 2}{2\ln 2 - \ln 3}$$
$$= 24.1\ \text{min} \qquad \text{(Answer)}$$

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REVIEW AND SUMMARY

Heat Heat Q is energy that is transferred between a system and its environment because of a temperature difference between them. It can be measured in **joules** (J), **calories** (cal), **kilocalories** (Cal or kcal), or **British thermal units** (Btu), with

$$1 \text{ cal} = 3.968 \times 10^{-3} \text{ Btu} = 4.1868 \text{ J}. \quad (19\text{-}1)$$

Heat Capacity and Specific Heat If heat Q is absorbed by an object, the object's temperature change $T_f - T_i$ is related to Q by

$$Q = C(T_f - T_i), \quad (19\text{-}2)$$

in which C is the **heat capacity** of the object. If the object has mass m, then

$$Q = cm(T_f - T_i), \quad (19\text{-}3)$$

where c is the **specific heat** of the material making up the object. The **molar specific heat** of a material is the heat capacity per mole, which means per 6.02×10^{23} elementary units of the material.

Heat of Transformation Matter can exist in three common states: solid, liquid, and vapor. Heat absorbed by a material may change the material's physical state—for example, from solid to liquid or from liquid to gas. The amount of energy required per unit mass to change the state (but not the temperature) of a particular material is its **heat of transformation** L. Thus,

$$Q = Lm. \quad (19\text{-}5)$$

The **heat of vaporization** L_V is the amount of energy per unit mass that must be added to vaporize a liquid or that must be removed to condense a gas. The **heat of fusion** L_F is the amount of energy per unit mass that must be added to melt a solid or that must be removed to freeze a liquid.

Calorimetry The amount of heat energy transfer (absorbed or released) of a material can be measured by a technique called calorimetry. The heat associated with the process is determined by measuring the change in temperature it causes in its surroundings by using an apparatus called a calorimeter.

The amount of energy that leaves the system (of unknown specific heat) must be equal the amount of energy that enters the system

$$Q_{\text{cold}} = -Q_{\text{hot}} \quad (19\text{-}8)$$

Conduction, Convection, and Radiation The rate P_{cond} at which energy is *conducted* through a slab for which one face is maintained at the higher temperature T_H and the other face is maintained at the lower temperature T_C is

$$P_{\text{cond}} = \frac{Q}{t} = kA\frac{T_H - T_C}{L} \quad (19\text{-}14)$$

Here each face of the slab has area A, the length of the slab (the distance between the faces) is L, and k is the thermal conductivity of the material.

Convection occurs when temperature differences cause an energy transfer by motion within a fluid.

Radiation is an energy transfer via the emission of electromagnetic energy. The rate P_{rad} at which an object emits energy via thermal radiation is

$$P_{\text{rad}} = \sigma\varepsilon AT^4, \quad (19\text{-}20)$$

where σ ($= 5.6704 \times 10^{-8}$ W/m$^2 \cdot$K^4) is the Stefan – Boltzmann constant, ε is the emissivity of the object's surface, A is its surface area, and T is its surface temperature (in kelvins). The rate P_{abs} at which an object absorbs energy via thermal radiation from its environment, which is at the uniform temperature T_{env} (in kelvins), is

$$P_{\text{abs}} = \sigma\varepsilon AT_{\text{env}}^4. \quad (19\text{-}21)$$

Wein's Displacement Law According to Wien's displacement law, the peak of the wavelength distribution shifts to shorter wavelengths as the temperature increases. This relationship is described by the following equation

$$\lambda_{\text{max}} T = 2.88 \times 10^{-3} \text{ m} \cdot \text{K}$$

where λ_{max} max is the wavelength at which the curve peaks and T is the absolute temperature of the surface of the object emitting the radiation.

Newton's Law of Cooling The rate of heat loss from a body is directly proportional to the excess temperature of the body over the surroundings. It can be written as

$$-\frac{\Delta Q}{\Delta t} = \text{constant} \times (T - T_{\text{env}})$$

where T_{env} is the temperature of the surroundings, and T is the temperature of the body.

PROBLEMS

1. An 8.00 g ice cube at its melting point is added to 130 cm^3 of tea (water) initially at 90.0°C in a well-insulated container. When equilibrium is reached, what has been the decrease in the tea's temperature?
2. A certain substance has a mass per mole of 50.0 g/mol. When 325 J is added as heat to a 30.0 g sample, the sample's temperature rises from 25.0°C to 45.0°C. What are the (a) specific heat and (b) molar specific heat of this substance? (c) How many moles are in the sample?
3. One way to keep the contents of a garage from becoming too cold on a night when a severe subfreezing temperature is forecast is to put a tub of water in the garage. If the mass of the water is 135 kg and its initial temperature is 20°C, (a) how much energy must the water transfer to its

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surroundings in order to freeze completely and (b) what is the lowest possible temperature of the water and its surroundings until that happens?

4. *Nonmetric version:* (a) How long does a 2.0×10^5 Btu/h water heater take to raise the temperature of 65 gal of water from 70°F to 100°F? *Metric version:* (b) How long does a 59 kW water heater take to raise the temperature of 246 L of water from 21°C to 38°C?

5. Samples A and B are at different initial temperatures when they are placed in a thermally insulated container and allowed to come to thermal equilibrium. Figure 19-15*a* gives their temperatures T versus time t. Sample A has a mass of 4.0 kg; sample B has a mass of 2.0 kg. Figure 19-15*b* is a general plot for the material of sample B. It shows the temperature change ΔT that the material undergoes when energy is transferred to it as heat Q. The change ΔT is plotted versus the energy Q per unit mass of the material, and the scale of the vertical axis is set by $\Delta T_s = 4.0$°C. What is the specific heat of sample A?

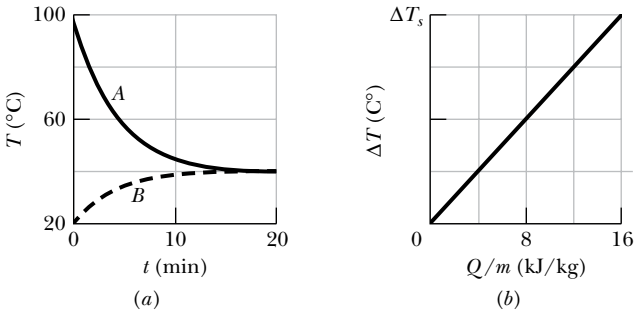

Figure 19-15 Problem 5.

6. A small electric immersion heater is used to heat 170 g of water for a cup of instant coffee. The heater is labeled "180 watts" (it converts electrical energy to thermal energy at this rate). Calculate the time required to bring all this water from 23.0°C to 100°C, ignoring any heat losses.

7. A 3.0 cm slab has formed on an outdoor tank of water (Fig. 19-16). The air is at –14°C. Find the rate of ice formation (centimeters per hour). The ice has thermal conductivity 0.0040 cal/s·cm·°C and density 0.92 g/cm³. Assume there is no energy transfer through the walls or bottom.

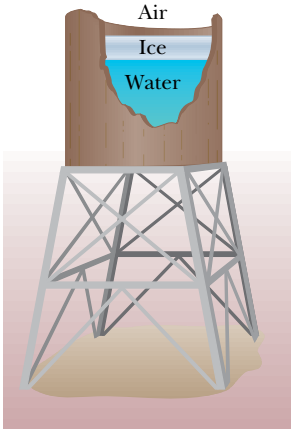

Figure 19-16 Problem 7.

8. The specific heat of a substance varies with temperature according to the function $c = 0.20 + 0.14T + 0.023T^2$, with T in °C and c in Cal/g·K. Find the energy required to raise the temperature of 1.0 g of this substance from 5.0°C to 15°C.

9. *Leidenfrost effect.* A water drop will last about 1 s on a hot skillet with a temperature between 100°C and about 200°C. However, if the skillet is much hotter, the drop can last several minutes, an effect named after an early investigator. The longer lifetime is due to the support of a thin layer of air and water vapor that separates the drop from the metal (by distance L in Fig. 19-17). Let $L = 0.0800$ mm, and assume that the drop is flat with height $h = 1.50$ mm and bottom face area $A = 5.00 \times 10^{-6}$ m². Also assume that the skillet has a constant temperature $T_s = 300$°C and the drop has a temperature of 100°C. Water has density $\rho = 1000$ kg/m³, and the supporting layer has thermal conductivity $k = 0.026$ W/m·K. (a) At what rate is energy conducted from the skillet to the drop through the drop's bottom surface? (b) If conduction is the primary way energy moves from the skillet to the drop, how long will the drop last?

Figure 19-17 Problem 9.

10. Ethyl alcohol has a boiling point of 78.0°C, a freezing point of –114°C, a heat of vaporization of 879 kJ/kg, a heat of fusion of 109 kJ/kg, and a specific heat of 2.43 kJ/kg·K. How much energy must be removed from 0.720 kg of ethyl alcohol that is initially a gas at 78.0°C so that it becomes a solid at –114°C?

11. Figure 19-18 shows the cross section of a wall made of three layers. The layer thicknesses are L_1, $L_2 = 0.750L_1$, and $L_3 = 0.350L_1$. The thermal conductivities are k_1, $k_2 = 0.900k_1$, and $k_3 = 0.800k_1$. The temperatures at the left side and right side of the wall are $T_H = 30.0$°C and $T_C = -15.0$°C, respectively. Thermal conduction is steady. (a) What is the temperature difference ΔT_2 across layer 2 (between the left and right sides of the layer)? If k_2 were, instead, equal to $1.1k_1$, (b) would the rate at which energy is conducted through the wall be greater than, less than, or the same as previously, and (c) what would be the value of ΔT_2?

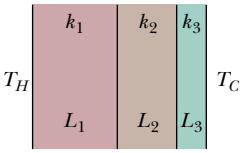

Figure 19-18 Problem 11.

12. Consider the slab shown in Fig. 19-2. Suppose that $L = 15.0$ cm, $A = 105$ cm², and the material is copper. If $T_H = 125$°C, $T_C = 10.0$°C, and a steady state is reached, find the conduction rate through the slab.

13. What mass of steam at 100°C must be mixed with 250 g of ice at its melting point, in a thermally insulated container, to produce liquid water at 50°C?

14. *Evaporative cooling of beverages.* A cold beverage can be kept cold even on a warm day if it is slipped into a porous ceramic container that has been soaked in water. Assume that energy lost to evaporation matches the net energy gained via the radiation exchange through the top and side surfaces. The container and beverage have temperature $T = 10$°C, the environment has temperature $T_{env} = 32$°C, and the container is a cylinder with radius $r = 2.2$ cm and height 10 cm. Approximate the emissivity

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as $\varepsilon = 1$, and neglect other energy exchanges. At what rate dm/dt is the container losing water mass?

15. A solid cylinder of radius $r_1 = 2.5$ cm, length $h_1 = 5.5$ cm, emissivity 0.85, and temperature 30°C is suspended in an environment of temperature 50°C. (a) What is the cylinder's net thermal radiation transfer rate P_1? (b) If the cylinder is stretched until its radius is $r_2 = 0.50$ cm, its net thermal radiation transfer rate becomes P_2. What is the ratio P_2/P_1?

16. Figure 19-19 shows (in cross section) a wall consisting of four layers, with thermal conductivities $k_1 = 0.060$ W/m·K, $k_3 = 0.040$ W/m·K, and $k_4 = 0.12$ W/m·K (k_2 is not known). The layer thicknesses are $L_1 = 1.2$ cm, $L_3 = 5.6$ cm, and $L_4 = 4.0$ cm (L_2 is not known). The known temperatures are $T_1 = 30$°C, $T_{12} = 25$°C, and $T_4 = -10$°C. Energy transfer through the wall is steady. What is interface temperature T_{34}?

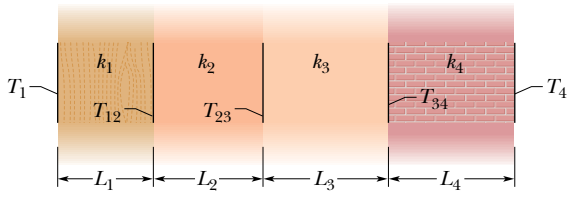

Figure 19-19 Problem 16.

17. A sphere of radius 0.350 m, temperature 27.0°C, and emissivity 0.850 is located in an environment of temperature 77.0°C. At what rate does the sphere (a) emit and (b) absorb thermal radiation? (c) What is the sphere's net change in energy in 3.50 min?

18. If you were to walk briefly in space without a spacesuit while far from the Sun (as an astronaut does in the movie *2001, A Space Odyssey*), you would feel the cold of space—while you radiated energy, you would absorb almost none from your environment. (a) At what rate would you lose energy? (b) How much energy would you lose in 44 s? Assume that your emissivity is 0.90, and estimate other data needed in the calculations.

19. Ice has formed on a shallow pond, and a steady state has been reached, with the air above the ice at –5.0°C and the bottom of the pond at 4.0°C. If the total depth of *ice + water* is 2.0 m, how thick is the ice? (Assume that the thermal conductivities of ice and water are 0.40 and 0.12 cal/m·°C·s, respectively.)

20. *Penguin huddling.* To withstand the harsh weather of the Antarctic, emperor penguins huddle in groups (Fig. 19-20). Assume that a penguin is a circular cylinder with a top surface area $a = 0.26$ m^2 and height $h = 90$ cm. Let P_r be the rate at which an individual penguin radiates energy to the environment (through the top and the sides); thus NP_r is the rate at which N identical, well-separated penguins radiate. If the penguins huddle closely to form a *huddled cylinder* with top surface area Na and height h, the cylinder radiates at the rate P_h. If $N = 1000$, (a) what is the value of the fraction P_h/NP_r and (b) by what percentage does huddling reduce the total radiation loss?

Alain Torterotot/Peter Arnold/Photolibrary

Figure 19-20 Problem 20.

21. The giant hornet *Vespa mandarinia japonica* preys on Japanese bees. However, if one of the hornets attempts to invade a beehive, several hundred of the bees quickly form a compact ball around the hornet to stop it. They do not sting, bite, crush, or suffocate it. Rather they overheat it by quickly raising their body temperatures from the normal 35°C to 47°C or 48°C, which is lethal to the hornet but not to the bees (Fig. 19-21). Assume the following: 500 bees form a ball of radius $R = 1.8$ cm for a time $t = 18$ min, the primary loss of energy by the ball is by thermal radiation, the ball's surface has emissivity $\varepsilon = 0.80$, and the ball has a uniform temperature. On average, how much additional energy must each bee produce during the 18 min to maintain 47°C?

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Figure 19-21 Problem 21.

22. Calculate the minimum amount of energy, in joules, required to completely melt 85.0 g of silver initially at 15.0°C.

23. What mass of butter, which has a usable energy content of 6.0 Cal/g (= 6000 cal/g), would be equivalent to the change in gravitational potential energy of a 78.0 kg man who ascends from sea level to the top of Mt. Everest, at elevation 8.84 km? Assume that the average g for the ascent is 9.80 m/s^2.

24 (a) Two 40 g ice cubes are dropped into 200 g of water in a thermally insulated container. If the water is initially at 25°C, and the ice comes directly from a freezer at –15°C, what is

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the final temperature at thermal equilibrium? (b) What is the final temperature if only one ice cube is used?

25. A 0.600 kg sample is placed in a cooling apparatus that removes energy as heat at a constant rate. Figure 19-22 gives the temperature T of the sample versus time t; the horizontal scale is set by $t_s = 80.0$ min. The sample freezes during the energy removal. The specific heat of the sample in its initial liquid phase is 3000 J/kg·K. What are (a) the sample's heat of fusion and (b) its specific heat in the frozen phase?

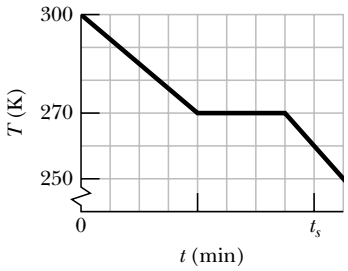

Figure 19-22 Problem 25.

26. A cylindrical copper rod of length 0.60 m and cross-sectional area 6.0 cm^2 is insulated along its side. The ends are held at a temperature difference of 100°C by having one end in a water–ice mixture and the other in a mixture of boiling water and steam. At what rate (a) is energy conducted by the rod and (b) does the ice melt?

27 A 0.485 kg sample of liquid water and a sample of ice are placed in a thermally insulated container. The container also contains a device that transfers energy as heat from the liquid water to the ice at a constant rate P, until thermal equilibrium is reached. The temperatures T of the liquid water and the ice are given in Fig. 19-23 as functions of time t; the horizontal scale is set by $t_s = 80.0$ min. (a) What is rate P? (b) What is the initial mass of the ice in the container? (c) When thermal equilibrium is reached, what is the mass of the ice produced in this process?

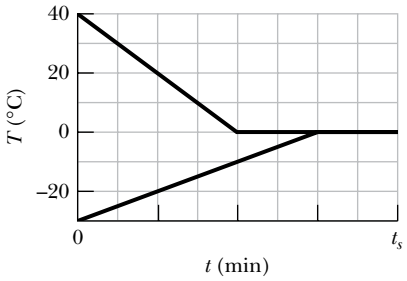

Figure 19-23 Problem 27.

28. In Fig. 19-24*a*, two identical rectangular rods of metal are welded end to end, with a temperature of $T_1 = 0$°C on the left side and a temperature of $T_2 = 100$°C on the right side. In 6.0 min, 43 J is conducted at a constant rate from the right side to the left side. How much time would be required to conduct 43 J if the rods were welded side to side as in Fig. 19-24*b*?

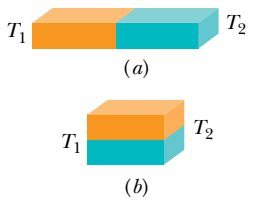

Figure 19-24 Problem 28.

29. In a solar water heater, energy from the Sun is gathered by water that circulates through tubes in a rooftop collector. The solar radiation enters the collector through a transparent cover and warms the water in the tubes; this water is pumped into a holding tank. Assume that the efficiency of the overall system is 25% (that is, 80% of the incident solar energy is lost from the system). What collector area is necessary to raise the temperature of 200 L of water in the tank from 20°C to 40°C in 1.0 h when the intensity of incident sunlight is 750 W/m^2?

30. A person makes a quantity of iced tea by mixing 250 g of hot tea (essentially water) with an equal mass of ice at its melting point. Assume the mixture has negligible energy exchanges with its environment. If the tea's initial temperature is $T_i = 90$°C, when thermal equilibrium is reached what are (a) the mixture's temperature T_f and (b) the remaining mass m_f of ice? If $T_i = 70$°C, when thermal equilibrium is reached what are (c) T_f and (d) m_f?

31. A 22.0 g copper ring at 0.000°C has an inner diameter of $D = 2.540\ 00$ cm. An aluminum sphere at 100.0°C has a diameter of $d = 2.545\ 08$ cm. The sphere is put on top of the ring (Fig. 19-25), and the two are allowed to come to thermal equilibrium, with no heat lost to the surroundings. The sphere just passes through the ring at the equilibrium temperature. What is the mass of the sphere?

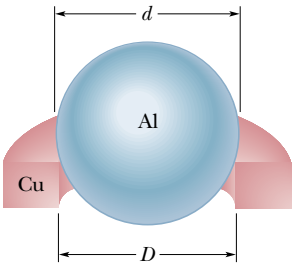

Figure 19-25 Problem 31.

32. A 150 g copper bowl contains 260 g of water, both at 20.0°C. A very hot 300 g copper cylinder is dropped into the water, causing the water to boil, with 5.00 g being converted to steam. The final temperature of the system is 100°C. Neglect energy transfers with the environment. (a) How much energy (in calories) is transferred to the water as heat? (b) How much to the bowl? (c) What is the original temperature of the cylinder?

33. A gas thermometer is constructed of two gas-containing bulbs, each in a water bath, as shown in Fig. 19-26. The pressure difference between the two bulbs is measured by a mercury manometer as shown. Appropriate reservoirs, not shown in the diagram, maintain constant gas volume in the two bulbs. There is no difference in pressure when both baths are at the triple point of water. The pressure difference is 120 torr when one bath is at the triple point and the other is at the boiling point of water. It is 80.0 torr when one bath is at the triple point and the other is at an unknown temperature to be measured. What is the unknown temperature?

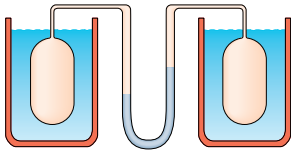

Figure 19-26 Problem 33.

34. The temperature of a body falls from 40°C to 36°C in 5 min when placed in a surrounding of constant temperature 16°C. Find the time taken for the temperature of the body to become 32°C.

35. The Earth receives solar radiation at a rate of 8.2 J/cm^2 min. Assuming that the Sun radiates like a blackbody, calculate

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the surface temperature of the Sun. The angle subtended by the Sun on the earth is 0.53° and the Stefan constant is $5.67 \times 10^{-8}\ W/m^2\ K^4$.

36. A hot body placed in air is cooled down according to Newton's law of cooling, the rate of decrease of temperature being k times the temperature difference from the surrounding. Starting from $t = 0$, find the time in which the body will lose half the maximum heat it can lose.

37. (a) Calculate the rate at which body heat is conducted through the clothing of a skier in a steady-state process, given the following data: the body surface area is 1.8 m^2, and the clothing is 1.0 cm thick; the skin surface temperature is 33°C and the outer surface of the clothing is at 1.0°C; the thermal conductivity of the clothing is 0.040 W/m·K. (b) If, after a fall, the skier's clothes became soaked with water of thermal conductivity 0.60 W/m·K, by how much is the rate of conduction multiplied?

38. Find the rate of heat flow through a cross-section of the rod shown in Fig. 19-27 ($\theta_2 > \theta_1$). Thermal conductivity of the material of the rod is K.

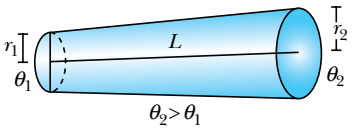

Figure 19-27 Problem 38.

39. A calorimeter contains 50 g of water at 50°C. The temperature falls to 45°C in 10 min. When the calorimeter contains 100 g of water at 50°C, it takes 18 min for the temperature to become 45°C. Find the water equivalent of the calorimeter.

40. A metal ball of mass 1 kg is heated by means of a 20 W heater in a room at 20°C. The temperature of the ball becomes steady at 50°C. (a) Find the rate of loss of heat to the surrounding when the ball is at 50°C. (b) Assuming Newtons's law of cooling, calculate the rate of loss of heat to the surrounding when the ball is at 30°C. (c) Assume that the temperature of the ball rises uniformly from 20°C to 30°C in 5 min. Find the total loss of heat to the surrounding during this period. (d) Calculate the specific heat capacity of the metal.

PRACTICE QUESTIONS

Single Correct Choice Type

1. A heat of transformation of a substance is
- **(a)** the energy absorbed as heat during a phase transformation.
- **(b)** the energy per unit mass absorbed as heat during a phase transformation.
- **(c)** the same as the heat capacity.
- **(d)** the same as the specific heat.

2. Heat is
- **(a)** energy transferred by virtue of a temperature difference.
- **(b)** energy transferred by macroscopic work.
- **(c)** energy content of an object.
- **(d)** a temperature difference.

3. An insulated container, filled with water, contains a thermometer and a paddle wheel. The paddle wheel can be rotated by an external source. This apparatus can be used to determine
- **(a)** specific heat of water.
- **(b)** relation between kinetic energy and absolute temperature.
- **(c)** thermal conductivity of water.
- **(d)** mechanical equivalent of heat.

4. For constant volume processes the heat capacity of gas A is greater than the heat capacity of gas B. We conclude that when they both absorb the same energy as heat at constant volume
- **(a)** the temperature of A increases more than the temperature of B.
- **(b)** the temperature of B increases more than the temperature of A.
- **(c)** the internal energy of A increases more than the internal energy of B.
- **(d)** the internal energy of B increases more than the internal energy of A.

5. The same energy Q enters five different substances as heat. Which of these has the greatest specific heat?
- **(a)** The temperature of 3 g of substance A increases by 10 K.
- **(b)** The temperature of 4 g of substance B increases by 4 K.
- **(c)** The temperature of 6 g of substance C increases by 15 K.
- **(d)** The temperature of 8 g of substance D increases by 5 K.

6. Take the mechanical equivalent of heat as 4 J/cal. A 10-gram bullet moving at 2000 m/s plunges into 1 kg of paraffin wax (specific heat 0.7 cal/(g°C). The wax was initially at 20°C. Assuming that all the bullet's energy heats the wax, its final temperature is
- **(a)** 20.14°C **(b)** 23.5°C
- **(c)** 20.006°C **(d)** 27.1°C

7. The specific heat of lead is 0.030 cal/(g°C). 300 g of lead shot at 100°C is mixed with 100 g of water at 70°C in an insulated container. The final temperature of the mixture is
- **(a)** 100°C **(b)** 85.5°C
- **(c)** 79.5°C **(d)** 72.5°C

8. The heat of fusion of water is 79.5 cal/g. This means 79.5 cal of energy are required to
- **(a)** raise the temperature of 1 g of water by 1 K.
- **(b)** turn 1 g of water to steam.

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(c) raise the temperature of 1 g of ice by 1 K.
(d) melt 1 g of ice.

9. The formation of ice from water is accompanied by
(a) absorption of energy as heat.
(b) temperature increase.
(c) decrease in volume.
(d) an evolution of heat.

10. How many calories are required to change one gram of 0°C ice to 100°C steam? The latent heat of fusion is 80 cal/g and the latent heat of vaporization is 540 cal/g. The specific heat of water is 1.00 cal/(g K).
(a) 100 cal (b) 540 cal
(c) 620 cal (d) 720 cal

11. Ten grams of ice at –20°C is to be changed to steam at 130°C. The specific heat of both ice and steam is 0.5 cal/(g°C). The specific heat of water is 1.00 cal/(g K). The heat of fusion is 80 cal/g and the heat of vaporization is 540 cal/g. The entire process requires
(a) 750 cal (b) 1250 cal
(c) 6950 cal (d) 7450 cal

12. Steam at 100°C enters a radiator and leaves as water (at 80°C). Take the heat of vaporization to be 540 cal/g. Of the total energy given off as heat, what percent arises from the cooling of the water?
(a) 100% (b) 54%
(c) 26% (d) 3.6%

13. Fifty grams of ice at 0°C is placed in a thermos bottle containing one hundred grams of water at 6°C. How many grams of ice will melt? The heat of fusion of water is 333 kJ/kg and the specific heat of water is 4190 J/(kg K).
(a) 7.5 g (b) 2.0 g
(c) 8.3 g (d) 17 g

14. A slab of material has area A, thickness L, and thermal conductivity k. One of its surfaces (P) is maintained at temperature T_1 and the other surface (Q) is maintained at a lower temperature T_2. The rate of heat flow from P to Q is:
(a) $\frac{kA(T_1 - T_2)}{L^2}$ (b) $\frac{kL(T_1 - T_2)}{A}$
(c) $\frac{kA(T_1 - T_2)}{L}$ (d) $\frac{k(T_1 - T_2)}{(LA)}$

15. The rate of heat flow by conduction through a slab does NOT depend upon the
(a) temperature difference between opposite faces of the slab.
(b) specific heat of the slab.
(c) slab thickness.
(d) cross-sectional area of the slab.

16. The rate of heat flow through a slab is P_{cond}. If the slab thickness is doubled, its cross-sectional area is halved, and the temperature difference across it is doubled, then the rate of heat flow becomes
(a) $2P_{cond}$ (b) $\frac{P_{cond}}{2}$
(c) P_{cond} (d) $\frac{P_{cond}}{8}$

17. An iron stove, used for heating a room by radiation, is more efficient if
(a) its inner surface is highly polished.
(b) its inner surface is covered with aluminum paint.
(c) its outer surface is covered with aluminum paint.
(d) its outer surface is rough and black.

18. An electric stove burner of diameter 20 cm is at a temperature of 250°C. If $\sigma = 5.67 \times 10^{-8}\ W/m^2 \cdot K^4$, at what rate is the burner radiating energy? Assume the emissivity $\varepsilon = 0.6$.
(a) 4 W (b) 80 W
(c) 320 W (d) 1600 W

19. A homeowner purchases insulation for her attic rated at R-15. She wants the attic insulated to R-30. If the insulation she purchased is 10 cm thick, what thickness does she need to use?
(a) 10 cm (b) 15 cm
(c) 20 cm (d) 30 cm

20. A 2.00 kg metal object requires 5.02×10^3 J of heat to raise its temperature from 20.0°C to 40.0°C. What is the specific heat capacity of the metal?
(a) 63.0 J/(kg·°C) (b) 126 J/(kg·°C)
(c) 251 J/(kg·°C) (d) 502 J/(kg·°C)

21. A 0.20 kg lead ball is heated to 90.0°C and dropped into an ideal calorimeter containing 0.50 kg of water initially at 20.0°C. What is the final equilibrium temperature of the lead ball? The specific heat capacity of lead is 128 J/(kg·°C); and the specific heat of water is 4186 J/(kg·°C).
(a) 4.8°C (b) 20.8°C
(c) 22.4°C (d) 27.8°C

22. A gold sphere has a radius of 1.000 cm at 25.0°C. If 7650 J of heat is added to the sphere, what will the final volume of the sphere be? Gold has a density of 19 300 kg/m³ at 25.0°C, a specific heat capacity of 129 J/(kg·°C), and a coefficient of volume expansion of 42.0×10^{-6}/°C.
(a) 2.88×10^{-6} m³ (b) 3.01×10^{-6} m³
(c) 4.32×10^{-6} m³ (d) 3.91×10^{-6} m³

23. Heat is added to a substance, but its temperature does not rise. Which one of the following statements provides the best explanation for this observation?
(a) The substance must be a gas.
(b) The substance must be a non-perfect solid.
(c) The substance undergoes a change of phase.
(d) The substance has unusual thermal properties.

24. Which would cause a more serious burn: 30 g of steam or 30 g of liquid water, both at 100°C; and why is this so?
(a) Water, because it is denser than steam.
(b) Steam, because of its specific heat capacity.
(c) Steam, because of its latent heat of vaporization.
(d) Water, because its specific heat is greater than that of steam.

25. Determine the latent heat of vaporization of unknown substance X in kcal/g if 3.0 g of boiling liquid X are completely vaporized in 1.5 hours by an input of 10 kcal/h into the system by an energy source.
(a) 5 kcal/g (b) 15 kcal/g
(c) 27 kcal/g (d) 30 kcal/g

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26. Which one of the following statements best explains why convection does not occur in solids?
- **(a)** Molecules in a solid are more closely spaced than in a gas.
- **(b)** The molecules in a solid are not free to move throughout the volume of the solid.
- **(c)** Molecules in a solid vibrate at a lower frequency than those in a liquid.
- **(d)** Solids are more compressible than liquids.

27. Suppose you are sitting next to a fireplace in which there is a fire burning. One end of a metal poker has been left in the fire. Which one of the following statements concerning this situation is true?
- **(a)** You can feel the heat of the fire primarily because of convection.
- **(b)** The end of the poker that is not in the fire is warmed through conduction.
- **(c)** Heat escapes through the chimney primarily through conduction.
- **(d)** You can feel the heat of the fire primarily because of conduction.

28. The ends of a cylindrical steel rod are maintained at two different temperatures. The rod conducts heat from one end to the other at a rate of 10 cal/s. At what rate would a steel rod twice as long and twice the diameter conduct heat between the same two temperatures?
- **(a)** 5 cal/s **(b)** 10 cal/s
- **(c)** 20 cal/s **(d)** 40 cal/s

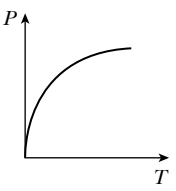
29. A cabin has a 0.159 m thick wooden floor [k = 0.141 W/(m·°C)] with an area of 13.4 m^2. A roaring fire keeps the interior of the cabin at a comfortable 18.0°C while the air temperature in the crawl space below the cabin is –20.6°C. What is the rate of heat conduction through the wooden floor?
- **(a)** 31 J/s **(b)** 459 J/s
- **(c)** 138 J/s **(d)** 245 J/s

30. Which object will emit more electromagnetic radiation than it absorbs from its surroundings?
- **(a)** 600°C lead sphere in a 700°C oven.
- **(b)** A girl scout sitting close to a campfire.
- **(c)** An ice cube in beaker of water at 50°C.
- **(d)** A 200°C copper coin in a beaker of water at 98°C.

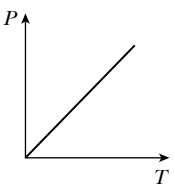
31. Two identical solid spheres have the same temperature. One of the spheres is cut into two identical pieces. These two hemispheres are then separated. The intact sphere radiates an energy Q during a given time interval. During the same interval, the two hemispheres radiate a total energy Q'. What is the ratio Q'/Q?
- **(a)** 2.0 **(b)** 4.0
- **(c)** 0.50 **(d)** 1.5

32. Which one of the following graphs shows the rate at which heat is emitted from a hot body as a function of its Kelvin temperature T?

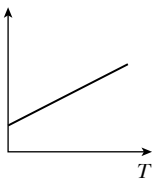
(a)



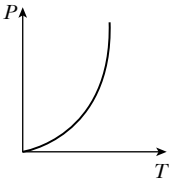
(b)



(c)



(d)



33. Assuming a filament in a 100 W light bulb acts like a perfect blackbody, what is the temperature of the hottest portion of the filament if it has a surface area of 6.3×10^{-5} m^2? The Stefan-Boltzmann constant is 5.67×10^{-8} W/(m$^2 \cdot$ K^2).
- **(a)** 130 K **(b)** 2300 K
- **(c)** 1100 K **(d)** 5800 K

34. Object A has an emissivity of 0.95; and its temperature is 25°C. At what temperature (in degrees Celsius) does object B, whose emissivity is 0.60, emit radiation at the same rate as object A if both objects have the same surface area?
- **(a)** 28°C **(b)** 40°C
- **(c)** 61°C **(d)** 73°C

35. Assume that the sun is a sphere of radius 6.96×10^8 m and that its surface temperature is 5.8×10^3 K. If the Sun radiates at a rate of 3.90×10^{26} W and is a perfect emitter, at what rate is energy emitted per square meter at the Sun's surface?
- **(a)** 5.6×10^7 W/m^2 **(b)** 6.4×10^7 W/m^2
- **(c)** 5.6×10^{17} W/m^2 **(d)** 12.8×10^7 W/m^2

36. A 1.5 kg steel sphere will not fit through a circular hole in a 0.85 kg aluminum plate, because the radius of the sphere is 0.10% larger than the radius of the hole. If both the sphere and the plate are always kept at the same temperature, how much heat must be put into the two so the ball just passes through the hole?
- **(a)** 6.2×10^4 J **(b)** 1.3×10^5 J
- **(c)** 7.0×10^4 J **(d)** 2.4×10^5 J

37. A 10.0 kg block of ice has a temperature of –10.0°C. The pressure is one atmosphere. The block absorbs 4.11×10^6 J of heat. What is the final temperature of the liquid water?
- **(a)** 0.58°C **(b)** 4.6°C
- **(c)** 9.2°C **(d)** 13°C

38. One end of a brass bar is maintained at 306°C, while the other end is kept at a constant, but lower, temperature. The cross-sectional area of the bar is 2.6×10^{-4} m^2. Because of insulation, there is negligible heat loss through the sides of the bar. Heat flows through the bar, however, at the rate of 3.6 J/s. What is the temperature of the bar at a point 0.15 m from the hot end?
- **(a)** 301°C **(b)** 297°C
- **(c)** 287°C **(d)** 265°C

39. Liquid helium is stored at its boiling-point temperature of 4.2 K in a spherical container (r = 0.30 m). The container is a perfect blackbody radiator. The container is surrounded by a spherical shield whose temperature is 77 K. A vacuum exists in the space between the container and the shield. The latent heat of vaporization for helium is 2.1×10^4 J/kg. What mass of liquid helium boils away through a venting valve in one hour?
- **(a)** 0.39 kg **(b)** 0.57 kg
- **(c)** 0.81 kg **(d)** 1.1 kg

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40. A small sphere (emissivity = 0.90, radius = r_1) is located at the center of a spherical asbestos shell (thickness = 1.0 cm, outer radius = r_2). The thickness of the shell is small compared to the inner and outer radii of the shell. The temperature of the small sphere is 800.0°C, while the temperature of the inner surface of the shell is 600.0°C, both temperatures remaining constant. Assuming that $r_2/r_1 = 10.0$ and ignoring any air inside the shell, find the temperature of the outer surface of the shell.

(a) 642.9°C **(b)** 557.7°C
(c) 689.3°C **(d)** 519.4°C

41. A solid sphere has a temperature of 773 K. The sphere is melted down and recast into a cube that has the same emissivity and emits the same radiant power as the sphere. What is the cube's temperature?

(a) 959 K **(b)** 816 K
(c) 732 K **(d)** 623 K

42. The ends of a thin bar are maintained at different temperatures. The temperature of the cooler end is 11°C, while the temperature at a point 0.13 m from the cooler end is 23°C and the temperature of the warmer end is 48°C. Assuming that heat flows only along the length of the bar (the sides are insulated), find the length of the bar.

(a) 0.50 m **(b)** 0.40 m
(c) 0.23 m **(d)** 0.64 m

43. A piece of glass has a temperature of 83.0°C. Liquid that has a temperature of 43.0°C is poured over the glass, completely covering it, and the temperature at equilibrium is 53.0°C. The mass of the glass and the liquid is the same. Ignoring the container that holds the glass and liquid and assuming that the heat lost to or gained from the surroundings is negligible, determine the specific heat capacity of the liquid.

(a) 280 J/(kg·°C) **(b)** 1300 J/(kg·°C)
(c) 2500 J/(kg·°C) **(d)** 3400 J/(kg·°C)

44. A thermos contains 150 cm^3 of coffee at 85°C. To cool the coffee, you drop two 11 – g ice cubes into the thermos. The ice cubes are initially at 0°C and melt completely. What is the final temperature of the coffee? Treat the coffee as if it were water.

(a) 59°C **(b)** 77°C
(c) 64°C **(d)** 73°C

45. A mass m of steam at 100°C is to be passed into a vessel containing 10 g of ice and 100 g of water at 0°C so that all the ice is melted and the temperature is raised to 5°C neglecting heat absorbed by the vessel, we get

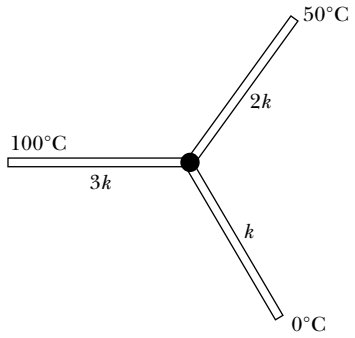
(a) $m = 2.1$ g **(b)** $m = 4.2$ g
(c) $m = 6.3$ g **(d)** $m = 8.4$ g

46. A substance of mass M kg requires a power input of P watts to remain in the molten state at its melting point. When the power source is turned off, the sample completely solidifies in time t seconds. The specific latent heat of fusion of the substance is

(a) Pt **(b)** $\frac{Pt}{M}$

(c) PtM **(d)** $\frac{PM}{t}$

47. Three rods of the same dimension have thermal conductivities $3k$, $2k$ and k. they are arranged as shown in the following figure with their ends at 100°C, 50°C and 0°C. the temperature of their junction is



(a) 75°C **(b)** 200/3°C
(c) 40°C **(d)** 100/3°C

48. A hot liquid is kept in a big room. The logarithm of the numerical value of the temperature difference between the liquid and the room is plotted against time. The plot will be very nearly

(a) a straight line. **(b)** a circular arc.
(c) a parabola. **(d)** an ellipse.

49. The temperature of an isolated black body falls from T_1 to T_2 in time t. Let c be a constant.

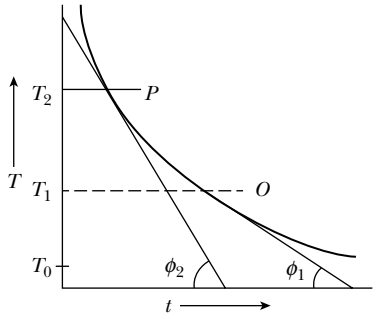
(a) $t = c\left[\frac{1}{T_2} - \frac{1}{T_1}\right]$ **(b)** $t = c\left[\frac{1}{T_2^2} - \frac{1}{T_1^2}\right]$

(c) $t = c\left[\frac{1}{T_2^3} - \frac{1}{T_1^3}\right]$ **(d)** $t = c\left[\frac{1}{T_2^4} - \frac{1}{T_1^4}\right]$

50. The power radiated by a black body is P, and it radiates maximum energy around the wavelength λ_1. If the temperature of the black body is now changed so that it radiates maximum energy around a wavelength $3\lambda_0/4$, the power radiated by it will increase by a factor of

(a) 4/3 **(b)** 16/9
(c) 64/27 **(d)** 256/81

51. A body cools in a surrounding which is at a constant temperature of T_0. Assume that it obeys Newton's law of cooling. Its temperature T is plotted against time t as shown in the following figure. Tangents are drawn to the curve at the points $P(T = T_1)$ and $Q(T = T_2)$. These tangents meet the time axis at angles of ϕ_2 and ϕ_1 as shown



(a) $\frac{\tan\phi_2}{\tan\phi_1} = \frac{T_1 - T_0}{T_2 - T_0}$ **(b)** $\frac{\tan\phi_2}{\tan\phi_1} = \frac{T_2 - T_0}{T_1 - T_0}$

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(c) $\frac{\tan\phi_1}{\tan\phi_2}=\frac{T_1}{T_2}$ (d) $\frac{\tan\phi_1}{\tan\phi_2}=\frac{T_2}{T_1}$

52. In solar radiation, the intensity of radiation is maximum around the wavelength λ_m. If R is the radius of the Sun and c is the velocity of light, the mass lost by the Sun per unit time is proportional to

(a) $\frac{R^2}{\lambda^4 c^2}$ (b) $\frac{R^2}{\lambda^2 c^2}$

(c) $\frac{R^3}{\lambda^4 c^3}$ (d) $\frac{R^3}{\lambda^4 c^2}$

53. When a body is placed in surroundings at a constant temperature of 20°C and heated by a 10 W heater, its temperature remains constant at 40°C. If the temperature of the body is now raised from 20°C to 80°C in 5 min at a uniform rate, the total heat it will lose to the surroundings will be

(a) 3000 J (b) 3600 J
(c) 4500 J (d) 5400 J

54. A point source of heat of power P is placed at the center of a spherical shell of mean radius R. The material of the shell has thermal conductivity k. The temperature difference between the outer and inner surfaces of the shell is not exceed T, the thickness of the shell should not be less than

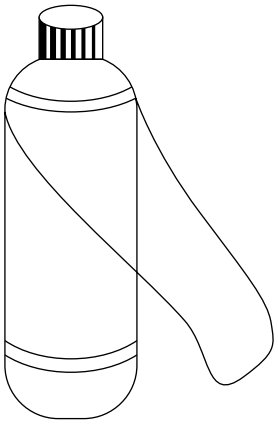
(a) $\frac{4\pi kR^2T}{p}$ (b) $\frac{4\pi kR^2}{Tp}$

(c) $\frac{4\pi R^2T}{kp}$ (d) $\frac{4\pi R^2P}{kT}$

55. There are three thermometers –one in contact with the skin of the man, other in between the vest and the shirt and coat. The readings of the thermometers are 30°C, 25°C, and 22°C, respectively. If the vest and shirt are of the same thickness, the ratio of their thermal conductivities is

(a) 9:25 (b) 25:9
(c) 5:3 (d) 3:5

56. The only possibility of the heat flow in a thermos flask is through its cork which is 75 cm^2 in area and 5 cm thick (see figure). Its thermal conductivity is 0.00075 cal/cm s°C. the outside temperature is 40°C and latent heat of ice is 80 cal/g. Time taken by 500 g of ice at 0°C in the flask to melt into water at 0°C is

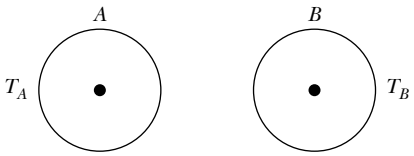


(a) 2.47 h (b) 4.27 h
(c) 7.42 h (d) 4.72 h

57. A black body at 200 K is found to emit maximum energy at a wavelength of 14 μm. When its temperature is raised to 1000 K, the wavelength at which maximum energy is emitted is

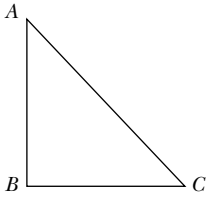
(a) 14 μm (b) 7 μm
(c) 2.8 μm (d) 7.4 μm

58. A and B are two isolated spheres kept in close proximity so that they can exchange energy by radiation as shown in the following figure. The two spheres have identical physical dimension, but the surface of A behaves like a perfectly black body while the surface of B reflects 20% of all the radiations it receives. They are isolated from all other sources of radiation.



(a) If they are in thermal equilibrium and exchange equal amounts of radiation per second, then they will be at same absolute temperature, $T_A = T_B$.
(b) If they are in thermal equilibrium and exchange equal amounts of radiation per second, then $T_A = (0.8)^{1/4}T_B$.
(c) If they are not in thermal equilibrium and are each at $t = 0$ at the same temperature $T_A = T_B = T$, then the sphere A will loss thermal energy and B gain thermal energy.
(d) If they are not in thermal equilibrium and are each at $t = 0$ at the same temperature $T_A = T_B = T$, then the sphere A will gain thermal energy and B lose thermal energy.

59. Three rods of identical cross-sectional area and made from the same metal form the sides of an isosceles triangles ABC right angled at B a shown in the following figure. The points A and B maintained at temperature T and$\sqrt{2}T$, respectively in the steady state. Assuming that only heat conduction takes place, temperature of point C is



(a) $\frac{3T}{\sqrt{2}+1}$ (b) $\frac{T}{\sqrt{2}+1}$

(c) $\frac{T}{3(\sqrt{2}-1)}$ (d) $\frac{T}{\sqrt{2}-1}$

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60. A sphere and a cube of same material and same total surface area are placed in the same evacuated space turn by turn after they are heated to the same temperature. Find the ratio of their initial rates of cooling in the enclosure

(a) $\sqrt{\frac{3\pi}{8}}:1$ **(b)** $\sqrt{\frac{\pi}{3}}:1$

(c) $\frac{\pi}{\sqrt{6}}:1$ **(d)** $\frac{\pi}{\sqrt{3}}:1$

More than One Correct Choice Type

61. A solid sphere and a hollow sphere of the same material and of equal radii are heated to the same temperature
- **(a)** both will emit equal amount of radiation per unit time in the beginning.
- **(b)** both will absorb equal amount of radiation from the surrounding in the beginning.
- **(c)** the initial rate of cooling (dT/dt) will be the same for the two spheres.
- **(d)** the two spheres will have equal temperatures at any instant.

62. The heat capacity of a body depends on
- **(a)** the heat given.
- **(b)** the temperature raised.
- **(c)** the mass of the body.
- **(d)** the material of the body.

63. A black body does not
- **(a)** emit radiation. **(b)** absorb radiation.
- **(c)** reflect radiation. **(d)** refract radiation.

64. Two identical objects A and B are at temperature T_A and T_B, respectively. Both objects are placed in a room with perfectly absorbing walls maintained at a temperature T ($T_A > T > T_B$). The objects A and B attain the temperature T eventually. Select the correct statements from the following:
- **(a)** A only emits radiation, while B only absorbs it until both attain temperature T.
- **(b)** A lose more heat by radiation than it absorbs while B absorbs more radiation than it emits until they attain temperature T.
- **(c)** Both A and B only absorb radiation, but do not emit it, until they attain temperature T.
- **(d)** Each object continues to emit and absorb radiation even after attaining temperature T.

65. Choose the correct statements from the following:
- **(a)** Good reflectors are good emitters of thermal radiation.
- **(b)** Burns caused by water at 100°C are more severe than those caused by steam at 100°C.
- **(c)** If the Earth did not have atmosphere, it would become intolerably cold.
- **(d)** It is impossible to construct a heat engine of a 100% efficiency.

66. Two bodies A and B have thermal emissivities of 0.01 and 0.81, respectively. The outer surface areas of the two bodies are the same. The two bodies radiate energy at the same rate. The wavelength λ_0, corresponding to the maximum spectral radiancy in the radiation from B, is shifted from the wavelength corresponding to the maximum spectral radiancy in the radiation from A by 1.00 μm. If the temperature of A is 5802 K, then
- **(a)** the temperature of B is 1934 K.
- **(b)** $\lambda_B = 1.5$ μm.
- **(c)** the temperature of B is 11604 K.
- **(d)** the temperature of B is 2901 K.

67. A piece of charcoal and a piece of shining steel of the same area are kept for a long time in an open lawn in bright Sun.
- **(a)** The steel will absorb more heat than the charcoal.
- **(b)** The temperature of the steel will be higher than that of the charcoal.
- **(c)** If both are picked up by bare hands, the steel will be felt hotter than the charcoal.
- **(d)** If the two are picked up from the lawn and kept in a cold chamber, the charcoal will lose heat at a faster rate than the steel.

68. In Newton's law of cooling, $d\theta/dt = -k(\theta - \theta_0)$, the constant k is proportional to
- **(a)** A, the surface area of the body.
- **(b)** S, the specific heat of the body.
- **(c)** $1/m$, where m is the mass of the body.
- **(d)** e, the emissivity of the body.

Linked Comprehension

Paragraph for Questions 69–71: A 0.0500 kg lead bullet of volume 5.00×10^{-6} m^3 at 20.0°C hits a block that is made of an ideal thermal insulator and comes to rest at its center. At that time, the temperature of the bullet is 327°C. Use the following information for lead:

Coefficient of linear expansion:	$\alpha = 2.0 \times 10^{-5}$/°C
Specific heat capacity:	$c = 128$ J/(kg · °C)
Latent heat of fusion:	$L_f = 23\,200$ J/kg
Melting point:	$T_{melt} = 327$°C

69. How much heat was needed to raise the bullet to its final temperature?
- **(a)** 963 J **(b)** 1960 J
- **(c)** 3640 J **(d)** 3880 J

70. What is the volume of the bullet when it comes to rest?
- **(a)** 5.00×10^{-6} m^3
- **(b)** 5.01×10^{-6} m^3
- **(c)** 5.09×10^{-6} m^3
- **(d)** 5.07×10^{-6} m^3

71. What additional heat would be needed to melt the bullet?
- **(a)** 420 J **(b)** 837 J
- **(c)** 628 J **(d)** 1160 J

Paragraph for Questions 72–75: Two bars, A and B, each of length 2.0 m and cross-sectional area 1.0 m^2, are placed end to end as shown in the figure.

The thermal conductivities of the bars are k_A = 339 J/(s · m · K) and k_B = 837 J/(s · m · K), respectively. The left end

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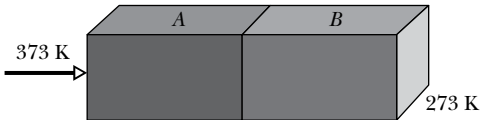
of bar A is maintained at 373 K while the right end of B is maintained at 273 K.



72. Which one of the following statements is true concerning the above situation?
- **(a)** For a given time t, the amount of heat transferred through bar A is the same as that through B.
- **(b)** For a given time t, the amount of heat transferred through bar A is smaller than that through B.
- **(c)** For a given time t, the amount of heat transferred through bar A is less than that through B.
- **(d)** The time for a quantity of heat Q to pass through bar A is less than that for B.

73. What is the temperature at the interface between A and B?
- **(a)** 332 K **(b)** 302 K
- **(c)** 323 K **(d)** 313 K

74. What is the rate of heat transfer through this combination?
- **(a)** 3.0×10^3 J/s **(b)** 2.4×10^4 J/s
- **(c)** 6.0×10^3 J/s **(d)** 1.2×10^4 J/s

75. How much heat passes the interface between A and B in 5.0 s?
- **(a)** 6.0×10^4 J **(b)** 1.5×10^3 J
- **(c)** 1.2×10^5 J **(d)** 4.5×10^4 J

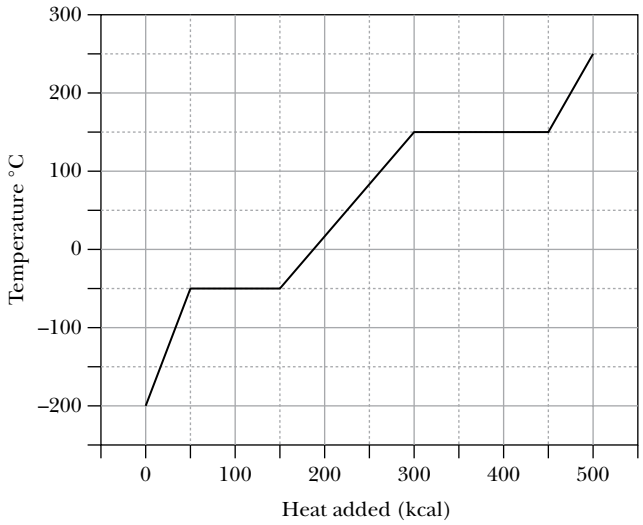
Matrix-Match

76. Match the laws given in Column I to the respective correct statements given in Column II.

Column I	Column II
(a) Wien's displacement law explains	**(p)** why days are hot and nights are cold in deserts
(b) Planck's law explains	**(q)** why a blackened platinum wire, when gradually heated, appears first dull red and then blue
(c) Kirchhoff's law explains	**(r)** the distribution of energy in blackbody spectrum at shorter as well as longer wavelengths
(d) Newton's law of cooling explains that	**(s)** the rate of cooling of a body is proportional to the excess temperature of the body over the surroundings

Directions for Questions 77: In the question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

77. Heat is added to a 1.0 kg solid sample of a material at –200°C. The figure shows the temperature of the material as a function of the heat added.



In the given table, Columns I, II and III have parts of the statements.

Column I	Column II	Column III
(I) The specific heat capacity of this substance	**(i)** in its liquid state	**(J)** 100 cal/g.
(II) The latent heat of	**(ii)** in its solid state	**(K)** 150 cal/g.
(III) The specific heat capacity of this substance	**(iii)** vaporization of this material	**(L)** 0.75 cal/g/°C.
(IV) The latent heat of	**(iv)** fusion of this material	**(M)** 0.33 cal/g/°C.

(1) Which one of the following combinations about the heat capacity of the substance is correct?
- **(a)** (I) (ii) (M) **(b)** (II) (i) (M)
- **(c)** (IV) (ii) (J) **(d)** (III) (iv) (L)

(2) Which one of the following combinations is correct about change of state of substance from solid to liquid?
- **(a)** (IV) (iii) (K)
- **(b)** (I) (ii) (L)
- **(c)** (III) (i) (M)
- **(d)** (II) (iv) (J)

(3) Which one of the following combinations is correct about change of state of substance from liquid to gas?
- **(a)** (II) (iv) (J)
- **(b)** (IV) (iii) (K)
- **(c)** (I) (i) (L)
- **(d)** (III) (ii) (M)

Integer Type

78. To help keep his barn warm on cold days, a farmer stores 840 kg of solar-heated water ($L_f = 3.35 \times 10^5$ J/kg) in barrels. For how many hours would a 2.0 kW electric space heater have to operate to provide the same amount of

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heat as the water does when it cools from 10.0 to 0.0°C and completely freezes?

79. A commonly used method of fastening one part to another part is called "shrink fitting." A steel rod has a diameter of 2.0026 cm, and a flat plate contains a hole whose diameter is 2.0000 cm. The rod is cooled so that it just fits into the hole. When the rod warms up, the enormous thermal stress exerted by the plate holds the rod securely to the plate. By how many Celsius degrees should the rod be cooled?

80. A steel ruler is calibrated to read true at 20.0°C. A draftsman uses the ruler at 40.0°C to draw a line on a 40.0°C copper plate. As indicated on the warm ruler, the length of the line is 0.50 m. To what temperature should the plate be cooled, such that the length of the line truly becomes 0.50 m?

81. In a house the temperature at the surface of a window is 25°C. The temperature outside at the window surface is 5.0°C. Heat is lost through the window via conduction, and the heat lost per second has a certain value. The temperature outside begins to fall, while the conditions inside the house remain the same. As a result, the heat lost per second increases. What is the temperature at the outside window surface when the heat lost per second doubles?

82. A body cools down from 60°C to 55°C in 30 s. Using Newton's law of cooling, calculate the approximate time taken by the same body to cool down from 55°C to 50°C. Assume that the temperature of the surroundings is 45°C.

ANSWER KEY

Checkpoints

1. A (see Eq. 19-3) **2.** *b* and *d* tie, then *a*, *c* (P_{cond} identical; see Eq. 19-14)

Problems

1. 9.82°C **2.** (a) 542 J/kg K; (b) 27.1 J/mol·K; (c) 0.600 mol

3. (a) 56 MJ; (b) the lowest temperature possible is 0°C

4. (a) 4.9 min; (b) 4.9 min **5.** 6.7×10^2 J/kg·K

6. 304.7 s ≈ 305 s **7.** 2.55 10 m/s 0.92 cm/h

8. 41 cal **9.** (a) 0.325 W ≈ 0.33 W; (b) 52 s

10. 1047.28 kJ ≈ 1.05 MJ

11. (a) 16.5°C; (b) greater conductivity should mean a larger rate of conductive heat transfer; (c) 14.5°C

12. $P_{cond} = 3.23 \times 10^3$ J/s; *R*- value = 3.74×10^{-4} 10 m^2 × K/W; low value of *R* is an indication that the copper slab is a good conductor

13. 55 g **14.** 8.61 10^{-7} kg/s ≈ 0.86 mg/s **15.** (a) 1.5W; (b) 3.45

16. –1.7°C. **17.** (a) 602 W; (b) 1115 W ≈ 1.12×10^3 W; (c) 1.08×10^5 J

18. (a) 8×10^2 W; (b) 3.52×10^4 J ≈ 4.0×10^4 J **19.** 1.6 m **20.** (a) 0.17; (b) 0.83 or 83%

21. 0.59 J **22.** 2.79×10^4 J = 27.9 kJ **23.** 269 g **24.** (a) 55 g; (b) 6.3°C

25. (a) 67500 J/kg ≈ 68 kJ/kg; (b) 2.3 kJ/kg·°C **26.** (a) 40 J/s; (b) 0.12 g/s

27. (a) 34 W; (b) 1.8 kg; (c) 0.12 kg **28.** 1.5 min **29.** 25 m^2

30. (a) 5.3°C; (b) 0; (c) 0.220 kg = 220 g **31.** 9.58×10^3 kg **32.** (a) 23.5 kcal; (b) 1.11 kcal; (c) 989°C

33. 340 K **34.** 6.1 min **35.** 5800 K **36.** $t_1 = \dfrac{\ln 2}{k}$

37. (a) 2.3×10^2 J/s; 23.5×10^3 J/s **38.** $\dfrac{k\pi r_1 r_2 (T_2 - T_1)}{L}$ **39.** 12.5 g

40. (a) 20 W; (b) 20/3 W; (c) 1000 J; (d) 500 J/kg K

Practice Questions

Single Correct Choice Type

1. (b)	**2.** (a)	**3.** (d)	**4.** (b)	**5.** (b)
6. (d)	**7.** (d)	**8.** (d)	**9.** (a)	**10.** (d)
11. (d)	**12.** (d)	**13.** (a)	**14.** (c)	**15.** (b)
16. (b)	**17.** (d)	**18.** (b)	**19.** (c)	**20.** (b)

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21. (b)	22. (c)	23. (c)	24. (c)	25. (a)
26. (b)	27. (b)	28. (c)	29. (b)	30. (d)
31. (d)	32. (d)	33. (b)	34. (c)	35. (b)
36. (b)	37. (d)	38. (c)	39. (a)	40. (b)
41. (c)	42. (b)	43. (c)	44. (c)	45. (a)
46. (b)	47. (b)	48. (a)	49. (d)	50. (d)
51. (b)	52. (a)	53. (c)	54. (a)	55. (d)
56. (a)	57. (c)	58. (a)	59. (a)	60. (a)

More than One Correct Choice Type

61. (a), (b)	62. (c), (d)	63. (c), (d)	64. (b), (d)	65. (c), (d)
66. (a), (b)	67. (c), (d)	68. (a), (b), (c)		

Linked Comprehension

69. (b)	70. (c)	71. (d)	72. (a)	73. (b)
74. (d)	75. (a)			

Matrix-Match

76. (a) $\rightarrow$ (q); (b) $\rightarrow$ (r); (c) $\rightarrow$ (p); (d) $\rightarrow$ (s)

77. (1) $\rightarrow$ (a); (2) $\rightarrow$ (d); (3) $\rightarrow$ (b)

Integer Type

78. 44	79. 110	80. 26	81. 15	82. 50

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CHAPTER 20

The Kinetic Theory of Gases

Contents

20.1 What is Physics?
20.2 Avogadro's Number
20.3 Ideal Gases
20.4 Pressure, Temperature, and RMS Speed
20.5 Translational Kinetic Energy
20.6 The Distribution of Molecular Speeds
20.7 The Equipartition of Energy and Degrees of Freedom
20.8 A Hint of Quantum Theory

20.1 | WHAT IS PHYSICS?

One of the main subjects in thermodynamics is the physics of gases. A gas consists of atoms (either individually or bound together as molecules) that fill their container's volume and exert pressure on the container's walls. We can usually assign a temperature to such a contained gas. These three variables associated with a gas—volume, pressure, and temperature—are all a consequence of the motion of the atoms. The volume is a result of the freedom the atoms have to spread throughout the container, the pressure is a result of the collisions of the atoms with the container's walls, and the temperature has to do with the kinetic energy of the atoms. The **kinetic theory of gases**, the focus of this chapter, relates the motion of the atoms to the volume, pressure, and temperature of the gas.

Applications of the kinetic theory of gases are countless. Automobile engineers are concerned with the combustion of vaporized fuel (a gas) in the automobile engines. Food engineers are concerned with the production rate of the fermentation gas that causes bread to rise as it bakes. Beverage engineers are concerned with how gas can produce the head in a glass of beer or shoot a cork from a champagne bottle. Medical engineers and physiologists are concerned with calculating how long a scuba diver must pause during ascent to eliminate nitrogen gas from the bloodstream (to avoid the *bends*). Environmental scientists are concerned with how heat exchanges between the oceans and the atmosphere can affect weather conditions.

The first step in our discussion of the kinetic theory of gases deals with measuring the amount of a gas present in a sample, for which we use Avogadro's number.

20.2 | AVOGADRO'S NUMBER

Key Concepts

- The kinetic theory of gases relates the macroscopic properties of gases (for example, pressure and temperature) to the microscopic properties of gas molecules (for example, speed and kinetic energy).

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- One mole of a substance contains N_A (Avogadro's number) elementary units (usually atoms or molecules), where N_A is found experimentally to be

$$N_A = 6.02 \times 10^{23}\ \text{mol}^{-1} \quad \text{(Avogadro's number)}.$$

  One molar mass M of any substance is the mass of one mole of the substance.
- A mole is related to the mass m of the individual molecules of the substance by

$$M = mN_A.$$

- The number of moles n contained in a sample of mass M_{sam}, consisting of N molecules, is related to the molar mass M of the molecules and to Avogadro's number N_A as given by

$$n = \frac{N}{N_A} = \frac{M_{sam}}{M} = \frac{M_{sam}}{mN_A}.$$

When our thinking is slanted toward atoms and molecules, it makes sense to measure the sizes of our samples in moles. If we do so, we can be certain that we are comparing samples that contain the same number of atoms or molecules. The *mole* is one of the seven SI base units and is defined as follows:

One mole is the number of atoms in a 12 g sample of carbon-12.

The obvious question now is: "How many atoms or molecules are there in a mole?" The answer is determined experimentally and, is

$$N_A = 6.02 \times 10^{23}\ \text{mol}^{-1} \quad \text{(Avogadro's number)}, \tag{20-1}$$

where mol^{-1} represents the inverse mole or "per mole," and mol is the abbreviation for mole. The number N_A is called **Avogadro's number** after Italian scientist Amedeo Avogadro (1776–1856), who suggested that all gases occupying the same volume under the same conditions of temperature and pressure contain the same number of atoms or molecules.

The number of moles n contained in a sample of any substance is equal to the ratio of the number of molecules N in the sample to the number of molecules N_A in 1 mol:

$$n = \frac{N}{N_A}. \tag{20-2}$$

(*Caution:* The three symbols in this equation can easily be confused with one another, so you should sort them with their meanings now, before you end in "N-confusion.") We can find the number of moles n in a sample from the mass M_{sam} of the sample and either the *molar mass* M (the mass of 1 mol) or the molecular mass m (the mass of one molecule):

$$n = \frac{M_{sam}}{M} = \frac{M_{sam}}{mN_A}. \tag{20-3}$$

In Eq. 20-3, we used the fact that the mass M of 1 mol is the product of the mass m of one molecule and the number of molecules N_A in 1 mol:

$$M = mN_A. \tag{20-4}$$

PROBLEM-SOLVING TACTICS

Tactic 1*: *Avogadro's Number of What? In Eq. 20-1, Avogadro's number is expressed in terms of mol^{-1}, which is the inverse mole, or 1/mol. We could instead explicitly state the elementary unit involved in a given situation. For example, if the elementary unit is an atom, we might write $N_A = 6.02 \times 10^{23}$ atoms/mol. If, instead, the elementary unit is a molecule, then we might write $N_A = 6.02 \times 10^{23}$ molecules/mol. Being explicit about the unit is usually the better style of writing.

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20.3 | IDEAL GASES

Key Concepts

- An ideal gas is one for which the pressure p, volume V, and temperature T are related by

$$pV = nRT \quad \text{(ideal gas law)}.$$

  Here n is the number of moles of the gas present and R is a constant (8.31 J/mol·K) called the gas constant.
- The ideal gas law can also be written as

$$pV = NkT,$$

  where the Boltzmann constant k is

$$k = \frac{R}{N_A} = 1.38 \times 10^{-23}\ \text{J/K}.$$

- The work done *by* an ideal gas during an isothermal (constant-temperature) change from volume V_i to volume V_f is

$$W = nRT \ln \frac{V_f}{V_i} \quad \text{(ideal gas, isothermal process)}.$$

Our goal in this chapter is to explain the macroscopic properties of a gas—such as its pressure and its temperature—in terms of the behavior of the molecules that make it up. However, there is an immediate problem: which gas? Should it be hydrogen, oxygen, or methane, or perhaps uranium hexafluoride? They are all different. Experimenters have found, though, that if we confine 1 mol samples of various gases in boxes of identical volume and hold the gases at the same temperature, then their measured pressures are almost the same, and at lower densities the differences tend to disappear.

Consider an ideal gas is confined to a rigid container whose volume can be varied by means of a movable piston as in Fig. 20-1. We also assume that container does not leak, that is, the amount of the gas remains constant. For such system, experiments provide the following information:

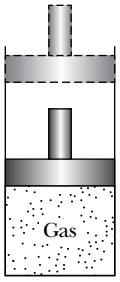

Figure 20-1 Gas confined in a cylinder under a piston.

1. When the gas is kept at a constant temperature, its pressure p is inversely proportional to the volume V.

$$p \propto \frac{1}{V} \quad \text{(Boyle's law)}$$

2. When the pressure of the gas is kept constant, the volume is directly proportional to the temperature T.

$$V \propto T \quad \text{(Charles's law)}$$

3. When the volume of the gas is kept constant, the pressure is directly proportional to the temperature.

$$p \propto T \quad \text{(Gay-Lussac's law)}$$

Further experiments show that, at low enough densities, all real gases tend to obey the relation

$$pV = nRT \quad \text{(Ideal gas law)}, \tag{20-5}$$

in which p is the absolute (not gauge) pressure, n is the number of moles of gas present, and T is the temperature in kelvins. The symbol R is a constant called the **gas constant** that has the same value for all gases—namely,

$$R = 8.31\ \text{J/mol·K}. \tag{20-6}$$

Equation 20-5 is called the **ideal gas law**. Provided the gas density is low, this law holds for any single gas or for any mixture of different gases. (For a mixture, n is the total number of moles in the mixture.)

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We can rewrite Eq. 20-5 in an alternative form, in terms of a constant called the **Boltzmann constant** k, which is defined as

$$k = \frac{R}{N_A} = \frac{8.31\ \text{J/mol}\cdot\text{K}}{6.02\times10^{23}\ \text{mol}^{-1}} = 1.38\times10^{-23}\ \text{J/K}. \tag{20-7}$$

This allows us to write $R = kN_A$. Then, with Eq. 20-2 ($n = N/N_A$), we see that

$$nR = Nk. \tag{20-8}$$

Substituting this into Eq. 20-5 gives a second expression for the ideal gas law:

$$pV = NkT \quad \text{(ideal gas law)}. \tag{20-9}$$

(*Caution*: Note the difference between the two expressions for the ideal gas law—Eq. 20-5 involves the number of moles n, and Eq. 20-9 involves the number of molecules N.)

You may well ask, "What is an *ideal gas*, and what is so 'ideal' about it?" The answer lies in the simplicity of the law (Eqs. 20-5 and 20-9) that governs its macroscopic properties. Using this law—as you will see—we can deduce many properties of the ideal gas in a simple way. Although there is no such thing in nature as a truly ideal gas, *all real* gases approach the ideal state at low enough densities—that is, under conditions in which their molecules are far enough apart that they do not interact with one another. Thus, the ideal gas concept allows us to gain useful insights into the limiting behavior of real gases.

Figure 20-2 gives a dramatic example of the ideal gas law. A stainless-steel tank with a volume of 18 m^3 was filled with steam at a temperature of 110°C through a valve at one end. The steam supply was then turned off and the valve closed, so that the steam was trapped inside the tank (Fig. 20-2*a*). Water from a fire hose was then poured onto the tank to rapidly cool it. Within less than a minute, the enormously sturdy tank was crushed (Fig. 20-2*b*), as if some giant invisible creature from a grade B science fiction movie had stepped on it during a rampage.

(*a*)

(*b*)

Courtesy www.doctorslime.com

Figure 20-2 (*a*) Before and (*b*) after images of a large steel tank crushed by atmospheric pressure after internal steam cooled and condensed.

Actually, it was the atmosphere that crushed the tank. As the tank was cooled by the water stream, the steam cooled and much of it condensed, which means that the number N of gas molecules and the temperature T of the gas inside the tank both decreased. Thus, the right side of Eq. 20-9 decreased, and because volume V was constant, the gas pressure p on the left side also decreased. The gas pressure decreased so much that the external atmospheric pressure was able to crush the tank's steel wall. Figure 20-2 was staged, but this type of crushing sometimes occurs in industrial accidents.

Variation of Atmospheric Pressure with Height

Figure 20-3 shows a thin slab in air at an altitude z. For equilibrium of the slab,

$$[p(z) - p(z + \Delta z)]A = \rho g A\, \Delta z$$

or

$$\frac{dp}{dz} = -\rho g$$

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From the ideal gas law,

$$pV = RT$$

$$p\frac{M}{\rho} = RT$$

Thus,

$$\rho = \frac{M}{RT}p$$

where M is the average molecular mass of air and ρ is its density. Thus,

$$\frac{dp}{dz} = \frac{M}{RT}pg$$

If we assume that the temperature does not vary with altitude, then

$$\int_{p_0}^{p}\frac{dp}{p} = -\frac{Mg}{RT}\int_0^z dz$$

and

$$p = p_0 e^{-(Mg/RT)z}$$

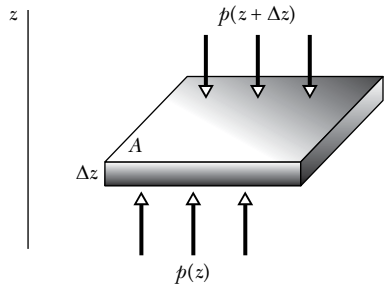

Figure 20-3 A small slice of atmospheric air is being supported by pressure forces acting on it.

SAMPLE PROBLEM 20.01

Application of gas laws

A vessel of volume V is evacuated by means of a piston air pump (Fig. 20-4). Upon each double stroke a piston pump sucks in a volume V_0 of air and expels it. The process is assumed to be isothermal, and the gas is ideal. When this pump is used to evacuate the air from a vessel with a volume V, it performs n double strokes. The initial pressure inside the vessel is p_0 which is equal to the atmospheric pressure.

(a) Calculate the final pressure inside the vessel.

KEY IDEA

In the Fig. 20-4 we can see that when the piston comes out, some of the air from the large vessel will come into the pump's chamber. Now when the piston goes in, the valve connecting chamber to the vessel will close and another valve will open to atmosphere. Thus, all the air inside the chamber will be thrown to the atmosphere. We cannot say that total volume thrown out is nV_0, as number of moles being expelled will vary with every stroke. As every stroke will reduce pressure so the number of moles in chamber in each stroke will be less than the previous stroke.

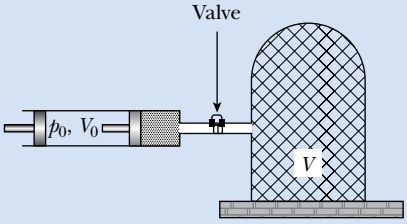

Figure 20-4 Pump used for evacuating or filling the large vessel.

Calculations: When air is being pumped out of the vessel, the volume occupied by the air after one stroke will become equal to $(V + V_0)$. As the process is assumed to be isothermal, we can write

$$p_0V = p_1(V + V_0)$$

The pressure after expelling the gas will also remain p_1, as closing of the valve will not allow change in the pressure. The pressure in the vessel after one double stroke will become equal to

$$p_1 = \frac{p_0V}{V + V_0}$$

After the second double stroke, $p_1V = p_2(V + V_0)$ and, consequently,

$$p_2 = p_0\left(\frac{V}{V + V_0}\right)^2, \quad \text{and so on.}$$

Thus, after n double strokes, the pressure in the vessel is

$$p' = p_0\left(\frac{V}{V + V_0}\right)^n$$

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We can see that pressure inside the vessel will never become zero. Every time the vessel connects to the chamber only a fraction of the total amount goes to the pump's chamber. Thus, some amount of the gas will always remain in the vessel.

(b) After that, another pump with the same chamber volume V_0 begins to suck in the atmospheric air, also making n double strokes. What will be the pressure inside the vessel?

KEY IDEA

In the figure we can see that when the piston comes out, the valve connecting chamber to the vessel will close and another valve will open to atmosphere. Thus, air from the atmosphere will come into the pump's chamber. Now when the piston goes in, the valve connecting chamber to the vessel will open and the other valve to atmosphere will close. So all the air inside the chamber will go to the vessel.

We can say that the total volume gone in is nV_0 as number of moles entering the chamber will remain constant in every stroke. Also, during delivery, the pump at each double stroke sucks in air with a constant pressure p_0.

Calculations: Total number of moles that will be present in the vessel are

$$\frac{(p'V + p_0 nV_0)}{RT}$$

If final pressure is p then we can write

$$\frac{(p'V + p_0 nV_0)}{RT} = \frac{pV}{RT}$$

Solving we get,

$$p = p' + \frac{p_0 nV_0}{V} = p_0 \left\{ \left(\frac{V}{V + V_0} \right)^n + \frac{nV_0}{V} \right\}$$

Here $p > p'$ for any n, since during delivery, the pump at each double stroke sucks in air with a pressure p', and during evacuation, volume V_0 of the air is being pumped out at pressures below p_0.

SAMPLE PROBLEM 20.02

Application of gas laws

A horizontally kept cylindrical vessel closed at both ends is separated into two equal (42 cm each) parts by an adiabatic piston. Both parts contain gas having equal mass at a temperature of 27°C and a pressure of 1 atm. How much should the gas be heated in one part of the cylinder to shift the piston by 2 cm? Find the pressure p of the gas after shifting of the piston.

KEY IDEA

Since the gas in both ends is having equal mass and the piston is in equilibrium,

$$\frac{V_2}{V_1} = \frac{T_2}{T_1}$$

Hence, $$T_2 = \frac{V_2}{V_1} T_1$$

Considering l_1 and l_2 as the new lengths of the cylinder,

$$V_2 = \pi r^2 l_2 \quad \text{and} \quad V_1 = \pi r^2 l_1$$

Calculations: Calculating T_2,

$$T_2 = \frac{V_2}{V_1} T_1 = \frac{l_2}{l_1} T_1$$

$$= \frac{44}{40} \times 27°\text{C} = 30°\text{C} = 330\ \text{K}$$

Applying Boyle's law to the volume of the gas whose temperature does not change, we obtain,

$$p = \frac{p_0 V_0}{V_1}$$

Using the values, $p_0 = 1$ atm, $V_0 = 44\pi r^2$ and $V_1 = 42\pi r^2$, we get

$$p = \frac{1 \times 44\pi r^2}{42\pi r^2} = 1.05\ \text{atm} \qquad \text{(Answer)}$$

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SAMPLE PROBLEM 20.03

Temperature of air in a tire

Before starting out on a long drive, you check the air in your car's tires to make sure they are properly inflated. The pressure gauge reads 209 kPa, and the temperature is 37°C. After a few hours of highway driving, you stop and check the pressure again. Now the gauge reads 239 kPa. What is the temperature of the air in the tires now? Assume that the volume of the tire remains constant.

KEY IDEAS

As a common experience we know that after driving a vehicle for a while the temperature of it's tires increase. We must use only absolute temperatures and absolute pressures when using the ideal gas law. The pressure gauge reads gauge pressure, hence to get absolute pressure we add, 1 atm = 101 kPa.

Given that expansion of tire is negligible, we can assume that it is isovolumetric process.

Calculations: The number of moles n are constant as long as the tire does not leak. The volume may actually change a bit as the tire warms up and expands, but this change is to be neglected. Since n and V are constant, we can rewrite the ideal gas law as a proportionality between P and T. We treat the air in the tire as an ideal gas. First convert the initial and final gauge pressures to absolute pressures:

$$p_i = 209\ \text{kPa} + 101\ \text{kPa} = 310\ \text{kPa}$$

$$p_f = 239\ \text{kPa} + 101\ \text{kPa} = 340\ \text{kPa}$$

Now convert the initial temperature to absolute temperature:

$$T_i = 37°\text{C} + 273\ \text{K} = 310\ \text{K}$$

According to the ideal gas law, pressure is proportional to temperature at constant volume (Gay-Lussac law), so

$$\frac{T_f}{T_i} = \frac{p_f}{p_i}$$

Then,
$$T_f = \frac{p_f}{p_i} T_i = \frac{340\ \text{kPa}}{310\ \text{kPa}} \times 310\ \text{K} = 340\ \text{K}$$

Now convert back to °C:

$$340\ \text{K} - 273\ \text{K} = 67°\text{C} \qquad \text{(Answer)}$$

Note: The final answer of 67°C seems reasonable since, after a long drive, the tires are noticeably hot.

CHECKPOINT 1

Suppose in the above problem, you decide to leak some air from the tires, since the manufacturer suggests keeping the pressure at 210 kPa. (The manufacturer's specification are for the tires which are "cold.") If you let out sufficient air so that the pressure returns to 210 kPa, what percentage of the air molecules did you let out of the tires? Suppose that you leave the car and go for a lunch and when you come back the tires have cooled down to 37°C. Now what is the gauge pressure?

20.4 | PRESSURE, TEMPERATURE, AND RMS SPEED

Key Concepts

- In terms of the speed of the gas molecules, the pressure exerted by n moles of an ideal gas is

$$p = \frac{nMv_{\text{rms}}^2}{3V},$$

where $v_{\text{rms}} = \sqrt{(v^2)_{\text{avg}}}$ is the root-mean-square speed of the molecules, M is the molar mass, and V is the volume.

- The rms speed can be written in terms of the temperature as

$$v_{\text{rms}} = \sqrt{\frac{3RT}{M}}.$$

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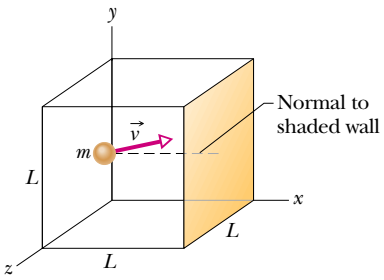

Figure 20-5 A cubical box of edge length L, containing n moles of an ideal gas. A molecule of mass m and velocity $\vec{v}$ is about to collide with the shaded wall of area L^2. A normal to that wall is shown.

Here is our first kinetic theory problem. Let n moles of an ideal gas be confined in a cubical box of volume V, as in Fig. 20-5. The walls of the box are held at temperature T. What is the connection between the pressure p exerted by the gas on the walls and the speeds of the molecules?

The molecules of gas in the box are moving in all directions and with various speeds, bumping into one another and bouncing from the walls of the box like balls in a racquetball court. We ignore (for the time being) collisions of the molecules with one another and consider only elastic collisions with the walls.

Figure 20-5 shows a typical gas molecule, of mass m and velocity $\vec{v}$, that is about to collide with the shaded wall. Because we assume that any collision of a molecule with a wall is elastic, when this molecule collides with the shaded wall, the only component of its velocity that is changed is the x component, and that component is reversed. This means that the only change in the particle's momentum is along the x axis, and that change is

$$\Delta p_x = (-mv_x) - (mv_x) = -2mv_x.$$

Hence, the momentum Δp_x delivered to the wall by the molecule during the collision is $+2mv_x$. (Because in this book the symbol p represents both momentum and pressure, we must be careful to note that here p represents momentum and is a vector quantity.)

The molecule of Fig. 20-5 will hit the shaded wall repeatedly. The time Δt between collisions is the time the molecule takes to travel to the opposite wall and back again (a distance $2L$) at speed v_x. Thus, Δt is equal to $2L/v_x$. (Note that this result holds even if the molecule bounces off any of the other walls along the way, because those walls are parallel to x and so cannot change v_x.) Therefore, the average rate at which momentum is delivered to the shaded wall by this single molecule is

$$\frac{\Delta p_x}{\Delta t} = \frac{2mv_x}{2L/v_x} = \frac{mv_x^2}{L}.$$

From Newton's second law ($\vec{F} = d\vec{p}/dt$), the rate at which momentum is delivered to the wall is the force acting on that wall. To find the total force, we must add up the contributions of all the molecules that strike the wall, allowing for the possibility that they all have different speeds. Dividing the magnitude of the total force F_x by the area of the wall ($= L^2$) then gives the pressure p on that wall, where now and in the rest of this discussion, p represents pressure. Thus, using the expression for $\Delta p_x/\Delta t$, we can write this pressure as

$$\begin{aligned} p &= \frac{F_x}{L^2} = \frac{mv_{x1}^2/L + mv_{x2}^2/L + \cdots + mv_{xN}^2/L}{L^2} \\ &= \left(\frac{m}{L^3}\right)(v_{x1}^2 + v_{x2}^2 + \cdots + v_{xN}^2), \end{aligned} \tag{20-10}$$

where N is the number of molecules in the box.

Since $N = nN_A$, there are nN_A terms in the second set of parentheses of Eq. 20-10. We can replace that quantity by $nN_A(v_x^2)_{avg}$, where $(v_x^2)_{avg}$ is the average value of the square of the x components of all the molecular speeds. Equation 20-10 then becomes

$$p = \frac{nmN_A}{L^3}(v_x^2)_{avg}.$$

However, mN_A is the molar mass M of the gas (that is, the mass of 1 mol of the gas). Also, L^3 is the volume of the box, so

$$p = \frac{nM(v_x^2)_{avg}}{V}. \tag{20-11}$$

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For any molecule, $v^2 = v_x^2 + v_y^2 + v_z^2$. Because there are many molecules and because they are all moving in random directions, the average values of the squares of their velocity components are equal, so that $v_x^2 = (1/3)v^2$. Thus, Eq. 20-11 becomes

$$p = \frac{nM(v^2)_{avg}}{3V}. \tag{20-12}$$

The square root of $(v^2)_{avg}$ is a kind of average speed, called the **root-mean-square speed** of the molecules and symbolized by v_{rms}. Its name describes it rather well: You *square* each speed, you find the *mean* (that is, the average) of all these squared speeds, and then you take the square *root* of that mean. With $\sqrt{(v^2)_{avg.}} = v_{rms}$, we can then write Eq. 20-12 as

$$p = \frac{nMv_{rms}^2}{3V}. \tag{20-13}$$

This tells us how the pressure of the gas (a purely macroscopic quantity) depends on the speed of the molecules (a purely microscopic quantity).

We can turn Eq. 20-13 around and use it to calculate v_{rms}. Combining Eq. 20-13 with the ideal gas law ($pV = nRT$) leads to

$$v_{rms} = \sqrt{\frac{3RT}{M}}. \tag{20-14}$$

Table 20-1 shows some rms speeds calculated from Eq. 20-14. The speeds are surprisingly high. For hydrogen molecules at room temperature (300 K), the rms speed is 1920 m/s, or 4300 mi/h—faster than a speeding bullet! On the surface of the Sun, where the temperature is 2×10^6 K, the rms speed of hydrogen molecules would be 82 times greater than at room temperature were it not for the fact that at such high speeds, the molecules cannot survive collisions among themselves. Remember too that the rms speed is only a kind of average speed; many molecules move much faster than this, and some much slower.

The speed of sound in a gas is closely related to the rms speed of the molecules of that gas. In a sound wave, the disturbance is passed on from molecule to molecule by means of collisions. The wave cannot move any faster than the "average" speed of the molecules. In fact, the speed of sound must be somewhat less than this "average" molecular speed because not all molecules are moving in exactly the same direction as the wave. As examples, at room temperature, the rms speeds of hydrogen and nitrogen molecules are 1920 m/s and 517 m/s, respectively. The speeds of sound in these two gases at this temperature are 1350 m/s and 350 m/s, respectively.

A question often arises: If molecules move so fast, why does it take as long as a minute or so before you can smell perfume when someone opens a bottle across a room? The answer is that, each perfume molecule may have a high speed but it moves away from the bottle only very slowly because its repeated collisions with other molecules prevent it from moving directly across the room to you.

Table 20-1 Some RMS Speeds at Room Temperature ($T = 300$ K)[a]

Gas	Molar Mass (10^{-3} kg/mol)	v_{rms} (m/s)
Hydrogen (H_2)	2.02	1920
Helium (He)	4.0	1370
Water vapor (H_2O)	18.0	645
Nitrogen (N_2)	28.0	517
Oxygen (O_2)	32.0	483
Carbon dioxide (CO_2)	44.0	412
Sulfur dioxide (SO_2)	64.1	342

[a]For convenience, we often set room temperature equal to 300 K even though (at 27°C or 81°F) that represents a fairly warm room.

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SAMPLE PROBLEM 20.04

Average and rms values

Here are five numbers: 5, 11, 32, 67, and 89.

(a) What is the average value n_{avg} of these numbers?

Calculation: We find this from

$$n_{avg} = \frac{5+11+32+67+89}{5} = 40.8. \qquad \text{(Answer)}$$

(b) What is the rms value n_{rms} of these numbers?

Calculation: We find this from

$$n_{rms} = \sqrt{\frac{5^2+11^2+32^2+67^2+89^2}{5}}$$

$$= 52.1. \qquad \text{(Answer)}$$

The rms value is greater than the average value because the larger numbers—being squared—are relatively more important in forming the rms value.

20.5 | TRANSLATIONAL KINETIC ENERGY

Key Concepts

- The average translational kinetic energy per molecule in an ideal gas is

$$K_{avg} = \frac{1}{2}mv_{rms}^2.$$

- The average translational kinetic energy is related to the temperature of the gas:

$$K_{avg} = \frac{3}{2}kT.$$

We again consider a single molecule of an ideal gas as it moves around in the box of Fig. 20-5, but we now assume that its speed changes when it collides with other molecules. Its translational kinetic energy at any instant is $1/2mv^2$. Its *average* translational kinetic energy over the time that we watch it is

$$K_{avg} = \left(\frac{1}{2}mv^2\right)_{avg} = \frac{1}{2}m(v^2)_{avg} = \frac{1}{2}mv_{rms}^2, \qquad (20\text{-}15)$$

in which we make the assumption that the average speed of the molecule during our observation is the same as the average speed of all the molecules at any given time. (Provided the total energy of the gas is not changing and provided we observe our molecule for long enough, this assumption is appropriate.) Substituting for v_{rms} from Eq. 20-14 leads to

$$K_{avg} = \left(\frac{1}{2}m\right)\frac{3RT}{M}.$$

However, M/m, the molar mass divided by the mass of a molecule, is simply Avogadro's number. Thus,

$$K_{avg} = \frac{3RT}{2N_A}.$$

Using Eq. 20-7 ($k = R/N_A$), we can then write

$$K_{avg} = \frac{3}{2}kT. \qquad (20\text{-}16)$$

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This equation tells us something unexpected:

At a given temperature T, all ideal gas molecules—no matter what their mass—have the same average translational kinetic energy—namely, $3/2kT$. When we measure the temperature of a gas, we are also measuring the average translational kinetic energy of its molecules.

CHECKPOINT 2

A gas mixture consists of molecules of types 1, 2, and 3, with molecular masses $m_1 > m_2 > m_3$. Rank the three types according to (a) average kinetic energy and (b) rms speed, greatest first.

20.6 | THE DISTRIBUTION OF MOLECULAR SPEEDS

Key Concepts

- The Maxwell speed distribution $P(v)$ is a function such that $P(v)\,dv$ gives the fraction of molecules with speeds in the interval dv at speed v:

$$P(v) = 4\pi\left(\frac{M}{2\pi RT}\right)^{3/2} v^2 e^{-Mv^2/2RT}.$$

- Three measures of the distribution of speeds among the molecules of a gas are

$$v_{avg} = \sqrt{\frac{8RT}{\pi M}} \quad \text{(average speed)},$$

$$v_P = \sqrt{\frac{2RT}{M}} \quad \text{(most probable speed)},$$

and

$$v_{rms} = \sqrt{\frac{3RT}{M}} \quad \text{(rms speed)}.$$

The root-mean-square speed v_{rms} gives us a general idea of molecular speeds in a gas at a given temperature. We often want to know more. For example, what fraction of the molecules have speeds greater than the rms value? What fraction have speeds greater than twice the rms value? To answer such questions, we need to know how the possible values of speed are distributed among the molecules. Figure 20-6*a* shows this distribution for oxygen molecules at room temperature ($T = 300$ K); Fig. 20-6*b* compares it with the distribution at $T = 80$ K.

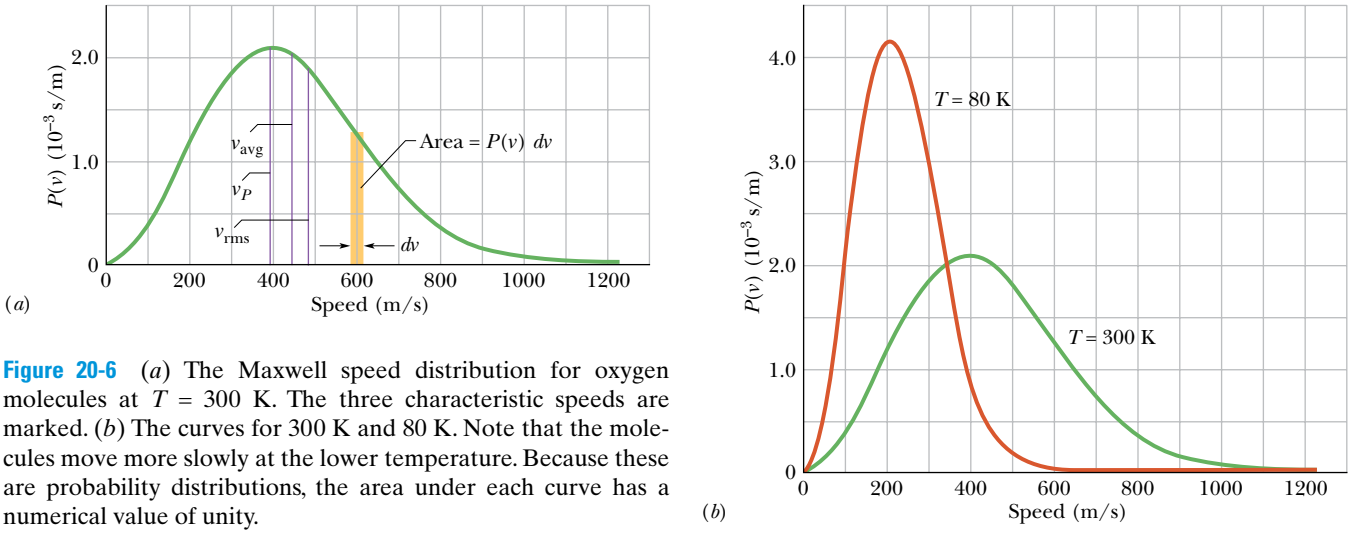

Figure 20-6 (*a*) The Maxwell speed distribution for oxygen molecules at $T = 300$ K. The three characteristic speeds are marked. (*b*) The curves for 300 K and 80 K. Note that the molecules move more slowly at the lower temperature. Because these are probability distributions, the area under each curve has a numerical value of unity.

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In 1852, Scottish physicist James Clerk Maxwell first solved the problem of finding the speed distribution of gas molecules. His result, known as **Maxwell's speed distribution law**, is

$$P(v) = 4\pi\left(\frac{M}{2\pi RT}\right)^{3/2} v^2 e^{-Mv^2/2RT}. \tag{20-17}$$

Here M is the molar mass of the gas, R is the gas constant, T is the gas temperature, and v is the molecular speed. It is this equation that is plotted in Fig. 20-6*a*, *b*. The quantity $P(v)$ in Eq. 20-17 and Fig. 20-6 is a *probability distribution function:* For any speed v, the product $P(v)\,dv$ (a dimensionless quantity) is the fraction of molecules with speeds in the interval dv centered on speed v.

As Fig. 20-6*a* shows, this fraction is equal to the area of a strip with height $P(v)$ and width dv. The total area under the distribution curve corresponds to the fraction of the molecules whose speeds lie between zero and infinity. All molecules fall into this category, so the value of this total area is unity; that is,

$$\int_0^\infty P(v)\,dv = 1. \tag{20-18}$$

The fraction (frac) of molecules with speeds in an interval of, say, v_1 to v_2 is then

$$\text{frac} = \int_{v_1}^{v_2} P(v)\,dv. \tag{20-19}$$

Average, RMS, and Most Probable Speeds

In principle, we can find the **average speed** v_{avg} of the molecules in a gas with the following procedure: We *weight* each value of v in the distribution; that is, we multiply it by the fraction $P(v)\,dv$ of molecules with speeds in a differential interval dv centered on v. Then we add up all these values of $v\,P(v)\,dv$. The result is v_{avg}. In practice, we do all this by evaluating

$$v_{avg} = \int_0^\infty v\,P(v)\,dv. \tag{20-20}$$

Substituting for $P(v)$ from Eq. 20-17 and using generic integral 20 from the list of integrals in Appendix E, we find

$$v_{avg} = \sqrt{\frac{8RT}{\pi M}} \quad \text{(average speed)}. \tag{20-21}$$

Similarly, we can find the average of the square of the speeds $(v^2)_{avg}$ with

$$(v^2)_{avg} = \int_0^\infty v^2 P(v)\,dv. \tag{20-22}$$

Substituting for $P(v)$ from Eq. 20-17 and using generic integral 16 from the list of integrals in Appendix E, we find

$$(v^2)_{avg} = \frac{3RT}{M}. \tag{20-23}$$

The square root of $(v^2)_{avg}$ is the root-mean-square speed v_{rms}. Thus,

$$v_{rms} = \sqrt{\frac{3RT}{M}} \quad \text{(rms speed)}, \tag{20-24}$$

which agrees with Eq. 20-14.

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The **most probable speed** v_P is the speed at which $P(v)$ is maximum (see Fig. 20-6*a*). To calculate v_P, we set $dP/dv = 0$ (the slope of the curve in Fig. 20-6*a* is zero at the maximum of the curve) and then solve for v. Doing so, we find

$$v_P = \sqrt{\frac{2RT}{M}} \quad \text{(most probable speed)}. \tag{20-25}$$

A molecule is more likely to have speed v_P than any other speed, but some molecules will have speeds that are many times v_P. These molecules lie in the *high-speed tail* of a distribution curve like that in Fig. 20-6*a*. Such higher speed molecules make possible both rain and sunshine (without which we could not exist):

Rain. The speed distribution of water molecules in, say, a pond at summer-time temperatures can be represented by a curve similar to that of Fig. 20-6*a*. Most of the molecules lack the energy to escape from the surface. However, a few of the molecules in the high-speed tail of the curve can do so. It is these water molecules that evaporate, making clouds and rain possible.

As the fast water molecules leave the surface, carrying energy with them, the temperature of the remaining water is maintained by heat transfer from the surroundings. Other fast molecules—produced in particularly favorable collisions—quickly take the place of those that have left, and the speed distribution is maintained.

Sunshine. Let the distribution function of Eq. 20-17 now refer to protons in the core of the Sun. The Sun's energy is supplied by a nuclear fusion process that starts with the merging of two protons. However, protons repel each other because of their electrical charges, and protons of average speed do not have enough kinetic energy to overcome the repulsion and get close enough to merge. Very fast protons with speeds in the high-speed tail of the distribution curve can do so, however, and for that reason the Sun can shine.

SAMPLE PROBLEM 20.05

Speed distribution in a gas

In oxygen (molar mass M = 0.0320 kg/mol) at room temperature (300 K), what fraction of the molecules have speeds in the interval 599 to 601 m/s?

KEY IDEAS

(1) The speeds of the molecules are distributed over a wide range of values, with the distribution $P(v)$ of Eq. 20-17.
(2) The fraction of molecules with speeds in a differential interval dv is $P(v)\,dv$.
(3) For a larger interval, the fraction is found by integrating $P(v)$ over the interval.
(4) However, the interval Δv = 2 m/s here is small compared to the speed v = 600 m/s on which it is centered.

Calculations: Because Δv is small, we can avoid the integration by approximating the fraction as

$$\text{frac} = P(v)\,\Delta v = 4\pi\left(\frac{M}{2\pi RT}\right)^{3/2} v^2 e^{-Mv^2/2RT}\,\Delta v.$$

The total area under the plot of $P(v)$ in Fig. 20-6*a* is the total fraction of molecules (unity), and the area of the thin gold strip (not to scale) is the fraction we seek. Let's evaluate frac in parts:

$$\text{frac} = (4\pi)(A)(v^2)(e^B)(\Delta v), \tag{20-26}$$

where

$$A = \left(\frac{M}{2\pi RT}\right)^{3/2} = \left(\frac{0.0320\ \text{kg/mol}}{(2\pi)(8.31\ \text{J/mol}\cdot\text{K})(300\ \text{K})}\right)^{3/2}$$
$$= 2.92\times10^{-9}\ \text{s}^3/\text{m}^3$$

and

$$B = -\frac{Mv^2}{2RT} = -\frac{(0.0320\ \text{kg/mol})(600\ \text{m/s})^2}{(2)(8.31\ \text{J/mol}\cdot\text{K})(300\ \text{K})}$$
$$= -2.31.$$

Substituting A and B into Eq. 20-26 yields

$$\text{frac} = (4\pi)(A)(v^2)(e^B)(\Delta v)$$
$$= (4\pi)(2.92\times10^{-9}\ \text{s}^3/\text{m}^3)(600\ \text{m/s})^2(e^{-2.31})(2\ \text{m/s})$$
$$= 2.62\times10^{-3} = 0.262\%. \quad \text{(Answer)}$$

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SAMPLE PROBLEM 20.06

Average speed, rms speed, most probable speed

The molar mass M of oxygen is 0.0320 kg/mol.

(a) What is the average speed v_{avg} of oxygen gas molecules at $T = 300$ K?

KEY IDEA

To find the average speed, we must weight speed v with the distribution function $P(v)$ of Eq. 20-17 and then integrate the resulting expression over the range of possible speeds (from zero to the limit of an infinite speed).

Calculation: We end up with Eq. 20-21, which gives us

$$v_{avg} = \sqrt{\frac{8RT}{\pi M}} = \sqrt{\frac{8(8.31\ \text{J/mol}\cdot\text{K})(300\ \text{K})}{\pi(0.0320\ \text{kg/mol})}} = 445\ \text{m/s}. \quad \text{(Answer)}$$

This result is plotted in Fig. 20-6*a*.

(b) What is the root-mean-square speed v_{rms} at 300 K?

KEY IDEA

To find v_{rms}, we must first find $(v^2)_{avg}$ by weighting v^2 with the distribution function $P(v)$ of Eq. 20-17 and then integrating the expression over the range of possible speeds. Then we must take the square root of the result.

Calculation: We end up with Eq. 20-24, which gives us

$$v_{rms} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3(8.31\ \text{J/mol}\cdot\text{K})(300\ \text{K})}{0.0320\ \text{kg/mol}}} = 483\ \text{m/s}. \quad \text{(Answer)}$$

This result, plotted in Fig. 20-6*a*, is greater than v_{avg} because the greater speed values influence the calculation more when we integrate the v^2 values than when we integrate the v values.

(c) What is the most probable speed v_P at 300 K?

KEY IDEA

Speed v_P corresponds to the maximum of the distribution function $P(v)$, which we obtain by setting the derivative $dP/dv = 0$ and solving the result for v.

Calculation: We end up with Eq. 20-25, which gives us

$$v_P = \sqrt{\frac{2RT}{M}} = \sqrt{\frac{2(8.31\ \text{J/mol}\cdot\text{K})(300\ \text{K})}{0.0320\ \text{kg/mol}}} = 395\ \text{m/s}. \quad \text{(Answer)}$$

This result is also plotted in Fig. 20-6*a*.

SAMPLE PROBLEM 20.07

Average speed and rms speed

Five molecules have speeds of 12, 16, 32, 40, and 48 m/s. Find (a) the average speed v_{avg}, and (b) the root-mean-square speed v_{rms}, for these molecules. (c) Show that for any distribution of speeds, $v_{rms} \geq v_{avg}$.

Calculations:

(a) $$v_{avg} = \frac{12 + 16 + 32 + 40 + 48}{5} = 29.6\ \text{m/s} \quad \text{(Answer)}$$

(b) $$v_{rms} = \left[\frac{12^2 + 16^2 + 32^2 + 40^2 + 48^2}{5}\right]^{1/2} = 32.6\ \text{m/s} \quad \text{(Answer)}$$

(c) Let v_i, for $i = 1 \ldots, N$, represent the various speeds.

Then, $$v_{avg} = \frac{1}{N}\sum_{i=1}^{N} v_i \quad \text{and} \quad v_{rms}^2 = \frac{1}{N}\sum_{i=1}^{N} v_i^2$$

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Let's consider a quantity

$$A = \frac{1}{N}\sum_{i=1}^{N}(v_i - v_{avg})^2$$

which is the average of the squares of the deviations of the individual speeds from the average speed. *This quantity can never be negative*. By squaring and opening the bracket, and noting

$$(v_i - v_{avg})^2 = v_i^2 - 2v_{avg}v_i + v_{avg}^2$$

we have

$$\begin{aligned} A &= \frac{1}{N}\sum_{i=1}^{N}v_i^2 - \frac{2v_{avg}}{N}\sum_{i=1}^{N}v_i + \frac{1}{N}(Nv_{avg}^2) \\ &= v_{rms}^2 - 2v_{avg}^2 + v_{avg}^2 \\ &= v_{rms}^2 - v_{avg}^2 \end{aligned}$$

This implies that

$$v_{rms}^2 - v_{avg}^2 \geq 0$$

Thus, $v_{rms} \geq v_{avg}$

When all the v_i are equal, then $v_{rms} \geq v_{avg}$.

SAMPLE PROBLEM 20.08

K_{avg} and v_{rms} of oxygen and nitrogen molecules present in air

(a) What are the average translation kinetic energies K_{avg} of the oxygen molecules and nitrogen molecules in the air that you breathe? Assume that air is an ideal gas at a temperature of 300 K, which is the traditional value taken for room temperature.

Calculation: The key idea here is that, for an ideal gas, K_{avg} depends only on the temperature and not on the type of molecule. From Eq. 20-16, the value of K_{avg} for both oxygen and nitrogen molecules is

$$\begin{aligned} K_{avg} &= \frac{3}{2}kT = \frac{3}{2}(1.38\times10^{-23}\ \text{J/K})(300\ \text{K}) \\ &= 6.21\times10^{-21}\ \text{J} \end{aligned} \qquad \text{(Answer)}$$

(b) The mass of an oxygen molecule differs from that of a nitrogen molecule. Do the oxygen molecules and the nitrogen molecules in air have the same rms speed v_{rms}?

KEY IDEA

We know that v_{rms} is related to Kavg by equation.

$$K_{avg} = \frac{1}{2}mv_{rms}^2$$

Although the two types of molecules have the same K_{avg}, they have different masses m and thus their rms speeds must differ.

To find their rms speeds, we solve Eq. 20-15 for v_{rms} and then substitute for molecular mass m with equation $M = mN_A$, because molar masses M can be found in Appendix F. We obtain

$$v_{rms} = \sqrt{\frac{2K_{avg}N_A}{M}} \qquad (20\text{-}27)$$

Calculations: In the air that we breathe, oxygen and nitrogen are diatomic molecules (O_2 and N_2) – each molecule consists of two atoms. The molar mass M_{O_2} of diatomic oxygen is 2(15.9994 g/mol) ≈ 0.0320 kg/mol and the molar mass M_{N_2} of diatomic nitrogen is 2(14.0067 g/mol) ≈ 0.0280 kg/mol.

Now we can substitute known data into Eq. 20-27. For oxygen, we find

$$\begin{aligned} v_{rms} &= \sqrt{\frac{2(6.21\times10^{-21}\ \text{J})(6.02\times10^{23}\ \text{mol}^{-1})}{0.0320\ \text{kg/mol}}} \\ &= 483\ \text{m/s} \end{aligned} \qquad \text{(Answer)}$$

Similarly, for nitrogen we find

$$v_{rms} = 517\ \text{m/s} \qquad \text{(Answer)}$$

The nitrogen molecules, with their smaller mass (and the same average kinetic energy), have the greater rms speed.

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20.7 | THE EQUIPARTITION OF ENERGY AND DEGREES OF FREEDOM

Key Concepts

- We find C_V by using the equipartition of energy theorem, which states that every degree of freedom of a molecule (that is, every independent way it can store energy) has associated with it—on average—an energy $(1/2)kT$ per molecule $(= (1/2)RT$ per mole).
- If f is the number of degrees of freedom, then $E_{int} = (f/2)nRT$ and

$$C_V = \left(\frac{f}{2}\right)R = 4.16f \text{ J/mol·K}.$$

- For monatomic gases $f = 3$ (three translational degrees); for diatomic gases $f = 5$ (three translational and two rotational degrees).

Internal Energy E_{int}

Let us first assume that our ideal gas is a *monatomic gas* (individual atoms rather than molecules), such as helium, neon, or argon. Let us also assume that the internal energy E_{int} is the sum of the translational kinetic energies of the atoms. (Quantum theory disallows rotational kinetic energy for individual atoms.)

The average translational kinetic energy of a single atom depends only on the gas temperature and is given by Eq. 20-16 as $K_{avg} = (3/2)kT$. A sample of n moles of such a gas contains nN_A atoms. The internal energy E_{int} of the sample is then

$$E_{int} = (nN_A)K_{avg} = (nN_A)\left(\frac{3}{2}kT\right). \tag{20-28}$$

Using Eq. 20-7 ($k = R/N_A$), we can rewrite this as

$$E_{int} = \frac{3}{2}nRT \quad \text{(monatomic ideal gas)}. \tag{20-29}$$

The internal energy E_{int} of an ideal gas is a function of the gas temperature *only*; it does not depend on any other variable.

Degrees of Freedom

As Table 20-2 shows, the prediction that $C_V = 3R/2$ agrees with experiment for monatomic gases but fails for diatomic and polyatomic gases. Let us try to explain the discrepancy by considering the possibility that molecules with more than one atom can store internal energy in forms other than translational kinetic energy.

Figure 20-7 shows common models of helium (a *monatomic* molecule, containing a single atom), oxygen (a *diatomic* molecule, containing two atoms), and methane (a *polyatomic* molecule). From such models, we would assume that all three types of molecules can have translational motions (say, moving left–right and up–down) and rotational motions (spinning about an axis like a top). In addition, we would assume that the diatomic and polyatomic molecules can have oscillatory motions, with the atoms oscillating slightly toward and away from one another, as if attached to opposite ends of a spring.

To keep account of the various ways in which energy can be stored in a gas, James Clerk Maxwell introduced the theorem of the **equipartition of energy**:

Every kind of molecule has a certain number f of *degrees of freedom,* which are independent ways in which the molecule can store energy. Each such degree of freedom has associated with it—on average—an energy of $(1/2)kT$ per molecule (or $(1/2)RT$ per mole).

Table 20-2 Molar Specific Heats at Constant Volume

Molecule	Example	C_V (J/mol·K)
Monatomic	Ideal	$\frac{3}{2}R = 12.5$
	Real	He 12.5
		Ar 12.6
Diatomic	Ideal	$\frac{5}{2}R = 20.8$
	Real	N_2 20.7
		O_2 20.8
Polyatomic	Ideal	$3R = 24.9$
	Real	NH_4 29.0
		CO_2 29.7

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Let us apply the theorem to the translational and rotational motions of the molecules in Fig. 20-7. (We discuss oscillatory motion below.) For the translational motion, superimpose an *xyz* coordinate system on any gas. The molecules will, in general, have velocity components along all three axes. Thus, gas molecules of all types have three degrees of translational freedom (three ways to move in translation) and, on average, an associated energy of $3(kT/2)$ per molecule.

For the rotational motion, imagine the origin of our *xyz* coordinate system at the center of each molecule in Fig. 20-7. In a gas, each molecule should be able to rotate with an angular velocity component along each of the three axes, so each gas should have three degrees of rotational freedom and, on average, an additional energy of $3(kT/2)$ per molecule. *However,* experiment shows this is true only for the polyatomic molecules. According to *quantum theory,* the physics dealing with the allowed motions and energies of molecules and atoms, a monatomic gas molecule does not rotate and so has no rotational energy (a single atom cannot rotate like a top). A diatomic molecule can rotate like a top only about axes perpendicular to the line connecting the atoms (the axes are shown in Fig. 20-7*b*) and not about that line itself. Therefore, a diatomic molecule can have only two degrees of rotational freedom and a rotational energy of only $2(kT/2)$ per molecule.

To extend our analysis of molar specific heats (C_p and C_V) to ideal diatomic and polyatomic gases, it is necessary to retrace the derivations of that analysis in detail. First, we replace Eq. 20-29 ($E_{int} = 3nRT/2$) with $E_{int} = (f/2)nRT$, where f is the number of degrees of freedom listed in Table 20-3. Doing so leads to the prediction.

$$C_V = \left(\frac{f}{2}\right)R = 4.16f \text{ J/mol}\cdot\text{K}, \qquad (20\text{-}30)$$

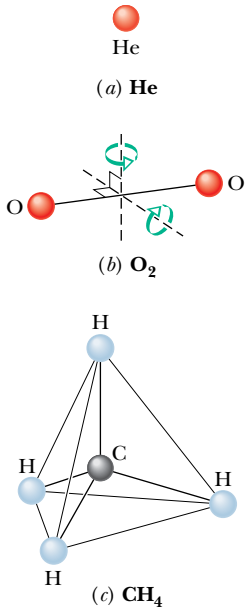

Figure 20-7 Models of molecules as used in kinetic theory: (*a*) helium, a typical monatomic molecule; (*b*) oxygen, a typical diatomic molecule; and (*c*) methane, a typical polyatomic molecule. The spheres represent atoms, and the lines between them represent bonds. Two rotation axes are shown for the oxygen molecule.

As Table 20-2 shows, this prediction also agrees with experiment for diatomic gases ($f = 5$), but it is too low for polyatomic gases ($f = 6$ for molecules comparable to CH_4).

Table 20-3 Degrees of Freedom for Various Molecules

Molecule	Example	Degrees of Freedom			Predicted Molar Specific Heats	
		Translational	Rotational	Total (f)	C_V (Eq. 20-30)	$C_p = C_V + R$
Monatomic	He	3	0	3	$\frac{3}{2}R$	$\frac{5}{2}R$
Diatomic	O_2	3	2	5	$\frac{5}{2}R$	$\frac{7}{2}R$
Polyatomic	CH_4	3	3	6	$3R$	$4R$

20.8 | A HINT OF QUANTUM THEORY

We can improve the agreement of kinetic theory with experiment by including the oscillations of the atoms in a gas of diatomic or polyatomic molecules. For example, the two atoms in the O_2 molecule of Fig. 20-7*b* can oscillate toward and away from each other, with the interconnecting bond acting like a spring. However, experiment shows that such oscillations occur only at relatively high temperatures of the gas—the motion is "turned on" only when the gas molecules have relatively large energies. Rotational motion is also subject to such "turning on," but at a lower temperature.

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Figure 20-8 is of help in seeing this turning on of rotational motion and oscillatory motion. The ratio C_V/R for diatomic hydrogen gas (H_2) is plotted there against temperature, with the temperature scale logarithmic to cover several orders of magnitude. Below about 80 K, we find that $C_V/R = 1.5$. This result implies that only the three translational degrees of freedom of hydrogen are involved in the specific heat.

As the temperature increases, the value of C_V/R gradually increases to 2.5, implying that two additional degrees of freedom have become involved. Quantum theory shows that these two degrees of freedom are associated with the rotational motion of the hydrogen molecules and that this motion requires a certain minimum amount of energy. At very low temperatures (below 80 K), the molecules do not have enough energy to rotate. As the temperature increases from 80 K, first a few molecules and then more and more of them obtain enough energy to rotate, and the value of C_V/R increases, until all of the molecules are rotating and $C_V/R = 2.5$.

Similarly, quantum theory shows that oscillatory motion of the molecules requires a certain (higher) minimum amount of energy. This minimum amount is not met until the molecules reach a temperature of about 1000 K, as shown in Fig. 20-8. As the temperature increases beyond 1000 K, more and more molecules have enough energy to oscillate and the value of C_V/R increases, until all of the molecules are oscillating and $C_V/R = 3.5$. (In Fig. 20-8, the plotted curve stops at 3200 K because there the atoms of a hydrogen molecule oscillate so much that they overwhelm their bond, and the molecule then *dissociates* into two separate atoms.)

The turning on of the rotation and vibration of the diatomic and polyatomic molecules is due to the fact that the energies of these motions are quantized, that is, restricted to certain values. There is a lowest allowed value for each type of motion. Unless the thermal agitation of the surrounding molecules provides those lowest amounts, a molecule simply cannot rotate or vibrate.

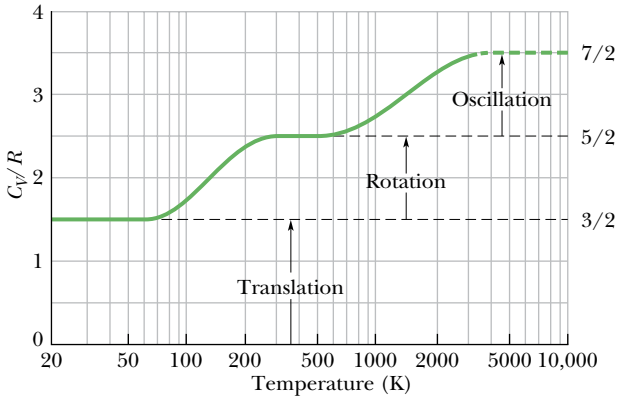

Figure 20-8 C_V/R versus temperature for (diatomic) hydrogen gas. Because rotational and oscillatory motions begin at certain energies, only translation is possible at very low temperatures. As the temperature increases, rotational motion can begin. At still higher temperatures, oscillatory motion can begin.

REVIEW AND SUMMARY

Kinetic Theory of Gases The *kinetic theory of gases* relates the *macroscopic* properties of gases (for example, pressure and temperature) to the *microscopic* properties of gas molecules (for example, speed and kinetic energy).

Avogadro's Number One mole of a substance contains N_A (*Avogadro's number*) elementary units (usually atoms or molecules), where N_A is found experimentally to be

$$N_A = 6.02 \times 10^{23}\ \text{mol}^{-1} \quad \text{(Avogadro's number)}. \quad (20\text{-}1)$$

One molar mass M of any substance is the mass of one mole of the substance. It is related to the mass m of the individual molecules of the substance by

$$M = mN_A. \quad (20\text{-}4)$$

The number of moles n contained in a sample of mass M_{sam}, consisting of N molecules, is given by

$$n = \frac{N}{N_A} = \frac{M_{sam}}{M} = \frac{M_{sam}}{mN_A}. \quad (20\text{-}2, 20\text{-}3)$$

Ideal Gas An *ideal gas* is one for which the pressure p, volume V, and temperature T are related by

$$pV = nRT \quad \text{(ideal gas law)}. \quad (20\text{-}5)$$

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Here n is the number of moles of the gas present and R is a constant (8.31 J/mol · K) called the **gas constant**. The ideal gas law can also be written as

$$pV = NkT, \tag{20-9}$$

where the **Boltzmann constant** k is

$$k = \frac{R}{N_A} = 1.38 \times 10^{-23} \text{ J/K}. \tag{20-7}$$

Pressure, Temperature, and Molecular Speed The pressure exerted by n moles of an ideal gas, in terms of the speed of its molecules, is

$$p = \frac{nMv_{rms}^2}{3V}, \tag{20-13}$$

where $v_{rms} = \sqrt{(v^2)_{avg}}$ is the **root-mean-square speed** of the molecules of the gas. With Eq. 20-5 this gives

$$v_{rms} = \sqrt{\frac{3RT}{M}}. \tag{20-14}$$

Temperature and Kinetic Energy The average translational kinetic energy K_{avg} per molecule of an ideal gas is

$$K_{avg} = \frac{3}{2}kT. \tag{20-16}$$

Maxwell Speed Distribution The *Maxwell speed distribution* $P(v)$ is a function such that $P(v)\,dv$ gives the *fraction* of molecules with speeds in the interval dv at speed v:

$$P(v) = 4\pi\left(\frac{M}{2\pi RT}\right)^{3/2} v^2 e^{-Mv^2/2RT}. \tag{20-17}$$

Three measures of the distribution of speeds among the molecules of a gas are

$$v_{avg} = \sqrt{\frac{8RT}{\pi M}} \quad \text{(average speed)}, \tag{20-21}$$

$$v_P = \sqrt{\frac{2RT}{M}} \quad \text{(most probable speed)}, \tag{20-25}$$

and the rms speed defined above in Eq. 20-14.

Degrees of Freedom and C_V The *equipartition of energy* theorem states that every *degree of freedom* of a molecule has an energy $(1/2)kT$ per molecule (= $(1/2)RT$ per mole). If f is the number of degrees of freedom, then $E_{int} = (f/2)nRT$ and

$$C_V = \left(\frac{f}{2}\right)R = 4.16f \text{ J/mol} \cdot \text{K} \tag{20-30}$$

For monatomic gases $f = 3$ (three translational degrees); for diatomic gases $f = 5$ (three translational and two rotational degrees).

PROBLEMS

1. The speeds of 22 particles are as follows (N_i represents the number of particles that have speed v_i):

N_i	2	4	6	8	2
v_i (cm/s)	1.0	2.0	3.0	5.0	6.0

What are (a) v_{avg}, (b) v_{rms}, and (c) v_P?

2. Figure 20-9 gives the probability distribution for nitrogen gas. The scale of the horizontal axis is set by $v_s = 2400$ m/s. What are the (a) gas temperature and (b) rms speed of the molecules?

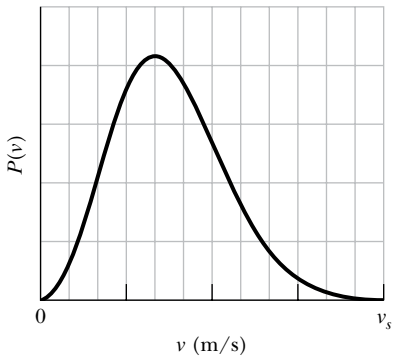

Figure 20-9 Problem 2.

3. Find the rms speed of argon atoms at 323 K. See Appendix F for the molar mass of argon atoms.

4. Water standing in the open at 10.0°C evaporates because of the escape of some of the surface molecules. The heat of vaporization (539 cal/g) is approximately equal to εn, where ε is the average energy of the escaping molecules and n is the number of molecules per gram. (a) Find ε. (b) What is the ratio of ε to the average kinetic energy of H_2O molecules, assuming the latter is related to temperature in the same way as it is for gases?

5. The speeds of 11 molecules are 2.0, 3.0, 4.0, …, 12 km/s. What are their (a) average speed and (b) rms speed?

6. A beam of hydrogen molecules (H_2) is directed toward a wall, at an angle of 32° with the normal to the wall. Each molecule in the beam has a speed of 1.0 km/s and a mass of 3.3×10^{-24} g. The beam strikes the wall over an area of 2.0 cm^2, at the rate of 4.0×10^{23} molecules per second. What is the beam's pressure on the wall?

7. Ten particles are moving with the following speeds: four at 300 m/s, two at 500 m/s, and four at 600 m/s. Calculate their (a) average and (b) rms speeds. (c) Is $v_{rms} > v_{avg}$?

8. What is the average translational kinetic energy of nitrogen molecules at 1500 K?

9. Suppose 10.0 g of oxygen (O_2) gas is heated at constant atmospheric pressure from 25.0°C to 125°C. (a) How many moles of oxygen are present? (See Table 20-1 for the molar mass.) (b) How much energy is transferred to the oxygen as heat? (The molecules rotate but do not oscillate.) (c) What fraction of the heat is used to raise the internal energy of the oxygen?

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10. The most probable speed of the molecules in a gas at temperature T_2 is equal to the average speed of the molecules at temperature T_1. Find T_2/T_1.

11. Two containers are at the same temperature. The first contains gas with pressure p_1, molecular mass m_1, and rms speed v_{rms1}. The second contains gas with pressure $1.5p_1$, molecular mass m_2, and average speed $v_{avg2} = 2.0v_{rms1}$. Find the mass ratio m_1/m_2.

12. Oxygen gas having a volume of 1200 cm^3 at 40.0°C and 1.01×10^5 Pa expands until its temperature is 493 k and its pressure is 1.06×10^5 Pa. Find (a) the number of moles of oxygen present and (b) the final volume of the sample.

13. *Water bottle in a hot car.* In the American Southwest, the temperature in a closed car parked in sunlight during the summer can be high enough to burn flesh. Suppose a bottle of water at a refrigerator temperature of 5.00°C is opened, then closed, and then left in a closed car with an internal temperature of 65.0°C. Neglecting the thermal expansion of the water and the bottle, find the pressure in the air pocket trapped in the bottle. (The pressure can be enough to push the bottle cap past the threads that are intended to keep the bottle closed.)

14. The lowest possible temperature in outer space is 2.7 K, which is associated with the microwave background radiation left over from the formation of the universe. What is the rms speed of hydrogen molecules at this temperature? (The molar mass is given in Table 20-1.)

15. A container encloses 1.5 mol of an ideal gas that has molar mass M_1 and 0.50 mol of a second ideal gas that has molar mass $M_2 = 3.0M_1$. What fraction of the total pressure on the container wall is attributable to the second gas? (The kinetic theory explanation of pressure leads to the experimentally discovered law of partial pressures for a mixture of gases that do not react chemically: *The total pressure exerted by the mixture is equal to the sum of the pressures that the several gases would exert separately if each were to occupy the vessel alone.* The molecule–vessel collisions of one type would not be altered by the presence of another type.)

16. Container A in Fig. 20-10 holds an ideal gas at a pressure of 8.0×10^5 Pa and a temperature of 250 K. It is connected by a thin tube (and a closed valve) to container B, with four times the volume of A. Container B holds the same ideal gas at a pressure of 1.0×10^5 Pa and a temperature of 450 K. The valve is opened to allow the pressures to equalize, but the temperature of each container is maintained. What then is the pressure?

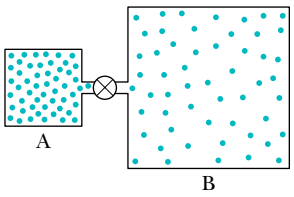

Figure 20-10 Problem 16.

17. Determine the average value of the translational kinetic energy of the molecules of an ideal gas at temperatures (a) 20.0°C and (b) 80.0°C. What is the translational kinetic energy per mole of an ideal gas at (c) 20.0°C and (d) 80.0°C?

18. At 546 K and 2.00×10^{-2} atm, the density of a gas is 1.24×10^{-5} g/cm^3. (a) Find v_{rms} for the gas molecules. (b) Find the molar mass of the gas and (c) identify the gas. See Table 20-1.

19. If vacuum techniques have reduced a gas to only 45 molecules/cm^3 when the temperature is 293 K, what is the pressure in the gas?

20. *Submarine rescue.* When the U.S. submarine *Squalus* became disabled at a depth of 80 m, a cylindrical chamber was lowered from a ship to rescue the crew. The chamber had a radius of 1.00 m and a height of 3.50 m, was open at the bottom, and held two rescuers. It slid along a guide cable that a diver had attached to a hatch on the submarine. Once the chamber reached the hatch and clamped to the hull, the crew could escape into the chamber. During the descent, air was released from tanks to prevent water from flooding the chamber. Assume that the interior air pressure matched the water pressure at depth h as given by $p_0 + \rho gh$, where $p_0 = 1.000$ atm is the surface pressure and $\rho = 1024$ kg/m^3 is the density of seawater. Assume a surface temperature of 20.0°C and a submerged water temperature of –40.0°C. (a) What is the air volume in the chamber at the surface? (b) If air had not been released from the tanks, what would have been the air volume in the chamber at depth $h = 80.0$ m? (c) How many moles of air were needed to be released to maintain the original air volume in the chamber?

21. An air bubble of volume 15 cm^3 is at the bottom of a lake 40 m deep, where the temperature is 4.0°C. The bubble rises to the surface, which is at a temperature of 20°C. Take the temperature of the bubble's air to be the same as that of the surrounding water. Just as the bubble reaches the surface, what is its volume?

22. What is the internal energy of 2.0 mol of an ideal monatomic gas at 273 K?

23. Gold has a molar mass of 197 g/mol. (a) How many moles of gold are in a 1.50 g sample of pure gold? (b) How many atoms are in the sample?

24. A quantity of ideal gas at 10.0°C and 100 kPa occupies a volume of 3.00 m^3. (a) How many moles of the gas are present? (b) If the pressure is now raised to 300 kPa and the temperature is raised to 30.0°C, how much volume does the gas occupy? Assume no leaks.

25. The temperature and pressure in the Sun's atmosphere are 2.00×10^6 K and 0.0300 Pa. Calculate the rms speed of free electrons (mass 9.11×10^{-31} kg) there, assuming they are an ideal gas.

26. Calculate the rms speed of helium atoms at 1200 K. See Appendix F for the molar mass of helium atoms.

27. Find the mass in kilograms of 3.23×10^{24} atoms of aluminum, which has a molar mass of 26.98 g/mol.

28. Compute (a) the number of moles and (b) the number of molecules in 1.00 cm^3 of an ideal gas at a pressure of 75.0 Pa and a temperature of 285 K.

29. (a) Compute the rms speed of a nitrogen molecule at 80.0°C. The molar mass of nitrogen molecules (N_2) is given in Table 20-1. At what temperatures will the rms speed be (b) half that value and (c) twice that value?

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PRACTICE QUESTIONS

Single Correct Choice Type

1. Which one of the following statements concerning the mole is false?
   (a) The mole is related to Avogadro's number.
   (b) The mole is defined in terms of the carbon-12 isotope.
   (c) The mole is the SI base unit for expressing the "amount" of a substance.
   (d) One mole of a substance has the same mass as one mole of any other substance.

2. According to the kinetic theory of gases, the pressure of a gas is due to
   (a) change of kinetic energy of molecules as they strike the wall.
   (b) change of momentum of molecules as they strike the wall.
   (c) average kinetic energy of the molecules.
   (d) force of repulsion between the molecules.

3. Which one of the following properties of a gas is not consistent with kinetic theory?
   (a) The average speed of the gas molecules is smaller at high temperatures.
   (b) Gas molecules are widely separated.
   (c) Gases fill whatever space is available to them.
   (d) Gas molecules move rapidly in a random fashion.

4. A closed vessel is maintained at a constant temperature. It is first evacuated and then vapor is injected into it continuously. The pressure of the vapor in the vessel
   (a) increases continuously.
   (b) first increases and then remains constant.
   (c) first increases and then decreases.
   (d) none of the above.

5. Which one of the following statements concerning a collection of gas molecules at a certain temperature is true?
   (a) All molecules move with the same velocity.
   (b) Most of the molecules have the same kinetic energy.
   (c) The lower the temperature, the greater are the molecular speeds.
   (d) The molecules have a range of kinetic energies.

6. Under which of the following conditions would you expect real gases to approach ideal behavior?
   (a) Low temperature and low pressure.
   (b) High temperature and high pressure.
   (c) High temperature and low pressure.
   (d) Low temperature and high pressure.

7. It is known that 28 grams of a certain ideal gas occupy 22.4 liters at standard conditions (0°C, 1 atm). The volume occupied by 42 grams of this gas at standard conditions is
   (a) 14.9 liters (b) 22.4 liters
   (c) 33.6 liters (d) 42 liters

8. The approximate number of air molecules in a 1 m^3 volume at room temperature (300 K) and atmospheric pressure is (Use $R = 8.2 \times 10^{-5}\ m^3 \cdot atm/mol\ K$ and $N_A = 6.02 \times 10^{23}\ mol^{-1}$.)
   (a) 41 (b) 450
   (c) 2.4×10^{25} (d) 2.7×10^{26}

9. What is the internal energy of 1.75 kg of helium (atomic mass = 4.00260 u) with a temperature of 100°C?
   (a) 4.65×10^3 J (b) 2.03×10^6 J
   (c) 5.44×10^5 J (d) 8.16×10^6 J

10. A runner weighs 580 N (about 130 lb), and 71% of this weight is water. How many moles of water are in the runner's body?
    (a) 5.3×10^2 mol (b) 1.6×10^3 mol
    (c) 9.8×10^2 mol (d) 2.3×10^3 mol

11. A 0.030 m^3 container is initially evacuated. Then, 4.0 g of water is placed in the container and, after some time, all the water evaporates. If the temperature of the water vapor is 388 K, what is its pressure?
    (a) 1.5×10^3 Pa (b) 2.4×10^4 Pa
    (c) 6.9×10^3 Pa (d) 8.2×10^4 Pa

12. What is the density (in kg/m^3) of nitrogen gas (molecular mass = 28 u) at a pressure of 2.0 atmospheres and a temperature of 310 K?
    (a) 0.88 kg/m^3 (b) 4.9 kg/m^3
    (c) 1.1 kg/m^3 (d) 2.2 kg/m^3

13. The capacity of a vessel is 31. It contains 6g O_2, 8g N_2 and 5g CO_2 mixture at 27°C. If $R = 8.31$ J/mol K, then the pressure in the vessel in N/m^2 will be (approx.)
    (a) 5×10^5 (b) 5×10^4
    (c) 1×10^6 (d) 1×10^5

14. If the translational rms speed of the water vapor molecules (H_2O) in air is 648 m/s, what is the translational rms speed of the carbon dioxide molecules (CO_2) in the same air? Both gases are at the same temperature.
    (a) 239 m/s (b) 338 m/s
    (c) 414 m/s (d) 307 m/s

15. Each molecule in a gas has an average kinetic energy. What is the total average kinetic energy of all the molecules in 3.0 mol of a gas whose temperature is 320 K?
    (a) 12 000 J (b) 10 000 J
    (c) 6000 J (d) 8000 J

16. Helium (He), a monatomic gas, fills a 0.010 m^3 container. The pressure of the gas is (6.2×10^5 Pa). How long would a 0.25 hp engine have to run (1 hp = 746 W) to produce an amount of energy equal to the internal energy of this gas?
    (a) 1.0×10^1 s (b) 5.0×10^1 s
    (c) 2.5×10^1 s (d) 4.3×10^1 s

17. A gas is confined to a cylindrical container of radius 1 cm and length 1 m. The pressure exerted on an end face, compared with the pressure exerted on the long-curved face, is
    (a) smaller because its area is smaller.
    (b) smaller because most molecules cannot traverse the length of the cylinder without undergoing collisions.
    (c) larger because the face is flat.
    (d) none of these.

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18. The equation of state for 5 g of oxygen at a pressure p and temperature T, when occupying a volume V, will be

(a) $pV = \frac{5RT}{32}$ **(b)** $pV = \frac{5RT}{16}$

(c) $pV = \frac{5RT}{2}$ **(d)** $pV = 5\,RT$

19. A tank contains 0.85 mol of molecular nitrogen (N_2). Determine the mass (in grams) of nitrogen that must be removed from the tank in order to lower the pressure from 38 to 25 atm. Assume that the volume and temperature of the nitrogen in the tank do not change.

(a) 2.1 g **(b)** 8.1 g
(c) 4.0 g **(d)** 11 g

20. The force on the walls of a vessel of a contained gas is due to

(a) repulsive force between gas molecules.
(b) slight loss in average speed of a gas molecule after collision with wall.
(c) change in momentum of a gas molecule due to collision with wall.
(d) elastic collisions between gas molecules.

21. Initially, the translational rms speed of a molecule of an ideal gas is 463 m/s. The pressure and volume of this gas are kept constant, while the number of molecules is doubled. What is the final translational rms speed of the molecules?

(a) 655 m/s **(b)** 463 m/s
(c) 327 m/s **(d)** 926 m/s

22. Two pure samples of atoms, labeled A and B, contain oxygen atoms and carbon atoms, respectively. Each sample contains the same number of atoms. What is the ratio of the mass of sample B to that of sample A, m_B/m_A? Note the following atomic masses: C = 12 u; O = 16 u.

(a) 1.0 **(b)** 0.75
(c) 0.5 **(d)** 1.5

23. An ideal gas with a fixed number of molecules is maintained at a constant pressure. At 30.0°C, the volume of the gas is 1.50 m^3. What is the volume of the gas when the temperature is increased to 75.0°C?

(a) 0.60 m^3 **(b)** 3.75 m^3
(c) 1.72 m^3 **(d)** 1.30 m^3

24. The density of diamond, a form of carbon, is 3520 kg/m^3. If the atomic mass of carbon is 12.011 u, how many carbon atoms are there in a solid diamond sphere with a radius of 0.033 m?

(a) 5.16×10^{26} atoms **(b)** 1.04×10^{27} atoms
(c) 2.08×10^{26} atoms **(d)** 2.66×10^{25} atoms

25. A sample of a monatomic ideal gas is originally at 20°C. What is the final temperature of the gas if both the pressure and volume are doubled?

(a) 5°C **(b)** 20°C
(c) 80°C **(d)** 900°C

26. Neon gas at 20°C is confined within a rigid vessel. The gas is then heated until its pressure is doubled. What is the final temperature of the gas?

(a) 10°C **(b)** 20°C
(c) 40°C **(d)** 313°C

27. Argon gas at 305 K is confined within a constant volume at a pressure p_1. If the gas has a pressure p_2 when it is cooled to 195 K, what is the ratio of p_2 to p_1?

(a) 0.410 **(b)** 0.717
(c) 0.639 **(d)** 1.28

28. The temperature of a monatomic ideal gas with a mass per mole of 0.00750 kg/mol is 294 K. The absolute pressure of the gas is 1.05×10^5 Pa when its volume is 1.31×10^{-3} m^3. What is the mass of the gas?

(a) 8.04×10^{-5} kg **(b)** 4.22×10^{-4} kg
(c) 1.92×10^{-4} kg **(d)** 7.66×10^{-4} kg

29. The temperature of low pressure hydrogen is reduced from 100°C to 20°C. The rms speed of its molecules decreases by, approximately

(a) 89% **(b)** 79%
(c) 46% **(d)** 11%

30. A mixture of two ideal gases A and B is in thermal equilibrium at 600 K. A molecule of A has one-fourth the mass of a molecule of B and the rms speed of molecules of A is 400 m/s. Determine the rms speed of molecules of B.

(a) 100 m/s **(b)** 200 m/s
(c) 400 m/s **(d)** 800 m/s

31. A canister containing 150 kg of an ideal gas has a volume of 8.0 m^3. If the gas exerts a pressure of 5.0×10^5 Pa, what is the rms speed of the molecules?

(a) 160 m/s **(b)** 280 m/s
(c) 350 m/s **(d)** 390 m/s

32. According to the Maxwellian speed distribution, as the temperature increases the most probable speed

(a) increases.
(b) decreases.
(c) increases at high temperatures and decreases at low.
(d) decreases at high temperatures and increases at low.

33. A tank contains 135 moles of the monatomic gas argon at a temperature of 15.3°C. How much energy must be added to the gas to increase its temperature to 45.0°C?

(a) 2.50×10^3 J **(b)** 5.00×10^4 J
(c) 3.33×10^4 J **(d)** 5.70×10^5 J

34. How many air molecules are in a room at temperature 23.8°C and standard pressure if the dimensions of the room are 3.66 m × 3.66 m × 2.43 m?

(a) 1330 **(b)** 16 600
(c) 3.03×10^{24} **(d)** 8.05×10^{26}

35. Platinum has a molar mass of 195 g/mol. If you have a ring that contains 2.3 g of platinum, how many moles does it contain?

(a) 0.012 mol **(b)** 85 mol
(c) 450 mol **(d)** 7.2×10^{21} mol

36. The pressures p and volumes V of the five ideal gases, with the same number of molecules, are given below. Which has the highest temperature?

(a) $p = 1 \times 10^5$ Pa and $V = 10$ cm^3
(b) $p = 3 \times 10^5$ Pa and $V = 6$ cm^3
(c) $p = 4 \times 10^5$ Pa and $V = 4$ cm^3
(d) $p = 6 \times 10^5$ Pa and $V = 2$ cm^3

37. The speeds of 25 molecules are distributed as follows: 5 in the range from 2 to 3 m/s, 10 in the range from 3 to 4 m/s,

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5 in the range from 4 to 5 m/s, 3 in the range from 5 to 6 m/s, 1 in the range from 6 to 7 m/s, and 1 in the range from 7 to 8 m/s. Their average speed is about

(a) 2 m/s **(b)** 3 m/s
(c) 4 m/s **(d)** 5 m/s

38. A sample of argon gas (molar mass 40 g) is at four times the absolute temperature of a sample of hydrogen gas (molar mass 2 g). The ratio of the rms speed of the argon molecules to that of the hydrogen is

(a) 1 **(b)** 5
(c) 1/5 **(d)** $1/\sqrt{5}$

39. In a certain gas the molecules are 5.0×10^{-9} m apart on average, have a mean free path of 5.0×10^{-6} m, and have an average speed of 500 m/s. The rate at which a molecule has collision with other molecules is about

(a) 10^{-11}/s **(b)** 10^{-8}/s
(c) 1/s **(d)** 10^{8}/s

40. A certain ideal gas has a temperature 300 K and a pressure 5.0×10^{4} Pa. The molecules have a mean free path of 4.0×10^{-7} m. If the temperature is raised to 350 K and the pressure is reduced to 1.0×10^{4} Pa the mean free path is then

(a) 6.9×10^{-8} m **(b)** 9.3×10^{-8} m
(c) 3.4×10^{-7} m **(d)** 2.3×10^{-6} m

41. The pressure of an ideal gas of diatomic molecules is doubled by halving the volume. The ratio of the new internal energy to the old, both measured relative to the internal energy at 0 K, is

(a) 1/4 **(b)** 1/2
(c) 1 **(d)** 2

42. Assume that helium behaves as an ideal monatomic gas. If 2 moles of helium undergo a temperature increase of 100 K at constant pressure, how much energy has been transferred to the helium as heat?

(a) 1700 J **(b)** 2500 J
(c) 4200 J **(d)** 5000 J

43. A 1.00×10^{-2} m^3 flask contains 0.0160 kg of oxygen gas, O_2, at 77.0°C. What is the pressure exerted on the inner walls of the flask by the oxygen gas? (The atomic mass of O is 15.9994 u.)

(a) 3.19×10^{4} Pa **(b)** 1.45×10^{5} Pa
(c) 2.90×10^{5} Pa **(d)** 5.79×10^{5} Pa

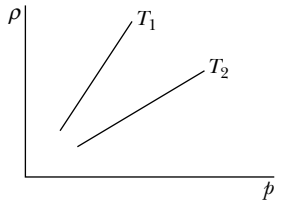
44. The volume of a carbon dioxide bubble rising in a glass of soda is observed to nearly double as the bubble rises from the bottom to the top of the glass. Why does the volume nearly double?

(a) The temperature at the bottom is cooler than it is at the top.
(b) The amount of carbon dioxide in the bubble increases.
(c) The fluid pressure of the soda is greater at the bottom of the glass than at the top.
(d) The pressure inside the bubble decreases as it rises.

45. The root mean square (rms) speed of oxygen molecules (O_2) at a certain absolute temperature is v. If the temperature is doubled and the oxygen gas dissociates into atomic oxygen, the rms speed would be

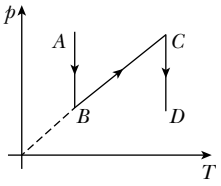
(a) v **(b)** $\sqrt{2}v$
(c) $2v$ **(d)** $2\sqrt{2}v$

46. Figure shows graphs of pressure versus density for an ideal gas at two temperatures T_1 and T_2. Then

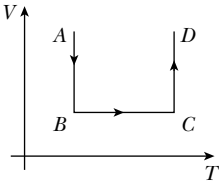


(a) $T_1 > T_2$ **(b)** $T_1 = T_2$
(c) $T_1 < T_2$ **(d)** any of these is possible

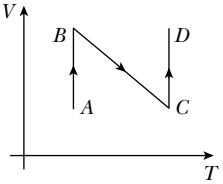
47. p–T diagram is shown in the figure. choose the corresponding V–T diagram.



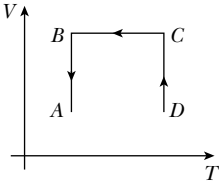
(a)



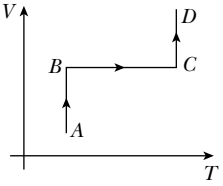
(b)



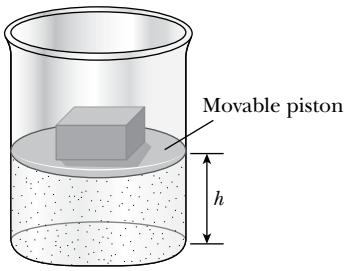
(c)



(d)



48. A frictionless gas-filled cylinder is fitted with a movable piston, as shown in the figure. The block resting on the top of the piston determines the constant pressure that the gas has. The height h is 0.120 m when the temperature is 273 K and increases as the temperature increases. What is the value of h when the temperature reaches 318 K?



(a) 0.140 m **(b)** 0.130 m
(c) 0.150 m **(d)** 0.125 m

49. A bulb contains 1 mol of hydrogen mixed with one mole of oxygen at temperature T. The ratio of rms

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values of velocity of hydrogen molecules to that of oxygen molecules is
(a) 1 : 16 (b) 1 : 4
(c) 4 : 1 (d) 16 : 1

50. A primitive diving bell consists of a cylindrical tank with one end open and one end closed. The tank is lowered into a freshwater lake, open end downward. Water rises into the tank, compressing the trapped air, whose temperature remains constant during the descent. The tank is brought to a halt when the distance between the surface of the water in the tank and the surface of the lake is 40.0 m. Atmospheric pressure at the surface of the lake is 1.01×10^5 Pa. Find the fraction of the tank's volume that is filled with air.
(a) 0.205 (b) 0.333
(c) 0.257 (d) 0.435

51. Near the surface of Venus, the rms speed of carbon dioxide molecules (CO_2) is 650 m/s. What is the temperature (in kelvins) of the atmosphere at that point?
(a) 250 K (b) 920 K
(c) 750 K (d) 1800 K

More than One Correct Choice Type

52. Consider a collision between an oxygen molecule and a hydrogen mole in a mixture of oxygen and hydrogen kept at room temperature. Which of the following are possible?
(a) The kinetic energy of both the molecules increase.
(b) The kinetic energies of both the molecules decrease.
(c) The kinetic energy of the oxygen molecule increases and that of the hydrogen molecule decreases.
(d) The kinetic energy of the hydrogen molecule increases and that of the oxygen molecule decreases.

53. Let v_{avg}, v_{rms} and v_P, respectively, denote the mean speed, the root square speed, and the most probable speed of the molecules in an ideal monoatomic gas at absolute temperature T. The mass of a molecules is m. Then
(a) no molecules can have speed greater than v_{rms}.
(b) no molecule can have speed less than $v_P/\sqrt{2}$.
(c) $v_p < v_{avg} < v_{rms}$
(d) the average kinetic energy of a molecule is $(3/4)mv_P^2$.

54. Which of the following statements are correct?
(a) At a given temperature, the internal energy of all ideal gases is the same.
(b) At a given temperature, the energy associated with each degree of freedom is $(1/2)kT$.
(c) At a given temperature, the energy associated with each degree of freedom is $(1/3)kT$.
(d) At a given temperature, all ideal gas molecules, irrespective of their mass, have average kinetic energy equal to $(3/2)kT$.

55. A box contains a mixture of H_2 and He gases. Which of the following statements are correct?
(a) The average translational kinetic energies of H_2 molecules and He atoms are same.
(b) The average energies of H_2 molecules and He atoms are same.
(c) H_2 molecules have greater average energy than that of He atoms.
(d) The average speed of H_2 molecules and He atoms is same.

56. Which of the following quantities is the same for all ideal gases at the same temperature?
(a) The kinetic energy of 1 mol.
(b) The kinetic energy of 1 g.
(c) The number of molecules in 1 mol.
(d) The number of molecules in 1 g.

57. Two vessels of the same volume contain the same gas at same temperature. If the pressure in the vessels be in the ratio of 1 : 2, then
(a) the ratio of the average energy is 1 : 2.
(b) the ratio of the root mean square velocity is 1 : 1.
(c) the ratio of the average velocity is 1 : 2.
(d) the ratio of number of molecules is 1 : 2.

58. The total kinetic energy of translational motion of all the molecules of 5 L of nitrogen exerting a pressure P is 3000 J.
(a) The total kinetic energy of 10 L of N_2 at a pressure of $2P$ is 3000 J.
(b) The total kinetic energy of 10 L of He at a pressure of $2P$ is 3000 J.
(c) The total kinetic energy of 10 L of O_2 at a pressure of $2P$ is 20,000 J.
(d) The total kinetic energy of 10 L of Ne at a pressure of $2P$ is 12,000 J.

59. If the atoms in a diatomic molecule can vibrate, the molecule has
(a) two rotational degrees of freedom.
(b) three rotational degrees of freedom.
(c) one vibrational degree of freedom.
(d) two vibrational degrees of freedom.

Linked Comprehension

Paragraph for Questions 60–62: A vessel contains an ideal gas at 27°C and pressure 200 kPa. The volume of the gas is 8.0 L. The molecules of the gas collide with the walls of the vessel and exert a pressure. The gas is allowed to leak till the pressure falls to 150 kPa and the temperature remains the same. The diameter of the ideal gas molecule is 300 pm. The average distance traveled by the molecule between collisions is termed as the mean free path. The mean free path of a gas molecule varies directly proportional to the temperature and inversely proportional to the pressure. Take, R = 8.3 J/mol K and $N_A = 6.02 \times 10^{23}$/mol.

60. The number of moles of the gas in the vessel before the gas was leaked is
(a) 0.64 (b) 0.48
(c) 0.16 (d) 0.8

61. The amount of gas leaked is
(a) 0.64 mol (b) 0.48 mol
(c) 0.16 mol (d) 0.8 mol

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62. The average translational kinetic energy of the molecules of the gas is

(a) 2.07×10^{-19} J **(b)** 2.07×10^{-20} J
(c) 2.07×10^{-21} J **(d)** 2.07×10^{-22} J

Paragraph for questions 63–65: X and Y are two equal size containers. X contains a gas A at a temperature 75°C and Y contains a gas B at a temperature 40°C. Each gas behaves as an ideal gas and specific heat at constant pressure for both gases has the same value.

63. From the given information, we can conclude that

(a) the molecules in container X are moving as fast as those in container Y.
(b) the molecules in container X are moving faster than those in container Y.
(c) the molecules in container Y are moving faster than those in container X.
(d) no possibility can be ruled out, that is, molecules in X could be moving faster than those in Y or the molecules in Y could be moving faster than those in X.

64. It can be concluded that the

(a) average translational kinetic energy of a molecule in container X is more than that of a molecule in container Y.
(b) average translational kinetic energy of a molecule in container Y is more than that of a molecule in container X.
(c) average translational kinetic energy of a molecule in container X could be more or could also be less than that of a molecule in container Y.
(d) average translational kinetic energy of a molecule in container X is equal to that of a molecule in container Y.

65. It can be concluded that the

(a) internal energy of a gas in container X is more than that of the gas in container Y.
(b) internal energy of a gas in container X is less than that of the gas in container Y.
(c) internal energy of a gas in container X could be more or could also be less than that of the gas in container Y.
(d) internal energy of a gas in container X has the same value as that of the gas in container.

Matrix-Match

66. Match the statements in Column I with those in Column II. Any given statement in Column I can have correct matching with *one or more* statements in Column II.

Column I	Column II
(a) Average translational kinetic energy per molecule	**(p)** $\sqrt{\frac{2RT}{M}}$
(b) Most probable speed	**(q)** $-\frac{p}{V}$
(c) Degrees of freedom	**(r)** $\frac{1}{\gamma - 1}$
(d) Molar specific heat at constant volume divided by R	**(s)** $\frac{3}{2}kT$

Directions for Questions 67: In the question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

67. Charles law

Column I	Column II	Column III
(I) $V_1 = 127$ mL	**(i)** $T_1 = 45°C$	**(J)** $T_2 = -73.05°C$
(II) $V_1 = 450$ mL	**(ii)** $T_1 = 27°C$	**(K)** $T_2 = -5°C$
(III) $V_1 = 300$ mL	**(iii)** $T_1 = 20°C$	**(L)** $T_2 = 476.04°C$
(IV) $V_1 = 1$ L	**(iv)** $T_1 = 40°C$	**(M)** $T_2 = -28.86°C$

(1) Which combination has 379.28 mL as value for V_2?
(a) (II) (i) (K) **(b)** (III) (iv) (J)
(c) (II) (ii) (M) **(d)** (II) (iii) (K)

(2) Which combination has 317 mL as value for V_2?
(a) (I) (iii) (K) **(b)** (IV) (iii) (J)
(c) (II) (iii) (M) **(d)** (I) (ii) (L)

(3) Which has combination 250 mL as value for V_2?
(a) (III) (i) (K) **(b)** (I) (i) (L)
(c) (III) (iii) (M) **(d)** (I) (iv) (L)

Integer Type

68. At what temperature is the rms velocity of a hydrogen molecule equal to that of an oxygen molecule at 47°C?

69. Find the temperature at which oxygen molecules have the same rms speed as N_2 molecules at 7°C.

70. Nitrogen is in equilibrium state at T = 421 K. Find the value of most probable speed, v_p.

71. What is the degree of freedom of gas? If at STP, the velocity of sound in it is 330 m/s, gas density = 1.3 mg/cm^3.

ANSWER KEY

Checkpoints

1. 8.82 %; 28.65 kPa **2.** (a) All are tie; (b) type 3, type 2, type 1

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Problems

1. (a) 3.6 cm/s; (b) 4.0 cm/s; (c) 5.0 cm/s
2. (a) 1078 K ≈ 1.1×10^2 K; (b) 9.8×10^2 m/s
3. 449 m/s
4. (a) 1.61×10^{-20} cal = 6.76×10^{-20} J; (b) 11.5
5. (a) 7.0 km/s; (b) 7.7 km/s
6. 1.1×10^4 Pa
7. (a) 460 m/s; (b) 480 m/s; (c) Yes, $v_{rms} > v_{avg}$
8. 3.11×10^{-20} J
9. (a) 0.313 mol; (b) 910 J; (c) ≈0.714
10. 1.27
11. 4.7
12. (a) 4.66×10^{-2} mol; (b) 1.80×10^{-3} m^3
13. 1.22 atm
14. 1.8×10^2 m/s
15. 0.25
16. 3.2×10^5 Pa
17. (a) 6.07×10^{-21} J; (b) 7.31×10^{-21} J; (c) 3.65×10^3 J; (d) 4.40×10^3 J
18. (a) 699 m/s; (b) 27.9 g/mol; (c) N_2
19. 1.82×10^{-13} Pa
20. (a) 10.996 m^3 ≈ 11.0 m^3; (b) 0.977 m^3; (c) 4.68×10^3 mol
21. 77 cm^3
22. 6.8×10^3 J
23. (a) 0.00761 mol; (b) 4.58×10^{21}
24. (a) 106 mol; (b) 1.07 m^3
25. 9.53×10^6 m/s
26. 2.74 km/s
27. 0.145 kg
28. (a) 3.17×10^{-8} mol; (b) 1.91×10^{16} molecules
29. (a) 561 m/s; (b) ≈ –185°C; (c) (1.14×10^3)°C

Practice Questions

Single Correct Choice Type

1. (d)
2. (b)
3. (a)
4. (b)
5. (d)
6. (c)
7. (c)
8. (c)
9. (b)
10. (d)
11. (b)
12. (d)
13. (a)
14. (c)
15. (a)
16. (b)
17. (d)
18. (a)
19. (b)
20. (c)
21. (c)
22. (b)
23. (c)
24. (d)
25. (d)
26. (d)
27. (c)
28. (b)
29. (d)
30. (b)
31. (b)
32. (a)
33. (b)
34. (d)
35. (a)
36. (b)
37. (c)
38. (d)
39. (d)
40. (d)
41. (c)
42. (c)
43. (b)
44. (b)
45. (c)
46. (a)
47. (d)
48. (a)
49. (c)
50. (a)
51. (c)

More than One Correct Choice Type

52. (c), (d)
53. (c), (d)
54. (b), (d)
55. (a), (c)
56. (a), (c)
57. (b), (d)
58. (c), (d)
59. (a), (d)

Linked Comprehension

60. (a)
61. (c)
62. (c)
63. (d)
64. (a)
65. (c)

Matrix-Match

66. (a) → (s); (b) → (p); (c) → (q); (d) → (r)
67. (1) → (a); (2) → (d); (3) → (c)

Integer Type

68. 20
69. 47
70. 500
71. 5

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CHAPTER 21

The First Law of Thermodynamics

21.1 | WHAT IS PHYSICS?

Thermodynamics is one of the principal branches of science and engineering. It is the study and application of the thermal (or internal) energy of the system.

Examples of how thermodynamics figures into everyday engineering and science are countless. Automobile engineers are concerned with the heating of a car engine, such as during a NASCAR race. Food engineers are concerned both with the proper heating of foods, such as pizzas being microwaved, and with the proper cooling of foods, such as TV dinners being quickly frozen at a processing plant. Geologists are concerned with the transfer of thermal energy in an El Niño event and in the gradual warming of ice expanses in the Arctic and Antarctic. Agricultural engineers are concerned with the weather conditions that determine whether the agriculture of a country thrives or vanishes. Medical engineers are concerned with how a patient's temperature might distinguish between a benign viral infection and a cancerous growth.

21.2 | A CLOSER LOOK AT HEAT AND WORK

Key Concept

- A gas may exchange energy with its surroundings through work. The amount of work W done *by* a gas as it expands or contracts from an initial volume V_i to a final volume V_f is given by

$$W = \int dw = \int_{V_i}^{V_f} p\, dV.$$

The integration is necessary because the pressure p may vary during the volume change.

Here we look in some detail at how energy can be transferred as heat and work between a system and its environment. Let us take as our system a gas confined to a cylinder with a movable piston, as in Fig. 21-1. The upward

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21.2 A Closer Look at Heat and Work
21.3 The First Law of Thermodynamics
21.4 Some Special Cases of the First Law of Thermodynamics
21.5 Some Applications of First Law of Thermodynamics
21.6 The Adiabatic Expansion of an Ideal Gas
21.7 The Carnot Engine

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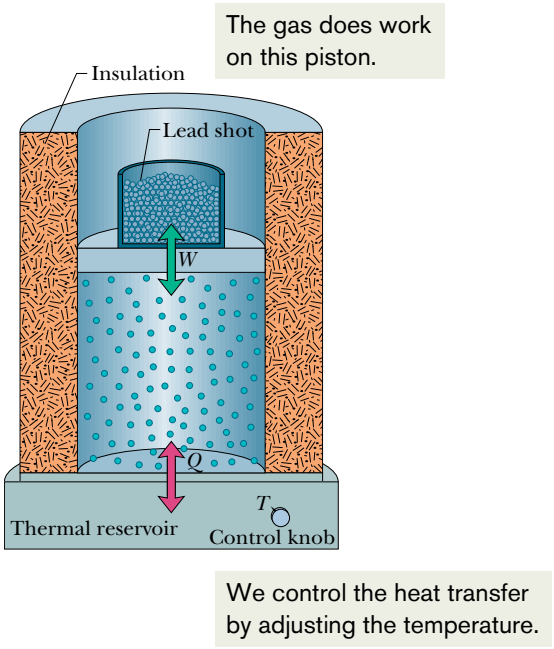

Figure 21-1 A gas is confined to a cylinder with a movable piston. Heat Q can be added to or withdrawn from the gas by regulating the temperature T of the adjustable thermal reservoir. Work W can be done by the gas by raising or lowering the piston.

force on the piston due to the pressure of the confined gas is equal to the weight of lead shot loaded onto the top of the piston. The walls of the cylinder are made of insulating material that does not allow any transfer of energy as heat. The bottom of the cylinder rests on a reservoir for thermal energy, a *thermal reservoir* (perhaps a hot plate) whose temperature T you can control by turning a knob.

The system (the gas) starts from an *initial state* i, described by a pressure p_i, a volume V_i, and a temperature T_i. You want to change the system to a *final state* f, described by a pressure p_f, a volume V_f, and a temperature T_f. The procedure by which you change the system from its initial state to its final state is called a *thermodynamic process*. During such a process, energy may be transferred into the system from the thermal reservoir (positive heat) or vice versa (negative heat). Also, work can be done by the system to raise the loaded piston (positive work) or lower it (negative work). We assume that all such changes occur slowly, with the result that the system is always in (approximate) thermal equilibrium (every part is always in thermal equilibrium).

Suppose that you remove a few lead shot from the piston of Fig. 21-1, allowing the gas to push the piston and remaining shot upward through a differential displacement $d\vec{s}$ with an upward force $\vec{F}$. Since the displacement is tiny, we can assume that $\vec{F}$ is constant during the displacement. Then $\vec{F}$ has a magnitude that is equal to pA, where p is the pressure of the gas and A is the face area of the piston. The differential work dW done by the gas during the displacement is

$$\begin{aligned} dW &= \vec{F}\cdot d\vec{s} = (pA)(ds) = p(A\,ds) \\ &= p\,dV, \end{aligned} \tag{21-1}$$

in which dV is the differential change in the volume of the gas due to the movement of the piston. When you have removed enough shot to allow the gas to change its volume from V_i to V_f, the total work done by the gas is

$$W = \int dW = \int_{V_i}^{V_f} p\,dV. \tag{21-2}$$

During the volume change, the pressure and temperature may also change. To evaluate Eq. 21-2 directly, we would need to know how pressure varies with volume for the actual process by which the system changes from state i to state f.

One Path. There are actually many ways to take the gas from state i to state f. One way is shown in Fig. 21-2*a*, which is a plot of the pressure of the gas versus its volume and which is called a p-V diagram. In Fig. 21-2*a*, the curve indicates that the pressure decreases as the volume increases. The integral in Eq. 21-2 (and thus the work W done by the gas) is represented by the shaded area under the curve between points i and f. Regardless of what exactly we do to take the gas along the curve, that work is positive, due to the fact that the gas increases its volume by forcing the piston upward.

Another Path. Another way to get from state i to state f is shown in Fig. 21-2*b*. There the change takes place in two steps—the first from state i to state a, and the second from state a to state f.

Step ia of this process is carried out at constant pressure, which means that you leave undisturbed the lead shot that ride on top of the piston in Fig. 21-1. You cause the volume to increase (from V_i to V_f) by slowly turning up the temperature control knob, raising the temperature of the gas to some higher value T_a. (Increasing the temperature increases the force from the gas on the piston, moving it upward.) During this step, positive work is done by the expanding gas (to lift the loaded piston) and heat is absorbed by the system from the thermal reservoir

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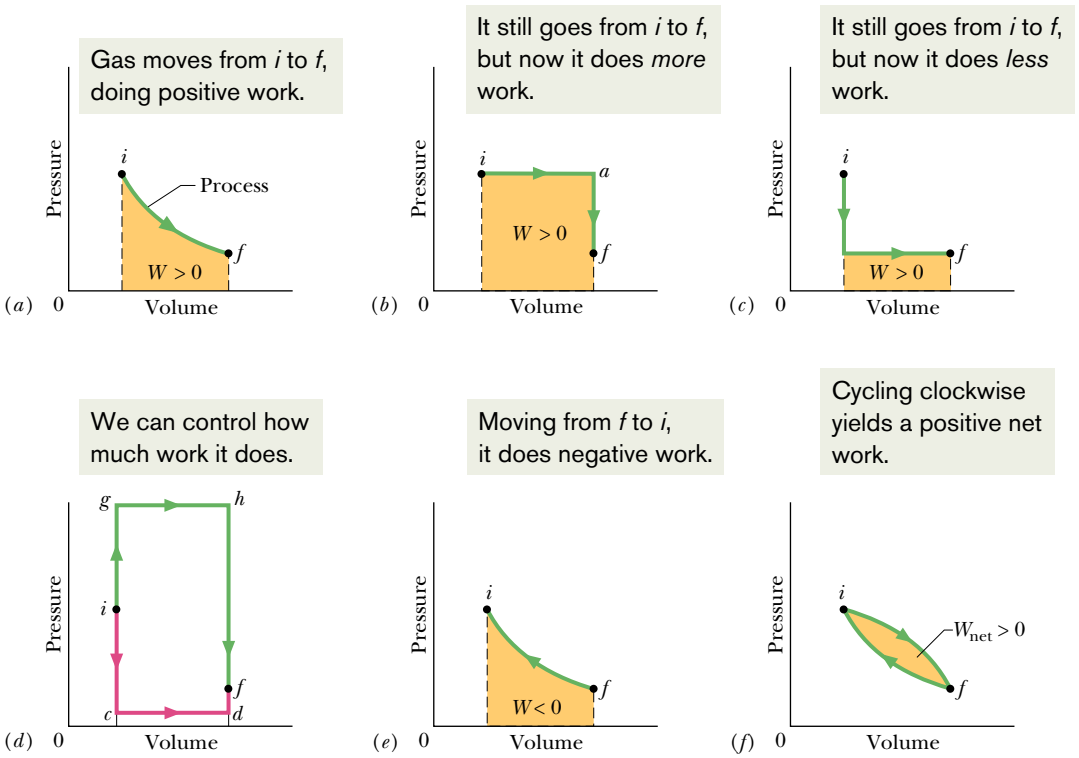

Figure 21-2 (*a*) The shaded area represents the work W done by a system as it goes from an initial state i to a final state f. Work W is positive because the system's volume increases. (*b*) W is still positive, but now greater. (*c*) W is still positive, but now smaller. (*d*) W can be even smaller (path *icdf*) or larger (path *ighf*). (*e*) Here the system goes from state f to state i as the gas is compressed to less volume by an external force. The work W done *by* the system is now negative. (*f*) The net work W_{net} done by the system during a complete cycle is represented by the shaded area.

(in response to the arbitrarily small temperature differences that you create as you turn up the temperature). This heat is positive because it is added to the system.

Step *af* of the process of Fig. 21-2*b* is carried out at constant volume, so you must wedge the piston, preventing it from moving. Then as you use the control knob to decrease the temperature, you find that the pressure drops from p_a to its final value p_f. During this step, heat is lost by the system to the thermal reservoir.

For the overall process *iaf*, the work W, which is positive and is carried out only during step *ia*, is represented by the shaded area under the curve. Energy is transferred as heat during both steps *ia* and *af*, with a net energy transfer Q.

Reversed Steps. Figure 21-2*c* shows a process in which the previous two steps are carried out in reverse order. The work W in this case is smaller than for Fig. 21-2*b*, as is the net heat absorbed. Figure 21-2*d* suggests that you can make the work done by the gas as small as you want (by following a path like *icdf*) or as large as you want (by following a path like *ighf*).

To sum up: A system can be taken from a given initial state to a given final state by an infinite number of processes. Heat may or may not be involved, and in general, the work W and the heat Q will have different values for different processes. We say that heat and work are *path-dependent* quantities.

Negative Work. Figure 21-2*e* shows an example in which negative work is done by a system as some external force compresses the system, reducing its volume. The absolute value of the work done is still equal to the area beneath the curve, but because the gas is *compressed*, the work done by the gas is negative.

Cycle. Figure 21-2*f* shows a *thermodynamic cycle* in which the system is taken from some initial state i to some other state f and then back to i. The net work done by the system during the cycle is the sum of the *positive* work done during the expansion and the *negative* work done during the compression. In Fig. 21-2*f*, the net work is positive because the area under the expansion curve (i to f) is greater than the area under the compression curve (f to i).

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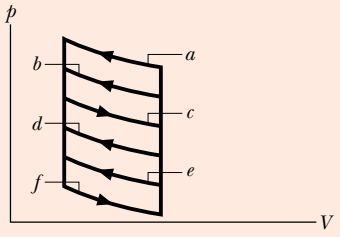
CHECKPOINT 1

The p-V diagram here shows six curved paths (connected by vertical paths) that can be followed by a gas. Which two of the curved paths should be part of a closed cycle (those curved paths plus connecting vertical paths) if the net work done by the gas during the cycle is to be at its maximum positive value?



Work Done by an Ideal Gas at Constant Temperature

Suppose we put an ideal gas in a piston–cylinder arrangement. Suppose also that we allow the gas to expand from an initial volume V_i to a final volume V_f while we keep the temperature T of the gas constant. Such a process, at *constant temperature*, is called an **isothermal expansion** (and the reverse is called an **isothermal compression**).

On a p-V diagram, an *isotherm* is a curve that connects points that have the same temperature. Thus, it is a graph of pressure versus volume for a gas whose temperature T is held constant. For n moles of an ideal gas, it is a graph of the equation

$$p = nRT\frac{1}{V} = (\text{a constant})\frac{1}{V}. \tag{21-3}$$

Figure 21-3 shows three isotherms, each corresponding to a different (constant) value of T. (Note that the values of T for the isotherms increase upward to the right.) Superimposed on the middle isotherm is the path followed by a gas during an isothermal expansion from state i to state f at a constant temperature of 310 K.

To find the work done by an ideal gas during an isothermal expansion, we start with Eq. 21-2,

$$W = \int_{V_i}^{V_f} p\,dV. \tag{21-4}$$

This is a general expression for the work done during any change in volume of any gas. For an ideal gas, we can use Eq. 20-5 ($pV = nRT$) to substitute for p, obtaining

$$W = \int_{V_i}^{V_f} \frac{nRT}{V}\,dV. \tag{21-5}$$

Because we are considering an isothermal expansion, T is constant, so we can move it in front of the integral sign to write

$$W = nRT\int_{V_i}^{V_f}\frac{dV}{V} = nRT\,[\ln V]_{V_i}^{V_f}. \tag{21-6}$$

The expansion is along an isotherm (the gas has constant temperature).

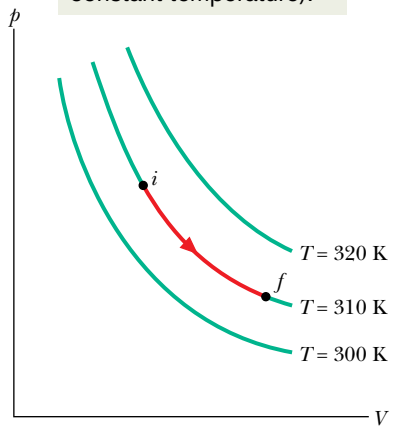


Figure 21-3 Three isotherms on a p-V diagram. The path shown along the middle isotherm represents an isothermal expansion of a gas from an initial state i to a final state f. The path from f to i along the isotherm would represent the reverse process—that is, an isothermal compression.

By evaluating the expression in brackets at the limits and then using the relationship $\ln a - \ln b = \ln(a/b)$, we find that

$$W = nRT\ln\frac{V_f}{V_i} \quad \text{(ideal gas, isothermal process)}. \tag{21-7}$$

Recall that the symbol ln specifies a *natural* logarithm, which has base e.

For an expansion, V_f is greater than V_i, so the ratio V_f/V_i in Eq. 21-7 is greater than unity. The natural logarithm of a quantity greater than unity is positive, and so the work W done by an ideal gas during an isothermal expansion is positive, as we expect. For a compression, V_f is less than V_i, so the ratio of volumes in Eq. 21-7 is less than unity. The natural logarithm in that equation—hence the work W—is negative, again as we expect.

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Work Done at Constant Volume and at Constant Pressure

Equation 21-7 does not give the work W done by an ideal gas during *every* thermodynamic process. Instead, it gives the work only for a process in which the temperature is held constant. If the temperature varies, then the symbol T in Eq. 21-5 cannot be moved in front of the integral symbol as in Eq. 21-6, and thus we do not end up with Eq. 21-7.

However, we can always go back to Eq. 21-4 to find the work W done by an ideal gas (or any other gas) during any process, such as a constant-volume process and a constant-pressure process. If the volume of the gas is constant, then Eq. 21-4 yields

$$W = 0 \quad \text{(constant-volume process)}. \tag{21-8}$$

If, instead, the volume changes while the pressure p of the gas is held constant, then Eq. 21-4 becomes

$$W = p(V_f - V_i) = p\,\Delta V \quad \text{(constant-pressure process)}. \tag{21-9}$$

CHECKPOINT 2

An ideal gas has an initial pressure of 3 pressure units and an initial volume of 4 volume units. The table gives the final pressure and volume of the gas (in those same units) in five processes. Which processes start and end on the same isotherm?

	a	*b*	*c*	*d*	*e*
p	12	6	5	4	1
V	1	2	7	3	12

SAMPLE PROBLEM 21.01

Ideal gas and changes of temperature, volume, and pressure

A cylinder contains 12 L of oxygen at 20°C and 15 atm. The temperature is raised to 35°C, and the volume is reduced to 8.5 L. What is the final pressure of the gas in atmospheres? Assume that the gas is ideal.

KEY IDEA

Because the gas is ideal, we can use the ideal gas law to relate its parameters, both in the initial state i and in the final state f.

Calculations: From Eq. 20-5 we can write

$$p_i V_i = nRT_i \quad \text{and} \quad p_f V_f = nRT_f.$$

Dividing the second equation by the first equation and solving for p_f yields

$$p_f = \frac{p_i T_f V_i}{T_i V_f}. \tag{21-10}$$

Note here that if we converted the given initial and final volumes from liters to the proper units of cubic meters, the multiplying conversion factors would cancel out of Eq. 21-10. The same would be true for conversion factors that convert the pressures from atmospheres to the proper pascals. However, to convert the given temperatures to kelvins requires the addition of an amount that would not cancel and thus must be included. Hence, we must write

$$T_i = (273 + 20)\ \text{K} = 293\ \text{K}$$

and

$$T_f = (273 + 35)\ \text{K} = 308\ \text{K}.$$

Inserting the given data into Eq. 21-10 then yields

$$p_f = \frac{(15\ \text{atm})(308\ \text{K})(12\ \text{L})}{(293\ \text{K})(8.5\ \text{L})} = 22\ \text{atm}. \quad \text{(Answer)}$$

SAMPLE PROBLEM 21.02

Work done by an ideal gas

One mole of oxygen (assume it to be an ideal gas) expands at a constant temperature T of 310 K from an initial volume V_i of 12 L to a final volume V_f of 19 L. How much work is done by the gas during the expansion?

KEY IDEA

Generally we find the work by integrating the gas pressure with respect to the gas volume, using Eq. 21-4. However, because the gas here is ideal and

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the expansion is isothermal, that integration leads to Eq. 21-7.

Calculations: Therefore, we can write

$$W = nRT \ln \frac{V_f}{V_i}$$

$$= (1\text{ mol})(8.31\text{ J/mol}\cdot\text{K})(310\text{ K}) \ln \frac{19\text{ L}}{12\text{ L}}$$

$$= 1180\text{ J}. \qquad \text{(Answer)}$$

The expansion is graphed in the p-V diagram of Fig. 21-4. The work done by the gas during the expansion is represented by the area beneath the curve *if*.

You can show that if the expansion is now reversed, with the gas undergoing an isothermal compression from 19 L to 12 L, the work done by the gas will be –1180 J. Thus, an external force would have to do 1180 J of work on the gas to compress it.

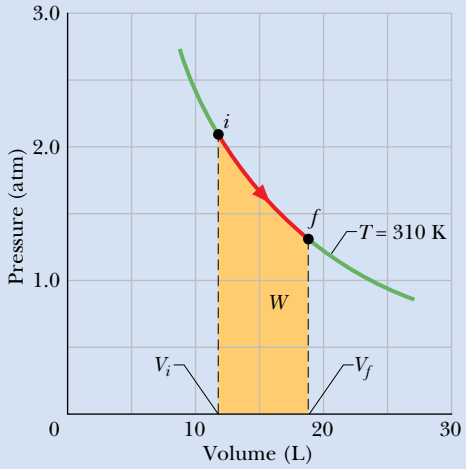

Figure 21-4 The shaded area represents the work done by 1 mol of oxygen in expanding from V_i to V_f at a temperature T of 310 K.

The Language of Thermodynamics. A rich but sometimes misleading language is used in scientific and engineering studies of thermodynamics. You may see statements that say heat is added, absorbed, subtracted, extracted, rejected, discharged, discarded, withdrawn, delivered, gained, lost, transferred, or expelled, or that it flows from one body to another (as if it were a liquid). You may also see statements that describe a body as *having* heat (as if heat can be held or possessed) or that its heat is increased or decreased. You should always keep in mind what is meant by the term *heat* in science and engineering:

Heat is energy that is transferred from one body to another body due to a difference in the temperatures of the bodies. When we identify one of the bodies as being our system, any such transfer of energy into it is positive heat Q and out of it is negative heat Q.

The term *work* also requires close attention. You may see statements that say work is produced or generated, or combined with heat or changed from heat. Here is what is meant by the term *work*:

Work is energy that is transferred from one body to another body due to a force that acts between them. When we identify one of the bodies as being our system of interest, any such transfer of energy out of the system is either positive work W done *by* the system or negative work W done *on* the system. Any such transfer of energy into the system is negative work done *by* the system or positive work done *on* the system. (The preposition that is used is important.) Obviously, this can be confusing–whenever you see the term *work*, read carefully.

21.3 | THE FIRST LAW OF THERMODYNAMICS

Key Concepts

- The principle of conservation of energy for a thermodynamic process is expressed in the first law of thermodynamics, which may assume either of the forms

  $$\Delta E_{int} = E_{int,f} - E_{int,i} = Q - W \quad \text{(first law)}$$

  or $$dE_{int} = dQ - dW \quad \text{(first law)}.$$

  E_{int} represents the internal energy of the material, which depends only on the material's state (temperature, pressure, and volume). Q represents the energy exchanged as heat between the system and its surroundings; Q is positive if the system absorbs heat and negative if the system loses heat. W is the work done *by* the system; W is positive if the system expands against an external force from the surroundings and negative if the system contracts because of an external force.
- Q and W are path dependent; ΔE_{int} is path independent.

You have just seen that when a system changes from a given initial state to a given final state, both the work W and the heat Q depend on the nature of the process. Experimentally, however, we find a surprising thing. *The quantity*

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$Q - W$ is the same for all processes. It depends only on the initial and final states and does not depend at all on how the system gets from one to the other. All other combinations of Q and W, including Q alone, W alone, $Q + W$, and $Q - 2W$, are *path dependent*; only the quantity $Q - W$ is not.

The quantity $Q - W$ must represent a change in some intrinsic property of the system. We call this property the *internal energy* E_{int} and we write

$$\Delta E_{int} = E_{int,f} - E_{int,i} = Q - W \quad \text{(first law)}. \tag{21-11}$$

Equation 21-11 is the **first law of thermodynamics**. If the thermodynamic system undergoes only a differential change, we can write the first law as*

$$dE_{int} = dQ - dW \quad \text{(first law)}. \tag{21-12}$$

The internal energy E_{int} of a system tends to increase if energy is added as heat Q and tends to decrease if energy is lost as work W done by the system.

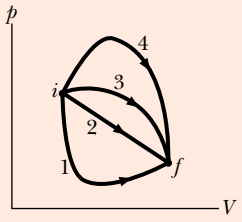
We have already discussed the principle of energy conservation as it applies to isolated systems—that is, to systems in which no energy enters or leaves the system. The first law of thermodynamics is an extension of that principle to systems that are *not* isolated. In such cases, energy may be transferred into or out of the system as either work W or heat Q. In our statement of the first law of thermodynamics above, we assume that there are no changes in the kinetic energy or the potential energy of the system as a whole; that is, $\Delta K = \Delta U = 0$.

Rules. In this chapter, we focus on the work done *by* a system, such as the gas in Fig. 21-1.

The work done *on* a system is always the negative of the work done *by* the system, so if we rewrite Eq. 21-11 in terms of the work W_{on} done *on* the system, we have $\Delta E_{int} = Q + W_{on}$. This tells us the following: The internal energy of a system tends to increase if heat is absorbed by the system or if positive work is done *on* the system. Conversely, the internal energy tends to decrease if heat is lost by the system or if negative work is done *on* the system.

CHECKPOINT 3

The figure here shows four paths on a p-V diagram along which a gas can be taken from state i to state f. Rank the paths according to (a) the change ΔE_{int} in the internal energy of the gas, (b) the work W done by the gas, and (c) the magnitude of the energy transferred as heat Q between the gas and its environment, greatest first.



21.4 | SOME SPECIAL CASES OF THE FIRST LAW OF THERMODYNAMICS

Key Concept

- The first law of thermodynamics can be applied to the following thermodynamic processes:

Isothermal:	$Q = W, \quad \Delta E_{int} = 0$
Adiabatic processes:	$Q = 0, \quad \Delta E_{int} = -W$
Constant-volume processes:	$W = 0, \quad \Delta E_{int} = Q$
Cyclical processes:	$\Delta E_{int} = 0, \quad Q = W$
Free expansions:	$Q = W = \Delta E_{int} = 0$

*Here dQ and dW, unlike dE_{int}, are not true differentials; that is, there are no such functions as $Q(p, V)$ and $W(p, V)$ that depend only on the state of the system. The quantities dQ and dW are called *inexact differentials* and are usually represented by the symbols $đQ$ and $đW$. For our purposes, we can treat them simply as infinitesimally small energy transfers.

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In the earlier section, we have discussed about work done at constant pressure which is an isobaric process. Here are five other thermodynamic processes as summarized in Table 21-1.

1. ***Isothermal Processes.*** The isothermal process occurs at constant temperature. Work is done at the same rate as heat is supplied, hence there is no increase of internal energy. Due to this, $\Delta E_{int} = 0$ and thus $\Delta Q = \Delta W$. We have already calculated the work done in an isothermal process. We have also seen that for isothermal process pV = constant. The practical way of realizing such a process is by keeping the gas in a highly conducting vessel and carrying out the process very slowly.
2. ***Adiabatic processes.*** An adiabatic process is one that occurs so rapidly or occurs in a system that is so well insulated that *no transfer of energy as heat* occurs between the system and its environment. Putting $Q = 0$ in the first law (Eq. 21-11) yields

$$\Delta E_{int} = -W \quad \text{(adiabatic process)}. \tag{21-13}$$

This tells us that if work is done *by* the system (that is, if W is positive), the internal energy of the system decreases by the amount of work. Conversely, if work is done *on* the system (that is, if W is negative), the internal energy of the system increases by that amount.

Figure 21-5 shows an idealized adiabatic process. Heat cannot enter or leave the system because of the insulation. Thus, the only way energy can be transferred between the system and its environment is by work. If we remove shot from the piston and allow the gas to expand, the work done by the system (the gas) is positive and the internal energy of the gas decreases. If, instead, we add shot and compress the gas, the work done by the system is negative and the internal energy of the gas increases.

3. ***Constant-volume (Isochoric) processes.*** If the volume of a system (such as a gas) is held constant, that system can do no work. Putting $W = 0$ in the first law (Eq. 21-11) yields

$$\Delta E_{int} = Q \quad \text{(constant-volume process)}. \tag{21-14}$$

Thus, if heat is absorbed by a system (that is, if Q is positive), the internal energy of the system increases. Conversely, if heat is lost during the process (that is, if Q is negative), the internal energy of the system must decrease.

4. ***Cyclical processes.*** There are processes in which, after certain interchanges of heat and work, the system is restored to its initial state. In that case, no intrinsic property of the system—including its internal energy—can possibly change. Putting $\Delta E_{int} = 0$ in the first law (Eq. 21-11) yields

$$Q = W \quad \text{(cyclical process)}. \tag{21-15}$$

Thus, the net work done during the process must exactly equal the net amount of energy transferred as heat; the store of internal energy of the system remains unchanged. Cyclical processes form a closed loop on a p-V plot, as shown in Fig. 21-2*f*.

5. ***Free expansions.*** These are adiabatic processes in which no transfer of heat occurs between the system and its environment and no work is done on or by the system. Thus, $Q = W = 0$, and the first law requires that

$$\Delta E_{int} = 0 \quad \text{(free expansion)}. \tag{21-16}$$

Figure 21-6 shows how such an expansion can be carried out. A gas, which is in thermal equilibrium within itself, is initially confined by a closed stopcock to one half of an insulated double chamber; the other half is evacuated.

Table 21-1 The First Law of Thermodynamics: Five Special Cases

The Law: $\Delta E_{int} = Q - W$ (Eq. 21-11)		
Process	**Restriction**	**Consequence**
Isothermal	$Q = W$	$\Delta E_{int} = 0$
Adiabatic	$Q = 0$	$\Delta E_{int} = -W$
Constant volume	$W = 0$	$\Delta E_{int} = Q$
Closed cycle	$\Delta E_{int} = 0$	$Q = W$
Free expansion	$Q = W = 0$	$\Delta E_{int} = 0$

We slowly remove lead shot, allowing an expansion without any heat transfer.

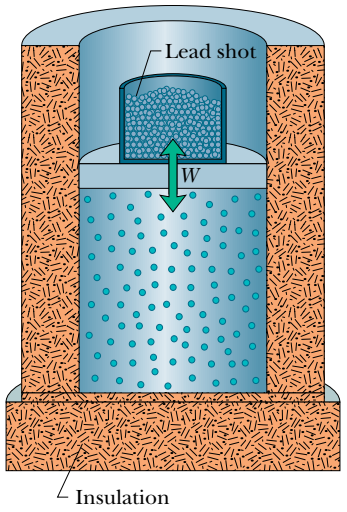

Figure 21-5 An adiabatic expansion can be carried out by slowly removing lead shot from the top of the piston. Adding lead shot reverses the process at any stage.

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The stopcock is opened, and the gas expands freely to fill both halves of the chamber. No heat is transferred to or from the gas because of the insulation. No work is done by the gas because it rushes into a vacuum and thus does not meet any pressure.

A free expansion differs from all other processes we have considered because it cannot be done slowly and in a controlled way. As a result, at any given instant during the sudden expansion, the gas is not in thermal equilibrium and its pressure is not uniform. Thus, although we can plot the initial and final states on a p-V diagram, we cannot plot the expansion itself.

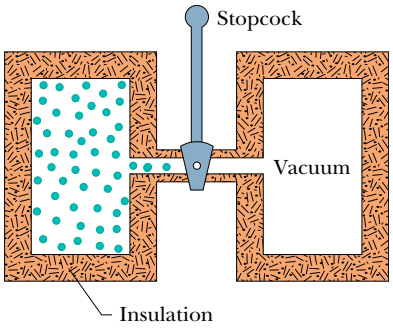
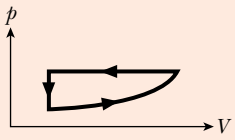


Figure 21-6 The initial stage of a free-expansion process. After the stopcock is opened, the gas fills both chambers and eventually reaches an equilibrium state.

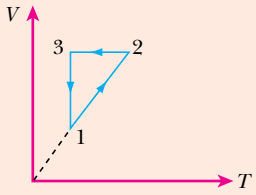
CHECKPOINT 4

For one complete cycle as shown in the p-V diagram here, are (a) ΔE_{int} for the gas and (b) the net energy transferred as heat Q positive, negative, or zero?



CHECKPOINT 5

A cyclic process 1-2-3-1, performed on an ideal gas is depicted in the given V-T diagram. Show the same process on a p-V diagram and indicate the stages when the gas receives and rejects heat.



SAMPLE PROBLEM 21.03

First law of thermodynamics: work, heat, and internal energy change

Let 1.00 kg of liquid water at 100°C be converted to steam at 100°C by boiling at standard atmospheric pressure (which is 1.00 atm or 1.01×10^5 Pa) in the arrangement of Fig. 21-7. The volume of that water changes from an initial value of 1.00×10^{-3} m^3 as a liquid to 1.671 m^3 as steam.

(a) How much work is done by the system during this process?

KEY IDEAS

(1) The system must do positive work because the volume increases. (2) We calculate the work W done by integrating the pressure with respect to the volume (Eq. 21-2).

Calculation: Because here the pressure is constant at 1.01×10^5 Pa, we can take p outside the integral. Thus,

$$W = \int_{V_i}^{V_f} p\,dV = p\int_{V_i}^{V_f} dV = p(V_f - V_i)$$
$$= (1.01\times10^5\ \text{Pa})(1.671\,\text{m}^3 - 1.00\times10^{-3}\,\text{m}^3)$$
$$= 1.69\times10^5\,\text{J} = 169\ \text{kJ}. \qquad \text{(Answer)}$$

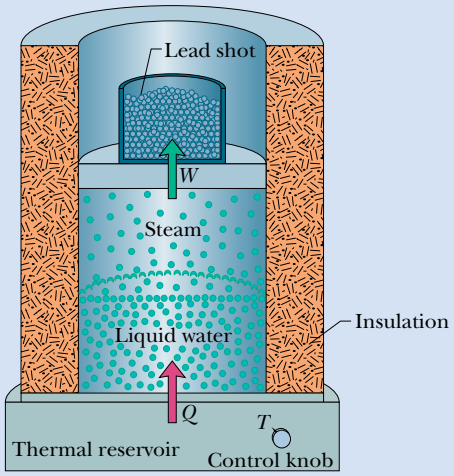


Figure 21-7 Water boiling at constant pressure. Energy is transferred from the thermal reservoir as heat until the liquid water has changed completely into steam. Work is done by the expanding gas as it lifts the loaded piston.

(b) How much energy is transferred as heat during the process?

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KEY IDEA

Because the heat causes only a phase change and not a change in temperature, it is given fully by Eq. 19-5 ($Q = Lm$).

Calculation: Because the change is from liquid to gaseous phase, L is the heat of vaporization L_V, with the value given in Eq. 19-6 and Table 19-2. We find

$$Q = L_V m = (2256 \text{ kJ/kg})(1.00 \text{ kg})$$
$$= 2256 \text{ kJ} \approx 2260 \text{ kJ}. \qquad \text{(Answer)}$$

(c) What is the change in the system's internal energy during the process?

KEY IDEA

The change in the system's internal energy is related to the heat (here, this is energy transferred into the system) and the work (here, this is energy transferred out of the system) by the first law of thermodynamics (Eq. 21-11).

Calculation: We write the first law as

$$\Delta E_{\text{int}} = Q - W = 2256 \text{ kJ} - 169 \text{ kJ}$$
$$\approx 2090 \text{ kJ} = 2.09 \text{ MJ}. \qquad \text{(Answer)}$$

Learn: This quantity is positive, indicating that the internal energy of the system has increased during the boiling process. The added energy goes into separating the H_2O molecules, which strongly attract one another in the liquid state. We see that, when water is boiled, about 7.5% (= 169 kJ/2260 kJ) of the heat goes into the work of pushing back the atmosphere. The rest of the heat goes into the internal energy of the system.

SAMPLE PROBLEM 21.04

Cyclic Process

The p-V diagram in Fig. 21-8*a* represents a series of processes, performed by an ideal gas, comprising a cyclic process of operations in which CD is an isothermal and DA is an adiabatic process. Fill in the table in Fig. 21-8*b* for this cycle, using "+" to indicate an increase in the quantity listed, "–" to indicate a decrease, and "0" to indicate no change.

Calculations:

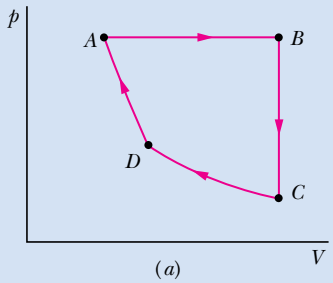
Path AB: The process represents an isobaric expansion (p remains constant while V increases). Clearly the gas will do some work on the environment, so $\Delta W > 0$. In order for the pressure to remain constant with a decrease in the gas density, the temperature must increase, so $\Delta T > 0$. For an ideal gas, the energy is a function of only temperature, and E increases with T, so $\Delta E > 0$. Because $\Delta H = \Delta E + \Delta W$, we must have $\Delta H > 0$.

Path BC: The process is isovolumetric (isochoric) and cooling (V remains constant while p decreases and T also decreases) with it. Because the volume does not vary anywhere along the path, we have $\Delta V = 0$ and therefore $\Delta W = p\Delta V = 0$. As we already noted, $\Delta T < 0$, since that is the only way that p can decrease at constant volume. Since E is directly dependent on T, we have $\Delta E < 0$. Finally, $\Delta E + \Delta W = \Delta H < 0$.

Path CD: The process is isothermal compression, so $\Delta T = 0$. Therefore, for an ideal gas, $\Delta E = 0$. Since there is a monotonic decrease in volume ($\Delta V < 0$), so the work $\Delta W < 0$. Therefore, $\Delta E + \Delta W = \Delta H < 0$.

Path DA: The process is adiabatic compression, so that $\Delta H = 0$. There is a monotonic decrease in volume ($\Delta V < 0$), so $\Delta W < 0$. Therefore, $\Delta H - \Delta W \equiv \Delta E > 0$. The temperature of the ideal gas must increase correspondingly and so, $\Delta T > 0$. These results are summarized below.

Path	ΔH	ΔW	ΔE	ΔT
AB	+	+	+	+
BC	–	0	–	–
CD	–	–	0	0
DA	0	–	+	+



(*a*)

Path	ΔH	ΔW	ΔE	ΔT
AB				
BC				
CD				
DA				

(*b*)

Figure 21-8 p-V diagram for ideal gas.

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21.5 | SOME APPLICATIONS OF FIRST LAW OF THERMODYNAMICS

Key Concepts

- The molar specific heat C_V of a gas at constant volume is defined as

$$C_V = \frac{Q}{n\Delta T} = \frac{\Delta E_{int}}{n\Delta T},$$

in which Q is the energy transferred as heat to or from a sample of n moles of the gas, ΔT is the resulting temperature change of the gas, and ΔE_{int} is the resulting change in the internal energy of the gas.

- For an ideal monatomic gas,

$$C_V = \frac{3}{2}R = 12.5 \text{ J/mol} \cdot \text{K}.$$

- The molar specific heat C_p of a gas at constant pressure is defined to be

$$C_p = \frac{Q}{n\Delta T},$$

in which Q, n, and ΔT are defined as above. C_p is also given by

$$C_p = C_V + R.$$

- For n moles of an ideal gas,

$$E_{int} = nC_V T \quad \text{(ideal gas)}.$$

- If n moles of a confined ideal gas undergo a temperature change ΔT due to *any* process, the change in the internal energy of the gas is

$$\Delta E_{int} = nC_V \Delta T \quad \text{(ideal gas, any process)}.$$

In Chapter 20, we have derived the expression for internal energy of an ideal gas,

$$E_{int} = \frac{3}{2}nRT \quad \text{(monatomic gas)}. \tag{21-17}$$

It is the energy associated with the random motion of atoms or molecules in the gas. This expression along with the application of first law of thermodynamics can be applied to derive the molar specific heats of an ideal gas. One is for the case in which the volume of the gas remains constant (isochoric process) as energy is transferred to or from it as heat. The other is for the case in which the pressure of the gas remains constant (isobaric process) as energy is transferred to or from it as heat. The symbols for these two molar specific heats are C_V and C_p, respectively. (By convention, the capital letter C is used in both cases, even though C_V and C_p represent types of specific heats and not heat capacities.)

Molar Specific Heat at Constant Volume

Figure 21-9*a* shows n moles of an ideal gas at pressure p and temperature T, confined to a cylinder of fixed volume V. This *initial state i* of the gas is marked on the p-V diagram of Fig. 21-9*b*. Suppose now that you add a small amount of energy to the gas as heat Q by slowly turning up the temperature of the thermal reservoir. The gas temperature rises a small amount to $T + \Delta T$, and its pressure rises to $p + \Delta p$, bringing the gas to *final state f*. In such experiments, we would find that the heat Q is related to the temperature change ΔT by

$$Q = nC_V \Delta T \quad \text{(constant volume)}, \tag{21-18}$$

where C_V is a constant called the **molar specific heat at constant volume**. Substituting this expression for Q into the first law of thermodynamics as given by Eq. 21-11 ($\Delta E_{int} = Q - W$) yields

$$\Delta E_{int} = nC_V \Delta T - W. \tag{21-19}$$

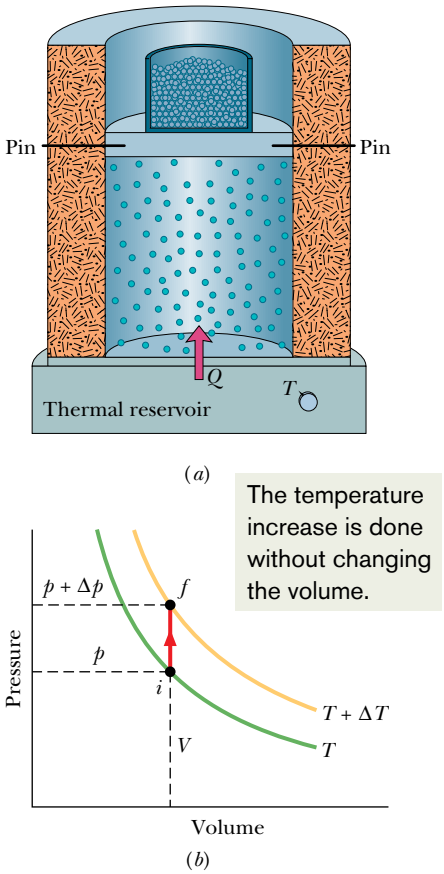

Figure 21-9 (*a*) The temperature of an ideal gas is raised from T to $T + \Delta T$ in a constant-volume process. Heat is added, but no work is done. (*b*) The process on a p-V diagram.

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With the volume held constant, the gas cannot expand and thus cannot do any work. Therefore, $W = 0$, and Eq. 21-19 gives us

$$C_V = \frac{\Delta E_{\text{int}}}{n\Delta T}. \tag{21-20}$$

From Eq. 21-17, the change in internal energy must be

$$\Delta E_{\text{imt}} = \frac{3}{2} nR\Delta T. \tag{21-21}$$

Substituting this result into Eq. 21-20 yields

$$C_V = \frac{3}{2} R = 12.5 \text{ J/mol} \cdot \text{K} \quad \text{(monatomic gas)}. \tag{21-22}$$

We have seen in Chapter 20 that this prediction of the kinetic theory (for ideal gases) agrees very well with experiment for real monatomic gases, the case that we have assumed. The (predicted and) experimental values of C_V for *diatomic gases* (which have molecules with two atoms) and *polyatomic gases* (which have molecules with more than two atoms) are greater than those for monatomic gases.

We can now generalize Eq. 21-17 for the internal energy of any ideal gas by substituting C_V for $(3/2)R$; we get

$$E_{\text{int}} = nC_V T \quad \text{(any ideal gas)}. \tag{21-23}$$

This equation applies not only to an ideal monatomic gas but also to diatomic and polyatomic ideal gases, provided the appropriate value of C_V is used. Just as with Eq. 21-17, we see that the internal energy of a gas depends on the temperature of the gas but not on its pressure or density.

When a confined ideal gas undergoes temperature change ΔT, then from either Eq. 21-20 or Eq. 21-23 the resulting change in its internal energy is

$$\Delta E_{\text{int}} = nC_V \Delta T \quad \text{(ideal gas, any process)}. \tag{21-24}$$

This equation tells us:

A change in the internal energy E_{int} of a confined ideal gas depends on only the change in the temperature, *not* on what type of process produces the change.

As an example, consider the three paths between the two isotherms in the p-V diagram of Fig. 21-10. Path 1 represents a constant-volume process. Path 2 represents a constant-pressure process (we examine it next). Path 3 represents a process in which no heat is exchanged with the system's environment. Although the values of heat Q and work W associated with these three paths differ, as do p_f and V_f, the values of ΔE_{int} associated with the three paths are identical and are all given by Eq. 21-24, because they all involve the same temperature change ΔT. Therefore, no matter what path is actually taken between T and $T + \Delta T$, we can *always* use path 1 and Eq. 21-24 to compute ΔE_{int} easily.

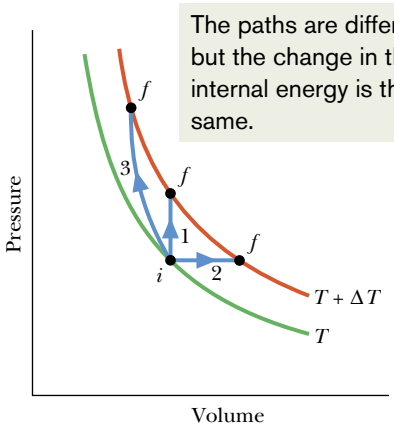

Figure 21-10 Three paths representing three different processes that take an ideal gas from an initial state i at temperature T to some final state f at temperature $T + \Delta T$. The change ΔE_{int} in the internal energy of the gas is the same for these three processes and for any others that result in the same change of temperature.

Molar Specific Heat at Constant Pressure

We now assume that the temperature of our ideal gas is increased by the same small amount ΔT as previously but now the necessary energy (heat Q) is added with the gas under constant pressure. An experiment for doing this is shown in Fig. 21-11*a*; the p-V diagram for the process is plotted in Fig. 21-11*b*. From such experiments we find that the heat Q is related to the temperature change ΔT by

$$Q = nC_p \Delta T \quad \text{(constant pressure)}, \tag{21-25}$$

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where C_p is a constant called the **molar specific heat at constant pressure**. This C_p is *greater* than the molar specific heat at constant volume C_V, because energy must now be supplied not only to raise the temperature of the gas but also for the gas to do work—that is, to lift the weighted piston of Fig. 21-11*a*.

To relate molar specific heats C_p and C_V, we start with the first law of thermodynamics (Eq. 21-11):

$$\Delta E_{int} = Q - W. \tag{21-26}$$

We next replace each term in Eq. 21-26. For ΔE_{int}, we substitute from Eq. 21-2. For Q, we substitute from Eq. 21-2. To replace W, we first note that since the pressure remains constant, Eq. 21-1 tells us that $W = p\,\Delta V$. Then we note that, using the ideal gas equation ($pV = nRT$), we can write

$$W = p\,\Delta V = nR\,\Delta T. \tag{21-27}$$

Making these substitutions in Eq. 21-26 and then dividing through by $n\,\Delta T$, we find

$$C_V = C_p - R$$

and then

$$C_p = C_V + R. \tag{21-28}$$

This prediction of kinetic theory agrees well with experiment, not only for monatomic gases but also for gases in general, as long as their density is low enough so that we may treat them as ideal.

The left side of Fig. 21-12 shows the relative values of Q for a monatomic gas undergoing either a constant-volume process ($Q = (3/2)nR\,\Delta T$) or a constant-pressure process ($Q = (5/2)nR\,\Delta T$). Note that for the latter, the value of Q is higher by the amount W, the work done by the gas in the expansion. Note also that for the constant-volume process, the energy added as Q goes entirely into the change in internal energy ΔE_{int} and for the constant-pressure process, the energy added as Q goes into both ΔE_{int} and the work W.

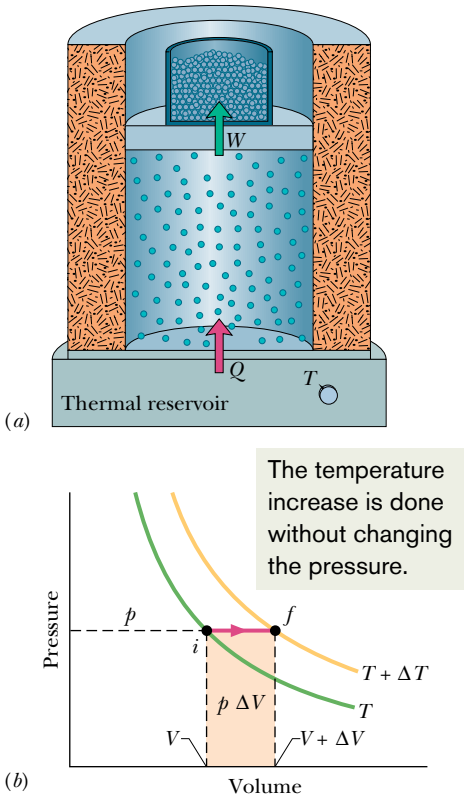

Figure 21-11 (*a*) The temperature of an ideal gas is raised from T to $T + \Delta T$ in a constant-pressure process. Heat is added and work is done in lifting the loaded piston. (*b*) The process on a p-V diagram. The work $p\,\Delta V$ is given by the shaded area.

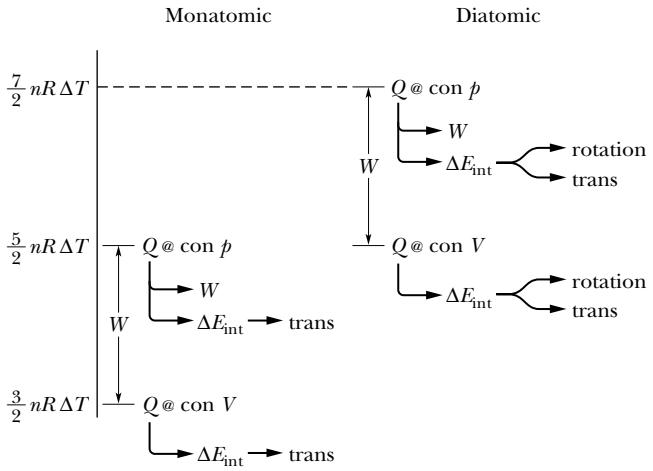

Figure 21-12 The relative values of Q for a monatomic gas (left side) and a diatomic gas undergoing a constant-volume process (labeled "con V") and a constant-pressure process (labeled "con p"). The transfer of the energy into work W and internal energy (ΔE_{int}) is noted.

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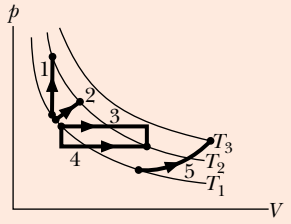
CHECKPOINT 6

The figure here shows five paths traversed by a gas on a p-V diagram. Rank the paths according to the change in internal energy of the gas, greatest first.



SAMPLE PROBLEM 21.05

Monatomic gas, heat, internal energy, and work

A bubble of 5.00 mol of helium is submerged at a certain depth in liquid water when the water (and thus the helium) undergoes a temperature increase ΔT of 20.0°C at constant pressure. As a result, the bubble expands. The helium is monatomic and ideal.

(a) How much energy is added to the helium as heat during the increase and expansion?

KEY IDEA

Heat Q is related to the temperature change ΔT by a molar specific heat of the gas.

Calculations: Because the pressure p is held constant during the addition of energy, we use the molar specific heat at constant pressure C_p and Eq. 21-25,

$$Q = nC_p\,\Delta T, \tag{21-29}$$

to find Q. To evaluate C_p we go to Eq. 21-28, which tells us that for any ideal gas, $C_p = C_V + R$. Then from Eq. 21-22, we know that for any *monatomic* gas (like the helium here), $C_V = (3/2)R$. Thus, Eq. 21-29 gives us

$$Q = n(C_V + R)\Delta T = n\left(\frac{3}{2}R + R\right)\Delta T = n\left(\frac{5}{2}R\right)\Delta T$$

$$= (5.00\text{ mol})(2.5)(8.31\text{ J/mol}\cdot\text{K})(20.0°\text{C})$$

$$= 2077.5\text{ J} \approx 2080\text{ J}. \qquad \text{(Answer)}$$

(b) What is the change ΔE_{int} in the internal energy of the helium during the temperature increase?

KEY IDEA

Because the bubble expands, this is not a constant-volume process. However, the helium is nonetheless confined (to the bubble). Thus, the change ΔE_{int} is the same as *would occur* in a constant-volume process with the same temperature change ΔT.

Calculation: We can now easily find the constant-volume change ΔE_{int} with Eq. 21-24:

$$\Delta E_{int} = nC_V\Delta T = n\left(\frac{3}{2}R\right)\Delta T$$

$$= (5.00\text{ mol})(1.5)(8.31\text{ J/mol}\cdot\text{K})(20.0°\text{C})$$

$$= 1246.5\text{ J} \approx 1250\text{ J}. \qquad \text{(Answer)}$$

(c) How much work W is done by the helium as it expands against the pressure of the surrounding water during the temperature increase?

KEY IDEAS

The work done by *any* gas expanding against the pressure from its environment is given by Eq. 21-2, which tells us to integrate $p\,dV$. When the pressure is constant (as here), we can simplify that to $W = p\,\Delta V$. When the gas is *ideal* (as here), we can use the ideal gas law (Eq. 20-5) to write $p\,\Delta V = nR\,\Delta T$.

Calculation: We end up with

$$W = nR\,\Delta T$$

$$= (5.00\text{ mol})(8.31\text{ J/mol}\cdot\text{K})(20.0°\text{C})$$

$$= 831\text{ J}. \qquad \text{(Answer)}$$

Another way: Because we happen to know Q and ΔE_{int}, we can work this problem another way: We can account for the energy changes of the gas with the first law of thermodynamics, writing

$$W = Q - \Delta E_{int} = 2077.5\text{ J} - 1246.5\text{ J}$$

$$= 831\text{ J}. \qquad \text{(Answer)}$$

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The transfers: Let's follow the energy. Of the 2077.5 J transferred to the helium as heat Q, 831 J goes into the work W required for the expansion and 1246.5 J goes into the internal energy E_{int}, which, for a monatomic gas, is entirely the kinetic energy of the atoms in their translational motion. These several results are suggested on the left side of Fig. 21-11.

SAMPLE PROBLEM 21.06

Diatomic gas, heat, temperature, internal energy

We transfer 1000 J as heat Q to a diatomic gas, allowing the gas to expand with the pressure held constant. The gas molecules each rotate around an internal axis but do not oscillate. How much of the 1000 J goes into the increase of the gas's internal energy? Of that amount, how much goes into ΔK_{tran} (the kinetic energy of the translational motion of the molecules) and ΔK_{rot} (the kinetic energy of their rotational motion)?

KEY IDEAS

(1) The transfer of energy as heat Q to a gas under constant pressure is related to the resulting temperature increase ΔT via Eq. 21-25 ($Q = nC_p\,\Delta T$).
(2) Because the gas is diatomic with molecules undergoing rotation but not oscillation, the molar specific heat is, from Fig. 21-12 and Table 20-3, $C_p = (7/2)R$.
(3) The increase ΔE_{int} in the internal energy is the same as would occur with a constant-volume process resulting in the same ΔT. Thus, from Eq. 21-24, $\Delta E_{int} = nC_V\,\Delta T$. From Fig. 21-12 and Table 20-3, we see that $C_V = (5/2)R$.
(4) For the same n and ΔT, ΔE_{int} is greater for a diatomic gas than for a monatomic gas because additional energy is required for rotation.

Increase in E_{int}: Let's first get the temperature change ΔT due to the transfer of energy as heat. From Eq. 21-25, substituting $(7/2)R$ for C_p, we have

$$\Delta T = \frac{Q}{\frac{7}{2}nR}. \tag{21-30}$$

We next find ΔE_{int} from Eq. 21-24, substituting the molar specific heat $C_V (= 5/2R)$ for a constant-volume process and using the same ΔT. Because we are dealing with a diatomic gas, let's call this change $\Delta E_{int,dia}$. Equation 21-24 gives us

$$\Delta E_{int,dia} = nC_V\Delta T = n\frac{5}{2}R\left(\frac{Q}{\frac{7}{2}nR}\right) = \frac{5}{7}Q$$
$$= 0.71428Q = 714.3\text{ J}. \qquad \text{(Answer)}$$

In words, about 71% of the energy transferred to the gas goes into the internal energy. The rest goes into the work required to increase the volume of the gas, as the gas pushes the walls of its container outward.

Increase in K: If we were to increase the temperature of a *monatomic* gas (with the same value of n) by the amount given in Eq. 21-30, the internal energy would change by a smaller amount, call it $\Delta E_{int,mon}$, because rotational motion is not involved. To calculate that smaller amount, we still use Eq. 21-24 but now we substitute the value of C_V for a monatomic gas—namely, $C_V = 3/2R$. So,

$$\Delta E_{int,mon} = n\frac{3}{2}R\,\Delta T.$$

Substituting for ΔT from Eq. 21-30 leads us to

$$\Delta E_{int,mon} = n\frac{3}{2}R\left(\frac{Q}{n\frac{7}{2}R}\right) = \frac{3}{7}Q$$
$$= 0.42857Q = 428.6\text{ J}.$$

Learn: For the monatomic gas, all this energy would go into the kinetic energy of the translational motion of the atoms. The important point here is that for a diatomic gas with the same values of n and ΔT, the same amount of energy goes into the kinetic energy of the translational motion of the molecules. The rest of $\Delta E_{int,dia}$ (that is, the additional 285.7 J) goes into the rotational motion of the molecules. Thus, for the diatomic gas,

$$\Delta K_{trans} = 428.6\text{ J} \quad \text{and} \quad \Delta K_{rot} = 285.7\text{ J}. \qquad \text{(Answer)}$$

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21.6 | THE ADIABATIC EXPANSION OF AN IDEAL GAS

Key Concepts

- When an ideal gas undergoes a slow adiabatic volume change (a change for which $Q = 0$),

  pV^{γ} = a constant (adiabatic process),

  in which $\gamma (= C_p/C_V)$ is the ratio of molar specific heats for the gas.
- For a free expansion, pV = a constant.

Sound waves are propagated through air and other gases as a series of compressions and expansions; these variations in the transmission medium take place so rapidly that there is no time for energy to be transferred from one part of the medium to another as heat. As we know, a process for which $Q = 0$ is an *adiabatic process*. We can ensure that $Q = 0$ either by carrying out the process very quickly (as in sound waves) or by doing it (at any rate) in a well-insulated container.

Figure 21-13*a* shows our usual insulated cylinder, now containing an ideal gas and resting on an insulating stand. By removing mass from the piston, we can allow the gas to expand adiabatically. As the volume increases, both the pressure and the temperature drop. We shall prove next that the relation between the pressure and the volume during such an adiabatic process is

$$pV^{\gamma} = \text{a constant} \quad \text{(adiabatic process)}, \tag{21-31}$$

in which $\gamma = C_p/C_V$, the ratio of the molar specific heats for the gas. On a p-V diagram such as that in Fig. 21-13*b*, the process occurs along a line (called an *adiabat*) that has the equation p = (a constant)/V^{γ}. Since the gas goes from an initial state i to a final state f, we can rewrite Eq. 21-31 as

$$p_i V_i^{\gamma} = p_f V_f^{\gamma} \quad \text{(adiabatic process)}. \tag{21-32}$$

To write an equation for an adiabatic process in terms of T and V, we use the ideal gas equation ($pV = nRT$) to eliminate p from Eq. 21-31, finding

$$\left(\frac{nRT}{V}\right) V^{\gamma} = \text{a constant}.$$

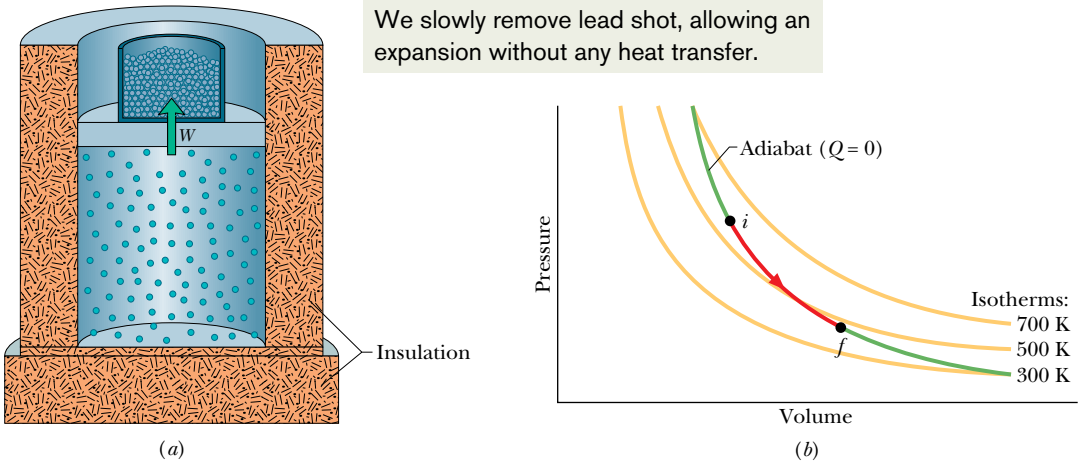

Figure 21-13 (*a*) The volume of an ideal gas is increased by removing mass from the piston. The process is adiabatic ($Q = 0$). (*b*) The process proceeds from i to f along an adiabat on a p-V diagram.

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Because n and R are constants, we can rewrite this in the alternative form

$$TV^{\gamma-1} = \text{a constant} \quad \text{(adiabatic process)}, \tag{21-33}$$

in which the constant is different from that in Eq. 21-31. When the gas goes from an initial state i to a final state f, we can rewrite Eq. 21-33 as

$$T_i V_i^{\gamma-1} = T_i V_f^{\gamma-1} \quad \text{(adiabatic process)}. \tag{21-34}$$

Understanding adiabatic processes allows you to understand why popping the cork on a cold bottle of champagne or the tab on a cold can of soda causes a slight fog to form at the opening of the container. At the top of any unopened carbonated drink sits a gas of carbon dioxide and water vapor. Because the pressure of that gas is much greater than atmospheric pressure, the gas expands out into the atmosphere when the container is opened. Thus, the gas volume increases, but that means the gas must do work pushing against the atmosphere. Because the expansion is rapid, it is adiabatic, and the only source of energy for the work is the internal energy of the gas. Because the internal energy decreases, the temperature of the gas also decreases and so does the number of water molecules that can remain as a vapor. So, lots of the water molecules condense into tiny drops of fog.

Proof of Eq. 21-31

Suppose that you remove some shot from the piston of Fig. 21-13*a*, allowing the ideal gas to push the piston and the remaining shot upward and thus to increase the volume by a differential amount dV. Since the volume change is tiny, we may assume that the pressure p of the gas on the piston is constant during the change. This assumption allows us to say that the work dW done by the gas during the volume increase is equal to $p\,dV$. From Eq. 21-12, the first law of thermodynamics can then be written as

$$dE_{\text{int}} = Q - p\,dV. \tag{21-35}$$

Since the gas is thermally insulated (and thus the expansion is adiabatic), we substitute 0 for Q. Then we use Eq. 21-24 to substitute $nC_V\,dT$ for dE_{int}. With these substitutions, and after some rearranging, we have

$$n\,dT = -\left(\frac{p}{C_V}\right)dV. \tag{21-36}$$

Now from the ideal gas law ($pV = nRT$) we have

$$p\,dV + V\,dp = nR\,dT. \tag{21-37}$$

Replacing R with its equal, $C_p - C_V$, in Eq. 21-37 yields

$$n\,dT = \frac{p\,dV + V\,dp}{C_p - C_V}. \tag{21-38}$$

Equating Eqs. 21-36 and 21-38 and rearranging then give

$$\frac{dp}{p} + \left(\frac{C_p}{C_V}\right)\frac{dV}{V} = 0.$$

Replacing the ratio of the molar specific heats with γ and integrating (see integral 5 in Appendix E) yield

$$\ln p + \gamma \ln V = \text{a constant}.$$

Rewriting the left side as $\ln pV^{\gamma}$ and then taking the antilog of both sides, we find

$$pV^{\gamma} = \text{a constant}. \tag{21-39}$$

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SAMPLE PROBLEM 21.07

Work done by a gas in an adiabatic expansion

Initially an ideal diatomic gas has pressure $p_i = 2.00 \times 10^5$ Pa and volume $V_i = 4.00 \times 10^{-6}$ m^3. How much work W does it do, and what is the change ΔE_{int} in its internal energy if it expands adiabatically to volume $V_f = 8.00 \times 10^{-6}$ m^3? Throughout the process, the molecules have rotation but not oscillation.

KEY IDEAS

(1) In an adiabatic expansion, no heat is exchanged between the gas and its environment, and the energy for the work done by the gas comes from the internal energy. (2) The final pressure and volume are related to the initial pressure and volume by Eq. 21-32 ($p_i V_i^\gamma = p_f V_f^\gamma$). (3) The work done by a gas in any process can be calculated by integrating the pressure with respect to the volume (the work is due to the gas pushing the walls of its container outward).

Calculations: We want to calculate the work by filling out this integration,

$$W = \int_{V_i}^{V_f} p\, dV, \tag{21-40}$$

but we first need an expression for the pressure as a function of volume (so that we integrate the expression with respect to volume). So, let's rewrite Eq. 21-32 with indefinite symbols (dropping the subscripts f) as

$$p = \frac{1}{V^\gamma} p_i V_i^\gamma = V^{-\gamma} p_i V_i^\gamma. \tag{21-41}$$

The initial quantities are given constants but the pressure p is a function of the variable volume V. Substituting this expression into Eq. 21-40 and integrating lead us to

$$\begin{aligned} W &= \int_{V_i}^{V_f} p\, dV = \int_{V_i}^{V_f} V^{-\gamma} p_i V_i^\gamma\, dV \\ &= p_i V_i^\gamma \int_{V_i}^{V_f} V^{-\gamma}\, dV = \frac{1}{-\gamma+1} p_i V_i^\gamma [V^{-\gamma+1}]_{V_i}^{V_f} \\ &= \frac{1}{-\gamma+1} p_i V_i^\gamma [V_f^{-\gamma+1} - V_i^{-\gamma+1}]. \end{aligned} \tag{21-42}$$

Before we substitute in given data, we must determine the ratio γ of molar specific heats for a gas of diatomic molecules with rotation but no oscillation. From Table 20-3 we find

$$\gamma = \frac{C_p}{C_V} = \frac{\frac{7}{2}R}{\frac{5}{2}R} = 1.4. \tag{21-43}$$

We can now write the work done by the gas as the following (with volume in cubic meters and pressure in pascals):

$$\begin{aligned} W &= \frac{1}{-1.4+1}(2.00\times10^5)(4.00\times10^{-6})^{1.4} \\ &\quad \times[(8.00\times10^{-6})^{-1.4+1} - (4.00\times10^{-6})^{-1.4+1}] \\ &= 0.48 \text{ J}. \end{aligned} \tag{Answer}$$

The first law of thermodynamics (Eq. 21-11) tells us that $\Delta E_{int} = Q - W$. Because $Q = 0$ in the adiabatic expansion, we see that

$$\Delta E_{int} = -0.48 \text{ J}. \tag{Answer}$$

With this decrease in internal energy, the gas temperature must also decrease because of the expansion.

PROBLEM-SOLVING TACTICS

A Graphical Summary of Four Gas Processes

In this chapter we have discussed four special processes that an ideal gas can undergo. An example of each (for a monatomic ideal gas) is shown in Fig. 21-14, and some associated characteristics are given in Table 21-2, including two process names (isobaric and isochoric) that we have not used but that you might see in other courses.

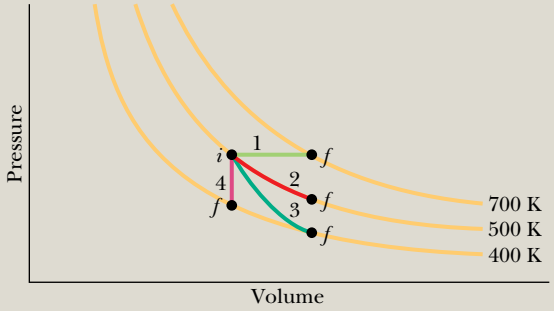

Figure 21-14 A p-V diagram representing four special processes for an ideal monatomic gas.

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Table 21-2 Four Special Processes

Path in Fig. 21-14	Constant Quantity	Process Type	Some Special Results ($\Delta E_{int} = Q - W$ and $\Delta E_{int} = nC_V\Delta T$ for all paths)
1	p	Isobaric	$Q = nC_p\Delta T; W = p\Delta V$
2	T	Isothermal	$Q = W = nRT\ln(V_f/V_i); \Delta E_{int} = 0$
3	$pV^\gamma, TV^{\gamma-1}$	Adiabatic	$Q = 0; W = -\Delta E_{int}$
4	V	Isochoric	$Q = \Delta E_{int} = nC_V\Delta T; W = 0$

CHECKPOINT 7

Rank paths 1, 2, and 3 in Fig. 21-14 according to the energy transfer to the gas as heat, greatest first.

21.7 | THE CARNOT ENGINE

Key Concepts

- In an ideal engine, all processes are reversible and no wasteful energy transfers occur due to, say, friction and turbulence.
- A Carnot engine is an ideal engine that follows the cycle of Fig. 21-16. Its efficiency is

$$\varepsilon_C = 1 - \frac{|Q_L|}{|Q_H|} = 1 - \frac{T_L}{T_H},$$

in which T_H and T_L are the temperatures of the high- and low-temperature reservoirs, respectively. Real engines always have an efficiency lower than that of a Carnot engine. Ideal engines that are not Carnot engines also have efficiencies lower than that of a Carnot engine.

A **heat engine**, or more simply, an **engine**, is a device that extracts energy from its environment in the form of heat and does useful work. At the heart of every engine is a *working substance*. In a steam engine, the working substance is water, in both its vapor and its liquid form. In an automobile engine the working substance is a gasoline–air mixture. If an engine is to do work on a sustained basis, the working substance must operate in a *cycle*; that is, the working substance must pass through a closed series of thermodynamic processes, called *strokes*, returning again and again to each state in its cycle. Let us see what the laws of thermodynamics can tell us about the operation of engines.

We have seen that we can learn much about real gases by analyzing an ideal gas, which obeys the simple law $pV = nRT$. Although an ideal gas does not exist, any real gas approaches ideal behavior if its density is low enough. Similarly, we can study real engines by analyzing the behavior of an **ideal engine**.

In an ideal engine, all processes are reversible and no wasteful energy transfers occur due to, say, friction and turbulence.

We shall focus on a particular ideal engine called a **Carnot engine** after the French scientist and engineer N. L. Sadi Carnot (pronounced "car-no"), who first proposed the engine's concept in 1824. This ideal engine turns out to be the best (in principle) at using energy as heat to do useful work. Surprisingly, Carnot was able to analyze the performance of this engine before the first law of thermodynamics and the concept of entropy had been discovered.

Figure 21-15 shows schematically the operation of a Carnot engine. During each cycle of the engine, the working substance absorbs energy $|Q_H|$ as heat from a thermal reservoir at constant temperature T_H and discharges energy $|Q_L|$ as heat to a second thermal reservoir at a constant lower temperature T_L.

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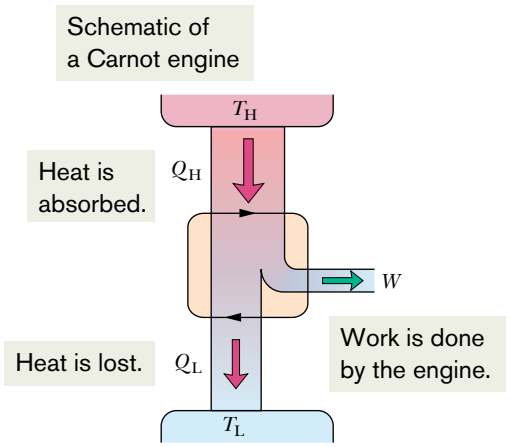

Figure 21-15 The elements of a Carnot engine. The two black arrowheads on the central loop suggest the working substance operating in a cycle, as if on a p-V plot. Energy $|Q_H|$ is transferred as heat from the high-temperature reservoir at temperature T_H to the working substance. Energy $|Q_L|$ is transferred as heat from the working substance to the low-temperature reservoir at temperature T_L. Work W is done by the engine (actually by the working substance) on something in the environment.

Figure 21-16 shows a p-V plot of the *Carnot cycle*—the cycle followed by the working substance. As indicated by the arrows, the cycle is traversed in the clockwise direction. Imagine the working substance to be a gas, confined to an insulating cylinder with a weighted, movable piston. The cylinder may be placed at will on either of the two thermal reservoirs, as in Fig. 21-17, or on an insulating slab. Figure 21-16*a* shows that, if we place the cylinder in contact with the high-temperature reservoir at temperature T_H, heat $|Q_H|$ is transferred *to* the working substance *from* this reservoir as the gas undergoes an isothermal *expansion* from volume V_a to volume V_b. Similarly, with the working substance in contact with the low-temperature reservoir at temperature T_L, heat $|Q_L|$ is transferred *from* the working substance *to* the low-temperature reservoir as the gas undergoes an isothermal *compression* from volume V_c to volume V_d (Fig. 21-16*b*).

In the engine of Fig. 21-15, we assume that heat transfers to or from the working substance can take place *only* during the isothermal processes *ab* and *cd* of Fig. 21-16. Therefore, processes *bc* and *da* in that figure, which connect the two isotherms at temperatures T_H and T_L, must be (reversible) adiabatic processes; that is, they must be processes in which no energy is transferred as heat. To ensure this, during processes *bc* and *da* the cylinder is placed on an insulating slab as the volume of the working substance is changed.

During the processes *ab* and *bc* of Fig. 21-16*a*, the working substance is expanding and thus doing positive work as it raises the weighted piston. This work is represented in Fig. 21-16*a* by the area under curve *abc*. During the processes *cd* and *da* (Fig. 21-16*b*), the working substance is being compressed, which means that it is doing negative work on its environment or, equivalently, that its environment is doing work on it as the loaded piston descends. This work is represented by the area under curve *cda*. The *net work per cycle*, which is represented by W in both Figs. 21-15 and 21-16, is the difference between these two areas and is a positive quantity equal to the area enclosed by cycle *abcda* in Fig. 21-16. This work W is performed on some outside object, such as a load to be lifted.

The Work. To calculate the net work done by a Carnot engine during a cycle, let us apply Eq. 21-11, the first law of thermodynamics ($\Delta E_{int} = Q - W$), to the working substance. That substance must return again and again to any

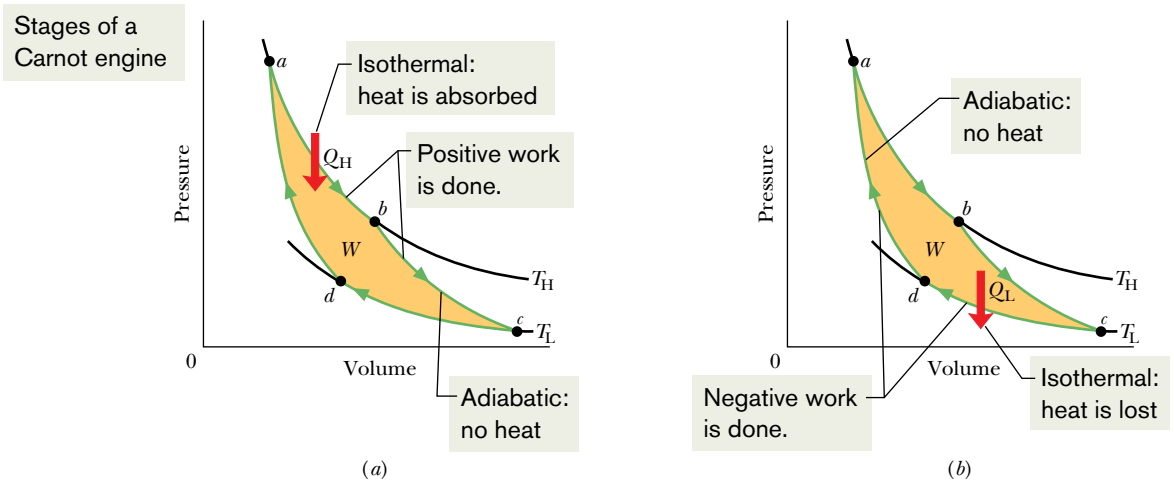

Figure 21-16 A pressure–volume plot of the cycle followed by the working substance of the Carnot engine in Fig. 21-15. The cycle consists of two isothermal (*ab* and *cd*) and two adiabatic processes (*bc* and *da*). The shaded area enclosed by the cycle is equal to the work W per cycle done by the Carnot engine.

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arbitrarily selected state in the cycle. Thus, if X represents any state property of the working substance, such as pressure, temperature, volume, or internal energy, we must have $\Delta X = 0$ for every cycle. It follows that $\Delta E_{int} = 0$ for a complete cycle of the working substance. Recalling that Q in Eq. 21-11 is the *net* heat transfer per cycle and W is the *net* work, we can write the first law for this cycle (or any ideal cycle) as

$$W = |Q_H| - |Q_L|. \tag{21-44}$$

Here $|Q_H|$ is the energy absorbed as heat at constant temperature T_H during expansion and $|Q_L|$ is the energy lost as heat at constant temperature T_L.

We shall now derive an expression for the efficiency of a Carnot engine.

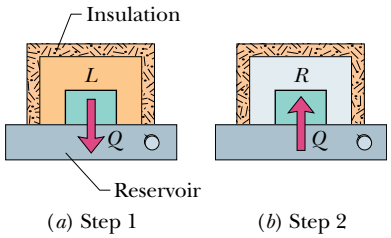

Figure 21-17 The copper blocks L and R are identical except for their temperatures are kept in an insulating box. They can proceed from their initial state to their final state in a reversible way if we use a reservoir with a controllable temperature (*a*) to extract heat reversibly from block L and (*b*) to add heat reversibly to block R.

Efficiency of a Carnot Engine

The purpose of any engine is to transform as much of the extracted energy Q_H into work as possible. We measure its success in doing so by its **thermal efficiency** ε, defined as the work the engine does per cycle ("energy we get") divided by the energy it absorbs as heat per cycle ("energy we pay for"):

$$\varepsilon = \frac{\text{energy we get}}{\text{energy we pay for}} = \frac{|W|}{|Q_H|} \quad \text{(efficiency, any engine)}. \tag{21-45}$$

For any ideal engine we substitute for W from Eq. 21-44 to write Eq. 21-45 as

$$\varepsilon = \frac{|Q_H| - |Q_L|}{Q_H} = 1 - \frac{|Q_L|}{|Q_H|}. \tag{21-46}$$

For a Carnot engine, we can write efficiency as

$$\varepsilon_C = 1 - \frac{T_L}{T_H} \quad \text{(efficiency, Carnot engine)}. \tag{21-47}$$

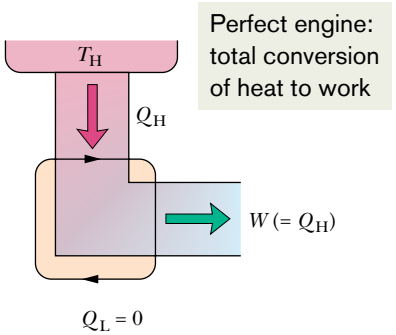

Figure 21-18 The elements of a perfect engine—that is, one that converts heat Q_H from a high-temperature reservoir directly to work W with 100% efficiency.

where the temperatures T_L and T_H are in kelvins. Because $T_L < T_H$, the Carnot engine necessarily has a thermal efficiency less than unity—that is, less than 100%. This is indicated in Fig. 21-15, which shows that only part of the energy extracted as heat from the high-temperature reservoir is available to do work, and the rest is delivered to the low-temperature reservoir. No real engine can have a thermal efficiency greater than that calculated from Eq. 21-47.

Inventors continually try to improve engine efficiency by reducing the energy $|Q_L|$ that is "thrown away" during each cycle. The inventor's dream is to produce the *perfect engine*, diagrammed in Fig. 21-18, in which $|Q_L|$ is reduced to zero and $|Q_H|$ is converted completely into work. Such an engine on an ocean liner, for example, could extract energy as heat from the water and use it to drive the propellers, with no fuel cost. An automobile fitted with such an engine could extract energy as heat from the surrounding air and use it to drive the car, again with no fuel cost. Alas, a perfect engine is only a dream: Inspection of Eq. 21-47 shows that we can achieve 100% engine efficiency (that is, $\varepsilon = 1$) only if $T_L = 0$ or $T_H \to \infty$, impossible requirements. Instead, experience gives the following alternative version of the second law of thermodynamics, which says in short, *there are no perfect engines:*

No series of processes is possible whose sole result is the transfer of energy as heat from a thermal reservoir and the complete conversion of this energy to work.

To summarize: The thermal efficiency given by Eq. 21-47 applies only to Carnot engines. Real engines, in which the processes that form the engine cycle are not reversible, have lower efficiencies. If your car were powered by

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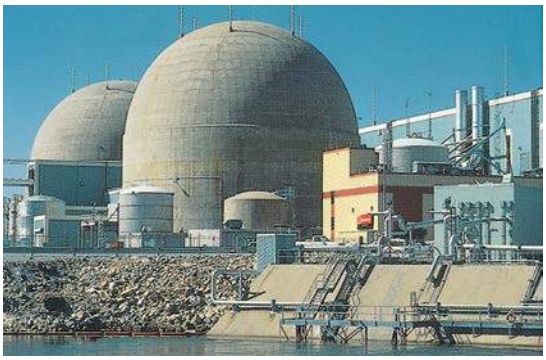

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Figure 21-19 The North Anna nuclear power plant near Charlottesville, Virginia, which generates electric energy at the rate of 900 MW. At the same time, by design, it discards energy into the nearby river at the rate of 2100 MW. This plant and all others like it throw away more energy than they deliver in useful form. They are real counterparts of the ideal engine of Fig. 21-15.

a Carnot engine, it would have an efficiency of about 55% according to Eq. 21-47; its actual efficiency is probably about 25%. A nuclear power plant (Fig. 21-19), taken in its entirety, is an engine. It extracts energy as heat from a reactor core, does work by means of a turbine, and discharges energy as heat to a nearby river. If the power plant operated as a Carnot engine, its efficiency would be about 40%; its actual efficiency is about 30%. In designing engines of any type, there is simply no way to beat the efficiency limitation imposed by Eq. 21-47.

CHECKPOINT 8

Three Carnot engines operate between reservoir temperatures of (a) 400 and 500 K, (b) 600 and 800 K, and (c) 400 and 600 K. Rank the engines according to their thermal efficiencies, greatest first.

SAMPLE PROBLEM 21.08

Carnot engine, efficiency, power

Imagine a Carnot engine that operates between the temperatures $T_H = 850$ K and $T_L = 300$ K. The engine performs 1200 J of work each cycle, which takes 0.25 s.

(a) What is the efficiency of this engine?

KEY IDEA

The efficiency ε of a Carnot engine depends only on the ratio T_L/T_H of the temperatures (in kelvins) of the thermal reservoirs to which it is connected.

Calculation: Thus, from Eq. 21-47, we have

$$\varepsilon = 1 - \frac{T_L}{T_H} = 1 - \frac{300\text{ K}}{850\text{ K}} = 0.647 \approx 65\%. \qquad \text{(Answer)}$$

(b) What is the average power of this engine?

KEY IDEA

The average power P of an engine is the ratio of the work W it does per cycle to the time t that each cycle takes.

Calculation: For this Carnot engine, we find

$$P = \frac{W}{t} = \frac{1200\text{ J}}{0.25\text{ s}} = 4800\text{ W} = 4.8\text{ kW}. \qquad \text{(Answer)}$$

(c) How much energy $|Q_H|$ is extracted as heat from the high-temperature reservoir every cycle?

KEY IDEA

The efficiency ε is the ratio of the work W that is done per cycle to the energy $|Q_H|$ that is extracted as heat from the high-temperature reservoir per cycle ($\varepsilon = W/|Q_H|$).

Calculation: Here we have

$$|Q_H| = \frac{W}{\varepsilon} = \frac{1200\text{ J}}{0.647} = 1855\text{ J}. \qquad \text{(Answer)}$$

(d) How much energy $|Q_L|$ is delivered as heat to the low-temperature reservoir every cycle?

KEY IDEA

For a Carnot engine, the work W done per cycle is equal to the difference in the energy transfers as heat: $|Q_H| - |Q_L|$, as in Eq. 21-44.

Calculation: Thus, we have

$$|Q_L| = |Q_H| - W$$
$$= 1855\text{ J} - 1200\text{ J} = 655\text{ J}. \qquad \text{(Answer)}$$

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SAMPLE PROBLEM 21.09

Impossibly efficient engine

An inventor claims to have constructed an engine that has an efficiency of 75% when operated between the boiling and freezing points of water. Is this possible?

KEY IDEA

The efficiency of a real engine must be less than the efficiency of a Carnot engine operating between the same two temperatures.

Calculation: From Eq. 21-47, we find that the efficiency of a Carnot engine operating between the boiling and freezing points of water is

$$\varepsilon = 1 - \frac{T_L}{T_H} = 1 - \frac{(0+273)\,K}{(100+273)\,K} = 0.268 \approx 27\%$$

Thus, for the given temperatures, the claimed efficiency of 75% for a real engine (with its irreversible processes and wasteful energy transfers) is impossible.

SAMPLE PROBLEM 21.10

Otto cycle

The cycle of a petrol engine can be approximated by an Otto cycle (Fig. 21-20*a*). $1 \rightarrow 2$ and $3 \rightarrow 4$ are adiabatic compression and expansion, respectively; $2 \rightarrow 3$ and $4 \rightarrow 1$ are constant-volume processes. Treat the working medium as an ideal gas with constant $\gamma = C_p/C_V$. Compute the efficiency of this cycle for $g = 1.4$ and compression ratio, $r = V_i/V_f = 10$.

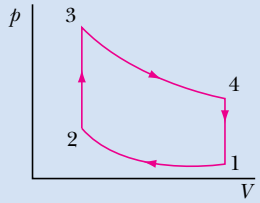

Figure 21-20 (*a*) *p-V* diagram of Otto cycle.

KEY IDEAS

We can see from the *p-V* graph (Fig. 21-20*b*) that it is a cyclic process as the working medium returns to its initial state. We can see from the *p-V* graph that in the process $3 \rightarrow 4$, the engine is doing positive work on the surroundings and in $1 \rightarrow 2$, it is doing negative work as the gas is compressed. In other two processes, $2 \rightarrow 3$ and $4 \rightarrow 1$, work done is zero, as $\Delta V = 0$.

The efficiency of a cyclic process is given by $\varepsilon = W/Q_1$, where W is the work done by the cycle and Q_1 is the amount of heat absorbed by the working medium (in this case, gas). Because the working medium returns to its initial state $W = Q_1 - Q_2$, where Q_2 is the amount of heat transferred from the gas. Therefore,

$$\varepsilon = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1}$$

As the processes $1 \rightarrow 2$ and $3 \rightarrow 4$ are adiabatic, heat transfer can take place only during the $2 \rightarrow 3$ and $1 \rightarrow 4$ stages.

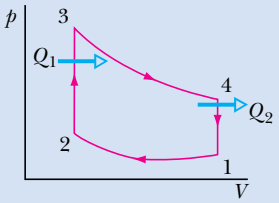

Figure 21-20 (*b*) Otto cycle showing the heat absorbed (Q_1) and heat transferred (Q_2) by the working medium.

Calculations: Let us calculate Q_1 and Q_2. Since both processes are occurring at a constant volume (Fig. 21-20*b*), we may write

$$Q_1 = \int_2^3 dQ = C_V \int_2^3 dT = C_V (T_3 - T_2)$$

$$Q_2 = C_V \int_4^1 dT = -C_V (T_1 - T_4) = C_V (T_4 - T_1)$$

and

$$\frac{Q_2}{Q_1} = \frac{T_4 - T_1}{T_3 - T_2}$$

We know that for an adiabatic process
pV^γ = constant or $TV^{\gamma-1}$ = constant

So,

$$T_1 V_i^{\gamma-1} = T_2 V_f^{\gamma-1} \qquad (21\text{-}48)$$

$$T_3 V_f^{\gamma-1} = T_4 V_i^{\gamma-1} \qquad (21\text{-}49)$$

Dividing Eq. 21-49 by Eq. 21-48, we get

$$\frac{T_4}{T_1} = \frac{T_3}{T_2}$$

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Subtracting 1 from both sides and solving, gives

$$\frac{T_4 - T_1}{T_1} = \frac{T_3 - T_2}{T_2}$$

Rearranging above relation and using Eq. 21-48 gives

$$\frac{T_4 - T_1}{T_3 - T_2} = \frac{T_1}{T_2} = \left(\frac{V_f}{V_i}\right)^{\gamma - 1}$$

and therefore, the efficiency is

$$\varepsilon = 1 - \left(\frac{1}{r}\right)^{\gamma - 1}$$

For $\gamma = 1.4$ and $r = 10$, the efficiency is

$$\varepsilon = 1 - (0.10)^{0.4} = 0.6 = 60\% \qquad \text{(Answer)}$$

The maximum possible efficiency, if the cycle would have been Carnot cycle, would be, $1 - T_2/T_4$, as T_2 and T_4 are the lowest and highest temperatures in the cycle. It can be seen that efficiency is less than the Carnot efficiency as the value of T_1/T_2 would be more than T_2/T_4.

SAMPLE PROBLEM 21.11

Joule cycle

Find the efficiency of the Joule cycle, consisting of two adiabats and two isobars (Fig. 21-21*a*). The cycle is being performed between pressures p_2 and p_1. Assume the gas to be ideal and adiabatic exponent to be γ.

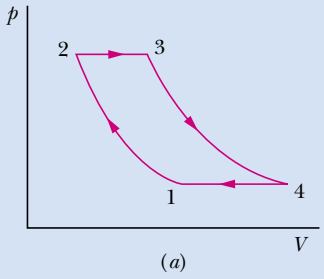

Figure 21-21 (*a*) *p-V* diagram of Joule cycle.

KEY IDEAS

It can be seen from the *p-V* graph that it is a cyclic process as the working medium returns to its initial state. We can see from the *p-V* graph that in processes $2 \to 3 \to 4$, the engine is doing positive work on the surroundings and in processes $4 \to 1 \to 2$, it is doing negative work as the gas is getting compressed.

The efficiency η of the cycle is given by the total work W, done during the cycle, divided by the heat Q absorbed in path $2 \to 3$ (Fig. 21-21*a*). Total work done W can be calculated by the area enclosed by the four paths of the *p-V* plot.

Calculations: The integral $W = \int p\, dV$ along the paths of constant pressure $2 \to 3$ and $4 \to 1$, is simply the difference in volume of the ends, multiplied by the pressure.

Consider the *p-V* graph shown in Fig. 21-21*b*. Here, the work done along the adiabats, where there is no heat transfer $\delta Q = 0$, is given by the change in internal energy $C_V \delta T$.

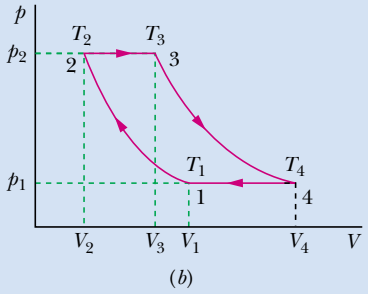

Figure 21-21 (*b*) *p-V* diagram of Joule cycle along the two adiabats and two isobars.

So, the total work done during the whole cycle $1 \to 2 \to 3 \to 4$, is given by

$$W = C_V[(T_3 - T_4) - (T_2 - T_1)] + p_2(V_3 - V_2) - p_1(V_4 - V_1) \qquad (21\text{-}50)$$

Substituting the ideal gas law $PV = nRT$ into Eq. 21-50 and rearranging, we find

$$W = (C_V + nR)[(T_3 - T_4) - (T_2 - T_1)]$$
$$= C_p[(T_3 - T_2) - (T_4 - T_1)] \qquad (21\text{-}51)$$

because, $$C_p = C_V + nR$$

In the process $2 \to 3$, the gas absorbs heat Q, hence

$$Q = C_p\,(T_3 - T_2)$$

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We now need to write W and Q in terms of p and form the quotient. Using the equation for an adiabatic process in an alternative form,

$$pV^{\gamma} = \text{constant}$$

$$Tp^{(1-\gamma)/\gamma} = \text{constant}$$

We have
$$\frac{T_1}{T_2} = \left(\frac{p_1}{p_2}\right)^{(\gamma-1)/\gamma} \tag{21-52}$$

and
$$\frac{T_4}{T_3} = \left(\frac{p_1}{p_2}\right)^{(\gamma-1)/\gamma} \tag{21-53}$$

Substituting for T_1 and T_4 from Eq. 21-52 and 21-53 into Eq. 21-50 gives

$$W = C_p\left[(T_3 - T_2) - (T_3 - T_2)\left(\frac{p_1}{p_2}\right)^{(\gamma-1)/\gamma}\right]$$

The efficiency ε, of the Joule cycle is therefore given by

$$\varepsilon = \frac{W}{Q} = 1 - \frac{T_4}{T_3} = 1 - \left(\frac{p_1}{p_2}\right)^{(\gamma-1)/\gamma}$$

The maximum possible efficiency, if the cycle would have been Carnot, would be, $1 - T_2/T_4$, as T_2 and T_4 are the lowest and highest temperatures in the cycle. It can be seen that efficiency is less than the Carnot efficiency as the value of T_4/T_3 would be more than T_2/T_4.

REVIEW AND SUMMARY

Work Associated with Volume Change A gas may exchange energy with its surroundings through work. The amount of work W done *by* a gas as it expands or contracts from an initial volume V_i to a final volume V_f is given by

$$W = \int dw = \int_{V_i}^{V_f} p\,dV. \tag{21-2}$$

The integration is necessary because the pressure p may vary during the volume change.

Work in an Isothermal Volume Change The work done *by* an ideal gas during an **isothermal** (constant-temperature) change from volume V_i to volume V_f is

$$W = nRT\ln\frac{V_f}{V_i} \quad \text{(ideal gas, isothermal process)}. \tag{21-7}$$

First Law of Thermodynamics The principle of conservation of energy for a thermodynamic process is expressed in the **first law of thermodynamics**, which may assume either of the forms

$$\Delta E_{\text{int}} = E_{\text{int},f} - E_{\text{int},i} = Q - W \quad \text{(first law)} \tag{21-11}$$

or
$$dE_{\text{int}} = dQ - dW \quad \text{(first law)}. \tag{21-12}$$

E_{int} represents the internal energy of the material, which depends only on the material's state (temperature, pressure, and volume). Q represents the energy exchanged as heat between the system and its surroundings; Q is positive if the system absorbs heat and negative if the system loses heat. W is the work done *by* the system; W is positive if the system expands against an external force from the surroundings and negative if the system contracts because of an external force. *Q and W are path dependent; ΔE_{int} is path independent.*

Special Cases of the First Law The first law of thermodynamics can be applied to the thermodynamic processes:

Isothermal process $Q = W, \quad \Delta E_{\text{int}} = 0$

Adiabatic processes: $Q = 0, \quad \Delta E_{\text{int}} = -W$

Constant-volume processes: $W = 0, \quad \Delta E_{\text{int}} = Q$

Cyclical processes: $\Delta E_{\text{int}} = 0, \quad Q = W$

Free expansions: $Q = W = \Delta E_{\text{int}} = 0$

Molar Specific Heats The molar specific heat C_V of a gas at constant volume is defined as

$$C_V = \frac{Q}{n\Delta T} = \frac{\Delta E_{\text{int}}}{n\Delta T}, \tag{21-18, 21-20}$$

in which Q is the energy transferred as heat to or from a sample of n moles of the gas, ΔT is the resulting temperature change of the gas, and ΔE_{int} is the resulting change in the internal energy of the gas. For an ideal monatomic gas,

$$C_V = \frac{3}{2}R = 12.5\ \text{J/mol}\cdot\text{K}. \tag{21-22}$$

The molar specific heat C_p of a gas at constant pressure is defined to be

$$C_p = \frac{Q}{n\Delta T}, \tag{21-25}$$

in which Q, n, and ΔT are defined as above. C_p is also given by

$$C_p = C_V + R. \tag{21-28}$$

For n moles of an ideal gas,

$$E_{\text{int}} = nC_VT \quad \text{(ideal gas)}. \tag{21-23}$$

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If n moles of a confined ideal gas undergo a temperature change ΔT due to *any* process, the change in the internal energy of the gas is

$$\Delta E_{int} = nC_V \Delta T \quad \text{(ideal gas, any process)}. \qquad (21\text{-}24)$$

Adiabatic Process When an ideal gas undergoes an adiabatic volume change (a change for which $Q = 0$),

$$pV^{\gamma} = \text{a constant} \quad \text{(adiabatic process)}, \qquad (21\text{-}31)$$

in which $\gamma (= C_p/C_V)$ is the ratio of molar specific heats for the gas. For a free expansion, however, pV = a constant.

Engines An **engine** is a device that, operating in a cycle, extracts energy as heat $|Q_H|$ from a high-temperature reservoir and does a certain amount of work $|W|$. The *efficiency* ε of any engine is defined as

$$\varepsilon = \frac{\text{energy we get}}{\text{energy we pay for}} = \frac{|W|}{|Q_H|}. \qquad (21\text{-}45)$$

In an **ideal engine**, all processes are reversible and no wasteful energy transfers occur due to, say, friction and turbulence. A **Carnot engine** is an ideal engine that follows the cycle of Fig. 21-16. Its efficiency is

$$\varepsilon_C = 1 - \frac{|Q_L|}{|Q_H|} = 1 - \frac{T_L}{T_H} \qquad (21\text{-}46, 21\text{-}47)$$

in which T_H and T_L are the temperatures of the high- and low-temperature reservoirs, respectively. Real engines always have an efficiency lower than that given by Eq. 21-47. Ideal engines that are not Carnot engines also have lower efficiencies.

A *perfect engine* is an imaginary engine in which energy extracted as heat from the high-temperature reservoir is converted completely to work. Such an engine would violate the second law of thermodynamics, which can be restated as follows: No series of processes is possible whose sole result is the absorption of energy as heat from a thermal reservoir and the complete conversion of this energy to work.

PROBLEMS

1. A lab sample of gas is taken through cycle *abca* shown in the *p-V* diagram of Fig. 21-22. The net work done is +1.5 J. Along path *ab*, the change in the internal energy is +3.0 J and the magnitude of the work done is 5.0 J. Along path *ca*, the energy transferred to the gas as heat is +2.5 J. How much energy is transferred as heat along (a) path *ab* and (b) path *bc*?

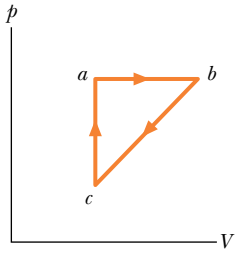


Figure 21-22 Problem 1.

2. When a system is taken from state i to state f along path *iaf* in Fig. 21-23, Q = 100 cal and W = 40 cal. Along path *ibf*, Q = 72 cal. (a) What is W along path *ibf*? (b) If W = –26 cal for the return path *fi*, what is Q for this path? (c) If $E_{int,i}$ = 20 cal, what is $E_{int,f}$? If $E_{int,b}$ = 44 cal, what is Q for (d) path *ib* and (e) path *bf*?

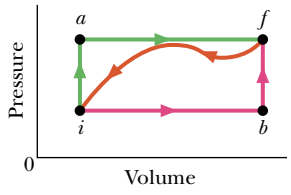
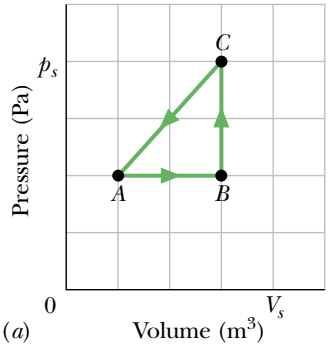


Figure 21-23 Problem 2.

3. A thermodynamic system is taken from state A to state B to state C, and then back to A, as shown in the *p-V* diagram of Fig. 21-24*a*. The vertical scale is set by p_s = 20 Pa, and the horizontal scale is set by V_s = 2.0 m³. (a)–(g) Complete the table in Fig. 21-24*b* by inserting a plus sign, a minus sign, or a zero in each indicated cell. (h) What is the net work done by the system as it moves once through the cycle *ABCA*?



	Q	W	ΔE_{int}
$A \rightarrow B$	(a)	(b)	+
$B \rightarrow C$	+	(c)	(d)
$C \rightarrow A$	(e)	(f)	(g)

Figure 21-24 Problem 3.

4. Suppose 200 J of work is done on a system and 80.0 cal is extracted from the system as heat. In the sense of the first law of thermodynamics, what are the values (including algebraic signs) of (a) W, (b) Q, and (c) ΔE_{int}?

5. In Fig. 21-25, a gas sample expands from V_0 to $4.0V_0$ while its pressure decreases from p_0 to $p_0/4.0$. If V_0 = 1.0 m³ and p_0 = 60 Pa, how much work is done by the gas if its pressure changes with volume via (a) path A, (b) path B, and (c) path C?

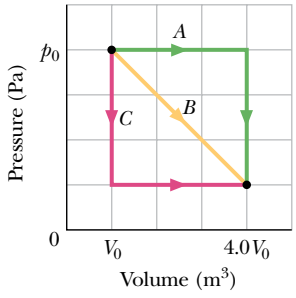


Figure 21-25 Problem 5.

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6. A gas within a closed chamber undergoes the cycle shown in the p-V diagram of Fig. 21-26. The horizontal scale is set by $V_s = 8.0\ m^3$. Calculate the net energy added to the system as heat during one complete cycle.

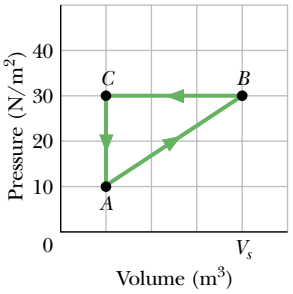

Figure 21-26 Problem 6.

7. As a gas is held within a closed chamber, it passes through the cycle shown in Fig. 21-27. Determine the energy transferred by the system as heat during constant-pressure process CA if the energy added as heat Q_{AB} during constant-volume process AB is 25.0 J, no energy is transferred as heat during adiabatic process BC, and the net work done during the cycle is 15.0 J.

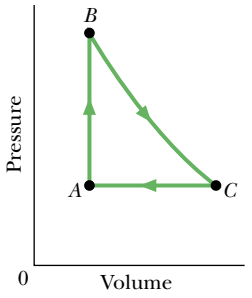

Figure 21-27 Problem 7.

8. Figure 21-28 represents a closed cycle for a gas (the figure is not drawn to scale). The change in the internal energy of the gas as it moves from a to c along the path abc is –50 J. As it moves from c to d, 45 J must be transferred to it as heat. An additional transfer of 20 J to it as heat is needed as it moves from d to a. How much work is done on the gas as it moves from c to d?

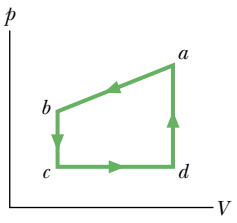

Figure 21-28 Problem 8.

9. Calculate the specific heat of a metal from the following data. A container made of the metal has a mass of 3.6 kg and contains 15 kg of water. A 1.8 kg piece of the metal initially at a temperature of 180°C is dropped into the water. The container and water initially have a temperature of 16.0°C, and the final temperature of the entire (insulated) system is 18.0°C.

10. When 0.40 mol of oxygen (O_2) gas is heated at constant pressure starting at 0°C, how much energy must be added to the gas as heat to triple its volume? (The molecules rotate but do not oscillate.)

11. A gas is to be expanded from initial state i to final state f along either path 1 or path 2 on a p-V diagram. Path 1 consists of three steps: an isothermal expansion (work is 23 J in magnitude), an adiabatic expansion (work is 35 J in magnitude), and another isothermal expansion (work is 16 J in magnitude). Path 2 consists of two steps: a pressure reduction at constant volume and an expansion at constant pressure. What is the change in the internal energy of the gas along path 2?

12. When 22.5 J was added as heat to a particular ideal gas, the volume of the gas changed from 50.0 cm^3 to 100 cm^3 while the pressure remained at 1.00 atm. (a) By how much did the internal energy of the gas change? If the quantity of gas present was 2.00×10^{-3} mol, find (b) C_p and (c) C_V.

13. Suppose 2.80 mol of an ideal diatomic gas, with molecular rotation but not oscillation, experienced a temperature increase of 45.0 K under constant-pressure conditions. What are (a) the energy transferred as heat Q, (b) the change ΔE_{int} in internal energy of the gas, (c) the work W done by the gas, and (d) the change ΔK in the total translational kinetic energy of the gas?

14. The temperature of 4.50 mol of an ideal monatomic gas is raised 27.0 K at constant volume. What are (a) the work W done by the gas, (b) the energy transferred as heat Q, (c) the change ΔE_{int} in the internal energy of the gas, and (d) the change ΔK in the average kinetic energy per atom?

15. *Opening champagne.* In a bottle of champagne, the pocket of gas (primarily carbon dioxide) between the liquid and the cork is at pressure of $p_i = 4.00$ atm. When the cork is pulled from the bottle, the gas undergoes an adiabatic expansion until its pressure matches the ambient air pressure of 1.00 atm. Assume that the ratio of the molar specific heats is $\gamma = 4/3$. If the gas has initial temperature $T_i = 5.00$°C, what is its temperature at the end of the adiabatic expansion?

16. One mole of an ideal diatomic gas goes from a to c along the diagonal path in Fig. 21-29. The scale of the vertical axis is set by $p_{ab} = 5.0$ kPa and $p_c = 0.50$ kPa, and the scale of the horizontal axis is set by $V_{bc} = 4.0\ m^3$ and $V_a = 2.0\ m^3$. During the transition, (a) what is the change in internal energy of the gas, and (b) how much energy is added to the gas as heat? (c) How much heat is required if the gas goes from a to c along the indirect path abc?

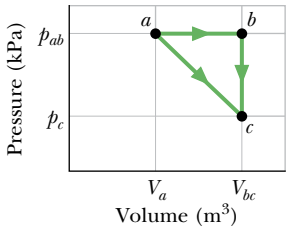

Figure 21-29 Problem 16.

17. *Adiabatic wind.* The normal airflow over the Rocky Mountains is west to east. The air loses much of its moisture content and is chilled as it climbs the western side of the mountains. When it descends on the eastern side, the increase in pressure toward lower altitudes causes the temperature to increase. The flow, then called a chinook wind, can rapidly raise the air temperature at the base of the mountains. Assume that the air pressure p depends on altitude y according to $p = p_0 \exp(-ay)$, where $p_0 = 1.00$ atm and $a = 1.16 \times 10^{-4}$/m. Also assume that the ratio of the molar specific heats is $\gamma = 4/3$. A parcel of air with an initial temperature of –13.0°C descends adiabatically from $y_1 = 4267$ m to $y = 1567$ m. What is its temperature at the end of the descent?

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18. We give 90 J as heat to a diatomic gas, which then expands at constant pressure. The gas molecules rotate but do not oscillate. By how much does the internal energy of the gas increase?

19. Suppose 0.825 mol of an ideal gas undergoes an isothermal expansion as energy is added to it as heat Q. If Fig. 21-30 shows the final volume V_f versus Q, what is the gas temperature? The scale of the vertical axis is set by $V_{fs} = 0.30\ m^3$, and the scale of the horizontal axis is set by $Q_s = 1800$ J.

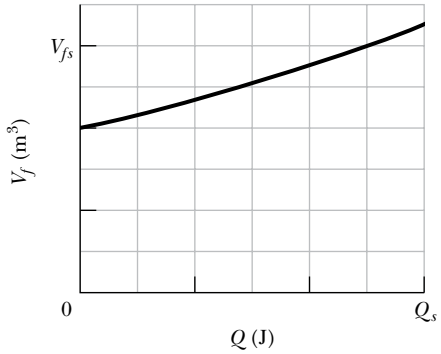

Figure 21-30 Problem 19.

20. Figure 21-31 shows a cycle undergone by 2.00 mol of an ideal monatomic gas. The temperatures are $T_1 = 300$ K, $T_2 = 600$ K, and $T_3 = 455$ K. For $1 \rightarrow 2$, what are (a) heat Q, (b) the change in internal energy ΔE_{int}, and (c) the work done W? For $2 \rightarrow 3$, what are (d) Q, (e) ΔE_{int}, and (f) W? For $3 \rightarrow 1$, what are (g) Q, (h) ΔE_{int}, and (i) W? For the full cycle, what are (j) Q, (k) ΔE_{int}, and (l) W? The initial pressure at point 1 is 1.00 atm ($= 1.013 \times 10^5$ Pa). What are the (m) volume and (n) pressure at point 2 and the (o) volume and (p) pressure at point 3?

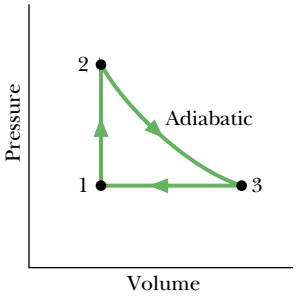

Figure 21-31 Problem 20.

21. Figure 21-32 shows two paths that may be taken by a gas from an initial point i to a final point f. Path 1 consists of an isothermal expansion (work is 60 J in magnitude), an adiabatic expansion (work is 45 J in magnitude), an isothermal compression (work is 30 J in magnitude), and then an adiabatic compression (work is 25 J in magnitude). What is the change in the internal energy of the gas if the gas goes from point i to point f along path 2?

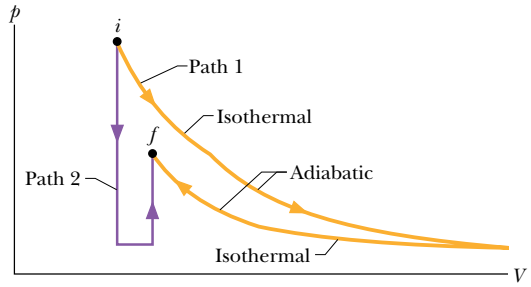

Figure 21-32 Problem 21.

22. An ideal diatomic gas, with rotation but no oscillation, undergoes an adiabatic compression. Its initial pressure and volume are 1.20 atm and 0.200 m³. Its final pressure is 3.60 atm. How much work is done by the gas?

23. The volume of an ideal gas is adiabatically reduced from 350 L to 130 L. The initial pressure and temperature are 2.00 atm and 380 K. The final pressure is 8.00 atm. (a) Is the gas monatomic, diatomic, or polyatomic? (b) What is the final temperature? (c) How many moles are in the gas?

24. Under constant pressure, the temperature of 3.00 mol of an ideal monatomic gas is raised 15.0 K. What are (a) the work W done by the gas, (b) the energy transferred as heat Q, (c) the change ΔE_{int} in the internal energy of the gas, and (d) the change ΔK in the average kinetic energy per atom?

25. An automobile tire has a volume of 1.64×10^{-2} m³ and contains air at a gauge pressure (pressure above atmospheric pressure) of 165 kPa when the temperature is 0.00°C. What is the gauge pressure of the air in the tires when its temperature rises to 37.0°C and its volume increases to 1.67×10^{-2} m³? Assume atmospheric pressure is 1.01×10^5 Pa.

26. Suppose 1.00 L of a gas with $\gamma = 1.30$, initially at 285 K and 1.00 atm, is suddenly compressed adiabatically to half its initial volume. Find its final (a) pressure and (b) temperature. (c) If the gas is then cooled to 273 K at constant pressure, what is its final volume?

27. A certain gas occupies a volume of 0.76 L at a pressure of 1.2 atm and a temperature of 4000 K. It is expanded adiabatically to a volume of 4.3 L. Determine (a) the final pressure and (b) the final temperature, assuming the gas to be an ideal gas for which $\gamma = 1.4$.

28. The mass of a gas molecule can be computed from its specific heat at constant volume c_V. (Note that this is not C_V.) Take $c_V = 0.1476$ cal/g·°C for neon and calculate (a) the mass of a neon atom and (b) the molar mass of neon.

29. Air that initially occupies 0.280 m³ at a gauge pressure of 103.0 kPa is expanded isothermally to a pressure of 101.3 kPa and then cooled at constant pressure until it reaches its initial volume. It then returns to its initial pressure in a constant-volume process. Compute the net work done by the air. (Gauge pressure is the difference between the actual pressure and atmospheric pressure.)

30. A sample of an ideal gas is taken through the cyclic process $abca$ shown in Fig. 21-33. The scale of the vertical axis is set by $p_b = 7.5$ kPa and $p_{ac} = 2.5$ kPa. At point a, $T = 280$ K. (a) How many moles of gas are in the sample? What are (b) the temperature of the gas at point b, (c) the temperature of the gas at point c, and (d) the net work done by the gas during the cycle, and (e) the net energy added to the gas as heat during the cycle?

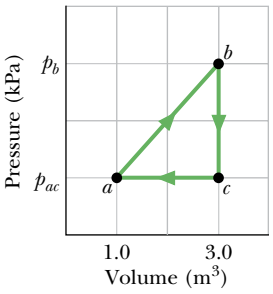

Figure 21-33 Problem 30.

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31. Suppose 1.80 mol of an ideal gas is taken from a volume of 3.00 m^3 to a volume of 5.50 m^3 via an isothermal expansion at 30°C. (a) How much energy is transferred as heat during the process, and (b) is the transfer *to* or *from* the gas?

32. A Carnot engine has an efficiency of 15.0%. It operates between constant-temperature reservoirs differing in temperature by 55.0°C What is the temperature of the lower-temperature reservoir?

33. Figure 21-34 shows a reversible cycle through which 1.00 mol of a monatomic ideal gas is taken. Assume that $p = 2p_0$, $V = 2V_0$, $p_0 = 1.01 \times 10^5$ Pa, and $V_0 = 0.0335$ m^3. Calculate (a) the work done during the cycle, (b) the energy added as heat during stroke *abc*, and (c) the efficiency of the cycle. (d) What is the efficiency of a Carnot engine operating between the highest and lowest temperatures that occur in the cycle?

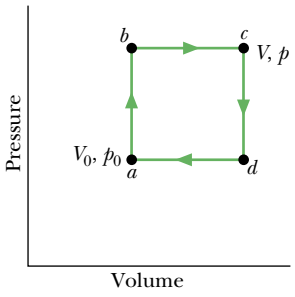

Figure 21-34 Problem 33.

34. A Carnot engine whose high-temperature reservoir is at 483 K has an efficiency of 40%. By how much should the temperature of the low-temperature reservoir be decreased to increase the efficiency to 50%?

35. In a hypothetical nuclear fusion reactor, the fuel is deuterium gas at a temperature of 7×10^8 K. If this gas could be used to operate a Carnot engine with $T_L = 50$°C, what would be the engine's efficiency? Take both temperatures to be exact and report your answer to eight significant figures.

36. An ideal gas (3.0 mol) is the working substance in an engine that operates on the cycle shown in Fig. 21-35. Processes *BC* and *DA* are reversible and adiabatic. (a) Is the gas monatomic, diatomic, or polyatomic? (b) What is the engine efficiency?

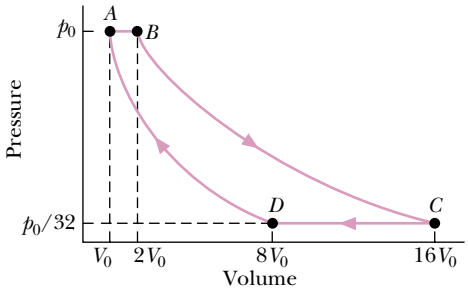

Figure 21-35 Problem 36.

37. (a) During each cycle, a Carnot engine absorbs 730 J as heat from a high-temperature reservoir at 360 K, with the low-temperature reservoir at 280 K. How much work is done per cycle? (b) The engine is then made to work in reverse to function as a Carnot refrigerator between those same two reservoirs. During each cycle, how much work is required to remove 1200 J as heat from the low-temperature reservoir?

38. (a) For 2.5 mol of a monatomic ideal gas taken through the cycle in Fig. 21-36, where $V_1 = 4.00V_0$, what is W/p_0V_0 as the gas goes from state *a* to state *c* along path *abc*? What is $\Delta E_{int}/p_0V_0$ in going (b) from *b* to *c* and (c) through one full cycle?

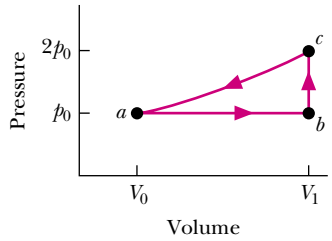

Figure 21-36 Problem 38.

39. A 600 W Carnot engine operates between constant-temperature reservoirs at 100°C and 60.0°C. What is the rate at which energy is (a) taken in by the engine as heat and (b) exhausted by the engine as heat?

40. A Carnot engine is set up to produce a certain work *W* per cycle. In each cycle, energy in the form of heat Q_H is transferred to the working substance of the engine from the higher-temperature thermal reservoir, which is at an adjustable temperature T_H. The lower-temperature thermal reservoir is maintained at temperature $T_L = 250$ K. Figure 21-37 gives Q_H for a range of T_H. The scale of the vertical axis is set by $Q_{Hs} = 12.0$ kJ. If T_H is set at 550 K, what is Q_H?

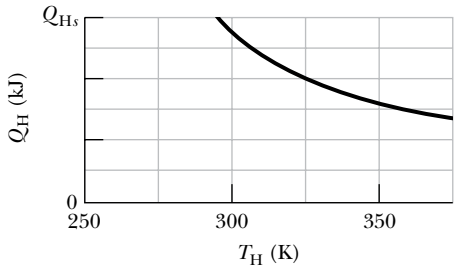

Figure 21-37 Problem 40.

41. Figure 21-38 shows a reversible cycle through which 1.00 mol of a monatomic ideal gas is taken. Volume $V_C = 8.00V_b$. Process *bc* is an adiabatic expansion, with $p_b = 5.00$ atm and $V_b = 1.00 \times 10^{-3}$ m^3. For the cycle, find (a) the energy added to the gas as heat, (b) the energy leaving the gas as heat, (c) the net work done by the gas, and (d) the efficiency of the cycle.

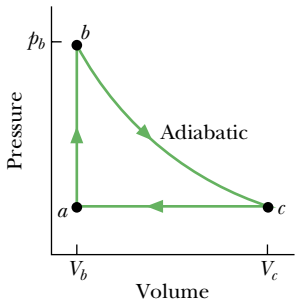

Figure 21-38 Problem 41.

42. A Carnot engine operates between 235°C and 115°C, absorbing 3.00×10^4 J per cycle at the higher temperature. (a) How much energy per cycle is discharged at the lower temperature? (b) How much work per cycle is this engine capable of performing?

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PRACTICE QUESTIONS

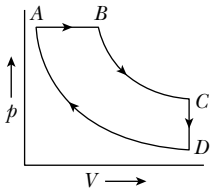
Single Correct Choice Type

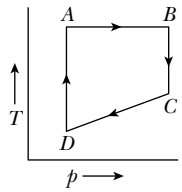
1. Which statement is incorrect?
- **(a)** All reversible cycles have same efficiency.
- **(b)** Reversible cycle has more efficiency than an irreversible one.
- **(c)** Carnot cycle is a reversible one.
- **(d)** Carnot cycle has the maximum efficiency in all cycles.

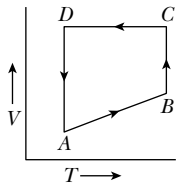
2. A mixture of n_1 moles of a monoatomic gas and n_2 moles of diatomic gas has $C_p/C_V = \gamma = 1.5$.
- **(a)** $n_1 = n_2$
- **(b)** $2n_1 = n_2$
- **(c)** $n_1 = 2n_2$
- **(d)** $2n_1 = 3n_2$

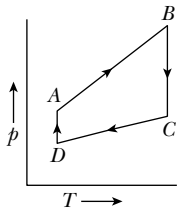
3. The molar heat capacity of an ideal gas (C)
- **(a)** cannot be negative.
- **(b)** must lie in the range starting from C_V and terminating at C_p, that is, $C_V \le C \le C_p$.
- **(c)** must equal either C_p or C_V.
- **(d)** may have any value lying between $-\infty$ and ∞.

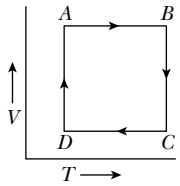
4. A cyclic process $ABCD$ is shown in the p–V diagram (see figure). Which of the following curves represent the same process?



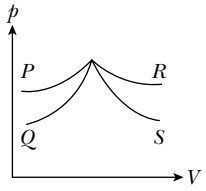
(a)


(b)


(c)


(d)


5. The given figure shows four p–V diagrams for the given sample of gas. In which case, no exchange of heat occurs with the sample?

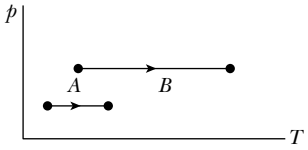


- **(a)** P
- **(b)** Q
- **(c)** R
- **(d)** S

6. A gas is contained in a metallic cylinder fitted with a piston. The piston is suddenly moved in to compress the gas and is maintained at this position. As time passes the pressure of the gas in the cylinder
- **(a)** increases.
- **(b)** decreases.
- **(c)** remains constant.
- **(d)** increases or decreases depending on the nature of the gas.

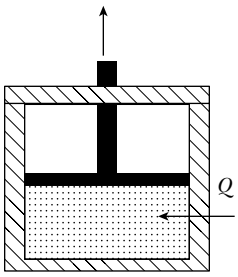
7. How much work has to be done in decreasing the volume of an ideal gas by an amount of 2.4×10^{-4} m^3 at normal temperature and constant normal pressure of 1×10^5 N/m^2?
- **(a)** 24 J
- **(b)** 25 J
- **(c)** 27 J
- **(d)** 28 J

8. Consider two processes on a system as shown in the figure.



The volumes in the initial states are the same in the two processes and the volumes in the final states are also the same. Let ΔW_1 and ΔW_2 be the work done by the system in the processes A and B, respectively.
- **(a)** $\Delta W_1 > \Delta W_2$
- **(b)** $\Delta W_1 = \Delta W_2$
- **(c)** $\Delta W_1 < \Delta W_2$
- **(d)** Nothing can be said about the relation between ΔW_1 and ΔW_2

9. When the gas enclosed beneath the piston shown in the figure receives 1930 J of heat, Q, from its surroundings, it performs 2250 J of work in raising the piston. What is the change in the internal energy of the gas?



- **(a)** –320 J
- **(b)** +320 J
- **(c)** –4180 J
- **(d)** +4180 J

10. A gas is compressed at a constant pressure of 50 N/m^2 from a volume of 10 m^3 to a volume of 4 m^3. Energy of 100 J is then added to the gas by heating. Its internal energy is
- **(a)** increased by 400 J
- **(b)** increased by 200 J
- **(c)** increased by 100 J
- **(d)** decreased by 200 J

11. Two samples A and B are initially kept in the same state. The sample A is expanded through an adiabatic process and the sample B through an isothermal process. The final

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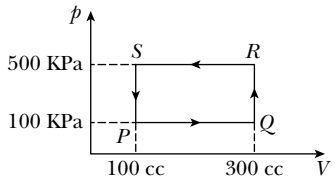
volume of the samples are the same. The final pressure in A and B are p_A and p_B, respectively.

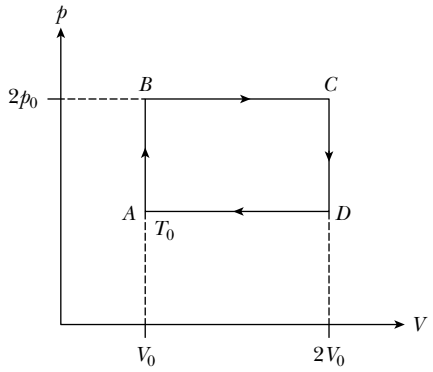
(a) $p_A > p_B$
(b) $p_A = p_B$
(c) $p_A < p_B$
(d) The relation between p_A and p_B cannot be deduced

12. A thermodynamic system is taken through the cyclic $PQRSP$ process (see figure). The net work done by the system is



(a) 20 J **(b)** –80 J
(c) 400 J **(d)** –374 J

13. N moles of a monoatomic gas are carried round the reversible rectangular cycle $ABCDA$ as shown in the figure. The temperature at A is T_0. The thermodynamic efficiency of the cycle is



(a) 15% **(b)** 50%
(c) 20% **(d)** 25%

14. In a given process for an ideal gas, $dW = 0$ and $dQ < 0$. Then, for the gas

(a) the temperature will decrease.
(b) the volume will increase.
(c) the pressure will remain constant.
(d) the temperature will increase.

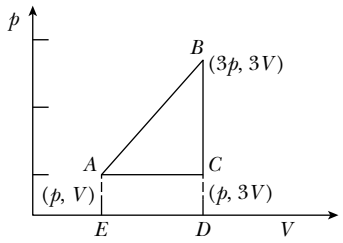
15. A match ignites within in an oxygen-filled cylinder that has a movable piston. The piston is moved so quickly that no heat escapes. What kind of change is demonstrated in this process?

(a) An isobaric change
(b) An adiabatic change
(c) An isochoric change
(d) A change of heat capacity

16. Ideal gas undergoes an adiabatic change in its state from (p_1, V_1, T_1) to (p_2, V_2, T_2). The work done (W) in the process is (n = number of moles, C_p and C_V are molar specific heats of gas)

(a) $W = n(T_1 - T_2)C_p$ **(b)** $W = n(T_1 - T_2)C_V$
(c) $W = n(T_1 + T_2)C_p$ **(d)** $W = n(T_1 + T_2)C_V$

17. An ideal gas is taken around $ABCA$ as shown in the p–V diagram (below). The work done during a cycle is



(a) $2pV$ **(b)** pV
(c) $1/2pV$ **(d)** zero

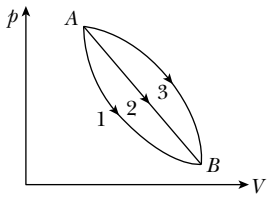
18. Work done by 0.1 mole of a gas at 27°C to double its volume at constant pressure is (use $R = 2$ cal/mol/°C)

(a) 54 cal **(b)** 600 cal
(c) 60 cal **(d)** 546 cal

19. A thermally isolated sample of an ideal gas at a fixed temperature is confined to one half of a container by an impermeable membrane. The other half of the container is evacuated. The membrane is then pierced and the gas is allowed to expand freely and to double its volume as shown. Which one of the following statements is true concerning this situation?

(a) The process is reversible.
(b) This is an isothermal process.
(c) The entropy of the gas decreases.
(d) The internal energy of the gas must decrease.

20. An ideal gas of mass m in a state A goes to another state B via three different processes as shown in figure. If Q_1, Q_2 and Q_3 denote the heat absorbed by the gas along the three paths, then



(a) $Q_1 < Q_2 < Q_3$
(b) $Q_1 < Q_2 = Q_3$
(c) $Q_1 = Q_2 > Q_3$
(d) $Q_1 > Q_2 > Q_3$

21. Starting with the same initial conditions, an ideal gas expands from volume V_1 to V_2 in three different ways: The work done by the gas is W_1 if the process is purely isothermal, W_2 if the process is purely isobaric, and W_3 if the process is purely adiabatic, then

(a) $W_1 > W_2 > W_3$
(b) $W_2 > W_3 > W_1$
(c) $W_3 > W_2 > W_1$
(d) $W_2 > W_1 > W_3$

22. Variation of internal energy with density of one mole of monatomic gas is depicted in figure. The corresponding variation of pressure with volume can be depicted as (Assume the curve is rectangular hyperbola.)

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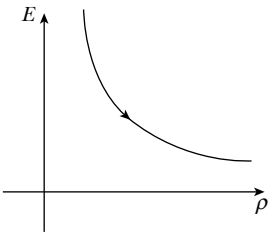
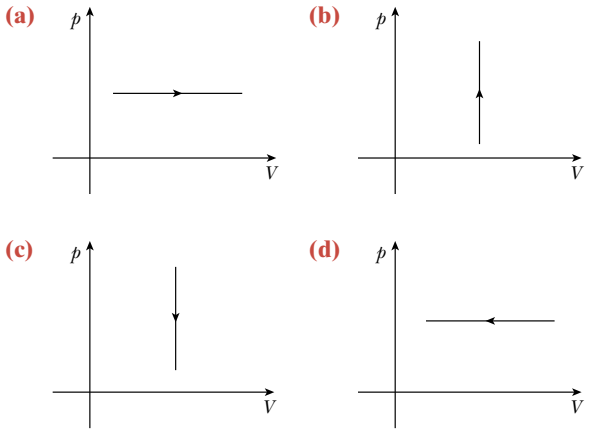




23. We have a jar A filled with gas characterized by parameters p, V, and T and another jar B filled with a gas with parameters $2p$, $V/4$, and $2T$, where the symbols have the usual meanings. The ratio of the number of molecules of jar A to those of jar B is

(a) 1:1 **(b)** 1:2
(c) 2:1 **(d)** 4:1

24. A Carnot engine operates between hot and cold reservoirs with temperatures 527°C and –73.0°C, respectively. If the engine performs 1000.0 J of work per cycle, how much heat is extracted per cycle from the hot reservoir?

(a) 878 J **(b)** 1333 J
(c) 1163 J **(d)** 1527 J

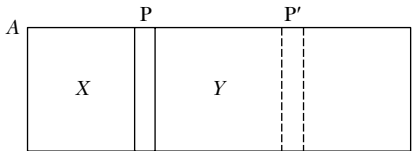
25. If in adiabatic change, the pressure p and temperature T of a monoatomic gas are related by the relation $p \propto T^c$, where c is equal to

(a) 5/2 **(b)** 5/3
(c) 2/5 **(d)** 3/5

26. Neon is a monatomic gas with a molar heat capacity at constant volume of 12.66 J/(mol K). Two moles of neon gas enclosed in a constant volume system receive 5250 J of heat. If the gas was initially at 293 K, what is the final temperature of the neon?

(a) 200 K **(b)** 300 K
(c) 400 K **(d)** 500 K

27. A cylindrical chamber A of uniform cross-section is divided into two parts X and Y by a movable piston P which can slide without friction inside the chamber (see figure). Initially part X contains 1 mol of a monoatomic gas and Y contains 2 mol of a diatomic gas, and the volumes of X and Y are in the ratio 1:2 with both parts X and Y being at the same temperature T. Assuming the gases to be ideal, the work W that will be done in moving the piston slowly to the position where the ratio of the volumes of X and Y is 2:1 will be



(a) $-5.8T$ **(b)** $8.3T$
(c) $12.3T$ **(d)** zero

28. Determine the quantity of heat added to 3.5 moles of the ideal gas argon if the temperature increases from 75°C to 225°C during an isobaric process.
(The molar specific heats of argon are $C_V = 3.0$ cal/K mol and $C_p = 5.0$ cal/K mol.)

(a) 2600 cal
(b) 1600 cal
(c) 2100 cal
(d) 1100 cal

29. 175 calories of heat is required to raise the temperature of 5 mol of an ideal gas at constant pressure from 20°C to 25°C. The amount of heat required to raise the temperature of same gas from 20°C to 25°C at constant volume will be

(a) 100 cal **(b)** 125 cal
(c) 150 cal **(d)** 175 cal

30. If one complete cycle of a reversible process is carried out on a sample of an ideal gas so that its final state is the same as its initial state, which one of the following quantities is the only one which can be non-zero?

(a) The change in volume of the sample
(b) The net heat absorbed by the sample
(c) The change in the entropy of the sample
(d) The change in temperature of the sample

31. What is the maximum possible efficiency of an engine operating between the boiling and freezing points of water at sea level?

(a) 0.27 **(b)** 0.61
(c) 0.49 **(d)** 0.85

32. Two cylinders A and B fitted with pistons contain equal amounts of an ideal diatomic gas at 300 K. The piston of A is free to move while that of B is held fixed. The same amount of heat is given to the gas in each cylinder. If the rise in temperature of the gas in A is 30 K, then the rise in temperature of the gas in B is

(a) 30 K **(b)** 18 K
(c) 50 K **(d)** 42 K

33. An ideal monatomic gas undergoes an adiabatic process; and its internal energy increases by 50 J. Which pair of choices below is correct for this process?

	Work done	**Heat exchanged**
(a)	50 J by the system	zero joules
(b)	50 J on the system	zero joules
(c)	50 J by the system	100 J supplied
(d)	zero joules	50 J removed

34. An engine is used to lift a 2700 kg truck to a height of 3.0 m at constant speed. In the lifting process, the engine

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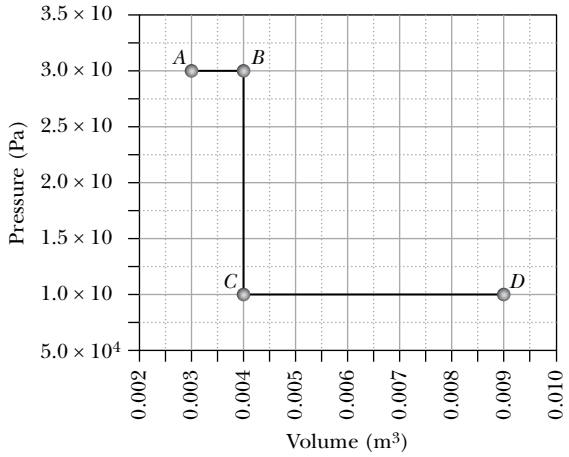
received 3.3×10^5 J of heat from the fuel burned in its interior. What is the efficiency of the engine?

(a) 0.19 **(b)** 0.29
(c) 0.24 **(d)** 0.34

35. The slope of isothermal and adiabatic curves are related as

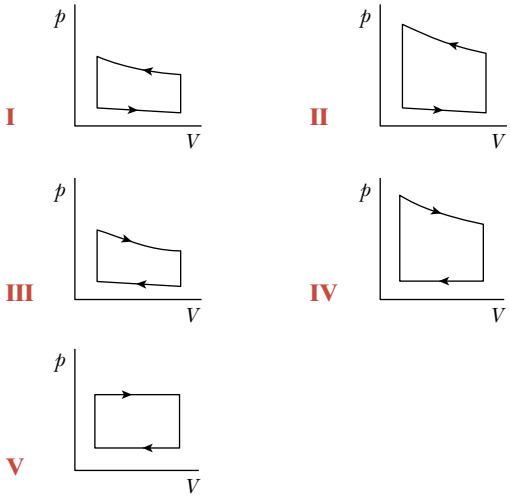
(a) isothermal curve slope = adiabatic curve slope
(b) isothermal curve slope = $\gamma\times$ adiabatic slope
(c) adiabatic curve slope = $\gamma\times$ isothermal slope
(d) adiabatic curve slope = (1/2) $\times$ isothermal curve slope

36. Two moles of a confined ideal monatomic gas begin at state A in the pressure-volume graph and follow the path shown to state D. If the temperature of the gas at A is 54 K, what is the temperature of the gas at D?



(a) 32 K **(b)** 46 K
(c) 54 K **(d)** 60 K

37. Pressure vs. volume graphs for a certain gas undergoing five different cyclic processes are shown below. During which cycle does the gas do the greatest positive work?



(a) I **(b)** II
(c) III **(d)** IV

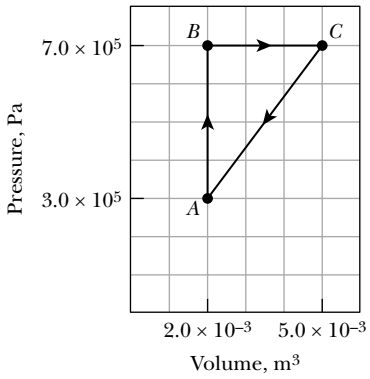
38. In an adiabatic process:

(a) the energy absorbed as heat equals the work done by the system on its environment.
(b) the energy absorbed as heat equals the work done by the environment on the system.
(c) the energy absorbed as heat equals the change in internal energy.
(d) the work done by the environment on the system equals the change in internal energy.

39. A jogger's internal energy changes because he performs 6.4×10^5 J of work and gives off 4.9×10^5 J of heat. However, to cause the same change in his internal energy while walking, he must do 8.2×10^5 J of work. Determine the magnitude of the heat given off while walking.

(a) 3.6×10^5 J **(b)** 4.0×10^5 J
(c) 2.4×10^5 J **(d)** 3.1×10^5 J

40. The pressure and volume of a gas are changed along the path $ABCA$. Using the data shown in the graph, determine the work done (including the algebraic sign) in each segment of the path B to C.



(a) -1.5×10^3 J **(b)** $+1.5 \times 10^3$ J
(c) -2.1×10^3 J **(d)** $+2.1 \times 10^3$ J

41. The temperature of a monatomic ideal gas remains constant during a process in which 4700 J of heat flows out of the gas. How much work (including the proper + or – sign) is done?

(a) zero J **(b)** +9400 J
(c) +4700 J **(d)** –4700 J

42. Heat is added isothermally to 2.5 mol of a monatomic ideal gas. The temperature of the gas is 430 K. How much heat must be added to make the volume of the gas double?

(a) 5500 J **(b)** 3100 J
(c) 6700 J **(d)** 2700 J

43. One mole of a monatomic ideal gas has an initial pressure, volume, and temperature of p_0, V_0, and 438 K, respectively. It undergoes an isothermal expansion that triples the volume of the gas. Then, the gas undergoes an isobaric compression back to its original volume. Finally, the gas undergoes an isochoric increase in pressure, so that the final pressure, volume, and temperature are p_0, V_0, and 438 K, respectively. Find the total heat for this three-step process, and state whether it is absorbed by or given off by the gas.

(a) 3640 J, absorbed **(b)** –1820 J, given off
(c) 1570 J, absorbed **(d)** –1110 J, given off

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44. Suppose a monatomic ideal gas is contained within a vertical cylinder that is fitted with a movable piston. The piston is frictionless and has a negligible mass. The area of the piston is 3.14×10^{-2} m^2, and the pressure outside the cylinder is 1.01×10^5 Pa. Heat (2093 J) is removed from the gas. Through what distance does the piston drop?
(a) 0.185 m **(b)** 0.389 m
(c) 0.539 m **(d)** 0.264 m

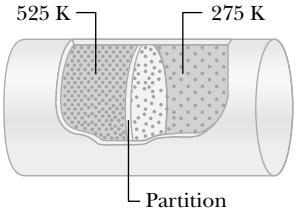
45. The efficiency of an automobile engine increases by 5.0%. For an input heat of 1300 J, how much more work does the engine produce?
(a) 26 J **(b)** 65 J
(c) 41 J **(d)** 88 J

46. Engine 1 has an efficiency of 0.18 and requires 5500 J of input heat to perform a certain amount of work. Engine 2 has an efficiency of 0.26 and performs the same amount of work. How much input heat does the second engine require?
(a) 7900 J **(b)** 3800 J
(c) 6700 J **(d)** 4400 J

47. A Carnot engine has an efficiency of 0.40. The Kelvin temperature of its hot reservoir is quadrupled, and the Kelvin temperature of its cold reservoir is doubled. What is the efficiency that results from these changes?
(a) 0.30 **(b)** 0.60
(c) 0.50 **(d)** 0.70

48. The hot reservoir for a Carnot engine has a temperature of 890 K, while the cold reservoir has a temperature of 670 K. The heat input for this engine is 4800 J. The 670 K reservoir also serves as the hot reservoir for a second Carnot engine. This second engine uses the rejected heat of the first engine as input and extracts additional work from it. The rejected heat from the second engine goes into a reservoir that has a temperature of 420 K. Find the total work delivered by the two engines.
(a) 3500 J **(b)** 2500 J
(c) 3000 J **(d)** 2000 J

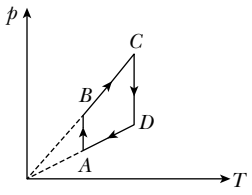
49. The drawing shows an adiabatically isolated cylinder that is divided initially into two identical parts by an adiabatic partition. Both sides contain one mole of a monatomic ideal gas ($\gamma = 5/3$), with the initial temperature being 525 K on the left and 275 K on the right. The partition is then allowed to move slowly (i.e., quasi-statically) to the right, until the pressures on each side of the partition are the same. Find the final temperatures on the left.

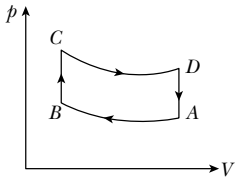


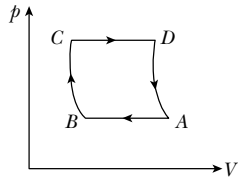
(a) 206 K **(b)** 524 K
(c) 254 K **(d)** 477 K

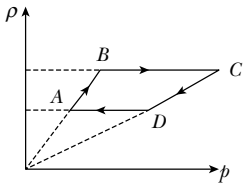
More than One Correct Choice Type

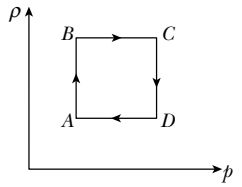
50. Pressure temperature (p–T) graph of n moles of an ideal gas is shown in the figure. Among the following four graphs, which are the corresponding pressure volume (p–V) and density pressure (ρ–p) graphs?



(a)


(b)


(c)


(d)


51. A gas may expand either adiabatically or isothermally. A number of p–V curves are drawn for the two processes over different ranges of pressure and volume. It will be found that
(a) two adiabatic curves do not intersect
(b) two isothermal curves do not intersect
(c) an adiabatic curve and an isothermal curve may intersect
(d) the magnitude of the slope of an adiabatic curve is greater than the magnitude of the slope of an isothermal curve for the same values of pressure and volume

52. Which of the following is true in case of isothermal changes?
(a) $\Delta T = 0$
(b) $\Delta U_{\text{int}} = 0$
(c) $\Delta Q = 0$
(d) $\Delta U_{\text{int}} + \Delta W = 0$

53. For an ideal gas,
(a) the change in internal energy in a constant pressure process from temperature T_1 to T_2 is equal to $nC_V(T_2 - T_1)$, where C_V is the molar specific heat at constant volume and n, the number of moles of the gas.
(b) the change in internal energy of the gas and the work done by the gas are equal in magnitude in an adiabatic process.
(c) no heat is added or removed in an adiabatic process.
(d) the internal energy does not change in an isothermal process.

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54. A system undergoes a cyclic process in which it absorbs Q_1 heat and gives out Q_2 heat. The efficiency of the process is η and the work out is W.

(a) $W = Q_1 - Q_2$ **(b)** $\eta = W/Q_1$

(c) $\eta = Q_2/Q_1$ **(d)** $\eta = 1 - (Q_2/Q_1)$

55. A steel drill making 180 rmp is used to drill a hole in a block of steel. The mass of the steel block and the drill is 180 g of each. The entire mechanical work is used up in producing heat such that the rate of rise of temperature of block is 0.5°C/s. P is the power rating of the drill and τ is the couple required to drive the drill. (Given specific heat of steel $C_{steel} = 0.10$ cal/g/°C.

(a) $P = 37.8$ W **(b)** $P = 9$ W

(c) $\tau = 2$ N·m **(d)** $\tau = 6.3$ N·m

56. An ideal gas is heated from temperature T_1 to T_2 under various conditions. The correct statement(s) are

(a) $\Delta E = nC_V(T_2 - T_1)$ for isobaric, isochoric and adiabatic process.

(b) Work is done at expense of internal energy in an adiabatic process and both have equal.

(c) $\Delta E_{int} = 0$ for an isothermal process.

(d) $C_p = 0$ for an adiabatic process.

57. The internal energy of a system remains constant when it undergoes

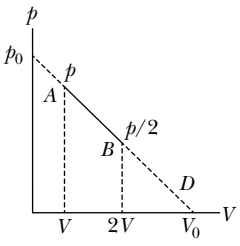
(a) a cyclic process.

(b) an isothermal process.

(c) an adiabatic process.

(d) any process in which the heat given out by the system is equal to a work done on the system.

58. An ideal gas is taken from the state A (pressure p, volume V) to the state B (pressure $p/2$ volume $2V_0$ along a straight-line path in the p–V diagram shown below. Select the correct statement(s) from the following:



(a) The work done by the gas in that would be done by it if the system were taken from A to B along the isotherm.

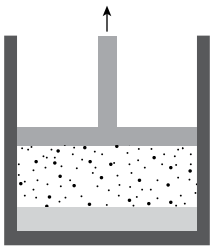
(b) In the T-V diagram, the path AB becomes a part of parabola.

(c) In the p-T diagram, the path AB becomes a part of hyperbola.

(d) In going from A to B, the temperature T of the gas first increases to a maximum value and then decreases.

Linked Comprehension

Paragraph for Questions 59–61: 5.00 kg of liquid water is heated to 100.0°C in a closed system. At this temperature, the density of liquid water is 958 kg/m^3. The pressure is maintained at atmospheric pressure of 1.01×10^5 Pa. A moveable piston of negligible weight rests on the surface of the water. The water is then converted to steam by adding an additional amount of heat to the system. When all of the water is converted, the final volume of the steam is 8.50 m^3. The latent heat of vaporization of water is 2.26×10^6 J/kg.



59. How much work is done by this closed system during this isothermal process?

(a) 8.37 J **(b)** 4.20×10^3 J

(c) 1.21×10^4 J **(d)** 8.58×10^5 J

60. How much heat is added to the system in the isothermal process of converting all of the water into steam?

(a) 2.17×10^3 J **(b)** 1.13×10^7 J

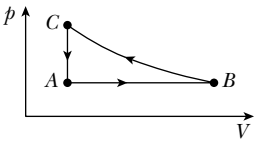
(c) 1.70×10^4 J **(d)** 3.78 × 106 J

61. What is the change in the internal energy during this isothermal process?

(a) zero joules **(b)** 1.04×10^7 J

(c) 1.28×10^4 J **(d)** 2.93×10^6 J

Paragraph for Questions 62–64: An ideal monatomic gas expands isobarically from state A to state B. It is then compressed isothermally from state B to state C and finally cooled at constant volume until it returns to its initial state A.



$V_A = 4 \times 10^{-3}$ m^3 $V_B = 8 \times 10^{-3}$ m^3

$P_A = P_B = 1 \times 10^6$ Pa $T_A = 600$ K

62. What is the temperature of the gas when it is in state B?

(a) 437 K **(b)** 927 K

(c) 573 K **(d)** 1200 K

63. How much work is done by the gas in expanding isobarically from A to B?

(a) 1×10^3 J **(b)** 3×10^3 J

(c) 2×10^3 J **(d)** 4×10^3 J

64. How much work is done on the gas in going from B to C?

(a) 2.5×10^6 J

(b) 4.5×10^6 J

(c) 5.5×10^6 J

(d) 6.5×10^6 J

Paragraph for Questions 65–67: Two moles of an ideal gas have an initial Kelvin temperature T_i and absolute pressure p_i. The gas undergoes a reversible isothermal compression from an initial volume V_i to a final volume $0.5V_i$.

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65. Which one of the following expressions represents the final pressure of the gas?

(a) $0.5p_i$ **(b)** $2p_i$
(c) $4p_i$ **(d)** $p_i/31.7$

66. How much heat is exchanged with the environment? If heat is exchanged, is it absorbed or released?

(a) p_iV_i (ln 2), released
(b) p_iV_i (ln 2), absorbed
(c) $(0.5)p_iV_i$, released
(d) $(0.5)p_iV_i$, absorbed

67. The ratio of the molar specific heat capacity at constant pressure to that at constant volume, γ, for diatomic hydrogen gas is 7/5. In an adiabatic compression, the gas, originally at atmospheric pressure, is compressed from an original volume of 0.30 m^3 to 0.15 m^3. What is the final pressure of the gas?

(a) 2.0×10^5 Pa
(b) 2.7×10^5 Pa
(c) 3.0×10^5 Pa
(d) 3.7×10^5 Pa

Paragraph for questions 68–71: Carnot engine is an ideal heat engine, which converts heat energy into mechanical energy. Efficiency of Carnot engine is given by $\eta = 1 - (T_2/T_1)$, where T_1 is temperature of source and T_1 is temperature of sink. If Q_1 is the amount of heat absorbed/cycle from the source; Q_2 is the amount of heat rejected/cycle to the sink and W is the amount of useful work done/cycle, then

$$W = Q_1 - Q_2 \quad \text{and} \quad \frac{Q_2}{Q_1} = \frac{T_2}{T_1}.$$

68. A Carnot engine absorbs 6×10^5 cal. at 227°C. Heat rejected to the sink at 127°C is

(a) 6×10^5 cal
(b) 4.8×10^5 cal
(c) 4×10^5 cal
(d) 5×10^5 cal

69. Work done/cycle by the engine in the above question is

(a) 5.04×10^5 J
(b) 5.04×10^5 cal
(c) 4.8×10^5 cal
(d) 4.8×10^5 J

70. Efficiency of the engine in the above question is

(a) 30% **(b)** 10%
(c) 20% **(d)** 80%

71. At what temperature should the sink be maintained to increase the efficiency by 10%?

(a) 77 K **(b)** 300 K
(c) 300°C **(d)** 77°C

Matrix-Match

72. There is ideal gas sample. The ratio of C_p and C_V for gas sample is γ. In its initial state its pressure is p_1 and volume is V_1. Now it is expanded isothermally from volume V_1 to V_2. Then, it is compressed adiabatically from volume V_2 to V.

Column I	Column II
(a) Heat given to system (that is, ideal gas sample) during isothermal expansion	**(p)** Positive
(b) Work done by gas during adiabatic compression	**(q)** $\frac{p_1V_1}{(\gamma-1)}\left[\left(\frac{V_2}{V_1}\right)^{\gamma-1}-1\right]$
(c) Change in internal energy of gas sample during	**(r)** $\frac{p_1V_1}{(1-\gamma)}\left[\left(\frac{V_2}{V_1}\right)^{\gamma-1}-1\right]$
(d) Change in internal energy of gas sample from most initial state to the final state	**(s)** Negative

Directions for Question 76: In the question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

73. First law of thermodynamics

Column I	Column II	Column III
(I) $R = 8.31$ J/mol·K	**(i)** $Q = 5150$ J	**(J)** $W = 7300$ J
(II) $R = 8.11$ J/mol·K	**(ii)** $Q = 4900$ J	**(K)** $W = 7150$ J
(III) $R = 8.23$ J/mol·K	**(iii)** $Q = 5000$ J	**(L)** $W = 7\,400$ J
(IV) $R = 8.29$ J/mol·K	**(iv)** $Q = 5100$ J	**(M)** $W = 7500$ J

(1) Which combination has –100 K as the change in temperature?

(a) (I) (iii) (L) **(b)** (IV) (i) (M)
(c) (II) (iv) (K) **(d)** (I) (iii) (M)

(2) Which combination has –98.6 K as the change in temperature?

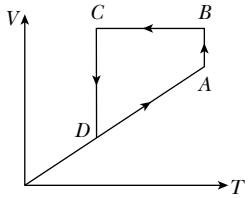
(a) (III) (ii) (L) **(b)** (II) (ii) (J)
(c) (II) (iii) (K) **(d)** (I) (i) (M)

(3) Which combination has –93.15 K as the change in temperature?

(a) (III) (iv) (L) **(b)** (I) (i) (K)
(c) (II) (iii) (L) **(d)** (I) (ii) (L)

Integer Type

74. Some gas ($C_p/C_V = 1.25$) follows the cycle $ABCDA$ shown in figure. Determine the ratio of the energy given by the gas to its surroundings during the isochoric section of the cycle to the expansion work done during the isobaric section of the circle.



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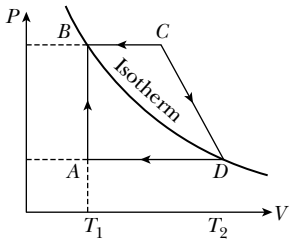
75. An electric heater supplies heat to a system at a rate of 100 W. If the system does work at a rate of 75 J per second, at what rate is the internal energy increasing?

76. An engine has an efficiency of 64% and produces 5500 J of work. Determine (a) the input heat.

77. Four moles of an ideal gas undergo the cyclic process $ABCDA$ as shown in figure. If $3T_C = 4T_B = 12T_A = 2400\text{K}$, determine the work done by the gas during the entire process.



78. The work done by one mole of a monatomic ideal gas ($\gamma = 5/3$) in expanding adiabatically is 825 J. The initial temperature and volume of the gas are 393 K and 0.100 m^3. Obtain the final temperature of the gas.

ANSWER KEY

Checkpoints

1. c and e (maximum area enclosed by a clockwise cycle)
2. a, b, d, e
3. (a) all tie (E_{int} depends on initial and final state and not the path); 4, 3, 2, 1 (compare all area under the curves); (c) 4, 3, 2, 1
4. (a) zero (closed cycle); (b) negative (W net is negative)
5. In paths $1 \rightarrow 2$ and $3 \rightarrow 1$, heat is supplied to the gas
6. 5 (greatest change in T), then tie of 1, 2, 3, and 4
7. 1, 2, 3 ($Q_3 = 0$ Q_2 goes into work W_2, but Q_1 goes into greater work W_1 and increases gas temperature)
8. c, b, a

Problems

1. (a) +8.0 J; (b) −9.0 J
2. (a) 12 cal; (b) −86 cal; (c) 80 cal; (d) 36 cal; (e) 36 cal
3. (a) $Q > 0$; (b) $W > 0$; (c) $W = 0$; (d) $\Delta E_{int} > 0$; (e) $Q < 0$; (f) $W < 0$; (g) $\Delta E_{int} < 0$; (h) −5.0 J
4. (a) −200 J; (b) −335 J; (c) −135 J
5. (a) 1.8×10^2 J; (b) 105 J ≈ 1.1×10^2 J.; (c) 45 J
6. −60 J, which means that 60 J of energy lost in the form of heat
7. −10.0 J; which means10.0 J of energy leaves the gas in the form of heat
8. 15 J
9. 0.44 kJ/kg K
10. 6.4×10^3 J
11. −20 J
12. (a) 17.5 J; (b) 37.0 J mol·K; (c) 28.7 J/mol·K
13. (a) 6.98×10^3 J.; (b) 4.99×10^3 J.; (c) 1.99×10^3 J; (d) 2.99×10^3 J
14. (a) $W = 0$; (b) 1.51×10^3 J; (c) $+1.51 \times 10^3$ J; (d) 5.59×10^{-22} J
15. −87°C
16. (a) -2.0×10^4 J.; (b) 5.5×10^3 J; (c) -14.5×10^3 0 J ≈ −15 kJ
17. 17°C
18. 64 J
19. 540 K
20. (a) 3.74×10^3 J; (b) 3.74×10^3 J; (c) word done is $W = 0$; (d) heat added is zero; (e) -1.81×10^3 J; (f) 1.81×10^3 J; (g) -3.22×10^3 J.; (h) -1.93×10^3; (i) -1.29×10^3; (j) 520 J.; (k) zero; (l) 520 J.; (m) 2.46×10^{-2} m^3; (n) 2.02×10^5 Pa ≈ 2.00 atm; (o); 3.73×10^{-2} m^3; (p) 1.013×10^5 Pa (1.00 atm)
21. − 20 J
22. 2.23×10^4 J
23. (a) gas is diatomic; (b) 565 K; (c) 22.4 mol
24. (a) 374 J; (b) +935; (c) +561 J; (d) $+3.11 \times 10^{-22}$ J.
25. 196 kPa
26. (a) 2.46 atm; (b) 376 K; (c) 0.406
27. (a) 13.6 atm ≈ 14 atm; (b) 6.2×10^2 K.
28. (a) 3.35×10^{-23} g = 3.35×10^{-26} kg; (b) 20.2 g/mol
29. 1.12×10^4 J
30. (a) 1.1 mol; (b) 2.5×10^3 K; (c) 8.4×10^2 K; (d) 5.0 kJ; (e) 5.0 kJ
31. (a) 2.75×10^3 J; (b) to
32. 1.312 K
33. (a) 3.38 kJ; (b) 22.0 kJ; (c) 15.4%; (d) 75.0%
34. 48 K
35. 99.999954%
36. (a) monoatomic; (b) 75%
37. (a) 162 J; (b) 343 J
38. (a) 3.00; (b) 6.0; (c) $\Delta E_{int} = 0$

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39. (a) 5.61 kJ s; (b) 5.01 kJ/s
40. 3.38 kJ ≈ 3.4 kJ
41. (a) 734 J; (b) 277 J; (c) 457 J; (d^3) 62.3%
42. (a) 2.29×10^4 J; (b) 7.09×10^3 J

Practice Questions

Single Correct Choice Type

1. (a)	2. (a)	3. (a)	4. (b)	5. (d)
6. (b)	7. (a)	8. (c)	9. (a)	10. (a)
11. (c)	12. (b)	13. (a)	14. (a)	15. (b)
16. (b)	17. (a)	18. (c)	19. (b)	20. (a)
21. (d)	22. (d)	23. (d)	24. (b)	25. (a)
26. (d)	27. (a)	28. (a)	29. (b)	30. (b)
31. (a)	32. (d)	33. (b)	34. (c)	35. (c)
36. (c)	37. (d)	38. (d)	39. (d)	40. (d)
41. (d)	42. (c)	43. (c)	44. (d)	45. (b)
46. (b)	47. (d)	48. (b)	49. (d)	

More than One Correct Choice Type

50. (a), (c)	51. (a), (b), (c)	52. (a), (b)	53. (a), (b), (c), (d)	54. (a), (b), (d)
55. (a), (c)	56. (a), (b), (c), (d)	57. (a), (b), (d)	58. (a), (b), (d)	

Linked Comprehension

59. (d)	60. (b)	61. (b)	62. (d)	63. (d)
64. (c)	65. (b)	66. (a)	67. (b)	68. (b)
69. (a)	70. (c)	71. (d)		

Matrix-Match

72. (a) → (p); (b) → (r), (s); (c) → (p), (q); (d) → (p), (q)
73. (1) → (d) (2) → (b) (3) → (a)

Integer Type

74. 4	75. 25	76. 8600	77. 800	78. 327

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APPENDIX

A

The International System of Units (SI)*

Table 1 The SI Base Units**

Quantity	Name	Symbol	Definition
length	meter	m	"… the length of the path traveled by light in vacuum in 1/299,792,458 of a second." (1983)
mass	kilogram	kg	"… this prototype [a certain platinum-iridium cylinder] shall henceforth be considered to be the unit of mass." (1889)
time	second	s	"… the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom." (1967)
electric current	ampere	A	"… that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross section, and placed 1 meter apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} newton per meter of length." (1946)
thermodynamic temperature	kelvin	K	"… the fraction 1/273.16 of the thermodynamic temperature of the triple point of water." (1967)
amount of substance	mole	mol	"… the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon-12." (1971)
luminous intensity	candela	cd	"… the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} hertz and that has a radiant intensity in that direction of 1/683 watt per steradian." (1979)

*Adapted from "The International System of Units (SI)," National Bureau of Standards Special Publication 330, 1972 edition.The definitions above were adopted by the General Conference of Weights and Measures, an international body, on the dates shown. In this book we do not use the candela.

**In November 2018, the 26th General Conference on Weights and Measures has adopted a resolution to redefine four of the seven base units—the kilogram, kelvin, mole and ampere.

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Table 2 Some SI Derived Units

Quantity	Name of Unit	Symbol	
area	square meter	m^2	
volume	cubic meter	m^3	
frequency	hertz	Hz	s^{-1}
mass density (density)	kilogram per cubic meter	kg/m^3	
speed, velocity	meter per second	m/s	
angular velocity	radian per second	rad/s	
acceleration	meter per second per second	m/s^2	
angular acceleration	radian per second per second	rad/s^2	
force	newton	N	$kg \cdot m/s^2$
pressure	pascal	Pa	N/m^2
work, energy, quantity of heat	joule	J	$N \cdot m$
power	watt	W	J/s
quantity of electric charge	coulomb	C	$A \cdot s$
potential difference, electromotive force	volt	V	W/A
electric field strength	volt per meter (or newton per coulomb)	V/m	N/C
electric resistance	ohm	Ω	V/A
capacitance	farad	F	$A \cdot s/V$
magnetic flux	weber	Wb	$V \cdot s$
inductance	henry	H	$V \cdot s/A$
magnetic flux density	tesla	T	Wb/m^2
magnetic field strength	ampere per meter	A/m	
entropy	joule per kelvin	J/K	
specific heat	joule per kilogram kelvin	$J/(kg \cdot K)$	
thermal conductivity	watt per meter kelvin	$W/(m \cdot K)$	
radiant intensity	watt per steradian	W/sr	

Table 3 The SI Supplementary Units

Quantity	Name of Unit	Symbol
plane angle	radian	rad
solid angle	steradian	sr

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APPENDIX

B

Some Fundamental Constants of Physics*

Constant	Symbol	Computational Value	Best (1998) Value	
			Value[a]	Uncertainty[b]
Speed of light in a vacuum	c	3.00×10^{8} m/s	2.997 924 58	exact
Elementary charge	e	1.60×10^{-19} C	1.602 176 487	0.025
Gravitational constant	G	6.67×10^{-11} m^3/s$^2 \cdot$kg	6.674 28	100
Universal gas constant	R	8.31 J/mol$\cdot$K	8.314 472	1.7
Avogadro constant	N_A	6.02×10^{23} mol^{-1}	6.022 141 79	0.050
Boltzmann constant	k	1.38×10^{-23} J/K	1.380 650 4	1.7
Stefan–Boltzmann constant	σ	5.67×10^{-8} W/m$^2 \cdot$K^4	5.670 400	7.0
Molar volume of ideal gas at STP[d]	V_m	2.27×10^{-2} m^3/mol	2.271 098 1	1.7
Permittivity constant	ε_0	8.85×10^{-12} F/m	8.854 187 817 62	exact
Permeability constant	μ_0	1.26×10^{-6} H/m	1.256 637 061 43	exact
Planck constant	h	6.63×10^{-34} J$\cdot$s	6.626 068 96	0.050
Electron mass[c]	m_e	9.11×10^{-31} kg	9.109 382 15	0.050
		5.49×10^{-4} u	5.485 799 094 3	4.2×10^{-4}
Proton mass[c]	m_p	1.67×10^{-27} kg	1.672 621 637	0.050
		1.0073 u	1.007 276 466 77	1.0×10^{-4}
Ratio of proton mass to electron mass	m_p/m_e	1840	1836.152 672 47	4.3×10^{-4}
Electron charge-to-mass ratio	e/m_e	1.76×10^{11} C/kg	1.758 820 150	0.025
Neutron mass[c]	m_n	1.68×10^{-27} kg	1.674 927 211	0.050
		1.0087 u	1.008 664 915 97	4.3×10^{-4}
Hydrogen atom mass[c]	$m_{^1\mathrm{H}}$	1.0078 u	1.007 825 031 6	0.0005
Deuterium atom mass[c]	$m_{^2\mathrm{H}}$	2.0136 u	2.013 553 212 724	3.9×10^{-5}
Helium atom mass[c]	$m_{^4\mathrm{He}}$	4.0026 u	4.002 603 2	0.067
Muon mass	m_μ	1.88×10^{-28} kg	1.883 531 30	0.056

(Contd.)

*The values in this table were selected from the 1998 CODATA recommended values (www.physics.nist.gov).

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Constant	Symbol	Computational Value	Best (1998) Value	
			Value[a]	Uncertainty[b]
Electron magnetic moment	μ_e	9.28×10^{-24} J/T	9.284 763 77	0.025
Proton magnetic moment	μ_p	1.41×10^{-26} J/T	1.410 606 662	0.026
Bohr magneton	μ_B	9.27×10^{-24} J/T	9.274 009 15	0.025
Nuclear magneton	μ_N	5.05×10^{-27} J/T	5.050 783 24	0.025
Bohr radius	a	5.29×10^{-11} m	5.291 772 085 9	6.8×10^{-4}
Rydberg constant	R	1.10×10^{7} m^{-1}	1.097 373 156 852 7	6.6×10^{-6}
Electron Compton wavelength	λ_C	2.43×10^{-12} m	2.426 310 217 5	0.0014

[a]Values given in this column should be given the same unit and power of 10 as the computational value.
[b]Parts per million.
[c]Masses given in u are in unified atomic mass units, where 1 u = $1.660\ 538\ 782 \times 10^{-27}$ kg.
[d]STP means standard temperature and pressure: 0°C and 1.0 atm (0.1 MPa).